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Three-dimensional CR submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

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Abstract
It is known that there exist only four six-dimensional homogeneous non-Kähler, nearly Kähler manifolds: the sphere $\mathbb{S}^6$, the complex projective space $\mathbb{C}P^3$, the flag manifold $\mathbb{F}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$. So far, most of the results about submanifolds have been obtained when the ambient space is the nearly Kähler $\mathbb{S}^6$. Recently, the investigation of almost complex and Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ has been initiated. Here we start the investigation of three-dimensional CR submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$. The tangent space of three-dimensional CR submanifold can be naturally split into two distributions $\mathscr{D}_1$ and $\mathscr{D}_1^{\perp}$. In this paper, we found conditions that three-dimensional CR submanifolds with integrable almost complex distribution $\mathscr{D}_1$ should satisfy, and we give some constructions which allow us to define a wide-range family of examples of this type of submanifolds. Our main result is classification of the three-dimensional CR submanifolds with totally geodesics both, almost complex distribution $\mathscr{D}_1$ and totally real distribution $\mathscr{D}_1^{\perp}$.

Keywords Totally geodesic distribution · CR submanifold · Nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ · Almost product structure

Mathematics Subject Classification 53B20 · 53B21 · 53B25 · 53C42

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1 Introduction

An almost Hermitian manifold $(\widetilde{M}, \mathbf{g}, \mathbf{J})$, with Levi-Civita connection $\widetilde{\nabla}$, is called a nearly Kähler manifold if for any tangent vector X it holds $(\widetilde{\nabla}_X \mathbf{J})X = 0$. If, moreover, $\widetilde{\nabla}\mathbf{J}$ is a vanishing tensor, $\widetilde{M}$ is said to be a Kähler manifold. It is known that there exist only four six-dimensional homogeneous nearly Kähler manifolds, that are not Kähler: the sphere $\mathbb{S}^6$, the complex projective space $\mathbb{C}P^3$, the flag manifold $\mathbb{F}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$, see [8]. One should also remark that the first examples of complete non-homogeneous Kähler manifolds were recently discovered by Foscolo and Haskins in [16].

It is natural to investigate for a submanifold M of an almost Hermitian manifold $(\widetilde{M}, \mathbf{g}, \mathbf{J})$ its relation with respect to the structure $\mathbf{J}$. If $\mathbf{J}T_pM = T_pM$ for any $p \in M$, M is called an almost complex submanifold and if $\mathbf{J}T_pM \subset T_pM^{\perp}$, for each $p \in M$, M is a totally real submanifold. Here, we denote by $T_pM^{\perp}$ the normal space of the submanifold at a point p. One of the natural generalisations of these two notions is the notion of a CR submanifold as introduced by Bejancu in [3].

In general, a submanifold M of $(\widetilde{M}, \mathbf{g}, \mathbf{J})$ is called a *CR submanifold* if there exists a C^{∞}-differential $\mathbf{J}$ invariant distribution $\mathscr{D}_1$ on M (i.e. $\mathbf{J}\mathscr{D}_1 = \mathscr{D}_1$), such that its orthogonal complement $\mathscr{D}_1^{\perp}$ in TM is totally real ($\mathbf{J}\mathscr{D}_1^{\perp} \subseteq T^{\perp}M$), where $T^{\perp}M$ is the normal bundle over M. We say that M is proper if it is neither almost complex, nor totally real. Note that in the specific case of a three-dimensional submanifold M of a six-dimensional (nearly) Kähler manifold, we have that M is a proper CR submanifold if and only if $\mathbf{J}T_pM \cap T_pM$ is a two-dimensional distribution. Note that a three-dimensional CR submanifold is automatically of maximal CR dimension, see [20].

In the past years, special types of submanifolds have been mostly investigated in the case of the nearly Kähler $\mathbb{S}^6$. Here we mention for example [6,7,9,10,15,17,18,23]. Recently, the investigation of the geometry of almost complex and three-dimensional totally real submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ has been initiated; we refer the reader to [4,5,11,12,19,21,24].

We investigate here three-dimensional CR submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$, and we are interested in the properties of the distribution $\mathscr{D}_1 = \mathbf{J}T_pM \cap T_pM$ and its complement. We investigate in particular when the distribution $\mathscr{D}_1$ is integrable or totally geodesic. We also classify the three-dimensional CR submanifolds for which the second fundamental form restricted to both $\mathscr{D}_1$ and $\mathscr{D}_1^{\perp}$ vanishes. Note that one has immediately from the fundamental equations that $\mathrm{h}(\mathscr{D}_1, \mathscr{D}_1^{\perp})$ cannot vanish identically. Similar problems for CR submanifolds of $\mathbb{S}^6$ and of the Sasakian $\mathbb{S}^7$ were, respectively, treated in [1,14]. Further interesting results were recently obtained on CR manifolds, see, for instance, [13].

2 The nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$

Let $\mathbb{S}^3$ be a unit sphere in the space $\mathbb{R}^4$ which we identify with the space of quaternions $\mathbb{H}$. Therefore, by using the isomorphism of the spaces $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$ we can represent an arbitrary tangent vector at a point $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$ by $Z = (p\alpha, q\beta)$, where α and β are imaginary quaternions. The almost complex structure on $\mathbb{S}^3 \times \mathbb{S}^3$ is given by, see, for example, [5,8]: $\mathbf{J}Z_{(p,q)} = \frac{1}{\sqrt{3}}(p(2\beta - \alpha), q(-2\alpha + \beta))$. Since the almost complex structure is not an isometry with respect to the standard product metric of $\mathbb{S}^3 \times \mathbb{S}^3$, inherited from the space $\mathbb{R}^8$, which we also denote by $\langle\cdot, \cdot\rangle$, we define a compatible metric $\mathbf{g}$ by $\mathbf{g}(Z, Z') = \frac{1}{2}(\langle Z, Z'\rangle + \langle \mathbf{J}Z, \mathbf{J}Z'\rangle)$. Let $\mathbf{G}$ denote the $(0, 2)$-type tensor $\mathbf{G}(X, Y) := (\widetilde{\nabla}_X \mathbf{J})Y$, where $\widetilde{\nabla}$

is the Levi-Civita connection of the metric $\mathbf{g}$. Then a straightforward calculation shows that $\mathbf{G}$ is skew-symmetric, which makes $(\mathbb{S}^3 \times \mathbb{S}^3, \mathbf{g}, \mathrm{J})$ a nearly Kähler manifold. For the basic formulas, we refer to [5,11,12]. We simply remark that in this case, as introduced in [5], see also [23], the following almost product structure P plays an important role: $\mathbf{P}(p\alpha, q\beta) = (p\beta, q\alpha)$. It is in particular compatible with the metric and it anticommutes with J.

Finally, for $X = (p\alpha, q\beta)$, $Y = (p\gamma, q\delta) \in T_{(p,q)}\mathbb{S}^3 \times \mathbb{S}^3$ it follows that

$$\begin{aligned}\mathbf{G}(X, Y) =& \frac{2}{3\sqrt{3}}(p(\beta \times \gamma + \alpha \times \delta + \alpha \times \gamma - 2\beta \times \delta),\\ & q(-\alpha \times \delta - \beta \times \gamma + 2\alpha \times \gamma - \beta \times \delta)).\end{aligned} \tag{1}$$

In [11], it was shown that the relation between the Euclidean connection ∇^E of $\mathbb{S}^3 \times \mathbb{S}^3$ and $\widetilde{\nabla}$ is given by

$$\nabla^E_X Y = \widetilde{\nabla}_X Y + \frac{1}{2}(\mathbf{JG}(X, \mathbf{P}Y) + \mathbf{JG}(Y, \mathbf{P}X)). \tag{2}$$

Also, we note the following. Since the connection D in the space $\mathbb{R}^8$ satisfies $\mathrm{D}_{E_i} f = \mathrm{d}f(E_i) = (p\alpha_i, q\beta_i)$, we have that

$$\nabla^E_{E_j} E_i = (p(\alpha_j \times \alpha_i + E_j(\alpha_i)), q(\beta_j \times \beta_i + E_j(\beta_i))). \tag{3}$$

3 Three-dimensional CR submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$

3.1 Some constructions

In order to show that the class of proper three-dimensional CR submanifolds is a large class, we first give some constructions which allow us to define a wide-range family of examples. The first construction of a family of three-dimensional CR submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$ starts with an almost complex surface. It is an immediate corollary of the fact that the maps $\mathscr{F}_{abc}(p, q) = (ap\bar{c}, bq\bar{c})$ where a, b, c are unitary quaternions are isometries preserving the almost complex structure J.

