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Local behavior of solutions to fractional Hardy–Hénon equations with isolated singularity

Yimei Li[1] · **Jiguang Bao**[1]

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Abstract In this paper, we study the local behaviors of positive solutions of

$$(-\Delta)^{\sigma} u = |x|^{\tau} u^{p}$$

with an isolated singularity at the origin, where $(-\Delta)^{\sigma}$ is the fractional Laplacian, $0 < \sigma < 1$, $\tau > -2\sigma$ and $p > 1$. Our first results provide a blowup rate estimate near an isolated singularity, and show that the solution is asymptotically radially symmetric.

Keywords Fractional Laplacian · Isolated singularity · Rate estimate · Asymptotically radially symmetric

Mathematics Subject Classification 35R11 · 35B40

1 Introduction

The Hardy–Hénon equation

$$-\Delta u = |x|^{\tau} u^{p} \quad \text{in} \;\; B_1 \backslash \{0\} \tag{1}$$

has been studied in many papers, where $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian, $\tau > -2$, $p > 1$ are parameters, the punctured unit ball $B_1 \setminus \{0\} \subset \mathbb{R}^n$ with $n \geq 3$.

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✉ Jiguang Bao
jgbao@bnu.edu.cn

Yimei Li
lym@mail.bnu.edu.cn

1 School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing Normal University, Beijing 100875, China

The blowup rate of solution of (1) has been very well understood. In the special case of $\tau = 0$, there exists a positive constant C such that

$$u(x) \le C|x|^{-\frac{2}{p-1}} \quad \text{near } \ x = 0, \quad 1 < p \le \frac{n+2}{n-2}, \ p \ne \frac{n}{n-2}. \tag{2}$$

See Lions [20] for $1 < p < \frac{n}{n-2}$, Gidas–Spruck [10] for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, Korevaar–Mazzeo–Pacard–Schoen [14] for $p = \frac{n+2}{n-2}$. Aviles [1,2] treated the case of $p = \frac{n}{n-2}$ and obtained that

$$u(x) \le C|x|^{2-n}(-\ln|x|)^{-\frac{n-2}{2}} \qquad \text{near } \ x = 0. \tag{3}$$

In the case of $-2 < \tau < 2$, the upper bound has the following forms near $x = 0$,

$$\begin{aligned} &u(x) \le C|x|^{2-n} && \text{if} \quad 1 < p < \tfrac{n+\tau}{n-2} \quad (\text{see [10,21,28]}),\\ &u(x) \le C|x|^{-\frac{2+\tau}{p-1}} && \text{if} \quad \tfrac{n+\tau}{n-2} < p < \tfrac{n+2}{n-2} \quad (\text{see [10]}),\\ &u(x) \le C|x|^{2-n}(-\ln|x|)^{-\frac{n-2}{2+\tau}} && \text{if} \quad p = \tfrac{n+\tau}{n-2} \quad (\text{see [1,2]}). \end{aligned} \tag{4}$$

Phan–Souplet [22] studied the case of $-2 < \tau$ and $1 < p < \frac{n+2}{n-2}$, and derived that

$$u(x) \le C|x|^{-\frac{2+\tau}{p-1}}, \quad |\nabla u(x)| \le C|x|^{-\frac{p+1+\tau}{p-1}} \quad \text{near } \ x = 0, \tag{5}$$

where ∇u denotes the gradient of u. We point out that the first estimate of (5) extends a number of previously results of [10] and [20] regarding them as consequence.

In the classical paper [6], Caffarelli–Gidas–Spruck considered the semilinear elliptic equations

$$-\Delta u = g(u) \quad \text{in } \ B_1\backslash\{0\},$$

and proved that every solution is asymptotically radially symmetric

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \qquad \text{as } \ x \to 0. \tag{6}$$

Here $g(u)$ is a locally Lipschitz function and $\bar{u}(|x|) := \fint_{\mathbb{S}^n} u(|x|\theta)\mathrm{d}\theta$ is the spherical average of u. A typical example is $g(u) = u^p$ with $\frac{n}{n-2} \le p \le \frac{n+2}{n-2}$. Li [15] obtained a weaker asymptotically radially symmetric for more general $g(x, u) = |x|^\tau u^p$, where $\tau \le 0$, and $1 < p \le \frac{n+2+\tau}{n-2}$.

Inspired by previous work, this paper is aiming at studying the local behaviors of positive solutions of

$$(-\Delta)^\sigma u = |x|^\tau u^p \qquad \text{in } \ B_1\backslash\{0\}, \tag{7}$$

where $0 < \sigma < 1$, $\tau > -2\sigma$, $p > 1$, the punctured unit ball $B_1\backslash\{0\} \subset \mathbb{R}^n$, $n \ge 2$, and $(-\Delta)^\sigma$ is the fractional Laplacian taking the form

$$(-\Delta)^\sigma u(x) := C_{n,\sigma}\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}}\mathrm{d}y = C_{n,\sigma} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n\backslash B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}}\mathrm{d}y, \tag{8}$$

here P.V. stands for the Cauchy principal value and

$$C_{n,\sigma} := \frac{2^{2\sigma}\sigma\Gamma(\frac{n}{2} + \sigma)}{\pi^{\frac{n}{2}}\Gamma(1 - \sigma)}$$

with the gamma function Γ. The operator $(-\Delta)^\sigma$ is well defined in the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$.

One can also define the fractional Laplacian acting on spaces of functions with weaker regularity. Considering the space

$$L_\sigma(\mathbb{R}^n) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}} \mathrm{d}x < \infty \right\},$$

endowed with the norm

$$\|u\|_{L_\sigma(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}} \mathrm{d}x.$$

We can verify that if $u \in C^2(B_1\backslash\{0\}) \cap L_\sigma(\mathbb{R}^n)$, the integral on the right hand side of (8) is well defined in $B_1\backslash\{0\}$. Moreover, from [24, Proposition 2.4], we have

$$\begin{aligned} &(-\Delta)^\sigma u \in C^{1,1-2\sigma}(B_1\backslash\{0\}), \qquad \text{if } \ 0 < \sigma < \tfrac{1}{2}, \\ &(-\Delta)^\sigma u \in C^{0,2-2\sigma}(B_1\backslash\{0\}), \qquad \text{if } \ \tfrac{1}{2} \le \sigma < 1. \end{aligned}$$

