Maximal functions associated with flat plane curves with mitigating factors

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Abstract

We study the boundedness problem for maximal operators $\mathbb{M}^\sigma$ associated with flat plane curves with mitigating factors, defined by

$$\mathbb{M}^\sigma f(x) := \sup_{1 \leq t \leq 2} \left| \int_0^1 f(x - t\Gamma(s)) \kappa(s)^\sigma ds \right|,$$

where $\kappa(s)$ denotes the curvature of the curve $\Gamma(s) = (s, g(s) + 1)$, $g(s) \in C^5[0, 1]$ in $\mathbb{R}^2$. Let Δ be the closed triangle with vertices $P = (\frac{2}{5}, \frac{1}{5})$, $Q = (\frac{1}{2}, \frac{1}{2})$, $R = (0, 0)$. In this paper, we prove that for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > \max\{\sigma^{-1}, 2\}\}$, there is a constant B such that $\|\mathbb{M}^\sigma f\|_{L^q(\mathbb{R}^2)} \leq B \|f\|_{L^p(\mathbb{R}^2)}$.

Keywords Maximal operator · Infinitely flat curve · Mitigating factors · Local smoothing

Mathematics Subject Classification Primary 42B25; Secondary 42B15 · 46T30

1 Introduction

Let $\mathcal{C}$ denote a smooth, compactly supported curve in the plane that does not pass through the origin, and denoting by $t\mathcal{C}$ the curve dilated by a factor $t > 0$, we consider the averaging operator defined for functions $f \in \mathcal{S}$, the Schwartz class of functions, by

$$M_t f(x) := \int_{\mathcal{C}} f(x - ty) d\sigma(y),$$

where $d\sigma$ denotes the normalized arc length measure over the curves $\mathcal{C}$. Consider now the maximal operator given by

$$\mathcal{M}f(x) := \sup_{t>0} |M_t f(x)|.$$

This work is dedicated to my late mother Smt. Dipali Manna

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It is not obvious that such averaging operators are well defined for f in L^p -spaces, since the curve $\mathcal{C}$ has measure zero in $\mathbb{R}^2$. Nevertheless, a priori $L^p \rightarrow L^q$ estimates are possible when $\mathcal{C}$ has suitable curvature properties. Therefore, a natural question we ask is for what range of the exponents p and q is the following a priori inequality satisfied:

$$\|\mathcal{M}f\|_{L^q(\mathbb{R}^2)} \leq B \|f\|_{L^p(\mathbb{R}^2)}, \quad f \in \mathcal{S}. \quad (1.1)$$

There is a vast literature on maximal and averaging operators over families of lower dimensional curves in the plane (see [15, 16]). The study of such a maximal operator over dilations of a fixed curves $\mathcal{C} \subset \mathbb{R}^2$ has its beginnings in the circular maximal theorem of Bourgain (see, [2]). Bourgain showed that when $\mathcal{C} = \mathcal{S}^1$, the unit circle, the corresponding maximal operator is bounded on $L^p(\mathbb{R}^2)$ for $p > 2$. His proof of the circular maximal theorem relies more directly on the geometry involved. The relevant geometry information concerns intersections of pairs of thin annuli, (for more details, see [2]). Another proof is due to Mockenhaupt, Seeger and Sogge (see, [10]) and proof of this result is based on their local smoothing estimates (see also, [11]). These local smoothing estimates, as well as Bourgain's original techniques, actually imply that if one modifies the definition so that the supremum is taken over $1 < t < 2$, then the resulting circular maximal operator is bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for some $q > p$. Here, the maximal operator $\mathbb{M}$ is defined by

$$\mathbb{M}f(x) := \sup_{1 \leq t \leq 2} \left| \int_{\mathcal{C}} f(x - ty) \, d\sigma(y) \right|. \quad (1.2)$$

Let Δ be the closed triangle with vertices $P = (\frac{2}{5}, \frac{1}{5})$, $Q = (\frac{1}{2}, \frac{1}{2})$, $R = (0, 0)$. In 1997, Schlag (see, [12]) showed that if $\mathcal{C}$ is unit circle, then the maximal operator $\mathbb{M}$ satisfies the inequality (1.1) if $(\frac{1}{p}, \frac{1}{q})$ lies in the interior of Δ . His result was obtained using the “combinatorial method” of Kolasa and Wolff cf. [7]. A different proof of this result was later obtained by Schlag and Sogge cf. [13], which was based on a simple application of Sobolev's theorem and the appropriate local smoothing estimates. Schlag also showed that except possibly for endpoints, this result is sharp (see, [12, 13]). Later, in 2002, Sanghyuk Lee (see, [8]) considered the remaining endpoint estimates for the circular maximal operator.

Therefore, a natural question we ask is if the same a priori maximal inequality (1.1) holds even if we consider a situation when the curvature is allowed to vanish of finite order on a finite set of isolated points. In this connections, Iosevich (see, [6]) had already shown that: If $\mathcal{C}$ is of finite-type curve, whose curvature vanishes to order at most $m - 2$ at a single point. Then, the inequality

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^2)} \leq B_p \|f\|_{L^p(\mathbb{R}^2)},$$

holds for $p > m$, also this result is sharp i.e., $\mathcal{M}$ is unbounded if $p = m$. If we study finite-type curves in the plane given by $\Gamma(s) = (s, g(s) + 1)$, $s \in [0, 1]$, for some suitably smooth g , where $0 = g(0) = g'(0) = \dots = g^{(m-1)}(0) \neq g^{(m)}(0) > 0$, then we can reinterpret his results as follows. Define

$$\mathcal{M}_k f(x) := \sup_{t > 0} \left| \int_{2^{-k}}^{2^{1-k}} f(x - t\Gamma(s)) \, ds \right|,$$

for Schwartz class of functions f . Iosevich proved that

$$\|\mathcal{M}_k f\|_{L^p \rightarrow L^p} \leq c_p 2^{-k(1-m/p)},$$

for $p > 2$. If we note that $\kappa(s)$, the curvature of the curve $\Gamma(s)$ is approximately $2^{-k(m-2)}$ whenever $s \in [2^{-k}, 2^{1-k}]$, then we have that the operator