Proposition 1 *Let $a(t)$, $b(t)$, $c(t)$ be curves in $\mathbb{S}^3$ and $g : U \subset \mathbb{R}^2 \to \mathbb{S}^3 \times \mathbb{S}^3 : (x, y) \mapsto (p(x, y), q(x, y))$ be an almost complex surface of $\mathbb{S}^3 \times \mathbb{S}^3$. Then, providing that the mapping $f(x, y, t) = (ap\bar{c}, bq\bar{c})$ is an immersion, it is a CR immersion, for which the almost complex distribution $\mathscr{D}_1$ is integrable.*

Example 1 If we start from the almost complex totally geodesic immersions introduced in [5] by

$$(p, q)(s, t) = (\cos s + i \sin s, \cos t + i \sin t). \tag{4}$$

and

$$(p, q)(x) = \frac{1}{2}\left(1 - \sqrt{3}x, 1 + \sqrt{3}x\right), x \in \mathbb{S}^2 \subset Im\mathbb{H}. \tag{5}$$

we obtain the following CR immersions:

$$\big(p(x_1, x_2, t), q(x_1, x_2, t)\big) = (a(x_3)(\cos x_1 + i \sin x_1)\overline{c}(x_3), b(x_3)(\cos x_2 + i \sin x_2)\overline{c}(x_3)), \tag{6}$$

$$\big(p(x,t), q(x,t)\big) = \left(a(x_3)\frac{1-\sqrt{3}x}{2}\bar{c}(x_3), b(x_3)\frac{1+\sqrt{3}x}{2}\bar{c}(x_3)\right), x \in \mathbb{S}^2 \subset Im\mathbb{H}, \tag{7}$$

where a, b, c are curves depending on x_3 in $\mathbb{S}^3$. Here, the distribution $\mathscr{D}_1$ is totally geodesic and satisfies $\mathbf{P}\mathscr{D}_1 = \mathscr{D}_1$ and $\mathbf{P}\mathscr{D}_1 \perp \mathscr{D}_1$, respectively.

Note that for a distribution $\mathscr{D}$ on M we say that M is *$\mathscr{D}$-totally geodesic* if and only if the second fundamental form restricted to vector fields belonging to $\mathscr{D}$ vanishes identically.

Proposition 2 *Let M be a three-dimensional, $\mathscr{D}_1$-geodesic, CR submanifold of $\mathbb{S}^3 \times \mathbb{S}^3$. Then M is locally congruent to one of the immersions* (6) *and* (7).

Proof Denote by ∇^{D_1} the orthogonal projection of the connection ∇ to the distribution $\mathscr{D}_1$ and denote by E_3 the unit vector field spanning the totally real distribution. Denote by $E_4 = \mathbf{J}E_3$, the vector field orthogonal to M. Then for the vector fields $X, Y \in \mathscr{D}_1$ we can write

$$\widetilde{\nabla}_X Y = \nabla_X^{D_1} Y + \mathbf{g}(\widetilde{\nabla}_X Y, E_3)E_3 + \mathbf{h}(X, Y),$$

and since the ambient manifold is nearly Kähler we have that $\widetilde{\nabla}_X(\mathbf{J}X) = \mathbf{J}(\widetilde{\nabla}_X X)$. Taking $\mathbf{h}_{\mathscr{D}_1} = 0$, this equality reduces to $\nabla_X^{D_1}(\mathbf{J}X) = \mathbf{J}(\nabla_X^{D_1} X)$ and $\mathbf{g}(\widetilde{\nabla}_X X, E_3) = \mathbf{g}(\widetilde{\nabla}_X(\mathbf{J}X), E_3) = 0$. Since, for a nonzero vector field $X \in \mathscr{D}_1$, X and $\mathbf{J}X$ span $\mathscr{D}_1$, we obtain that $\mathscr{D}_1$ is integrable with totally geodesic leaves in $\mathbb{S}^3 \times \mathbb{S}^3$. Therefore, each of the leaves is locally congruent either to (4) or to (5).

Note also that for $X \in \mathscr{D}_1$, the angle $\theta = \angle(\mathbf{P}X, \mathscr{D}_1)$ is independent of the choice of X and is a differentiable function and therefore a continuous function. Since (4) and (5), respectively, have the tangent spaces invariant for $\mathbf{P}$ or orthogonal to its image under $\mathbf{P}$, the function θ is also discrete and therefore a constant. Hence, all the leaves of one immersion are mutually congruent. More precisely, they are congruent to either one of (4) or (5) which we denote by (p, q).

We can take the local coordinates x_1, x_2, x_3 of the submanifold M such that x_1, x_2 span $\mathscr{D}$. Then, for an arbitrary point x along the coordinate curve for x_3 there exist unit quaternions a, b, c, depending on x_3, such that $\mathscr{F}_{a,b,c}$ maps (p, q) into the corresponding leaf through x. The functions a, b, c are clearly differentiable. Moreover, we can then write the immersion as $(ap\bar{c}, bq\bar{c})$. This concludes the proof. □

Proposition 3 *Let $\mu(t), \nu(t)$ be mappings into unit quaternions $\mathbb{S}^3$ and let*

$$f(x_1, x_2, t) = (p(x_1, x_2, t), q(x_1, x_2, t))$$

be a three-dimensional CR immersion with integrable almost complex distribution $\mathscr{D}_1$, parameterised by x_1, x_2. Then $(\mu(t)p, \nu(t)q)$, provided that it is an immersion, is also a CR immersion of the same type.

Proof For a three-dimensional CR submanifold, it is sufficient to check that M admits a two-dimensional invariant distribution. As for fixed t, $(\mu(t)p, \nu(t)q)$ is congruent by an isometry $\mathscr{F}_{\mu(t)\nu(t)1}$ to the almost complex surface $(p(x_1, x_2, t), q(x_1, x_2, t))$ this is immediate as the isometry $\mathscr{F}_{\mu(t)\nu(t)1}$ preserves the complex structure. □

3.2 The suitable moving frame for three-dimensional CR submanifolds

Now we will construct a moving frame along a three-dimensional proper CR submanifold M suitable for computing. We have that the almost complex distribution $\mathscr{D}_1$ is two-dimensional, while the totally real distribution $\mathscr{D}_1^\perp$ is of dimension one. We can take unit vector fields E_1 and $E_2 = \mathbf{J}E_1$ that span $\mathscr{D}_1$, and E_3 that spans $\mathscr{D}_1^\perp$. We consider the nearly Kähler metric $\mathbf{g}$ throughout the paper, if it is not explicitly stated otherwise. We have that $E_4 = \mathbf{J}E_3$ is a unit normal vector field. If we then put $E_5 = \sqrt{3}\mathbf{G}(E_1, E_3)$ and $E_6 = \sqrt{3}\mathbf{G}(E_2, E_3) = -\mathbf{J}E_5$, we obtain an orthonormal moving frame. Moreover, we obtain the following equalities

$$
\begin{aligned}
&\mathbf{G}(E_1, E_2) = 0, && \mathbf{G}(E_1, E_3) = \frac{1}{\sqrt{3}}E_5, && \mathbf{G}(E_1, E_4) = \frac{1}{\sqrt{3}}E_6,\\
&\mathbf{G}(E_1, E_5) = -\frac{1}{\sqrt{3}}E_3, && \mathbf{G}(E_1, E_6) = -\frac{1}{\sqrt{3}}E_4, && \mathbf{G}(E_2, E_3) = \frac{1}{\sqrt{3}}E_6,\\
&\mathbf{G}(E_2, E_4) = -\frac{1}{\sqrt{3}}E_5, && \mathbf{G}(E_2, E_5) = \frac{1}{\sqrt{3}}E_4, && \mathbf{G}(E_2, E_6) = -\frac{1}{\sqrt{3}}E_3,\\
&\mathbf{G}(E_3, E_4) = 0, && \mathbf{G}(E_3, E_5) = \frac{1}{\sqrt{3}}E_1, && \mathbf{G}(E_3, E_6) = \frac{1}{\sqrt{3}}E_2,\\
&\mathbf{G}(E_4, E_5) = -\frac{1}{\sqrt{3}}E_2, && \mathbf{G}(E_4, E_6) = \frac{1}{\sqrt{3}}E_1, && \mathbf{G}(E_5, E_6) = 0.
\end{aligned} \tag{8}
$$