Problems concerning the fractional Laplacian $(-\Delta)^\sigma$ with an internal isolated singularity have attracted a lot of attention. In particular, Caffarelli–Jin–Sire–Xiong [7] studied the local behaviors of positive solutions of the fractional Yamabe equations

$$(-\Delta)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}} \qquad \text{in } \ B_1\backslash\{0\} \tag{9}$$

with an isolated singularity at the origin. They obtained sharp blowup rate,

$$u(x) \le C|x|^{-\frac{n-2\sigma}{2}} \qquad \text{near } \ x = 0, \tag{10}$$

and proved that every local solution of (9) is asymptotically radially symmetric

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \qquad \text{as } \ x \to 0. \tag{11}$$

It is consistent with the result of Korevaar–Mazzeo–Pacard–Schoen [14] work on Laplacian. Jin-de Queiroz–Sire–Xiong [13] further studied Eq. (9) in $\Omega \setminus \Lambda$, where Ω is an open set in $\mathbb{R}^n$, and Λ is a singular set other than a single point. More work related isolated singular problem see Chen–Quaas [9] for $1 < p < \frac{n+2\sigma}{n-2\sigma}$, Sun–Jin [25] for higher order fractional case.

Our first result provides a blowup rate estimate near an isolated singularity.

Theorem 1.1 *Let $-2\sigma < \tau$, $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Suppose that $u \in C^2(B_1\backslash\{0\}) \cap L_\sigma(\mathbb{R}^n)$ is a positive solution of* (7)*, then there exists a positive constant $C = C(n, \sigma, \tau, p)$ such that*

$$u(x) \le C|x|^{-\frac{2\sigma+\tau}{p-1}}, \quad |\nabla u(x)| \le C|x|^{-\frac{2\sigma+\tau+p-1}{p-1}} \quad \text{near } x = 0. \tag{12}$$

The result can be understood as an extension of the work (5) and (10). We obtain the blowup rate estimate (12) by using the method of blowing-up and rescaling argument. (For more details and references, see, e.g., [7,11,13,22,27]). In particular, the doubling property (Proposition 2.1) plays a key role in our proof. The idea, by contradiction, is that if an estimate fails, the violating sequence of solutions u_k will be increasingly large along a sequence of points x_k such that each x_k has a suitable neighborhood, where the relative growth of u_k remains controlled. After appropriate rescaling, one can blow up the sequence of neighborhoods and pass to the limit to obtain a bounded solution of a limiting problem in the whole of $\mathbb{R}^n$. At last, by Liouville theorem, we get a contradiction.

With the help of the estimate (12), we are able to show that the solution of (7) is asymptotically radially symmetric.

Theorem 1.2 *Let $-2\sigma < \tau \le 0$, $\frac{n+\tau}{n-2\sigma} < p \le \frac{n+2\sigma+2\tau}{n-2\sigma}$. Suppose that $u \in C^2(B_1\backslash\{0\}) \cap L_\sigma(\mathbb{R}^n)$ is a positive solution of (7), then*

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \qquad \text{as } x \to 0,$$

where $\bar{u}(|x|) := \fint_{\mathbb{S}^n} u(|x|\theta)\mathrm{d}\theta$ is the spherical average of u.

The result can be understood as an extension of the work (6) and (11). Notice that we do not use any special structure of the unite ball B_1, B_1 can be replaced by general open sets containing the origin. We get the result by the method of moving spheres with Kelvin transformation, a variant of the method of moving planes, which has been widely used and has become a powerful and user-friendly tool in the study of nonlinear partial differential equations (see [8,16–19]). In addition, by Kelvin transformation, the difference between Eqs. (7) and (9) is that Eq. (9) is a conformally invariant equations but Eq. (7) is not. To deal with the problem, we need the following useful proposition.

Proposition 1.3 *For $x \in \mathbb{R}^n\backslash\{0\}$, $y \in \mathbb{R}^n \setminus \{x\}$, let $y_\lambda := x + \frac{\lambda^2(y-x)}{|y-x|^2}$, we have*

$$|x - y_\lambda||y| \le |x - y||y_\lambda|. \tag{13}$$

if and only if

$$\begin{aligned} &\lambda^2 \le |x-y|^2 && \text{if } x\cdot y \ge |x|^2/2, \\ &\lambda^2 \le \tfrac{|x|^2|x-y|^2}{|x|^2+(-2x\cdot y)^+} \ \text{ or } \ \lambda^2 \ge \tfrac{|x|^2|x-y|^2}{|x|^2-(2x\cdot y)^+} && \text{if } x\cdot y < |x|^2/2. \end{aligned} \tag{14}$$

One consequence of this proposition is the following corollary.

Corollary 1.4 *For $x \in \mathbb{R}^n\backslash\{0\}$, $0 < \lambda < \min\{|x|, |y-x|\}$, we have*

$$|x - y_\lambda||y| \le |x - y||y_\lambda|. \tag{15}$$

The outline of this paper is arranged as follows. In Sect. 2, we will obtain the blowup upper bound (12). Then with the help of the estimate we can devote to asymptotically radially symmetry property of solutions of (7) in Sect. 3. In Appendix, for readers' convenience, we not only prove some preliminaries, but also collect some basic propositions which will be used in our proof.

2 Upper bound near an isolated singularity

First, we recall the doubling property [23, Lemma 5.1] and denote $B_R(x)$ as the ball in $\mathbb{R}^n$ with radius R and center x. For convenience, we write $B_R(0)$ as B_R for short.

Proposition 2.1 *Suppose that $\emptyset \ne D \subset \Sigma \subset \mathbb{R}^n$, Σ is closed and $\Gamma = \Sigma \setminus D$. Let $M : D \to (0,\infty)$ be bounded on compact subset of D. If for a fixed positive constant k, there exists $y \in D$ satisfying*

$$M(y)\mathrm{dist}(y,\Gamma) > 2k,$$

then there exists $x \in D$ such that

$$M(x) \ge M(y), \qquad M(x)\mathrm{dist}(x,\Gamma) > 2k,$$

and for all $z \in D \cap B_{kM^{-1}(x)}(x)$,

$$M(z) \le 2M(x).$$

The second one is called the interior Schauder estimates. See [12, Theorem 2.11] for the proof. Readers can see [4] for more regularity issues.