$$\mathcal{M}^\sigma f(x) := \sup_{t>0} \left| \int_0^1 f(x - t\Gamma(s)) \kappa(s)^\sigma ds \right|,$$

is bounded on L^p for some $p > 2$, if σ is sufficiently large, since

$$\begin{aligned} \|\mathcal{M}^\sigma f\|_{L^p \rightarrow L^p} &\leq C \sum_{k \geq 0} 2^{-k(m-2)\sigma} \|\mathcal{M}_k f\|_{L^p \rightarrow L^p} \\ &\leq C \sum_{k \geq 0} 2^{-k((m-2)\sigma + 1 - m/p)} \end{aligned} \quad (1.3)$$

which is finite so long as $\sigma > (m/p - 1)(m - 2)^{-1}$. Now, we are interested to study the maximal operator $\mathcal{M}^\sigma$ for curves with curvature vanishing of infinite order at a given point. For that, first we need to choose σ independent of m such that the summation in (1.3) converges for the finite-type curves of order m . In fact, for each fixed $p > 2$, $\sigma = \frac{1}{p}$ is indeed one such choice and satisfies the condition $\sigma > (m/p - 1)(m - 2)^{-1}$ for $m > 2$. Hence, for the finite-type curves of order $m > 2$, $\mathcal{M}^{\frac{1}{p}}$ is bounded on L^p for some fixed $p > 2$. In this connection, Marletta [9] proved that $\mathcal{M}^\sigma$ is bounded on L^p for all $p > \max\{\sigma^{-1}, 2\}$, for a class of infinitely flat, convex curves in the plane. Counterexamples in [9] showed that this is the best possible result, in the sense that there exist flat curves for which $\mathcal{M}^\sigma$ is unbounded for $2 < p \leq \sigma^{-1}$.

Remark 1.1 As has been observed by J. G. Bak, such estimates for maximal functions with mitigating factors give rise to Orlicz space estimates for the corresponding maximal functions with no mitigating factor, with the help of Bak's interpolation lemma. See, for example [1] for the higher dimensional equivalents.

In this paper, we consider the maximal operator $\mathbb{M}^\sigma$, given by

$$\mathbb{M}^\sigma f(x) := \sup_{1 \leq t \leq 2} \left| \int_0^1 f(x - t\Gamma(s)) \kappa(s)^\sigma ds \right|.$$

Here, we shall extend the result of Marletta [9] to $L^p \rightarrow L^q$ estimates for the corresponding maximal operator $\mathbb{M}^\sigma$ associated with families of smooth, compactly supported curves in the plane. The proof of our main result, Theorem 2.1 will strongly make use of the results in [9] by Marletta.

We first decompose the operator $\mathbb{M}^\sigma$ into a family of operators $\{\mathbb{M}_k\}_{k \geq 0}$. We can actually obtain the $L^p \rightarrow L^q$ estimate from the corresponding $L^p \rightarrow L^q$ estimate for each of the operators $\mathbb{M}_k$, $k \geq 0$, by summing a geometric series. We make a decomposition of the s -space into intervals where the curvature $\kappa(s)$ of $\Gamma(s)$ is relatively constant, see Sect. 4. This is markedly different from the decomposition used in Fourier analysis (e.g., averages over flat curves in the plane), where dealing with dyadic decomposition in the space variable yields required estimates for the maximal operator over finite-type curves [6].

Our estimate for $\mathbb{M}_k f$ relies on stationary phase method and the local smoothing estimates of S. Lee [8]. The $L^p - L^p$ boundedness of the operators $\mathbb{M}_k f$, $k \geq 0$ is known by the work of Iosevich. Here, we will extend this estimate to $L^p \rightarrow L^q$ and also in the case where we consider the maximal operator over infinitely flat curves.

A key fact which leads to the proof is that the norm of the operator $\mathbb{M}_k$ will remain the same even the curve $\Gamma(s)$ is replaced by $L(\Gamma(s))$, where L is a unimodular linear map on $\mathbb{R}^2$, see Lemma 3.5.

Next, we formulate the main result of this paper in Sect. 2.

2 Main result and preliminaries

In this section, we shall state our main result of this paper.

Theorem 2.1 *Let $\Gamma(s) = (s, g(s) + 1)$, $s \in [0, 1]$, be a curve in $\mathbb{R}^2$ such that $g \in C^5[0, 1]$, $g(0) = g'(0) = 0$, $g^{(r)}(s)$ is single-signed and monotonic for $r = 1, 2, 3, 4, 5$, and both $\log g''(s)$ and $\log g'''(s)$ are concave. Let,*

$$\mathbb{M}^\sigma f(x) = \sup_{1 \leq t \leq 2} \left| \int_0^1 f(x - t\Gamma(s)) \kappa(s)^\sigma ds \right|,$$

where $\kappa(s)$ denotes the curvature of the curve $\Gamma(s)$.

Then, for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > \max\{\sigma^{-1}, 2\}\}$, there is a constant B such that the following inequality

$$\|\mathbb{M}^\sigma f\|_{L^q(\mathbb{R}^2)} \leq B \|f\|_{L^p(\mathbb{R}^2)}, \quad f \in \mathcal{S}(\mathbb{R}^2) \quad (2.1)$$

holds.

Remark 2.2 As $|\Gamma'(s)|$ is bounded between two constants independent of everything, we have

$$0 \leq c g''(s) \leq \kappa(s) \leq g''(s),$$

and so we may replace $\kappa(s)$ by $g''(s)$ in the maximal function. Notice also that, as $g''(s)$ is bounded, we may assume that $0 < c \leq g''(s) \leq 1$. The strange conditions concerning $\log g''(s)$ and $\log g'''(s)$ are more than we require here, see Sect. 4.

We now detail the dyadic decomposition of the dual space that is needed.