Note that, under the assumption that E_1, E_2, E_3 is a positively oriented tangent frame of M, the vector field E_3 is uniquely determined. However, we have a freedom to rotate E_1 in the almost complex distribution $\mathscr{D}_1$. Then, for some rotation angle φ, we have

$$
\begin{aligned}
&\widetilde{E}_1 = \cos\varphi E_1 + \sin\varphi E_2, && \widetilde{E}_2 = \mathbf{J}E_1 = -\sin\varphi E_1 + \cos\varphi E_2,\\
&\widetilde{E}_3 = E_3, && \widetilde{E}_4 = E_4,\\
&\widetilde{E}_5 = \cos\varphi E_5 + \sin\varphi E_6, && \widetilde{E}_6 = -\sin E_5 + \cos\varphi E_6.
\end{aligned}
$$

Now, let us denote the following

$$
\Gamma_{ij}^k = \mathbf{g}(\widetilde{\nabla}_{E_i}E_j, E_k), \quad h_{ij}^k = \mathbf{g}(\widetilde{\nabla}_{E_i}E_j, E_{k+3}), \quad b_{ij}^k = \mathbf{g}(\widetilde{\nabla}_{E_i}E_{j+3}, E_{k+3}),
$$

for $1 \le i, j, k \le 3$. Since the second fundamental form is symmetric, and $\widetilde{\nabla}$ is the Levi-Civita connection, we have that

$$
\Gamma_{ij}^k = -\Gamma_{ik}^j, \quad b_{ij}^k = -b_{ik}^j, \quad h_{ij}^k = h_{ji}^k.
$$

Similarly, using that M is a 3-dimensional CR submanifold, together with the properties of the nearly Kaehler sphere we get that (see [2]):

Lemma 1 *The coefficients $\Gamma_{ij}^k, h_{ij}^k, b_{ij}^k$ satisfy*

$$
\begin{aligned}
&\Gamma_{11}^3 = h_{12}^1, \quad \Gamma_{12}^3 = -h_{11}^1, \quad \Gamma_{21}^3 = h_{22}^1, \quad \Gamma_{22}^3 = -h_{12}^1, \quad \Gamma_{31}^3 = h_{23}^1,\\
&\Gamma_{32}^3 = -h_{13}^1, \quad h_{11}^2 = -h_{12}^3, \quad h_{12}^2 = h_{11}^3, \quad h_{13}^3 = h_{23}^2 + \frac{1}{\sqrt{3}}, \quad h_{22}^2 = h_{12}^3,\\
&h_{22}^3 = -h_{11}^3, \quad h_{23}^3 = -h_{13}^2, \quad b_{11}^2 = h_{13}^3 + \frac{1}{\sqrt{3}}, \quad b_{11}^3 = -h_{13}^2,\\
&b_{21}^2 = -h_{13}^2, \quad b_{21}^3 = -h_{13}^3 + \frac{2}{\sqrt{3}}, \quad b_{31}^2 = h_{33}^3, \quad b_{31}^3 = -h_{33}^2.
\end{aligned}
$$

Lemma 2 *It holds*

$$
b_{12}^3 = \Gamma_{11}^2 - \Gamma_{32}^3, \quad b_{22}^3 = \Gamma_{21}^2 + \Gamma_{31}^3, \quad b_{32}^3 = h_{33}^1 + \Gamma_{31}^2.
$$

Now, let us investigate the tensor field $\mathbf{P}$.

Lemma 3 *On an open dense subset of M, we can chose the orthonormal frame for $\mathscr{D}_1$ so that the tensor field $\boldsymbol{P}$ is given in the frame $E_1, \dots, E_6$ by*

$$\begin{aligned}
\boldsymbol{P}E_1 &= \cos\theta E_1 + a_1 \sin\theta E_3 + a_2 \sin\theta E_4 + a_3 \sin\theta E_5 + a_4 \sin\theta E_6,\\
\boldsymbol{P}E_2 &= -\cos\theta E_2 + a_2 \sin\theta E_3 - a_1 \sin\theta E_4 - a_4 \sin\theta E_5 + a_3 \sin\theta E_6,\\
\boldsymbol{P}E_3 &= a_1 \sin\theta E_1 + a_2 \sin\theta E_2 + (a_3^2 - a_4^2 + (a_2^2 - a_1^2)\cos\theta)E_3\\
&\quad + 2(a_3a_4 - a_1a_2\cos\theta)E_4 - (a_1a_3 + a_2a_4)(1+\cos\theta)E_5\\
&\quad + (a_2a_3 - a_1a_4)(-1+\cos\theta)E_6\\
\boldsymbol{P}E_4 &= a_2 \sin\theta E_1 - a_1 \sin\theta E_2 + 2(a_3a_4 - a_1a_2\cos\theta)E_3\\
&\quad + (a_4^2 - a_3^2 + (a_1^2 - a_2^2)\cos\theta)E_4 - (a_2a_3 - a_1a_4)(-1+\cos\theta)E_5\\
&\quad + (a_1a_3 + a_2a_4)(1+\cos\theta)E_6,\\
\boldsymbol{P}E_5 &= a_3 \sin\theta E_1 - a_4 \sin\theta E_2 - (a_1a_3 + a_2a_4)(1+\cos\theta)E_3\\
&\quad - (a_2a_3 - a_1a_4)(-1+\cos\theta)E_4 + (a_1^2 - a_2^2 + (a_4^2 - a_3^2)\cos\theta)E_5\\
&\quad + 2(a_1a_2 - a_3a_4\cos\theta)E_6,\\
\boldsymbol{P}E_6 &= a_4 \sin\theta E_1 + a_3 \sin\theta E_2 + (a_2a_3 - a_1a_4)(-1+\cos\theta)E_3\\
&\quad - (a_1a_3 + a_2a_4)(1+\cos\theta)E_4 + 2(a_1a_2 - a_3a_4\cos\theta)E_5\\
&\quad + (a_2^2 - a_1^2 + (a_3^2 - a_4^2)\cos\theta)E_6,
\end{aligned} \tag{9}$$

for some differentiable functions $\theta, a_1, a_2, a_3, a_4$ such that $\sum a_i^2 = 1$.

Proof The function $u \mapsto \mathbf{g}(\mathbf{P}u, u)$ attains the maximum on a unit sphere in $\mathscr{D}_1(p)$ at every point p of the submanifold. Since we have the freedom for rotating the orthonormal frame E_1, E_2, we can assume that this maximum is attained for $E_1(p)$. Then, the differentiable function $f(t) = \mathbf{g}(\mathbf{P}(\cos tE_1 + \sin tE_2), \cos tE_1 + \sin tE_2)(p)$ attains the maximum for $t = 0$. Moreover, the equality $f'(0) = 0$ reduces to $2\mathbf{g}(\mathbf{P}E_1, E_2) = 0$. Also, we have that $\mathbf{g}(\mathbf{P}E_2, E_2) = -\mathbf{g}(\mathbf{P}E_1, E_1)$. Therefore, if we denote by $\cos\theta = \mathbf{g}(\mathbf{P}E_1, E_1)$ we have that $\cos\theta \geq 0$.

Assume first, that $\sin\theta \neq 0$. Then, there exists a unit vector field F_1 orthogonal to $\mathscr{D}_1$ such that

$$\mathbf{P}E_1 = \cos\theta E_1 + \sin\theta F_1. \tag{10}$$

Then, for $F_2 = \mathbf{J}F_1$ we have that

$$\mathbf{P}E_2 = -\cos\theta E_2 - \sin\theta F_2, \tag{11}$$

and also

$$\mathbf{P}F_1 = \sin\theta E_1 - \cos\theta F_1, \quad \mathbf{P}F_2 = -\sin\theta E_2 + \cos\theta F_2. \tag{12}$$

Further on, we denote by $F_3 = \sqrt{3}\mathbf{G}(E_1, \mathbf{P}E_1) = \sqrt{3}/\sin\theta\mathbf{G}(E_1, F_1)$. Straightforward computations show that F_3 and $F_4 = \mathbf{J}F_3$ are unit vector fields, orthogonal to E_1, E_2, F_1, F_2, such that

$$\mathbf{P}F_3 = F_3, \mathbf{P}F_4 = -F_4. \tag{13}$$

If $\sin\theta = 0$, then E_1, E_2 are eigenvector fields for $\mathbf{P}$, so in the distribution $\mathscr{D}_1^\perp$ invariant for P we can choose $F_1, F_2 = \mathbf{J}F_1, F_3, F_4 = \mathbf{J}F_3$ such that (10), (11), (12), (13) hold.