Proposition 2.2 *Suppose that $g \in C^\gamma(B_R)$, $\gamma > 0$ and u is a nonnegative solution of*

$$(-\Delta)^\sigma u = g(x) \qquad \text{in } \ B_R.$$

If $2\sigma + \gamma \le 1$, then $u \in C^{0,2\sigma+\gamma}(B_{R/2})$. Moreover,

$$\|u\|_{C^{0,2\sigma+\gamma}(B_{R/2})} \le C\left(\|u\|_{L^\infty(B_{3R/4})} + \|g\|_{C^\gamma(B_{3R/4})}\right),$$

where C is a positive constant depending on n, σ, γ, R.

If $2\sigma + \gamma > 1$, then $u \in C^{1,2\sigma+\gamma-1}(B_{R/2})$. Moreover,

$$\|u\|_{C^{1,2\sigma+\gamma-1}(B_{R/2})} \le C\left(\|u\|_{L^\infty(B_{3R/4})} + \|g\|_{C^\gamma(B_{3R/4})}\right),$$

where C is a positive constant depending on n, σ, γ, R.

Next, in order to prove Theorem 1.1, we start with the following lemma.

Lemma 2.3 *Let $1 < p < \frac{n+2\sigma}{n-2\sigma}$, $0 < \alpha \le 1$ and $c(x) \in C^{2,\alpha}(\overline{B_1})$ satisfy*

$$\|c\|_{C^{2,\alpha}(\overline{B_1})} \le C_1, \quad c(x) \ge C_2 \ \text{ in } \overline{B_1} \tag{16}$$

for some positive constants C_1, C_2. Suppose that $u \in C^2(B_1) \cap L_\sigma(\mathbb{R}^n)$ is a nonnegative solution of

$$(-\Delta)^\sigma u = c(x)u^p \qquad \text{in } \ B_1, \tag{17}$$

then there exists a positive constant C depending only on n, σ, p, C_1, C_2 such that

$$|u(x)|^{\frac{p-1}{2\sigma}} + |\nabla u(x)|^{\frac{p-1}{p+2\sigma-1}} \le C[\mathrm{dist}(x, \partial B_1)]^{-1} \qquad \text{in } \ B_1.$$

Proof Arguing by contradiction, for $k = 1, 2, \cdots$, we assume that there exist nonnegative functions u_k satisfying (17) and points $y_k \in B_1$ such that

$$|u_k(y_k)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(y_k)|^{\frac{p-1}{p+2\sigma-1}} > 2k[\mathrm{dist}(y_k, \partial B_1)]^{-1}. \tag{18}$$

Define

$$M_k(x) := |u_k(x)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x)|^{\frac{p-1}{p+2\sigma-1}}.$$

Via Proposition 2.1, for $D = B_1$, $\Gamma = \partial B_1$, there exists $x_k \in B_1$ such that

$$M_k(x_k) \ge M_k(y_k), \quad M_k(x_k) > 2k[\mathrm{dist}(x_k, \partial B_1)]^{-1} \ge 2k, \tag{19}$$

and for any $z \in B_1$ and $|z - x_k| \le kM_k^{-1}(x_k)$,

$$M_k(z) \le 2M_k(x_k). \tag{20}$$

It follows from (19) that

$$\lambda_k := M_k^{-1}(x_k) \to 0 \qquad \text{as } k \to \infty, \tag{21}$$

$$\mathrm{dist}(x_k, \partial B_1) > 2k\lambda_k, \qquad \text{for } k = 1, 2, \cdots. \tag{22}$$

Consider

$$w_k(y) := \lambda_k^{\frac{2\sigma}{p-1}} u_k(x_k + \lambda_k y) \qquad \text{in } \ B_k.$$

Combining (22), we obtain that for any $y \in B_k$,

$$|x_k + \lambda_k y - x_k| \le \lambda_k |y| \le \lambda_k k < \frac{1}{2}\mathrm{dist}(x_k, \partial B_1),$$

that is,

$$x_k + \lambda_k y \in B_{\frac{1}{2}\mathrm{dist}(x_k, \partial B_1)}(x_k) \subset B_1.$$

Therefore, w_k is well defined in B_k and

$$\begin{aligned} |w_k(y)|^{\frac{p-1}{2\sigma}} &= \lambda_k |u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma}}, \\ |\nabla w_k(y)|^{\frac{p-1}{2\sigma+p-1}} &= \lambda_k |\nabla u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma+p-1}}. \end{aligned}$$

From (20), we find that for all $y \in B_k$,

$$|u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma+p-1}} \le 2\left(|u_k(x_k)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x_k)|^{\frac{p-1}{p+2\sigma-1}}\right).$$

That is,

$$|w_k(y)|^{\frac{p-1}{2\sigma}} + |\nabla w_k(y)|^{\frac{p-1}{2\sigma+p-1}} \le 2\lambda_k M_k(x_k) = 2. \tag{23}$$

Moreover, w_k satisfies

$$(-\triangle)^\sigma w_k = c_k(y) w_k^p \qquad \text{in } \ B_k, \tag{24}$$

and

$$|w_k(0)|^{\frac{p-1}{2\sigma}} + |\nabla w_k(0)|^{\frac{p-1}{2\sigma+p-1}} = 1,$$

where $c_k(y) := c(x_k + \lambda_k y)$.

By condition (16), we obtain that $\{c_k\}$ is uniformly bounded in $\mathbb{R}^n$. For each $R > 0$, and for all $y, z \in B_R$, we have

$$|D^\beta c_k(y) - D^\beta c_k(z)| \le C_1 \lambda_k^{|\beta|} |\lambda_k(y - z)|^\alpha \le C_1 |y - z|^\alpha, \quad |\beta| = 0, 1, 2,$$

for k is large enough. Therefore, by Arzela–Ascoli's Theorem, there exists a function $c \in C^2(\mathbb{R}^n)$, after extracting a subsequence, $c_k \to c$ in $C^2_{loc}(\mathbb{R}^n)$. Moreover, by (21), we obtain

$$|c_k(y) - c_k(z)| \to 0 \qquad \text{as } \ k \to \infty. \tag{25}$$

This implies that the function c actually is a constant C. By (16) again, $c_k \ge C_2 > 0$, we conclude that C is a positive constant.