2.1 The dyadic decomposition

The proof of our main result, as well as many other arguments that involve explicitly (or implicitly) the Fourier transform, makes use of the division of the dual (frequency) space into dyadic shells. Dyadic decomposition, whose ideas originated in the work of Littlewood and Paley, and others, will now be described in the form most suitable for us (see, [15]).

Let β be a non-negative radial function in $C_c^\infty(\mathbb{R}^2)$ supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ such that

$$\sum_{j=-\infty}^{\infty} \beta(2^{-j}\xi) = 1 \text{ for } \xi \neq 0.$$

For example, we shall take,

$$\Phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{2} \\ 0, & \text{if } |\xi| \geq 1. \end{cases}$$

and

$$\beta(\xi) = \Phi\left(\frac{\xi}{2}\right) - \Phi(\xi).$$

Then, one can easily see that $\sum_j \beta(2^{-j}\xi) = 1$, $\xi \neq 0$ (see [4]).

We shall use C as a constant independent of j , several times without mention it.

3 Scaling

Lemma 3.1 is crucial in the proof of our theorem. The idea of the proof of the Lemma 3.1 is based on Iosevich's approach in [6].

Lemma 3.1 *Let $\tilde{\Omega}(s) = (s + \alpha, \omega(s) + \beta)$, $s \in [0, 1]$, be a curve in $\mathbb{R}^2$ such that $\omega \in C^5[0, 1]$, $\omega(0) = \omega'(0) = 0$, $\omega'(s) \leq 1/2$ and $0 < c \leq \omega'' \leq C < \infty$, where β and α are constants satisfying $|\beta| \geq \max\{|\alpha|, 2\}$. Let,*

$$\tilde{M}_t f(x) = \int_0^1 f(x - t\tilde{\Omega}(s)) \, ds,$$

and

$$\tilde{\mathbb{M}}f(x) = \sup_{1 \leq t \leq 2} |\tilde{M}_t f(x)|,$$

be the averaging operator and the maximal function corresponding to $\tilde{\Omega}$, respectively.

Then, for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > 2\}$, the following inequality

$$\|\tilde{\mathbb{M}}f\|_{L^q(\mathbb{R}^2)} \leq B |\beta|^{1/q} \|f\|_{L^p(\mathbb{R}^2)}, \quad f \in \mathcal{S}(\mathbb{R}^2) \quad (3.1)$$

holds, where the constant B depends only on the c, C and the C^5 norm of $\omega(s)$.

Remark 3.2 In view of Iosevich's theorem (see, [6]), the maximal operator $\tilde{\mathbb{M}}$ is also of course bounded when the exponents lie on the half-open segment connecting $(\frac{1}{2}, \frac{1}{2})$ and $(0, 0)$.

Proof of Lemma 3.1: First, we assume that ω is defined in $[-2, 2]$ with $\omega'' \neq 0$ away from 0. Let M_t be the corresponding averaging operator. Now, choose a bump function $\phi \in C_c^\infty(\mathbb{R})$ supported in $[-2, 2]$ such that $\phi \geq 0$ and $\phi \equiv 1$ on $[0, 1]$. We may assume that $f \geq 0$. Consider the linear operator

$$\tilde{M}_t f(x) = \int_{\mathbb{R}} f(x - t\tilde{\Omega}(s)) \phi(s) \, ds.$$

For $f \geq 0$, we have $M_t f \lesssim \tilde{M}_t f$ and in the opposite direction, $\tilde{M}_t f \lesssim M_t f +$ a maximal operator on a curve with nonvanishing curvature.

Note that, the inequality (3.1) is equivalent to the following estimate for $\tilde{\mathbb{M}}$:

$$\|\tilde{\mathbb{M}}f\|_{L^q(\mathbb{R}^2)} \leq B (|\sigma| + 1)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^2)}, \quad f \in \mathcal{S}(\mathbb{R}^2), \quad (3.2)$$

for every $\sigma = (\alpha, \beta) \in \mathbb{R}^2$, where B is an admissible constant.

By means of the Fourier inversion formula, we can write

$$\tilde{M}_t f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-t\sigma) \cdot \xi} H(t\xi) \hat{f}(\xi) \, d\xi,$$

where

$$H(\xi_1, \xi_2) := \int_{\mathbb{R}} e^{-i(\xi_1 s + \xi_2 \omega(s))} \phi(s) \, ds.$$

By a stationary phase computation (for instance, see Stein [15], p. 360, Iosevich [6], Lemma 1.2, Sogge [14], see also [5]), we obtain

$$H(\xi) = e^{i\psi(\xi)} \frac{\chi(\frac{\xi_1}{\xi_2})A(\xi)}{(1 + |\xi|)^{\frac{1}{2}}} + B(\xi),$$

where χ is a smooth function supported on a small neighborhood of the origin. Moreover, $\psi(\xi)$ is a smooth function of ξ which is homogeneous of degree 1 in ξ and also the Hessian $D_{\xi}^2 \psi(\xi)$ has rank 1. Moreover, A is a symbol of order zero such that $A(\xi) = 0$, if $|\xi| \leq C$, and

$$|\xi^{\alpha} D_{\xi}^{\alpha} A(\xi)| \leq C_{\alpha}, \quad \alpha \in \mathbb{N}^2, \quad |\alpha| \leq 3, \quad (3.3)$$

where the C_{α} are admissible constants. Finally, B is a remainder term satisfying

$$|D_{\xi}^{\alpha} B(\xi)| \leq C_{\alpha, N} (1 + |\xi|)^{-N}, \quad |\alpha| \leq 3, \quad 0 \leq N \leq 3, \quad (3.4)$$

again with admissible constants $C_{\alpha, N}$. If we put

$$\tilde{M}_t^0 f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-t\sigma) \cdot \xi} B(t\xi) \hat{f}(\xi) \, d\xi,$$

then by (3.4) $\tilde{M}_t^0 f(x) = f \star k_t^{\sigma}(x)$, where $k_t^{\sigma}(x) = t^{-2} k^{\sigma}(\frac{x}{t})$ and where k^{σ} is the translate