Note that there exist differentiable functions a_1, a_2, a_3, a_4 such that $\sum a_i^2 = 1$ and

$$F_1 = a_1E_3 + a_2E_4 + a_3E_5 + a_4E_6, \quad F_2 = -a_2E_3 + a_1E_4 + a_4E_5 - a_3E_6.$$

By a straightforward computation, we obtain

$$F_3 = -a_3E_3 - a_4E_4 + a_1E_5 + a_2E_6, \quad F_4 = a_4E_3 - a_3E_4 + a_2E_5 - a_1E_6.$$

Now, the expressions for the tensor P in the frame $E_1, E_2, F_1, F_2, F_3, F_4$ are easily transformed into the given ones for the frame $E_1, \dots, E_6$. □

4 $\mathscr{D}_1$ integrable

Theorem 1 *Let M be a three-dimensional CR submanifold of $\mathbb{S}^3 \times \mathbb{S}^3$ with $\cos\theta \neq 0$ and with integrable almost complex distribution $\mathscr{D}_1$. Then M is of the form $(p(u, v, t), q(u, v, t))$ where p and q are solutions of the system of differential equations*

$$p_u = p\alpha, \qquad p_v = p\beta, \qquad p_t = p\gamma, \tag{14}$$

$$q_u = q\left(\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta\right), \qquad q_v = q\left(-\frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\beta\right), \qquad q_t = q\delta. \tag{15}$$

Here α and β are family of solutions of

$$\alpha_v - \beta_u = 2\alpha \times \beta, \qquad \alpha_u + \beta_v = \frac{2}{\sqrt{3}}\alpha \times \beta. \tag{16}$$

depending on u, v, t, and γ and δ are solutions of the system of differential equations

$$\gamma_u = \alpha_t + 2\gamma \times \alpha, \qquad \gamma_v = \beta_t + 2\gamma \times \beta, \tag{17}$$

$$\delta_u = \frac{1}{2}\alpha_t + \frac{\sqrt{3}}{2}\beta_t + 2\delta \times \left(\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta\right), \quad \delta_v = -\frac{\sqrt{3}}{2}\alpha_t + \frac{1}{2}\beta_t + 2\delta \times \left(-\frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\beta\right). \tag{18}$$

Proof We can choose a local coordinate system (u, v, t) such that $\mathscr{D}_1$ is spanned by ∂_u, ∂_v. First, let us show that there exist coordinates u, v which are isothermal on each leaf of $\mathscr{D}_1$. We suppose that $\mathscr{D}_1$ is integrable, so from equation $[E_1, E_2] = -\Gamma_{11}^2 E_1 - \Gamma_{21}^2 E_2 - (h_{11}^1 + h_{22}^1)E_3$ we get $h_{22}^1 = -h_{11}^1$. Also, we can assume that the operator P is defined as in (9). Taking $X \in \{E_1, E_2\}$ and $Y = E_1$ in

$$\mathbf{G}(X, \mathbf{P}Y) + \mathbf{P}\mathbf{G}(X, Y) = -2\mathbf{J}((\tilde{\nabla}_X \mathbf{P})Y), \tag{19}$$

we obtain the equations:

$$\Gamma_{11}^2 \cos\theta - \left(-h_{11}^1 a_1 + h_{12}^1 a_2 + h_{11}^3 a_3 + h_{12}^3 a_4\right)\sin\theta = 0,$$
$$\Gamma_{21}^2 \cos\theta + \left(h_{12}^1 a_1 + h_{11}^1 a_2 - h_{12}^3 a_3 + h_{11}^3 a_4\right)\sin\theta = 0.$$

Now, if we suppose that $\cos\theta \neq 0$ and $\sin\theta \neq 0$ we have

$$\Gamma_{11}^2 = \left(-h_{11}^1 a_1 + h_{12}^1 a_2 + h_{11}^3 a_3 + h_{12}^3 a_4\right)\tan\theta,$$
$$\Gamma_{21}^2 = -\left(h_{12}^1 a_1 + h_{11}^1 a_2 - h_{12}^3 a_3 + h_{11}^3 a_4\right)\tan\theta.$$

Note that for the function $f(\theta) = \frac{1}{\sqrt{\cos\theta}}$, the Lie bracket $[f(\theta)E_1, f(\theta)E_2]$ vanishes, so there exist local coordinates (u, v) such that $f(\theta)E_1 = \partial_u$ and $f(\theta)E_2 = \partial_v$. We get $\mathbf{g}(\partial_u, \partial_v) = \mathbf{g}(f(\theta)E_1, f(\theta)E_2) = f^2(\theta)\mathbf{g}(E_1, E_2) = 0$, so ∂_u and ∂_v are orthogonal. Also, $\mathbf{g}(\partial_u, \partial_u) = \mathbf{g}(f(\theta)E_1, f(\theta)E_1) = f^2(\theta)$. Analogously, $\mathbf{g}(\partial_v, \partial_v) = f^2(\theta)$. We get that ∂_u and ∂_v are orthogonal and have the same length, so (u, v) are isothermal coordinates on each leaf of $\mathscr{D}_1$. If we suppose that $\sin\theta = 0$, taking $X \in \{E_1, E_2\}$ and $Y = E_1$ in (19) we obtain $\Gamma_{11}^2 = \Gamma_{21}^2 = 0$, so the Lie brackets for vectors E_1 and E_2 vanish and we can conclude that the coordinates that correspond to them are isothermal.

Now, up to a possible permutation of u and v we can say that $\mathbf{J}\partial_u = \partial_v$. If $f(u, v, t) = (p, q)(u, v, t)$ is the immersion, we then have that (16) hold. We also denote

$$\partial_t p = p_t = p\gamma, \qquad \partial_t q = q_t = q\delta,$$

where γ and δ are also purely imaginary mappings satisfying (17) and (18). Moreover, the remaining integrability conditions are obtained from

$$p_{ut} = p\gamma\alpha + p\alpha_t, \qquad p_{tu} = p\alpha\gamma + p\gamma_u,$$
$$p_{vt} = p\gamma\beta + p\beta_t, \qquad p_{tv} = p\beta\gamma + p\gamma_v$$

and

$$q_{ut} = q\left(\delta\left(\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta\right) + \frac{1}{2}\alpha_t + \frac{\sqrt{3}}{2}\beta_t\right), \qquad q_{tu} = q\left(\left(\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta\right)\delta + \delta_u\right),$$
$$q_{vt} = q\left(\delta\left(-\frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\beta\right) - \frac{\sqrt{3}}{2}\alpha_t + \frac{1}{2}\beta_t\right), \quad q_{tv} = q\left(\left(-\frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\beta\right)\delta + \delta_v\right).$$

They reduce, respectively, to (17) and (18).

Conversely, assume we have a family of solutions of (16) α, β depending on u, v, t. Then, we need functions γ, δ satisfying (17) and (18). If we use the first relation of (16) and the Jacobi identity for the cross product, we easily get that the integrability condition for γ, given by $\gamma_{uv} - \gamma_{vu} = 0$, is satisfied. Similarly, a straightforward computation shows that the integrability conditions are also satisfied for δ. So with a prescribed initial condition $\gamma(0, 0, t) = \gamma_0(t)$, $\delta(0, 0, t) = \delta_0(t)$ we have solutions. Moreover, system (15) has a unique solution for given initial conditions $(p(0, 0, 0), q(0, 0, 0))$ which is a CR immersion of required type. □

Remark 1 We note that for the previous theorem to hold it is sufficient that the submanifold admits local coordinates such that u, v are isothermal on each leaf of $\mathscr{D}_1$. In the particular case of $\cos\theta = 0$, when $\mathbf{P}\mathscr{D}_1 = \mathrm{Span}\{E_3, \mathbf{J}E_3\}$, we can choose E_1 such that $\mathbf{P}E_1 = E_3$. Taking $(X, Y) \in \{(E_1, E_1), (E_2, E_2)\}$ in (19) we obtain $\Gamma_{11}^2 = -h_{13}^1$, $\Gamma_{21}^2 = -h_{23}^1$ and if we take $(X, Y) = (E_3, E_2)$ in (19) we get $h_{13}^1 = 0$, $h_{23}^1 = 0$. Also, we have that $[E_1, E_2] = -\Gamma_{11}^2 E_1 - \Gamma_{21}^2 E_2 - (h_{11}^1 + h_{22}^1)E_3$, so, we get that in this case E_1 and E_2 correspond to coordinate vector fields. One may notice that such submanifolds exist.