On the other hand, applying Proposition 2.2 a finite number of times to (23) and (24), there exists some positive $\gamma \in (0, 1)$ such that for every $R \in (1, k)$,

$$\|w_k\|_{C^{2,\gamma}(\overline{B_{R/2}})} \le C\left(\|w_k\|_{L^\infty(B_{3R/4})} + \|c_k w_k^p\|_{C^\gamma(B_{3R/4})}\right) \le C(R),$$

where $C(R)$ is a positive constant independent of k. Thus, after passing to a subsequence, we have, for some nonnegative function $w \in C^2_{\mathrm{loc}}(\mathbb{R}^n)$,

$$w_k \to w \qquad \text{in } \ C^2_{\mathrm{loc}}(\mathbb{R}^n).$$

Moreover, w satisfies

$$(-\triangle)^\sigma w = C w^p \qquad \text{in } \ \mathbb{R}^n \tag{26}$$

and

$$|w(0)|^{\frac{p-1}{2\sigma}} + |\nabla w(0)|^{\frac{p-1}{2\sigma+p-1}} = 1.$$

Since $p < \frac{n+2\sigma}{n-2\sigma}$, this contradicts the Liouville-type result [12, Remark 1.9] that the only nonnegative entire solution of (26) is $w = 0$. Then we conclude the lemma. $\square$

We now turn to prove Theorem 1.1.

Proof For $x_0 \in B_{1/2}\backslash\{0\}$, we denote $R := \frac{1}{2}|x_0|$. Then for any $y \in B_1$, we have $\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}$, and deduce that $x_0 + Ry \in B_1\backslash\{0\}$. Define

$$w(y) := R^{\frac{2\sigma+\tau}{p-1}} u(x_0 + Ry).$$

Therefore, we obtain that

$$(-\Delta)^\sigma w = c(y) w^p \qquad \text{in } \ B_1,$$

where $c(y) := |y + \frac{x_0}{R}|^\tau$. Notice that

$$1 < |y + \frac{x_0}{R}| < 3 \quad \text{in } \ \overline{B_1}.$$

Moreover,

$$\|c\|_{C^3(\overline{B_1})} \le C, \quad c(y) \ge 3^{-2\sigma} \quad \text{in } \ \overline{B_1}.$$

Applying Lemma 2.3, we obtain that

$$|w(0)|^{\frac{p-1}{2\sigma}} + |\nabla w(0)|^{\frac{p-1}{p+2\sigma-1}} \le C.$$

That is,

$$(R^{\frac{2\sigma+\tau}{p-1}} u(x_0))^{\frac{p-1}{2\sigma}} + (R^{\frac{2\sigma+\tau}{p-1}+1} |\nabla u(x_0)|)^{\frac{p-1}{p+2\sigma-1}} \le C.$$

Hence,

$$\begin{aligned} u(x_0) &\le C R^{-\frac{2\sigma+\tau}{p-1}} \le C|x_0|^{-\frac{2\sigma+\tau}{p-1}}, \\ |\nabla u(x_0)| &\le C R^{-\frac{2\sigma+\tau+p-1}{p-1}} \le C|x_0|^{-\frac{2\sigma+\tau+p-1}{p-1}}. \end{aligned}$$

Since $x_0 \in B_{1/2} \setminus \{0\}$ is arbitrary, Theorem 1.1 is proved. $\square$

3 Asymptotical radial symmetry

3.1 Proof of Theorem 1.2

Proof Assume that there exists some positive constant $\varepsilon \in (0, 1)$ such that for all $0 < \lambda < |x| \le \varepsilon$, $y \in B_{3/4}\backslash B_\lambda(x)$ and $y \ne 0$,

$$u_{x,\lambda}(y) \le u(y), \tag{27}$$

where

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y - x|}\right)^{n-2\sigma} u\left(x + \frac{\lambda^2(y - x)}{|y - x|^2}\right).$$

Let $r > 0$ and $x_1, x_2 \in \partial B_r$ be such that

$$u(x_1) = \max_{\partial B_r} u, \qquad u(x_2) = \min_{\partial B_r} u,$$

and define

$$x_3 := x_1 + \frac{\varepsilon(x_1 - x_2)}{4|x_1 - x_2|}, \quad \lambda := \sqrt{\frac{\varepsilon}{4}\Big(|x_1 - x_2| + \frac{\varepsilon}{4}\Big)}.$$

Then

$$|x_3| = \left| x_1 + \frac{\varepsilon(x_1 - x_2)}{4|x_1 - x_2|} \right| \le r + \frac{\varepsilon}{4}. \tag{28}$$

Via some direct computations and $|x_1|^2 = |x_2|^2 = r^2$, we find that

$$\begin{aligned} \lambda^2 - |x_3|^2 &= \frac{\varepsilon}{4}\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right) - \left| x_1 + \frac{\varepsilon(x_1 - x_2)}{4|x_1 - x_2|} \right|^2 \\ &= \frac{\varepsilon(|x_2|^2 - |x_1|^2)}{4|x_1 - x_2|} - x_1^2 = -x_1^2 < 0, \end{aligned}$$

which follows from this and (28) that $\lambda < |x_3| < \varepsilon$ by choosing $r < \frac{3\varepsilon}{4}$.

It follows from (27) that

$$u_{x_3,\lambda}(x_2) \le u(x_2).$$

Since

$$x_2 - x_3 = x_2 - x_1 + \frac{\varepsilon(x_2 - x_1)}{4|x_1 - x_2|} = \frac{x_2 - x_1}{|x_1 - x_2|}\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right),$$

then

$$\begin{aligned} |x_2 - x_3| &= |x_1 - x_2| + \frac{\varepsilon}{4}, \\ \frac{x_2 - x_3}{|x_2 - x_3|^2} &= \frac{x_2 - x_1}{|x_1 - x_2|\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right)}, \end{aligned}$$

and

$$\frac{\lambda^2(x_2 - x_3)}{|x_2 - x_3|^2} = \frac{\varepsilon(x_2 - x_1)}{4|x_1 - x_2|}.$$

Hence,

$$\begin{aligned} u_{x_3,\lambda}(x_2) &= \left(\frac{\lambda}{|x_2 - x_3|}\right)^{n-2\sigma} u\left(x_3 + \frac{\lambda^2(x_2 - x_3)}{|x_2 - x_3|^2}\right) \\ &= \left(\frac{\lambda}{|x_1 - x_2| + \frac{\varepsilon}{4}}\right)^{n-2\sigma} u\left(x_3 + \frac{\varepsilon(x_2 - x_1)}{4|x_1 - x_2|}\right) \\ &= \left(\frac{\lambda}{|x_1 - x_2| + \frac{\varepsilon}{4}}\right)^{n-2\sigma} u(x_1). \end{aligned}$$