$$k^{\sigma}(x) := k(x - \sigma) \quad (3.5)$$

of k by the vector σ of a fixed function k satisfying an estimate of the form

$$|k(x)| \leq C(1 + |x|)^{-3}. \quad (3.6)$$

Let $\tilde{\mathbb{M}}^0 f(x) := \sup_{1 \leq t \leq 2} |\tilde{M}_t^0 f(x)|$ denote the corresponding maximal operator. Then, (3.5) and the inequality (3.6) show that $\|\tilde{\mathbb{M}}^0\|_{L^{\infty} \rightarrow L^{\infty}} \leq C$, with a constant C which does not depend on σ . Moreover, scaling by the factor $(|\sigma| + 1)^{-1}$ in direction of the vector σ , we see that

$$|\tilde{M}_t^0 f(x)| \lesssim (|\sigma| + 1) M,$$

where M is the Hardy-Littlewood maximal operator. Hence, (3.2) holds for $\tilde{\mathbb{M}}^0$ in place of $\tilde{\mathbb{M}}$, for every $1 \leq p \leq q$.

The maximal operator $\tilde{\mathbb{M}}^1$, corresponding to the family of averaging operators

$$\tilde{M}_t^1 f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i[\xi \cdot x - t(\sigma \cdot \xi + \psi(\xi))]} \frac{\chi(\frac{\xi_1}{\xi_2})A(\xi)}{(1 + |\xi|)^{\frac{1}{2}}} \hat{f}(\xi) \, d\xi,$$

remains to be studied.

Hence, it is enough to show that

$$\left\| \sup_{1 \leq t \leq 2} |\tilde{M}_t^1 f(x)| \right\|_{L^q(\mathbb{R}^2)} \leq B(|\sigma| + 1)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^2)}, \quad (3.7)$$

for p, q as in the Lemma 3.1.

Using this Fourier integral representation, we shall break up the operators dyadically. For this purpose, let us fix $\beta \in C_c^\infty(\mathbb{R} \setminus \{0\})$ satisfying $\sum_{-\infty}^{\infty} \beta(2^{-j}s) = 1$, $s \neq 0$, as in Sect. 2.

We then define the dyadic operator $A_{j,t}$ by

$$A_{j,t}f(x,t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i[\xi \cdot x - t(\sigma \cdot \xi + \psi(\xi))]} \frac{\chi(\frac{\xi_1}{\xi_2})A(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} \beta(2^{-j}|t\xi|) \hat{f}(\xi) d\xi.$$

Since we may assume that A vanishes on a sufficiently large neighborhood of the origin, we have $A_{j,t}f = 0$, if $j \leq 0$, so that

$$\tilde{\mathbb{M}}_t^1 f(x) = \sum_{j=1}^{\infty} A_{j,t}f(x).$$

Therefore, the inequality (3.2), would follow from showing that when $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > 2\}$, there is a constant B such that

$$\left\| \sup_{1 \leq t \leq 2} \sum_{j=1}^{\infty} |A_{j,t}f(x)| \right\|_{L^q(\mathbb{R}^2)} \leq B(|\sigma| + 1)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.8)$$

Now, choose a bump function $\tilde{\psi} \in C_c^\infty(\mathbb{R})$ supported in $[\frac{1}{2}, 4]$ such that $\tilde{\psi}(t) = 1$ if $1 \leq t \leq 2$.

In order to estimate (3.8), we use the following well-known estimate (see e.g., [6], Lemma 1.3),

$$\begin{aligned} & \sup_{t \in \mathbb{R}} |\tilde{\psi}(t) A_{j,t}f(x)|^q \\ & \leq q \left(\int_{-\infty}^{\infty} |\tilde{\psi}(t) A_{j,t}f(x)|^q dt \right)^{\frac{q-1}{q}} \left(\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t} (\tilde{\psi}(t) A_{j,t}f(x)) \right|^q dt \right)^{\frac{1}{q}} \end{aligned} \quad (3.9)$$

which follows by using the fundamental theorem of calculus and Hölder's inequality. By Hölder's inequality, this implies

$$\begin{aligned} & \left\| \sup_{1 \leq t \leq 2} A_{j,t}f(x) \right\|_{L^q(\mathbb{R}^2)}^q \\ & \leq C q \left(\int_{\frac{1}{2}}^4 \int_{\mathbb{R}^2} |A_{j,t}f(x)|^q dx dt \right)^{\frac{q-1}{q}} \left(\int_{\frac{1}{2}}^4 \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial t} (A_{j,t}f(x)) \right|^q dx dt \right)^{\frac{1}{q}} \\ & \quad + C \int_{\frac{1}{2}}^4 \int_{\mathbb{R}^2} |A_{j,t}f(x)|^q dx dt. \end{aligned} \quad (3.10)$$

Now,

$$\frac{\partial}{\partial t} (A_{j,t}f(x)) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i[\xi \cdot x - t(\sigma \cdot \xi + \psi(\xi))]} \chi(\frac{\xi_1}{\xi_2}) h(t, j, \xi) \hat{f}(\xi) d\xi,$$

where,

$$h(t, j, \xi) = -i \frac{[\sigma \cdot \xi + \psi(\xi)] A(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} \beta(2^{-j}|t\xi|) + \frac{\partial}{\partial t} \left(\frac{A(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} \right) \beta(2^{-j}|t\xi|) \\ + \frac{A(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} (2^{-j}|\xi|) \beta'(2^{-j}|t\xi|).$$

Now, if $t \sim 1$, since A vanishes near the origin, we see that the amplitude in the integral in $A_{j,t}$ can be written as $2^{-j/2} a_{j,t}(\xi)$, where $a_{j,t}$ is a symbol of order 0 localized where $|\xi| \sim 2^j$. Similarly, the amplitude of $\frac{\partial}{\partial t} A_{j,t}$ can be written as $2^{j/2}(|\sigma| + 1) b_{j,t}$, where $b_{j,t}$ is a symbol of order 0 localized where $|\xi| \sim 2^j$.