Remark 2 Note that for a mapping $k(t)$ into unit quaternions $\mathbb{S}^3$ and $\alpha(u, v, t)$ and $\beta(u, v, t)$ solutions of (16), we have that $\alpha^* = k(t)\alpha k(t)^{-1}$, $\beta^* = k(t)\beta k(t)^{-1}$ are also solutions of (16).

5 $\mathscr{D}_1$ and $\mathscr{D}_1^{\perp}$ totally geodesic

The main result that we prove in this section is the following.

Theorem 2 *Let M be a three-dimensional CR submanifold of $\mathbb{S}^3 \times \mathbb{S}^3$, with $\mathscr{D}_1$ and $\mathscr{D}_1^{\perp}$ being totally geodesic distributions. Then M is locally congruent to the immersions (p_1, q_1), given by*

$$\begin{aligned} p_1 &= (\cos(c_1 t)\cos x_1, \cos(c_2 t)\sin x_1, \sin(c_2 t)\sin x_1, -\sin(c_1 t)\cos x_1), \\ q_1 &= (\cos(d_1 t)\cos x_2, \cos(d_2 t)\sin x_2, \sin(d_2 t)\sin x_2, -\sin(d_1 t)\cos x_2), \end{aligned} \tag{20}$$

where

$$c_1 = \frac{\sqrt{3-\chi_1^2-\chi_2^2}-\chi_2}{4\sqrt{3}}, \qquad c_2 = \frac{\sqrt{3-\chi_1^2-\chi_2^2}+\chi_2}{4\sqrt{3}},$$
$$d_1 = \frac{\sqrt{3-\chi_1^2-\chi_2^2}-\chi_1}{4\sqrt{3}}, \qquad d_2 = \frac{\sqrt{3-\chi_1^2-\chi_2^2}+\chi_1}{4\sqrt{3}},$$

for $\chi_1, \chi_2 \geq 0$, $\chi_1^2+\chi_2^2 \leq 3$.

Proof From the assumption that $\mathscr{D}_1$ and $\mathscr{D}_1^{\perp}$ are totally geodesic, we obtain a first set of relations:

$$\begin{aligned} &h_{11}^1 = 0, \quad h_{11}^2 = 0, \quad h_{11}^3 = 0, \quad h_{12}^1 = 0, \quad h_{12}^2 = 0, \quad h_{12}^3 = 0, \\ &h_{22}^1 = 0, \quad h_{22}^2 = 0, \quad h_{22}^3 = 0, \quad h_{33}^1 = 0, \quad h_{33}^2 = 0, \quad h_{33}^3 = 0. \end{aligned}$$

Notice that this makes $\mathscr{D}_1$ integrable as well. Next, we evaluate the curvature tensor $\mathbf{R}(E_1, E_2)E_1$ once using the definition and once using its expression from [2]. Then, take the difference between these two identities for the curvature tensor. For convenience, further on in this section we will refer to this procedure for vector fields E_i, E_j, E_k, as to the *two identities for the curvature*. In this case, for $\mathbf{R}(E_1, E_2)E_1$, as a_1, a_2, a_3 and a_4 do not vanish simultaneously, we obtain that $\cos\theta \sin\theta = 0$. Therefore, we will have to treat two cases: $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Case 1. $\theta = 0$. We make the following notation, in the definition of $\mathbf{P}$:

$$b_1 := -a_1^2 + a_2^2 + a_3^2 - a_4^2, \quad b_2 := 2a_3a_4 - 2a_1a_2, \quad b_3 := 2(a_1a_3 + a_2a_4).$$

We evaluate Eq. (19) successively for $X = E_1, Y = E_1$; $X = E_3, Y = E_1$; $X = E_2, Y = E_1$ and obtain, respectively, that $\Gamma_{11}^2 = 0$, $\Gamma_{31}^2 = 0$ and $\Gamma_{21}^2 = 0$. We will determine the derivatives w.r.t. E_1, E_2 and E_3 of the remaining unknown functions h_{ij}^k. In order to do so, we use the two identities for the curvature. We evaluate them for E_2, E_3, E_1; E_1, E_3, E_1; E_1, E_3, E_5 and replace successively every value found for each derivative, until we finally obtain:

$$\begin{aligned} E_2(h_{13}^1) &= \frac{1}{12}\left(-4b_1 + 12\left(h_{13}^1\right)^2 - 12\left(h_{13}^2\right)^2 - 12\left(h_{23}^1\right)^2 - 12\left(h_{23}^2\right)^2 + 5\right), \\ E_2(h_{13}^2) &= \frac{1}{3}\left(6h_{13}^1h_{13}^2 + \sqrt{3}h_{23}^1\right), \quad E_2(h_{23}^1) = \frac{1}{3}\left(b_2 + 6h_{13}^1h_{23}^1 - \sqrt{3}h_{13}^2\right), \\ E_2(h_{23}^2) &= \frac{1}{3}(6h_{13}^1h_{23}^2 - b_3); \quad E_1(h_{13}^1) = \frac{1}{3}\left(-b_2 - 6h_{13}^1h_{23}^1 + \sqrt{3}h_{13}^2\right), \\ E_1(h_{23}^1) &= \frac{1}{12}\big(-4b_1 + 12(h_{13}^1)^2 + 12(h_{13}^2)^2 - 12(h_{23}^1)^2 + 12(h_{23}^2)^2 + 8\sqrt{3}h_{23}^2 - 1\big), \end{aligned}$$

$$E_1(h^2_{13}) = \frac{1}{3}\left(b_3 - \sqrt{3}h^1_{13} - 6h^2_{13}h^1_{23}\right), \quad E_1(h^2_{23}) = -\frac{2}{3}h^1_{23}\left(3h^2_{23} + \sqrt{3}\right);$$
$$E_3\left(h^1_{13}\right) = 0, \quad E_3\left(h^2_{13}\right) = 0, \quad E_3\left(h^1_{23}\right) = 0, \quad E_3\left(h^2_{23}\right) = 0. \tag{21}$$

We may as well find the derivatives of b_1, b_2, b_3 as following. Use Eq. (19) for E_3, E_3 and E_1, E_3, respectively, in order to determine

$$E_3(b_1) = 0, \quad E_3(b_2) = 0, \quad E_3(b_3) = 0; \quad E_1(b_1) = 2b_2h^1_{13} - 2b_3h^2_{13},$$
$$E_1(b_2) = -2b_1h^1_{13} - b_3\left(2h^2_{23} + \sqrt{3}\right), \quad E_1(b_3) = 2b_1h^2_{13} + b_2\left(2h^2_{23} + \sqrt{3}\right). \tag{22}$$

Provided that den $:= 12(h^1_{13})^2 + 12(h^2_{13})^2 + 12(h^1_{23})^2 + 12(h^2_{23})^2 + 4\sqrt{3}h^2_{23} + 1$ is different than zero, we can express b_1, b_2 and b_3 w.r.t. h^k_{ij}, by using (19) for E_3, E_1:

$$\begin{aligned} b_1 &= -\tfrac{1}{\text{den}}\left(12\left(h^1_{13}\right)^2 + 12\left(h^2_{13}\right)^2 - 12\left(h^1_{23}\right)^2 - 12\left(h^2_{23}\right)^2 - 4\sqrt{3}h^2_{23} - 1\right), \\ b_2 &= \tfrac{1}{\text{den}}(4\left(6h^1_{13}h^1_{23} - 6h^2_{13}h^2_{23} - \sqrt{3}h^2_{13}\right)), \\ b_3 &= -\tfrac{1}{\text{den}}(4\left(6h^1_{13}h^2_{23} + \sqrt{3}h^1_{13} + 6h^2_{13}h^1_{23}\right)). \end{aligned} \tag{23}$$

In fact, the denominator is always different than zero, as it follows. Suppose it was not. Then, we would have $h^1_{13} = 0$, $h^2_{13} = 0$, $h^1_{23} = 0$ and $h^2_{23} = -\frac{1}{2\sqrt{3}}$. From the identities of the curvature, it follows on the one hand that for E_1, E_3, E_1 we have $b_2 = b_3 = 0$, $b_1 = -1$ and then for E_1, E_2, E_3, we get that $\frac{2}{3} = 0$. This is a contradiction. We shall continue then from Eq. (23).