On the other hand,

$$u_{x_3,\lambda}(x_2) = \left(\frac{\lambda}{|x_1 - x_2| + \frac{\varepsilon}{4}}\right)^{n-2\sigma} u(x_1) = \frac{u(x_1)}{\left(\frac{4|x_1 - x_2|}{\varepsilon} + 1\right)^{\frac{n-2\sigma}{2}}} \ge \frac{u(x_1)}{\left(\frac{8r}{\varepsilon} + 1\right)^{\frac{n-2\sigma}{2}}},$$

then

$$u(x_1) \le \left(\frac{8r}{\varepsilon} + 1\right)^{\frac{n-2\sigma}{2}} u_{x_3,\lambda}(x_2) \le (1 + Cr)^{\frac{n-2\sigma}{2}} u(x_2),$$

for some $C = C(\varepsilon)$. That is,

$$\max_{\partial B_r} u \leq (1 + Cr) \min_{\partial B_r} u.$$

Hence for any $x \in \partial B_r$,

$$\frac{u(x)}{\bar{u}(|x|)} - 1 \leq \frac{\max_{\partial B_r} u}{\min_{\partial B_r} u} - 1 \leq Cr,$$
$$\frac{u(x)}{\bar{u}(|x|)} - 1 \geq \frac{\min_{\partial B_r} u}{\max_{\partial B_r} u} - 1 \geq \frac{1}{1 + Cr} - 1 > -Cr,$$

In conclusion, we have

$$\left| \frac{u(x)}{\bar{u}(|x|)} - 1 \right| \leq Cr.$$

It follows that

$$u(x) = \bar{u}(|x|)(1 + O(r)) \quad \text{as} \quad x \to 0.$$

Therefore, in order to complete the proof of Theorem 1.2, it suffices to prove (27). □

3.2 Proof of (27)

Since the operator $(-\Delta)^\sigma$ is nonlocal, the traditional methods on local differential operators, such as on Laplacian, may not work on this nonlocal operator. To circumvent this difficulty, Caffarelli and Silvestre [5] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions with the connormal derivative boundary condition.

In order to describe the method in a more precise way, let us give some notations. We use capital letters, such as $X = (x, t)$ to denote points in $\mathbb{R}^{n+1}_+$. We denote $\mathcal{B}_R(X)$ as the ball in $\mathbb{R}^{n+1}$ with radius R and center X, and $\mathcal{B}^+_R(X)$ as $\mathcal{B}_R(X) \cap \mathbb{R}^{n+1}_+$. We also write $\mathcal{B}_R(0)$, $\mathcal{B}^+_R(0)$ as $\mathcal{B}_R$, $\mathcal{B}^+_R$ for short respectively. For a domain $D \subset \mathbb{R}^{n+1}_+$ with boundary ∂D, we denote $\partial' D := \partial D \cap \partial \mathbb{R}^{n+1}_+$ and $\partial'' D := \partial D \cap \mathbb{R}^{n+1}_+$. In particular, $\partial' \mathcal{B}^+_R(X) := \partial \mathcal{B}^+_R(X) \cap \partial \mathbb{R}^{n+1}_+$ and $\partial'' \mathcal{B}^+_R(X) := \partial \mathcal{B}^+_R(X) \cap \mathbb{R}^{n+1}_+$.

More precisely, for $u \in C^2(B_1 \backslash \{0\}) \cap L_\sigma(\mathbb{R}^n)$, define

$$U(x, t) := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - y, t) u(y) \mathrm{d}y, \tag{29}$$

where

$$\mathcal{P}_\sigma(x, t) := \frac{\beta(n, \sigma) t^{2\sigma}}{(|x|^2 + t^2)^{(n+2\sigma)/2}}$$

with a constant $\beta(n, \sigma)$ such that $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x, 1) \mathrm{d}x = 1$. Then

$$U \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathcal{B}^+_1} \backslash \{0\}), \quad t^{1-2\sigma} \partial_t U(x, t) \in C(\overline{\mathcal{B}^+_1} \backslash \{0\})$$

satisfying

$$\mathrm{div}(t^{1-2\sigma} \nabla U) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \tag{30}$$
$$U = u \quad \text{on} \quad \partial' \mathcal{B}^+_1 \backslash \{0\}. \tag{31}$$

In order to study the behaviors of the solution u of (7), we just need to study the behaviors of U defined by (29). In addition, by works of Caffarelli and Silvestre [5], it is known that up to a constant,

$$\frac{\partial U}{\partial \nu^\sigma} = (-\Delta)^\sigma u \quad \text{on} \quad \partial' \mathcal{B}_1^+ \backslash \{0\},$$

where the connormal derivative

$$\frac{\partial U}{\partial \nu^\sigma} := - \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x,t).$$

From this and (7), we have

$$\frac{\partial U}{\partial \nu^\sigma} = |x|^\tau u^p \quad \text{on} \quad \partial' \mathcal{B}_1^+ \backslash \{0\}. \tag{32}$$

For all $0 < |x| < \frac{1}{4}$, $X = (x, 0)$ and $\lambda > 0$, define the Kelvin transformation of U as

$$U_{X,\lambda}(\xi) = \left(\frac{\lambda}{|\xi - X|} \right)^{n-2\sigma} U \left(X + \frac{\lambda^2 (\xi - X)}{|\xi - X|^2} \right) \quad \text{in} \ \mathbb{R}_+^{n+1}.$$

The aim is to show that there exists some positive constant $\varepsilon \in (0, 1)$ such that for $0 < \lambda < |x| \leq \varepsilon$,

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{in} \ \mathcal{B}_{3/4}^+ \backslash \mathcal{B}_\lambda^+(X). \tag{33}$$

In particular, choose $\xi = (y, 0)$, $y \in \mathbb{R}^n \setminus \{0\}$, then for $0 < \lambda < |x| \leq \varepsilon$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{in} \ B_{3/4} \backslash B_\lambda(x),$$

that is (27).

3.3 Proof of (33)

To prove (33), for fixed $x \in B_{1/4} \backslash \{0\}$, we first define

$$\bar{\lambda}(x) := \sup \left\{ 0 < \mu \leq |x| \ \middle| \ U_{X,\lambda}(\xi) \leq U(\xi) \text{ in } \mathcal{B}_{3/4}^+ \backslash \mathcal{B}_\lambda^+(X), \ \forall \, 0 < \lambda < \mu \right\},$$

and then show $\bar{\lambda}(x) = |x|$.