Since, $\psi(\xi) \approx |\xi| \approx 2^j$, on the support of β , we can calculate the orders of the symbols to see that $\|\sup_{1 \leq t \leq 2} A_{j,t} f(x)\|_{L^q(\mathbb{R}^2)}$ in (3.10) is dominated by

$$C 2^{-j(\frac{1}{2} - \frac{1}{q})} (|\sigma| + 1)^{1/q} \|\mathcal{F}_j f\|_{L^q(\mathbb{R}^3)},$$

where

$$\mathcal{F}_j f = \tilde{\psi}(t) \int e^{ix \cdot \xi} e^{-it\psi(\xi)} \beta(2^{-j}|\xi|) a(t, \xi) \hat{f}(\xi) d\xi,$$

with $a(t, \xi)$ is a symbol of order 0 in ξ . A similar argument can also be seen in [5].

Now, we need a local smoothing estimates for the operators of the form

$$P_j f(x, t) = \int e^{ix \cdot \xi} e^{it\psi(\xi)} a(t, \xi) \beta(2^{-j}|\xi|) \hat{f}(\xi) d\xi, \quad (3.11)$$

where $a(t, \xi)$ is a symbol of order 0 in ξ and the Hessian matrix of ψ has rank 1 everywhere.

We will get the smoothing estimates by using some sharp Carleson-Sjölin type estimates for the 2-dimensional wave equation. For this, let us define,

$$\mathcal{F}_t f(x) = \int_{\mathbb{R}^2} e^{i[x \cdot \xi + t|\xi|]} \hat{f}(\xi) d\xi.$$

We wish to use the following $L^p \rightarrow L^q$ local smoothing estimates (see, [8]), which we shall only use for $N = 2^j$.

Theorem 3.3 *If $\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$, then for $\frac{1}{p} + \frac{3}{q} = 1$, $\frac{14}{3} < q \leq \infty$, we have*

$$\left(\int_{\mathbb{R}^2} \int_1^2 |\mathcal{F}_t f(x)|^q dt dx \right)^{\frac{1}{q}} \leq C N^{\frac{3}{2} - \frac{6}{q}} \|f\|_{L^p(\mathbb{R}^2)}.$$

In the proof of Theorem 3.3, S. Lee has used the bilinear cone restriction estimate of Wolff [19], and Tao [17] together with a decomposition technique which was used in [18], Sect. 4.

Now, for our operator $P_j f(x, t)$ (as can be seen by simple modification of the proof of Lee with $\psi(\xi) = C|\xi| + O(1)$, $|\xi| \sim 2^j$, see [8]), we get for $\frac{1}{p} + \frac{3}{q} = 1$, $\frac{14}{3} < q \leq \infty$,

$$\left(\int_{\mathbb{R}^2} \int_1^2 |P_j f(x)|^q dt dx \right)^{\frac{1}{q}} \leq C 2^{j(\frac{3}{2} - \frac{6}{q})} \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.12)$$

Let,

$$\mathbb{M}_j f(x) = \sup_{1 \leq t \leq 2} |A_{j,t} f(x)|.$$

Now, using the local smoothing estimates (3.12), from (3.10), it is easy to see that for $\frac{1}{p} + \frac{3}{q} = 1$, and $\frac{14}{3} < q \leq \infty$,

$$\|\mathbb{M}_j f\|_q \leq C 2^{j(1-\frac{5}{q})} (|\sigma| + 1)^{1/q} \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.13)$$

By Plancherel's theorem and from the estimate (3.9), it is easy to see that for $j \geq 1$,

$$\|\mathbb{M}_j f\|_2 \leq C (|\sigma| + 1)^{1/2} \|f\|_{L^2(\mathbb{R}^2)}. \quad (3.14)$$

A complex interpolation between (3.13) and (3.14) shows that if $(\frac{1}{p}, \frac{1}{q})$ is contained in the closed triangle with vertices $(1, 0)$, $(\frac{5}{14}, \frac{3}{14})$, $(\frac{1}{2}, \frac{1}{2})$ but is not on the closed line segment $[(\frac{5}{14}, \frac{3}{14}), (\frac{1}{2}, \frac{1}{2})]$, then

$$\|\mathbb{M}_j f\|_q \leq C (|\sigma| + 1)^{1/q} 2^{j(\frac{3}{p} - \frac{1}{q} - 1)} \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.15)$$

To sum up the last estimates, we use the following interpolation lemma ([8], Lemma 2.6), which is an extension to multilinear operators of a theorem in [3]. Now, we denote by $L^{p,r}$ the Lorentz spaces.

Lemma 3.4 (An interpolation lemma) *Let $\epsilon_1, \epsilon_2 > 0$. Suppose that $\{T_j\}$ is a sequence of linear (or sublinear) operators such that for some $1 \leq p_1, p_2 < \infty$, and $1 \leq q_1, q_2 < \infty$,*

$$\|T_j f\|_{q_1} \leq M_1 2^{\epsilon_1 j} \|f\|_{p_1}, \quad \|T_j f\|_{q_2} \leq M_2 2^{-\epsilon_2 j} \|f\|_{p_2}.$$

Then, $T = \sum T_j$ is bounded from $L^{p,1}$ to $L^{q,\infty}$ with

$$\|T f\|_{L^{q,\infty}} \leq C M_1^\theta M_2^{1-\theta} \|f\|_{L^{p,1}},$$

where $\theta = \frac{\epsilon_2}{(\epsilon_1 + \epsilon_2)}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{(1-\theta)}{q_2}$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{(1-\theta)}{p_2}$.