Let $\rho = \frac{1}{\sqrt{8+\text{den}}}$ and choose to work with the frame E_1, E_2, ρE_3. One may see that the Lie brackets vanish $[E_1, E_2] = 0$, $[E_1, \rho E_3] = 0$ and $[E_2, \rho E_3] = 0$, which means that there exist coordinate vector fields on the three-dimensional submanifold satisfying $\partial u = E_1$, $\partial v = E_2$, $\partial t = \rho E_3$. We have that $\mathbf{P}E_1 = E_1$, so we can write

$$\begin{aligned} \partial u &= (p_u, q_u) = (p\alpha_1, q\alpha_1), \ \partial v = (p_v, q_v) \\ &= \frac{1}{\sqrt{3}}(p\alpha_1, -q\alpha_1), \ \partial t = (p_t, q_t) = (p\alpha_3, q\beta_3). \end{aligned} \tag{24}$$

Also, we have that

$$\begin{aligned} \mathbf{P}E_3 =& b_1E_3 + b_2E_4 - b_3E_5, \\ =& \left(p\frac{1}{\rho}\left(b_1\alpha_3 + \frac{b_2}{\sqrt{3}}(2\beta_3 - \alpha_3) - \frac{2b_3}{3}(2\alpha_1 \times \alpha_3 - \alpha_1 \times \beta_3)\right),\right. \\ & \left. q\frac{1}{\rho}\left(b_1\beta_3 + \frac{b_2}{\sqrt{3}}(-2\alpha_3 + \beta_3) - \frac{2b_3}{3}(-2\alpha_1 \times \beta_3 + \alpha_1 \times \alpha_3)\right)\right) \end{aligned}$$

and at the same time, by definition of $\mathbf{P}$, we have $\mathbf{P}E_3 = (p\frac{\beta_3}{\rho}, q\frac{\alpha_3}{\rho})$. It gives:

$$\beta_3 = \frac{1}{2 + b_1 - \sqrt{3}b_2}\Big((1 + 2b_1)\alpha_3 - 2b_3\alpha_1 \times \alpha_3\Big), \tag{25}$$

when $2 + b_1 - \sqrt{3}b_2 \neq 0$. By using (23), we get that $2 + b_1 - \sqrt{3}b_2 = 0$ only in case when $h^1_{13} = 0$, $h^2_{13} = \frac{1}{2}$, $h^1_{23} = 0$, $h^2_{23} = -\frac{1}{\sqrt{3}}$. Denote with $d_p(X)$ and $d_q(X)$ projections of vector X on tangent space of both spheres. If we use (2), we get:

$$\nabla^E_{\partial u} d_p(\partial u) = 0, \quad \nabla^E_{\partial u} d_p(\partial v) = 0, \quad \nabla^E_{\partial v} d_p(\partial v) = 0, \quad \nabla^E_{\partial t} d_p(\partial t) = \frac{4}{3}f_1\left(\frac{1}{2}E_1 + \frac{\sqrt{3}}{2}E_2\right),$$

$$\nabla^E_{\partial u} d_q(\partial u) = 0, \quad \nabla^E_{\partial u} d_q(\partial v) = 0, \quad \nabla^E_{\partial v} d_q(\partial v) = 0, \quad \nabla^E_{\partial t} d_q(\partial t) = \frac{4}{3} g_1 \left(\frac{1}{2} E_1 - \frac{\sqrt{3}}{2} E_2 \right) \tag{26}$$

and from we have that:

$$\langle \alpha_1, \alpha_1 \rangle = \tfrac{3}{4}, \quad \langle \alpha_3, \alpha_3 \rangle = f_2, \quad \langle \beta_3, \beta_3 \rangle = g_2, \quad \langle \alpha_1, \alpha_3 \rangle = 0, \quad \langle \alpha_1, \beta_3 \rangle = 0, \tag{27}$$

where we denote with:

$$f_1 = \frac{1}{8} \left(\frac{h^1_{13}\left(\sqrt{3} - 6h^2_{23}\right) - 6(h^2_{13}+1)h^1_{23}}{8 + \text{den}} + \frac{h^1_{13}\left(6h^2_{23} + \sqrt{3}\right) + 6h^2_{13}h^1_{23}}{\text{den}} \right), \tag{28}$$

$$f_2 = \frac{3\left(4\sqrt{3}(2h^2_{13}+1)h^2_{23} + (2h^2_{13}+1)^2 + 4(h^1_{13} - \sqrt{3}h^1_{23})^2 + 12(h^2_{23})^2\right)}{4\text{den}(\text{den}+8)}, \tag{29}$$

$$g_1 = \frac{1}{8} \left(\frac{-h^1_{13}\left(\sqrt{3} - 6h^2_{23}\right) + 6(h^2_{13}-1)h^1_{23}}{8 + \text{den}} - \frac{h^1_{13}\left(6h^2_{23} + \sqrt{3}\right) + 6h^2_{13}h^1_{23}}{\text{den}} \right), \tag{30}$$

$$g_2 = \frac{3\left(4\sqrt{3}(1-2h^2_{13})h^2_{23} + (1-2h^2_{13})^2 + 4(h^1_{13} + \sqrt{3}h^1_{23})^2 + 12(h^2_{23})^2\right)}{4\text{den}(\text{den}+8)}. \tag{31}$$

Directly we obtain:

$$\begin{aligned} p_{uu} &= -\frac{3}{4}p, \quad p_{uv} = -\frac{\sqrt{3}}{4}p, \quad p_{tt} = \frac{4}{3} f_1 p_u - f_2 p, \\ q_{uu} &= -\frac{3}{4}q, \quad q_{uv} = \frac{\sqrt{3}}{4}q, \quad q_{tt} = \frac{4}{3} g_1 q_u - g_2 q \end{aligned} \tag{32}$$

so, the general solutions for immersions p and q are:

$$\begin{aligned} p(u,v,t) &= a_1(t)\cos\left(\frac{\sqrt{3}u+v}{2}\right) + a_2(t)\sin\left(\frac{\sqrt{3}u+v}{2}\right), \\ q(u,v,t) &= b_1(t)\cos\left(\frac{\sqrt{3}u-v}{2}\right) + b_2(t)\sin\left(\frac{\sqrt{3}u-v}{2}\right), \end{aligned} \tag{33}$$

where $a_1(t)$, $a_2(t)$ $b_1(t)$, $b_2(t) \in \mathbb{H}$. A straightforward computation gives us the following relations: $\partial_{uu} f_1 = -3f_1$, $\partial_{vv} f_1 = -f_1$, $\partial_t f_1 = 0$, $\partial u f_2 = -2f_1$, $\partial t f_2 = 0$, $-\partial u f_1 + \frac{3}{2} f_2 = c_3$; $\partial_{uu} g_1 = -3g_1$, $\partial_{vv} g_1 = -g_1$, $\partial_t g_1 = 0$, $\partial u g_2 = -2g_1$, $\partial_t g_2 = 0$, $\partial u g_1 - \frac{3}{2} g_2 = d_3$. General solutions of these functions are:

$$\begin{aligned} &f_1(u,v) = c_1\cos(\sqrt{3}u+v) + c_2\sin(\sqrt{3}u+v), \\ &f_2(u,v) = -\tfrac{2}{\sqrt{3}}c_1\sin(\sqrt{3}u+v) + \tfrac{2}{\sqrt{3}}c_2\cos(\sqrt{3}u+v) + \tfrac{2}{3}c_3, \\ &g_1(u,v) = d_1\cos(\sqrt{3}u-v) + d_2\sin(\sqrt{3}u-v), \\ &g_2(u,v) = -\tfrac{2}{\sqrt{3}}d_1\sin(\sqrt{3}u-v) + \tfrac{2}{\sqrt{3}}d_2\cos(\sqrt{3}u-v) - \tfrac{2}{3}d_3, \end{aligned} \tag{34}$$

for some real constants $c_1, c_2, c_3, d_1, d_2, d_3$. As they are constants, we can rewrite them on a following way:

$$c_1 = \xi_1\cos w_1, \quad c_2 = \xi_1\sin w_1, \quad d_1 = \xi_2\cos w_2, \quad d_2 = \xi_2\sin w_2, \tag{35}$$

for some constants $\xi_1, \xi_2 \geq 0$ and $w_1, w_2 \in [0, 2\pi)$. Expressions of f_1, f_2, g_1, g_2 depend on $h^1_{13}, h^2_{13}, h^1_{23}, h^2_{23}$, and using relations among them we get following equation:

$$-12\left(8d_3^2 - 8d_3 g_2 + d_3 - 8c_3^2 - 8c_3 f_2 + c_3 - 6g_2^2 + 6f_2^2\right)^2 - 768(f_1+g_1)^4 - 4(f_1+g_1)^2\cdot \left(640d_3^2 - 16d_3(64c_3 - 24g_2 - 9) + 640c_3^2 - 48c_3(8f_2+3) + 9\left(32g_2^2 + 32f_2^2 + 1\right)\right) = 0. \tag{36}$$