For sake of clarity, the proof of (33) is divided into three steps. For the first step, we need the following Claim 1 to make sure that $\bar{\lambda}(x)$ is well defined.

Claim 1: There exists $\lambda_0(x) < |x|$ such that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{in} \ \mathcal{B}_{3/4}^+ \backslash \mathcal{B}_\lambda^+(X).$$

Second, we give that

Claim 2: There exists a positive constant $\varepsilon \in (0, 1)$ sufficiently small such that for all $0 < \lambda < |x| \leq \varepsilon$,

$$U_{X,\lambda}(\xi) < U(\xi) \quad \text{on} \ \partial \mathcal{B}_{3/4}^+.$$

Last, we are going to prove that

Claim 3:

$$\bar{\lambda}(x) = |x|.$$

Proof of Claim 1 First of all, we are going to show that there exist μ and $\lambda_0(x)$ satisfying $0 < \lambda_0(x) < \mu < |x|$ such that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \quad \text{in } \overline{\mathcal{B}_\mu^+(X)} \backslash \mathcal{B}_\lambda^+(X). \tag{34}$$

Then we will prove that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \quad \text{in } \mathcal{B}_{3/4}^+ \backslash \overline{\mathcal{B}_\mu^+(X)}. \tag{35}$$

For every $0 < \lambda < \mu < \frac{1}{2}|x|$, $\xi \in \partial'' \mathcal{B}_\mu^+(X)$, we have $X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \in \mathcal{B}_\mu^+(X)$. Thus we can choose

$$\lambda_0(x) = \mu \left(\frac{\inf\limits_{\partial'' \mathcal{B}_\mu^+(X)} U}{\sup\limits_{\mathcal{B}_\mu^+(X)} U} \right)^{\frac{1}{n-2\sigma}},$$

such that for every $0 < \lambda < \lambda_0(x) < \mu$,

$$\begin{aligned} U_{X,\lambda}(\xi) &= \left(\frac{\lambda}{|\xi - X|} \right)^{n-2\sigma} U\left(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \right) \\ &\le \left(\frac{\lambda_0}{\mu} \right)^{n-2\sigma} \sup_{\mathcal{B}_\mu^+(X)} U = \inf_{\partial'' \mathcal{B}_\mu^+(X)} U \le U(\xi). \end{aligned}$$

The above inequality, together with

$$U_{X,\lambda}(\xi) = U(\xi) \qquad \text{on } \partial'' \mathcal{B}_\lambda^+(X),$$

implies that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \qquad \text{on } \partial'' \mathcal{B}_\mu^+(X) \cup \partial'' \mathcal{B}_\lambda^+(X). \tag{36}$$

We will make use of the narrow domain technique of Berestycki and Nirenberg from [3], and show that, for sufficiently small μ, that for $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \qquad \text{in } \mathcal{B}_\mu^+(X) \backslash \mathcal{B}_\lambda^+(X).$$

It is a straightforward computation to show that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U_{X,\lambda}) = 0 & \text{in } \mathcal{B}_\mu^+(X) \backslash \overline{\mathcal{B}_\lambda^+(X)}, \\ \frac{\partial}{\partial \nu^\sigma} U_{X,\lambda} = \left(\frac{\lambda}{|y-x|} \right)^{p^*} |y_\lambda|^\tau \, u_{x,\lambda}^p(y) & \text{on } \partial'(\mathcal{B}_\mu^+(X) \backslash \overline{\mathcal{B}_\lambda^+(X)}), \end{cases}$$

which yield

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla (U_{X,\lambda} - U)) = 0 & \text{in } \mathcal{B}_\mu^+(X) \backslash \overline{\mathcal{B}_\lambda^+(X)}, \\ \frac{\partial}{\partial \nu^\sigma} (U_{X,\lambda} - U) = \left(\frac{\lambda}{|y-x|} \right)^{p^*} |y_\lambda|^\tau \, u_{x,\lambda}^p(y) - |y|^\tau u^p(y) & \text{on } \partial'(\mathcal{B}_\mu^+(X) \backslash \overline{\mathcal{B}_\lambda^+(X)}), \end{cases} \tag{37}$$

where $y_\lambda := x + \frac{\lambda^2(y-x)}{|y-x|^2}$, $p^* := n + 2\sigma - p(n - 2\sigma)$.

Let $(U_{X,\lambda} - U)^+ := \max(0, U_{X,\lambda} - U)$ which equals 0 on $\partial''(\mathcal{B}_\mu^+(X) \backslash \mathcal{B}_\lambda^+(X))$. Multiplying the equation in (37) by $(U_{X,\lambda} - U)^+$ and integrating by parts in $\mathcal{B}_\mu^+(X) \backslash \overline{\mathcal{B}_\lambda^+(X)}$, we have

$$\int_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2$$
$$=\int_{B_\mu(x)\setminus B_\lambda(x)}\left[\left(\frac{\lambda}{|y-x|}\right)^{p^*}\left|y_\lambda\right|^\tau u_{x,\lambda}^p(y)-|y|^\tau u^p(y)\right](u_{x,\lambda}-u)^+.$$

Combining Corollary 1.4 with $\lambda^2=|x-y_\lambda||x-y|$, we have

$$\left(\frac{\lambda}{|x-y|}\right)^2\le\frac{|y_\lambda|}{|y|},$$

which implies that

$$\left(\frac{\lambda}{|x-y|}\right)^{p^*}\le\left(\frac{|y|}{|y_\lambda|}\right)^\tau,$$

due to $-2\tau\le p^*$ and $\frac{\lambda}{|x-y|}\le 1$. Therefore,

$$\left(\frac{\lambda}{|y-x|}\right)^{p^*}\left|y_\lambda\right|^\tau\le|y|^\tau,\tag{38}$$

and

$$\int_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2$$
$$\le\int_{B_\mu(x)\setminus B_\lambda(x)}|y|^\tau(u_{x,\lambda}^p(y)-u^p(y))(u_{x,\lambda}-u)^+.$$