Using (3.15) and Lemma 3.4, we have for $(\frac{1}{p}, \frac{1}{q}) \in [P, Q)$,

$$\|\tilde{\mathbb{M}}^1 f\|_{L^{q,\infty}} \leq C (|\sigma| + 1)^{1/q} \|f\|_{L^{p,1}}. \quad (3.16)$$

Since $\mathbb{M}$ is a local operator, an interpolation (real interpolation) between these estimates and the trivial $L^\infty - L^\infty$ estimate, we get for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\})$,

$$\|\tilde{\mathbb{M}}^1 f\|_{L^q} \leq C (|\sigma| + 1)^{1/q} \|f\|_{L^p}. \quad (3.17)$$

For $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > 2\}$, we thus get,

$$\|\tilde{\mathbb{M}}^1 f\|_{L^q(\mathbb{R}^2)} \leq B (|\sigma| + 1)^{1/q} \|f\|_{L^p(\mathbb{R}^2)}.$$

Hence, we finish our proof of the lemma. $\square$

We shall need one more lemma in our proof of Theorem 2.1.

Lemma 3.5 *Let L be an invertible linear map from $\mathbb{R}^2$ to itself with $\det(J_L) = 1$, J_L = Jacobian matrix of L , and let $\Gamma(s)$ and $\tilde{\Gamma}(s)$ be two curves in the plane, related by $\tilde{\Gamma}(s) = L(\Gamma(s))$. Then, the following two maximal functions have identical $L^p \rightarrow L^q$ operator norms:*

$$Mf(x) = \left| \sup_{1 \leq t \leq 2} \int f(x - t\Gamma(s)) \, ds \right|$$

$$\tilde{M}f(x) = \left| \sup_{1 \leq t \leq 2} \int f(x - t\tilde{\Gamma}(s)) \, ds \right|$$

Proof We see that,

$$\begin{aligned}\tilde{M}f(x) &= \left| \sup_{1 \leq t \leq 2} \int f(x - t\tilde{\Gamma}(s)) \, ds \right| \\ &= \left| \sup_{1 \leq t \leq 2} \int f(L(L^{-1}x - t\Gamma(s))) \, ds \right| = \left| \sup_{1 \leq t \leq 2} \int f_L(L^{-1}x - t\Gamma(s)) \, ds \right| \\ &= M(f_L)(L^{-1}x),\end{aligned}$$

where $f_L(x) = f(Lx)$.

Thus, using change of variable formula with $\det(J_L) = 1$, we get

$$\begin{aligned}\|\tilde{M}f(x)\|_{L^q}^q &= \|Mf_L(L^{-1}x)\|_{L^q}^q = \|Mf_L(x)\|_{L^q}^q \\ &\leq \|M\|_{L^p \rightarrow L^q}^q \|f_L(x)\|_{L^p}^q = \|M\|_{L^p \rightarrow L^q}^q \|f(x)\|_{L^p}^q.\end{aligned}$$

Hence, we get conclusion of the Lemma. $\square$

4 Decomposition and proof of Theorem 2.1

In this section, we will prove our main Theorem 2.1. The main idea of the proves of the Theorem 2.1 is to decompose the maximal operator $\mathbb{M}f$ into a family of maximal operators $\mathbb{M}_k f$. In this paper, we use the well-known decomposition into parts where the curvature is approximately 2^{-k} . In this connection, Iosevich [6] was able to decompose his operator into parts where $s \sim 2^{-k}$, because he was interested only in finite-type curves. Such a decomposition is not appropriate when one considers infinitely flat curves. Clearly, these two methods are equivalent for finite-type curves, and indeed, for such curves we shall obtain essentially the same estimates as Iosevich. Thus, for non-negative f , we have

$$\mathbb{M}^\sigma f(x) \leq C \sum_{k \geq 0} 2^{-k\sigma} \mathbb{M}_k f(x)$$

where

$$\mathbb{M}_k f(x) = \sup_{1 \leq t \leq 2} \left| \int_{I_k} f(x - t\Gamma(s)) \, ds \right|,$$

with $I_k = [s_{k+1}, s_k]$ is the interval where $2^{-k-1} \leq g''(s) \leq 2^{-k}$. By scaling and a change of variable, we see that $\mathbb{M}_k$ will have the same $L^p \rightarrow L^q$ operator norms as the operator $\tilde{\mathbb{M}}_k$, given by

$$\tilde{\mathbb{M}}_k f(x) = |I_k| \sup_{1 \leq t \leq 2} \left| \int_0^1 f\left(x - t\left(s + s_{k+1}|I_k|^{-1}, \rho_k^{-1}g(s|I_k| + s_{k+1}) + \rho_k^{-1}\right)\right) \, ds \right|$$

for any fixed $\rho_k > 0$. Suppose there exist constants $\eta_k, \beta_k, \alpha_k$ such that

$$\begin{aligned}&\left(s + s_{k+1}|I_k|^{-1}, \rho_k^{-1}g(s|I_k| + s_{k+1}) + \rho_k^{-1}\right) \\ &= (s + \alpha_k, \omega_k(s) + \beta_k + \eta_k(s + \alpha_k))\end{aligned}\tag{4.1}$$

for some curve ω_k of the type dealt with in Lemma 3.1, and with C^5 norms bounded independent of k .

By Lemma 3.5 (with $Lv = (v_1, v_2 - \eta_k v_1)$), we would then have that $\tilde{\mathbb{M}}_k$ has operator norms controlled by $|I_k|$ times those of the maximal function corresponding to the curve

$$(s + \alpha_k, \omega_k(s) + \beta_k)$$

which by Lemma 3.1 has operator norms bounded by a constant times

$$B|I_k||\beta_k|^{1/q},$$

for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > 2\}$, with B independent of k , if $|\beta_k| \geq \max\{|\alpha_k|, 2\}$.

The question is, how can we arrange that (4.1) be true? For completeness, we shall briefly give the answer to this question. An explicit proof can also be found in [9].

A Taylor expansion of $g(s)$ about $s = s_{k+1}$ suggests writing

$$g(s|I_k| + s_{k+1}) = g(s_{k+1}) + s|I_k|g'(s_{k+1}) + \tilde{\omega}_k(s)$$

where $\tilde{\omega}_k(0) = \tilde{\omega}'_k(0) = 0$ and $\tilde{\omega}_k$ and its derivatives inherit their monotonicity from g .