On the other hand, when we compute it in the equivalent way, by using (34), we obtain a polynomial in $\sin(2\sqrt{3}u + 2v)$, $\cos(2\sqrt{3}u + 2v)$, $\sin(2\sqrt{3}u - 2v)$, $\cos(2\sqrt{3}u - 2v)$, $\sin(2\sqrt{3}u)$, $\cos(2\sqrt{3}u)$, $\sin(2v)$, $\cos(2v)$ for which all the coefficients must vanish. Therefore, we obtain nine expressions which are all zero. By using them we get:

$$\begin{aligned} \xi_1^2\left((-3+32c_3)(-3+32c_3-32d_3) + 768\xi_2^2\right) &= 0,\\ \xi_2^2\left((3+32d_3)(3-32c_3+32d_3) + 768\xi_1^2\right) &= 0. \end{aligned} \tag{37}$$

Consider now the case when ξ_1, ξ_2 do not vanish. We solve the previous equation for ξ_1^2 and ξ_2^2 and get

$$\xi_1^2 = -\frac{1}{768}(3+32d_3)(3-32c_3+32d_3), \quad \xi_2^2 = -\frac{1}{768}(-3+32c_3)(-3+32c_3-32d_3).$$

As these expressions are positive, we need to have $3 + 32d_3 > 0$, $3 - 32c_3 + 32d_3 < 0$ and $-3 + 32c_3 < 0$. In order to simplify the previous equations, we introduce constants $\chi_1 > 0$ and $\chi_2 > 0$ such that $\chi_1^2 + \chi_2^2 < 3$ and $c_3 := \frac{-\chi_1^2+3}{32}$, $d_3 := \frac{\chi_2^2-3}{32}$. Then from the previous two equations, we obtain

$$\xi_1 = \frac{\chi_2\sqrt{3-\chi_1^2-\chi_2^2}}{16\sqrt{3}}, \quad \xi_2 = \frac{\chi_1\sqrt{3-\chi_1^2-\chi_2^2}}{16\sqrt{3}}.$$

Notice that f_1, f_2, g_1, g_2 become now in terms of $u, v, w_1, w_2, \chi_1, \chi_2$.

$$\begin{aligned} f_1 &= \tfrac{1}{16\sqrt{3}}\chi_2\sqrt{3-\chi_1^2-\chi_2^2}\cos(\sqrt{3}u + v - w_1),\\ f_2 &= \tfrac{1}{48}(3-\chi_1^2 - 2\chi_2\sqrt{3-\chi_1^2-\chi_2^2}\sin(\sqrt{3}u + v - w_1)),\\ g_1 &= \tfrac{1}{16\sqrt{3}}\chi_1\sqrt{3-\chi_1^2-\chi_2^2}\sin(\sqrt{3}u - v - w_2),\\ g_2 &= \tfrac{1}{48}(3-\chi_1^2 - 2\chi_1\sqrt{3-\chi_1^2-\chi_2^2}\sin(\sqrt{3}u - v - w_2)). \end{aligned} \tag{38}$$

As w_1 and w_2 are constants, we will keep the same notation for $\sqrt{3}u + v := \sqrt{3}u + v - w_1$ and $\sqrt{3}u - v := \sqrt{3}u - v - w_2$. Further on, we would like to find explicitly the immersion f. We replace f_1, f_2, g_1, g_2 from (38), together with general solution of p and g in expression of p_{tt} and q_{tt} from (32), and we get the following system of differential equations:

$$\begin{aligned} a_1''(t) &= \frac{1}{48}\left(\left(-3+\chi_1^2\right)a_1(t) + 2\chi_2\sqrt{3-\chi_1^2-\chi_2^2}a_2(t)\right),\\ a_2''(t) &= \frac{1}{48}\left(\left(-3+\chi_1^2\right)a_2(t) + 2\chi_2\sqrt{3-\chi_1^2-\chi_2^2}a_1(t)\right);\\ b_1''(t) &= \frac{1}{48}\left(\left(-3+\chi_2^2\right)b_1(t) + 2\chi_1\sqrt{3-\chi_1^2-\chi_2^2}b_2(t)\right),\\ b_2''(t) &= \frac{1}{48}\left(\left(-3+\chi_2^2\right)b_2(t) + 2\chi_1\sqrt{3-\chi_1^2-\chi_2^2}b_1(t)\right). \end{aligned}$$

We solve these systems for $a_1(t)$, $a_2(t)$, $b_1(t)$ and $b_2(t)$, and we find

$$\begin{aligned}
a_1(t) &= C_1 \cos\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_2|}{4\sqrt{3}}t\right) + C_2 \sin\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_2|}{4\sqrt{3}}t\right) \\
&\quad + C_3 \cos\left(\frac{\sqrt{3-\chi_1^2+\chi_2^2}+\chi_2}{4\sqrt{3}}t\right) + C_4 \sin\left(\frac{\sqrt{3-\chi_1^2-\chi_2^2}+\chi_2}{4\sqrt{3}}t\right), \\
a_2(t) &= C_1 \cos\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_2|}{4\sqrt{3}}t\right) + C_2 \sin\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_2|}{4\sqrt{3}}t\right) \\
&\quad - C_3 \cos\left(\frac{\sqrt{3-\chi_1^2+\chi_2^2}+\chi_2}{4\sqrt{3}}t\right) - C_4 \sin\left(\frac{\sqrt{3-\chi_1^2-\chi_2^2}+\chi_2}{4\sqrt{3}}t\right); \\
b_1(t) &= D_1 \cos\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_1|}{4\sqrt{3}}t\right) + D_2 \sin\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_1|}{4\sqrt{3}}t\right) \\
&\quad + D_3 \cos\left(\frac{\sqrt{3-\chi_1^2+\chi_2^2}+\chi_1}{4\sqrt{3}}t\right) + D_4 \sin\left(\frac{\sqrt{3-\chi_1^2-\chi_2^2}+\chi_1}{4\sqrt{3}}t\right), \\
b_2(t) &= D_1 \cos\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_1|}{4\sqrt{3}}t\right) + D_2 \sin\left(\frac{|\sqrt{3-\chi_1^2-\chi_2^2}-\chi_1|}{4\sqrt{3}}t\right) \\
&\quad - D_3 \cos\left(\frac{\sqrt{3-\chi_1^2+\chi_2^2}+\chi_1}{4\sqrt{3}}t\right) - D_4 \sin\left(\frac{\sqrt{3-\chi_1^2-\chi_2^2}+\chi_1}{4\sqrt{3}}t\right).
\end{aligned} \tag{39}$$

Therefore, in order to determine the immersion p we need to determine the quaternion constants C_i and D_i, $i = 1, 2, 3, 4$. From (32), we obtain the following derivatives:

$$\alpha_{1u} = 0, \quad \alpha_{1v} = 0, \quad \alpha_{3t} = \frac{4}{3} f_1 \alpha_1, \quad \beta_{3t} = \frac{4}{3} g_1 \alpha_1. \tag{40}$$

Further on, as $2 + b_1 - \sqrt{3} b_2 = 0$ is equivalent with $f_2 = 0$, which here is not case because $\xi_1 \neq 0$, we take the derivatives with respect to t both in the left- and right-hand sides of the equal sign in (25) and then cross product at right with α_3 gives α_{1t} as

$$\alpha_{1t} = \frac{f_1}{2b_3 f_2}\alpha_3 + \frac{1}{2b_3 f_2}\frac{4}{3}\left(g_1 - \frac{1+2b_1}{2+b_1-\sqrt{3}b_2} f_1\right)\alpha_1 \times \alpha_3. \tag{41}$$

b_3 vanish in case when $\xi_1 = \xi_2 = 0$, so here we can divide with it. Taking the derivative with respect to t in the above equation, we obtain that

$$\alpha_{1tt} = -\frac{1}{12}(3 - \chi_1^2 - \chi_2^2)\alpha_1.$$

Therefore, if necessary, we can always apply an isometry $\mathscr{F}_{abc}$ such that the choice of c, for new tangent vector $(\widetilde{p}\widetilde{\alpha}_1, \widetilde{q}\widetilde{\beta}_1)$, must satisfy that $\widetilde{\alpha}_1 = c\alpha_1\bar{c}$ is imaginary quaternion with components i and j, only. Therefore, for initial conditions $\alpha_1(u_0, v_0, 0) = \frac{\sqrt{3}}{2} i$ and $\alpha_1'(u_0, v_0, 0) = \frac{1}{4}\sqrt{3 - \chi_1^2 - \chi_2^2}\, j$, we obtain that

$$\alpha_1(t) = \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3-\chi_1^2-\chi_2^2}}{2\sqrt{3}}t\right) i + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3-\chi_1^2-\chi_2^2}}{2\sqrt{3}}t\right) j. \tag{42}$$