For any $y\in B_\mu(x)\setminus B_\lambda(x)$,

$$u_{x,\lambda}(y)=\left(\frac{\lambda}{|y-x|}\right)^{n-2\sigma}u(y_\lambda)\le u(y_\lambda),$$

where $y_\lambda:=x+\frac{\lambda^2(y-x)}{|y-x|^2}$. Combining $y_\lambda\in B_\lambda(x)\subset\overline{B}_{\frac{|x|}{2}}(x)$ with $u\in C^2(B_1\setminus\{0\})$, we deduce that there exists a positive constant C depending on x, such that for any $y\in B_\mu(x)\setminus B_\lambda(x)$,

$$u_{x,\lambda}(y)\le C(x).$$

With the help of mean value theorem and $\lambda<\frac{1}{2}|x|$, we obtain

$$\int_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2$$
$$\le\int_{B_\mu(x)\setminus B_\lambda(x)}2^{-\tau}|x|^\tau p u_{x,\lambda}^{p-1}(y)[(u_{x,\lambda}-u)^+]^2$$
$$\le 2^{-\tau}|x|^\tau p\Big(\int_{B_\mu(x)\setminus B_\lambda(x)}(u_{x,\lambda})^{\frac{n(p-1)}{2\sigma}}\Big)^{\frac{2\sigma}{n}}\left(\int_{B_\mu(x)\setminus B_\lambda(x)}[(u_{x,\lambda}-u)^+]^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}}$$
$$\le C(n,p,\sigma,\tau,|x|)|B_\mu(x)|^{\frac{2\sigma}{n}}\left(\int_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2\right),$$

where the trace inequality (Proposition 4.1) is used in the last inequality and $C(n, p, \sigma, \tau, |x|)$ is a positive constant.

We can fix μ sufficiently small such that

$$C(n, p, \sigma, \tau, |x|)|B_\mu(x)|^{\frac{2\sigma}{n}} \leq \frac{1}{2}.$$

Then

$$\nabla(U_{X,\lambda}(\xi) - U(\xi))^+ = 0 \quad \text{ in } \ \mathcal{B}_\mu^+(X)\backslash\mathcal{B}_\lambda^+(X).$$

Since (36), we deduce that

$$(U_{X,\lambda}(\xi) - U(\xi))^+ = 0 \quad \text{ in } \ \mathcal{B}_\mu^+(X)\backslash\mathcal{B}_\lambda^+(X).$$

Therefore,

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{ in } \ \mathcal{B}_\mu^+(X)\backslash\mathcal{B}_\lambda^+(X).$$

After that, we are going to prove (35). Let

$$\phi(\xi) := \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_\mu^+(X)} U,$$

which satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\phi) = 0 & \text{in } \mathbb{R}_+^{n+1} \setminus \mathcal{B}_\mu^+(X), \\ \dfrac{\partial\phi}{\partial\nu^\sigma} = 0 & \text{on } \partial'(\mathbb{R}_+^{n+1} \setminus \mathcal{B}_\mu^+(X)), \end{cases}$$

and

$$\phi(\xi) = \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_\mu^+(X)} U = \inf_{\partial''\mathcal{B}_\mu^+(X)} U \leq U(\xi) \quad \text{on } \ \partial''\mathcal{B}_\mu^+(X).$$

In addition, combining Proposition 4.2, we can choose μ small enough such that

$$\liminf_{\xi\to 0} U(\xi) \geq \left(\frac{\mu}{|X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_\mu^+(X)} U,$$

and

$$U(\xi) \geq \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_\mu^+(X)} U \quad \text{on } \ \partial''\mathcal{B}_{3/4}^+,$$

By the standard maximum principle (Proposition 4.3), we have

$$U(\xi) \geq \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_\mu^+(X)} U \quad \text{in } \ \mathcal{B}_{3/4}^+\backslash\overline{\mathcal{B}_\mu^+(X)}. \tag{39}$$

Then for all $\xi \in \mathcal{B}_{3/4}^+\backslash\mathcal{B}_\mu^+(X)$ and $\lambda \in (0, \lambda_0) \subset (0, \mu)$, we have

$$\begin{aligned} U_{X,\lambda}(\xi) &= \left(\frac{\lambda}{|\xi - X|}\right)^{n-2\sigma} U(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}) \\ &\leq \left(\frac{\lambda_0}{|\xi - X|}\right)^{n-2\sigma} \sup_{\mathcal{B}_\mu^+(X)} U = \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_\mu^+(X)} U \\ &\leq U(\xi), \end{aligned}$$

where (39) is used in the last inequality. Then Claim 1 is proved. □

Proof of Claim 2 For $y \in B_1$, $\frac{3}{8} \le |y| \le \frac{7}{8}$ and $0 < \lambda < |x| < \frac{1}{8}$, we have

$$|y-x| \ge |y| - |x| \ge \frac{1}{4} > 2|x|.$$

Hence

$$\left| x + \frac{\lambda^2(y-x)}{|y-x|^2} \right| \le |x| + \frac{|x|^2}{|y-x|} \le \frac{3|x|}{2},$$

and

$$\left| x + \frac{\lambda^2(y-x)}{|y-x|^2} \right| \ge |x| - \frac{|x|^2}{|y-x|} \ge \frac{|x|}{2}.$$

It follows from Theorem 1 that

$$u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right) \le C|x|^{-\frac{2\sigma+\tau}{p-1}},$$

Thus, for $0 < \lambda < |x| < \frac{1}{8}$, $\frac{3}{8} \le |y| \le \frac{7}{8}$, we conclude that

$$\begin{aligned} u_{x,\lambda}(y) = U_{X,\lambda}(y,0) &\le \left(\frac{\lambda}{|y-x|} \right)^{n-2\sigma} C|x|^{-\frac{2\sigma+\tau}{p-1}} \\ &\le C4^{n-2\sigma}\lambda^{n-2\sigma}|x|^{-\frac{2\sigma+\tau}{p-1}} \le C|x|^{\frac{p(n-2\sigma)-n-\tau}{p-1}}. \end{aligned}$$

Since $\frac{n+\tau}{n-2\sigma} < p \le \frac{n+2\sigma+2\tau}{n-2\sigma}$, we have $\frac{p(n-2\sigma)-n-\tau}{p-1} > 0$. By Harnack inequality (Proposition 4.4), $\varepsilon > 0$ can be chosen sufficiently small to guarantee that for all $0 < \lambda < |x| \le \varepsilon$ and $|\xi| = \frac{3}{4}$,

$$U_{X,\lambda}(\xi) \le C|x|^{\frac{p(n-2\sigma)-n-\tau}{p-1}} < U(\xi). \tag{40}$$

□

Proof of Claim 3 We prove Claim 3 by contradiction. Assume $\bar{\lambda}(x) < |x| \le \varepsilon$ for some $x \ne 0$. We want to show that there exists a positive constant $\widetilde{\varepsilon} \in (0, \frac{|x|-\bar{\lambda}(x)}{2})$ such that for $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x)+\widetilde{\varepsilon})$,

$$U_{X,\lambda}(\xi) \le U(\xi) \quad \text{in } \mathcal{B}^+_{3/4} \backslash \mathcal{B}^+_\lambda(X), \tag{41}$$

which contradicts the definition of $\bar{\lambda}(x)$, then we obtain $\bar{\lambda}(x) = |x|$.