Notice that for $s \in [0, 1]$, $\tilde{\omega}_k(s) \sim 2^{-k}|I_k|^2$. Hence, it is natural to choose

$$\omega_k(s) = 2^k|I_k|^{-2} \{g(s|I_k| + s_{k+1}) - g(s_{k+1}) - s|I_k|g'(s_{k+1})\} \quad (4.2)$$

so that $1/2 \leq \omega''_k(s) \leq 1$. Notice that ω_k inherits the monotonicity properties of $\tilde{\omega}_k$.

Now, we see that

$$\rho_k^{-1}g(s|I_k| + s_{k+1}) + \rho_k^{-1} = \rho_k^{-1} \left\{ 2^{-k}|I_k|^2\omega_k(s) + g(s_{k+1}) + 1 + s|I_k|g'(s_{k+1}) \right\}$$

and so choosing $\rho_k = 2^{-k}|I_k|^2$, we see that (4.1) will hold so long as

$$\alpha_k = s_{k+1}|I_k|^{-1}, \quad \eta_k = 2^k|I_k|^{-1}g'(s_{k+1})$$

and

$$\beta_k = 2^k|I_k|^{-2} \{g(s_{k+1}) + 1 - s_{k+1}g'(s_{k+1})\}.$$

Now, we will show that β_k, α_k satisfies the conditions of Lemma 3.1, i.e., $|\beta_k| \geq \max\{\alpha_k, 2\}$. Recall that, $I_k = [s_{k+1}, s_k]$ is the interval where $2^{-k-1} \geq g'' \geq 2^{-k}$. Hence, by mean value theorem, for all but finitely many k , we have

$$s_{k+1}g'(s_{k+1}) \leq 1/2.$$

So that, we have $|\beta_k| \geq 2^{k-1}|I_k|^{-2}$, for all but finitely many k . Thus, for all but finitely many k , we have $|\beta_k| \geq \max\{\alpha_k, 2\}$, since

$$2^{k-1}|I_k|^{-2} \geq 2 \Leftrightarrow |I_k|^2 \leq 2^{k-2}$$

and

$$2^{k-1}|I_k|^{-2} \geq \frac{s_{k+1}}{|I_k|} \Leftrightarrow |I_k| \leq \frac{2^{k-1}}{s_{k+1}},$$

both hold for all but finitely many k .

We notice that, $|\beta_k| \leq 2^k|I_k|^{-2}$. Assuming we can show that $\omega_k^{(r)}(s) \leq C$, for $r = 1, 2, 3, 4, 5$ and C independent of k , then we have that $\mathbb{M}_k$ has $L^p \rightarrow L^q$ operator norms bounded by

$$B|\beta_k|^{1/q}|I_k| \leq B2^{k/q}|I_k|^{1-2/q}$$

so long as $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q > 2\}$ (see, Lemma 3.1). This would then complete the proof of Theorem 2.1, as

$$\begin{aligned} \|\mathbb{M}^\sigma\|_{L^p \rightarrow L^q} &\leq B \sum_{k \geq 0} 2^{-k\sigma} \|\mathbb{M}_k\|_{L^p \rightarrow L^q} \\ &\leq B \sum_{k \geq 0} 2^{-k(\sigma-1/q)} |I_k|^{1-2/q} < \infty \end{aligned}$$

if $\sigma > 1/q$. We return now to those derivative estimates for ω_k . We shall show that ω_k has C^5 norms independent of k . The point here is that although ω_k depends on k , we shall show that it has C^5 norm independent of k and satisfies all the conditions of Lemma 3.1.

From (4.2), we notice that because the second derivative of ω_k is small, and because $\omega_k(0) = \omega'_k(0) = 0$, we have that $\omega'_k(s) \leq s$ and $\omega_k(s) \leq s^2/2$.

We have

$$\omega_k^{(r)}(s) = 2^k |I_k|^{r-2} g^{(r)}(s |I_k| + s_{k+1})$$

for $r = 3, 4, 5$. Given the monotonicity of $g^{(r)}$, and given that $2^{-k} = g''(s_k)$, we clearly require only that

$$|I_k|^{r-2} \frac{g^{(r)}(s_k)}{g''(s_k)} \leq C, \quad (4.3)$$

where C independent of k . This is where we shall use those conditions on $\log g''$ and $\log g'''$. The proof of (4.3) can be found in [9]. We briefly give the outline of the proof for the case $r = 3$. For the cases $r = 4$ and $r = 5$ will follow from our assumption that $\log g''$ and $\log g'''$ are concave, respectively. To proof (4.3) for the case $r = 3$, it is enough to show that

$$|I_k| \sim \frac{g''(s_k)}{g'''(s_k)}.$$

Now the mean value theorem gives us the following:

$$|I_k| = (g'')^{-1}(2^{-k}) - (g'')^{-1}(2^{-k-1}) = 2^{-k-1} \frac{1}{g'''(z)}$$

for some $z \in (s_{k+1}, s_k)$, and we get

$$c \frac{g''(s_k)}{g'''(s_k)} \leq |I_k| \leq C \frac{g''(s_{k+1})}{g'''(s_{k+1})}. \quad (4.4)$$

As our assumptions imply that $\frac{g''(s)}{g'''(s)}$ is monotonic increasing, and as $s_{k+1} < s_k$, the right-hand side of (4.4) is bounded by a constant times the left-hand side giving the stated result, that is $|I_k| \sim \frac{g''(s_k)}{g'''(s_k)}$. This completes the proof for the case $r = 3$ and hence the proof of Theorem 2.1.

5 The case $\sigma = 1/q$ and sharpness of results

In this section, we will see that a power $\sigma < 1/q$ of the curvature will not guarantee $L^p \rightarrow L^q$, ($q > p$) boundedness for our maximal function. For example, consider $\Gamma(s) =$

$(s, s^m + 1)$ and the test function

$$f(x) = \eta(x) |x_2|^{-1/q}$$

for some $q > p$, where $\eta \in C_c^\infty$ is a smooth bump function such that $\eta(x) = 1$ if $|x| \leq 1$, and $\eta(x) = 0$ if $|x| \geq 2$. Notice that $f \in L^r$ for all $r < q$, and hence f is in L^p .