Next, we compute the cross product between α_{1t} and α_3

$$\alpha_{1t} \times \alpha_3 = \frac{1}{24}\left(-3 + \chi_1^2 + \chi_2^2 + \chi_2\sqrt{3 - \chi_1^2 - \chi_2^2}\sin\left(\sqrt{3}u + v - w_1\right)\right)\alpha_1.$$

Multiplying at left with α_{1t} in the above relation and, considering that $\alpha_{1t} \times (\alpha_{1t} \times \alpha_3) = -f_1\alpha_{1t} + \frac{1}{16}(-3 + \chi_1^2 + \chi_2^2)\alpha_3$, we obtain that α_3 is given by

$$\alpha_3(t) = -\frac{4f_1}{\sqrt{3 - \chi_1^2 - \chi_2^2}}\left(-\sin\left(\frac{\sqrt{3 - \chi_1^2 - \chi_2^2}}{2\sqrt{3}}t\right)i + \cos\left(\frac{\sqrt{3 - \chi_1^2 - \chi_2^2}}{2\sqrt{3}}t\right)j\right)$$
$$- \frac{\sqrt{3}}{12}\left(\sqrt{3 - \chi_1^2 - \chi_2^2} - \chi_2\sin(\sqrt{3}u + v - w_1)\right)k.$$

By a convenient choice of a and b, we can fix the immersion p such that for initial conditions at the point (u_0, v_0, t_0), where $\sqrt{3}u_0 + v_0 - w_1 = \frac{\pi}{2}$, $\sqrt{3}u_0 - v_0 - w_2 = \frac{\pi}{2}$, $t = 0$, we have $p(u_0, v_0, 0) = \sqrt{2}C_1$, for $C_1 = \frac{1}{\sqrt{2}}(1, 0, 0, 0)$. We then denote the real coefficients of C_i by $C_i = (C_{i1}, C_{i2}, C_{i3}, C_{i4})$, for $i \in 2, 3, 4$.

Then p becomes

$$\begin{aligned} p = {} & C_3\cos(tk_2)\Big(\cos\tfrac{\sqrt{3}u+v-w_1}{2} - \sin\tfrac{\sqrt{3}u+v-w_1}{2}\Big) \\ & + \Big(C_1\cos(t(k_1 - k_2)) + C_2\sin(t(k_1 - k_2))\Big)\Big(\cos\tfrac{\sqrt{3}u+v-w_1}{2} + \sin\tfrac{\sqrt{3}u+v-w_1}{2}\Big) \\ & + C_4\sin(tk_2)\Big(\cos\tfrac{\sqrt{3}u+v-w_1}{2} - \sin\tfrac{\sqrt{3}u+v-w_1}{2}\Big), \end{aligned} \tag{43}$$

where k_1 and k_2 stand for $k_1 = \frac{\sqrt{3-\chi_1^2-\chi_2^2}}{2\sqrt{3}}$, $k_2 = \frac{\sqrt{3-\chi_1^2+2\chi_2\sqrt{3-\chi_1^1-\chi_2^2}}}{4\sqrt{3}}$. Having in mind the expression for α_1 in (42), we compute $\alpha_1(t) = \bar{p}p_u$. We compare its component in i, with the one from (42), and this gives a polynomial in $\cos((k_1 - 2k_2)t)$, $\sin((k_1 - 2k_2)t)$, $\cos(k_1)t$, $\sin(k_1 t)$ which vanishes identically. This implies $C_{42} = 0$ and $C_{32} = -\frac{1}{\sqrt{2}}$. By a similar reasoning for the component of α_1 in j, we find $C_{33} = 0$, $C_{43} = -\frac{1}{\sqrt{2}}$. The fact that p has constant length implies $C_{21} = 0$ and then $C_{41} = 0$, $C_{34} = 0$, $C_{22} = 0$. We see that $C_{31}^2 = C_{44}^2$ and $C_{23}^2 = \frac{1}{2} - C_{24}^2$, which leads to obtaining that $C_{44} = 0$, $C_{31} = 0$ and $C_{23} = 0$. Finally, we find $C_{24} = -\frac{1}{\sqrt{2}}$ and determine the immersion p:

$$\begin{aligned} p = {} & \tfrac{1}{\sqrt{2}}\cos\frac{t\left(2\sqrt{3-\chi_1^2-\chi_2^2}-\sqrt{3-\chi_1^2+2\sqrt{-\chi_2^2(\chi_1^2+\chi_2^2-3)}}\right)}{4\sqrt{3}}\left(\cos\tfrac{\sqrt{3}u+v-w_1}{2} + \sin\tfrac{\sqrt{3}u+v-w_1}{2}\right) \\ & -\tfrac{1}{\sqrt{2}}\cos\frac{t\sqrt{3-\chi_1^2+2\sqrt{-\chi_2^2(\chi_1^2+\chi_2^2-3)}}}{4\sqrt{3}}\left(\cos\tfrac{\sqrt{3}u+v-w_1}{2} - \sin\tfrac{\sqrt{3}u+v-w_1}{2}\right)i \\ & -\tfrac{1}{\sqrt{2}}\sin\frac{t\sqrt{3-\chi_1^2+2\sqrt{-\chi_2^2(\chi_1^2+\chi_2^2-3)}}}{4\sqrt{3}}\left(\cos\tfrac{\sqrt{3}u+v-w_1}{2} - \sin\tfrac{\sqrt{3}u+v-w_1}{2}\right)j \\ & -\tfrac{1}{\sqrt{2}}\sin\frac{t\left(2\sqrt{3-\chi_1^2-\chi_2^2}-\sqrt{3-\chi_1^2+2\sqrt{-\chi_2^2(\chi_1^2+\chi_2^2-3)}}\right)}{4\sqrt{3}}\left(\cos\tfrac{\sqrt{3}u+v-w_1}{2} + \sin\tfrac{\sqrt{3}u+v-w_1}{2}\right)k. \end{aligned} \tag{44}$$

It then follows that q is given by

$$
\begin{aligned}
q = &\frac{1}{\sqrt{2}} \cos \frac{t\left(2\sqrt{3-\chi_2^2-\chi_1^2}-\sqrt{3-\chi_2^2+2\sqrt{-\chi_1^2(\chi_2^2+\chi_1^2-3)}}\right)}{4\sqrt{3}} \left(\cos \frac{\sqrt{3}u-v-w2}{2} + \sin \frac{\sqrt{3}u-v-w2}{2}\right)\\
&-\frac{1}{\sqrt{2}} \cos \frac{t\sqrt{3-\chi_2^2+2\sqrt{-\chi_1^2(\chi_2^2+\chi_1^2-3)}}}{4\sqrt{3}} \left(\cos \frac{\sqrt{3}u-v-w2}{2} - \sin \frac{\sqrt{3}u-v-w2}{2}\right) i\\
&-\frac{1}{\sqrt{2}} \sin \frac{t\sqrt{3-\chi_2^2+2\sqrt{-\chi_1^2(\chi_2^2+\chi_1^2-3)}}}{4\sqrt{3}} \left(\cos \frac{\sqrt{3}u-v-w2}{2} - \sin \frac{\sqrt{3}u-v-w2}{2}\right) j\\
&-\frac{1}{\sqrt{2}} \sin \frac{t\left(2\sqrt{3-\chi_2^2-\chi_1^2}-\sqrt{3-\chi_2^2+2\sqrt{-\chi_1^2(\chi_2^2+\chi_1^2-3)}}\right)}{4\sqrt{3}} \left(\cos \frac{\sqrt{3}u-v-w2}{2} + \sin \frac{\sqrt{3}u-v-w2}{2}\right) k.
\end{aligned}
\tag{45}
$$

A reparametrisation then completes the proof. We also note that the other cases following from (35) can be treated in a similar way leading to the same result.

Case 2. $\theta = \frac{\pi}{2}$. Now we will still split into two subcases, according to whether $h_{13}^1 = h_{23}^1 = 0$ or $(h_{13}^1)^2 + (h_{23}^1)^2 \neq 0$. However following similar arguments as in the previous case, we obtain in both subcases a contradiction. □

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