Consider only points x that lie within distance $1/2$ of the origin, so that $|x - x_2 \Gamma(s)| \leq 1$ for all $s \in [0, 1]$. Then, we have for $1 \leq x_2 \leq 2$,

$$\begin{aligned} \mathbb{M}^\sigma f(x) &\geq C \int_0^1 f(x - x_2 \Gamma(s)) s^{(m-2)\sigma} ds \\ &\geq C \int_0^1 |x_2|^{-1/q} s^{(m-2)\sigma - m/q} ds, \end{aligned} \quad (5.1)$$

which diverges unless $(m-2)\sigma - m/q > -1$. So, if we wish to choose σ independent of $m \geq 2$, we clearly need $\sigma \geq 1/q$.

It is more interesting to ask what happens when we consider $\sigma = 1/q < 1/2$. In this case, we know by our earlier work that

$$\|\mathbb{M}^\sigma f\|_{L^p} \leq B \|f\|_{L^p} \sum_{k \geq 0} |I_k|^{1-2/q},$$

so if $|I_k|$ decays sufficiently fast, then we will have that $\mathbb{M}^\sigma$ is $L^p \rightarrow L^q$ bounded. For example, we have the following corollary.

Corollary 5.1 *Suppose Γ and g are as in Theorem 2.1, and suppose in addition that there exists an $\epsilon > 0$ such that*

$$|\log g''(s)|^{1+\epsilon} \left(\frac{g''(s)}{g'''(s)} \right)$$

is bounded.

Then, we have $|I_k| \leq Bk^{-1-\epsilon}$, and for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{(\frac{1}{p}, \frac{1}{q}) : q = \sigma^{-1} > 2(\epsilon^{-1} + 1)\}$, the maximal function of Theorem 2.1 is bounded on $L^p \rightarrow L^q$.

Proof The proof of Corollary 5.1 is a matter of simple calculus, working with the function $\gamma : \mathbb{R}^+ \rightarrow [0, 1]$, defined implicitly by

$$\log g''(\gamma(x)) = -(x^{-1/\epsilon}) \log 2.$$

This is chosen so that

$$\gamma(k^{-\epsilon}) = s_k,$$

where s_k is as before, defined by $g''(s_k) = 2^{-k}$. We have that $|I_k| = \gamma(k^{-\epsilon}) - \gamma((k+1)^{-\epsilon})$, which is $(k^{-\epsilon} - (k+1)^{-\epsilon}) \gamma'(z)$, for some $z \in (k, k+1)$. We ask that γ' be uniformly bounded, this being the hypothesis of Corollary 5.1, and hence $|I_k| \leq Ck^{-\epsilon-1}$, from which the desired result follows easily. $\square$

Remark 5.2 Notice that, the Corollary 5.1 applies to the family of flat curves $g(s) = \exp(-s^{-N})$, for $N \geq 1$, with $\epsilon = 1/N$.

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References

1. Bak, J.G.: Averages over surfaces with infinitely flat points. *J. Funct. Anal.* **129**, 455–470 (1995)
2. Bourgain, J.: Averages in the plane over convex curves and maximal operators. *J. Anal. Math.* **47**, 69–85 (1986)
3. Bourgain, J.: Estimations de certaines fonctions maximales. *C. R. Acad. Sci. Paris Ser. I Math.* **301**(10), 499–502 (1985)
4. Duoandikoetxea, J.: *Fourier Analysis*. Graduate Studies in Mathematics, vol. 29. American Mathematical Society, New York (2001)
5. Ikromov, I.A., Kempe, M., Müller, D.: Estimates for maximal functions associated with hypersurfaces in $\mathbb{R}^3$ and related problems of harmonic analysis. *Acta Math.* **204**(2), 151–271 (2010)
6. Iosevich, A.: Maximal operators associated to families of flat curves in the plane. *Duke Math. J.* **76**(2), 633–644 (1994)
7. Kolasa, L., Wolff, T.: on some variants of the keakeya problem. *Pacific J. Math.* **190**(1), 111–154 (1999)
8. Lee, S.: Endpoint estimates for the circular maximal function. *Proc. Am. Math. Soc.* **131**(5), 1433–1442 (2003)
9. Marletta, G.: Maximal functions with mitigating factors in the plane. *J. Lond. Math. Soc. (2)* **59**(2), 647–656 (1999)
10. Mockenhaupt, G., Seeger, A., Sogge, C.D.: Wave front sets, local smoothing and Bourgain’s circular maximal theorem. *Ann. Math.* **136**, 207–218 (1992)
11. Mockenhaupt, G., Seeger, A., Sogge, C.D.: Local smoothing for Fourier integral operators and Carleson–Sölin estimates. *J. Am. Math. Soc.* **6**, 65–130 (1993)
12. Schlag, W.: A generalization of Bourgain’s circular maximal theorem. *J. Am. Math. Soc.* **10**, 103–122 (1997)
13. Schlag, W., Sogge, C.D.: Local smoothing estimates related to the circular maximal theorem. *Math. Res. Lett.* **4**, 1–15 (1997)
14. Sogge, C.D.: *Fourier Integrals in Classical Analysis*, vol. 105. Cambridge Tracts In Mathematics, Cambridge (1993)
15. Stein, E.M.: *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton University Press, Princeton (1993)
16. Stein, E.M., Wainger, S.: Problem in harmonic analysis related to curvature. *Bull. Am. Math. Soc.* **84**, 1239–1295 (1978)
17. Tao, T.: Endpoint bilinear restriction theorems for the cone and some sharp null form estimates. *Math. Z.* **238**, 215–268 (2001)
18. Tao, T., Vargas, A.: A bilinear approach to cone multipliers. II. *Geom. Funct. Anal.* **10**, 216–258 (2000)
19. Wolff, T.: A sharp cone restriction estimate. *Ann. Math.* **153**, 661–698 (2001)