Divide the region $\mathcal{B}^+_{3/4} \backslash \mathcal{B}^+_\lambda(X)$ into three parts,

$$\begin{aligned} K_1 &= \left\{ \xi \in \mathcal{B}^+_{3/4} \,\middle|\, 0 < |\xi| < \delta_1 \right\}, \\ K_2 &= \left\{ \xi \in \mathcal{B}^+_{3/4} \,\middle|\, \delta_1 \le |\xi|,\ |\xi - X| \ge \bar{\lambda}(x) + \delta_2 \right\}, \\ K_3 &= \left\{ \xi \in \mathcal{B}^+_{3/4} \,\middle|\, \lambda \le |\xi - X| \le \bar{\lambda}(x) + \delta_2 \right\}, \end{aligned}$$

where δ_1, δ_2 will be fixed later. To obtain (41) it suffices to prove that it established on K_1, K_2, K_3. And then we are going to prove it respectively. By Claim 1, we have

$$U_{X,\bar{\lambda}(x)}(\xi) \le U(\xi) \quad \text{in } \mathcal{B}^+_{3/4} \backslash \mathcal{B}^+_{\bar{\lambda}(x)}(X).$$

With the help of Claim 2 and the strong maximum principle, we deduce that

$$U_{X,\bar{\lambda}(x)}(\xi) < U(\xi) \quad \text{in } \mathcal{B}^+_{3/4} \backslash \overline{\mathcal{B}^+_{\bar{\lambda}(x)}(X)}. \tag{42}$$

Choose $R \in (0, \frac{|x|-\bar{\lambda}(x)}{2})$, then

$$\begin{cases} \mathrm{div}(t^{1-2\sigma}\nabla(U - U_{X,\bar{\lambda}(x)})) = 0 & \text{in } \mathcal{B}^+_R, \\ \frac{\partial(U-U_{X,\bar{\lambda}(x)})}{\partial\nu^\sigma} = |y|^\tau u^p(y) - \left(\frac{\bar{\lambda}(x)}{|y-x|}\right)^{p^*} \left|y_{\bar{\lambda}(x)}\right|^\tau u^p_{x,\bar{\lambda}(x)}(y) \geq 0 & \text{on } \partial'\mathcal{B}^+_R \setminus \{0\}, \end{cases}$$

where $y_{\bar{\lambda}(x)} := x + \frac{\bar{\lambda}(x)^2(y-x)}{|y-x|^2}$.

By (42) and Proposition 4.2, we get

$$\liminf_{\xi\to 0}(U(\xi) - U_{X,\bar{\lambda}(x)}(\xi)) > 0.$$

Thus, there exists a positive constant $\delta_1 \in (0, \frac{|x|-\bar{\lambda}(x)}{2})$ and a positive constant C_1 such that

$$U(\xi) - U_{X,\bar{\lambda}(x)}(\xi) > C_1 \quad \text{in } K_1.$$

By the uniform continuity of U on compact sets, there exists a positive constant ε_1 small enough such that for $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_1)$,

$$U_{X,\bar{\lambda}(x)}(\xi) - U_{X,\lambda}(\xi) > -\frac{C_1}{2} \quad \text{in } K_1.$$

From the above argument, we conclude that for $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_1)$,

$$U(\xi) - U_{X,\lambda}(\xi) > \frac{C_1}{2} \quad \text{in } K_1. \tag{43}$$

Next, choose δ_2 to be small, the aim is to show that there exist positive constants C_2 and ε_2, such that for $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_2)$,

$$U(\xi) - U_{X,\lambda}(\xi) > \frac{C_2}{2} \quad \text{in } K_2. \tag{44}$$

From (42) and K_2 is compact, there exist a positive constant C_2 such that

$$U(\xi) - U_{X,\bar{\lambda}(x)}(\xi) > C_2 \quad \text{in } K_2.$$

By the uniform continuity of U on compact sets, there exists a positive constant ε_2 sufficiently small such that for all $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_2)$,

$$U_{X,\bar{\lambda}(x)}(\xi) - U_{X,\lambda}(\xi) > -\frac{C_1}{2} \quad \text{in } K_2.$$

Hence for all $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_2)$, we have

$$U(\xi) - U_{X,\lambda}(\xi) > \frac{C_1}{2} \quad \text{in } K_2.$$

Then we obtain (44).

Last, let us focus on the region K_3. We can choose a positive constant $\widetilde{\varepsilon}$ as small as we want (less then ε_1 and ε_2) such that for $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \widetilde{\varepsilon})$,

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{on } \partial''K_3.$$

Using the narrow domain technique as that the proof of (34) in Claim 1, we can choose δ_2 to be small such that

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{ in } \ K_3. \tag{45}$$

Together with (43), (44) and (45), we can see that the moving sphere procedure may continue beyond $\bar{\lambda}(x)$ where we reach a contradiction. □

At last, we give a proof for Proposition 1.3 and Corollary 1.4, which have been used in our proof.

Proof of Proposition 1.3 Via a straightforward calculation, (13) is rewritten as

$$\lambda^2|y| \leq \left|x(x-y)^2 + \lambda^2(y-x)\right|,$$

that is,

$$\lambda^4(|x|^2 - 2x\cdot y) + 2\lambda^2|x-y|^2(x\cdot y - |x|^2) + |x|^2|x-y|^4 \geq 0.$$

Let

$$s := \lambda^2,$$
$$f(s) := s^2(|x|^2 - 2x\cdot y) + 2s|x-y|^2(x\cdot y - |x|^2) + |x|^2|x-y|^4.$$

If $|x|^2 - 2x\cdot y = 0$, it is easy to see that $f(s)$ is a affine function, and

$$f(0) = |x|^2|x-y|^4 > 0, \quad f(|x-y|^2) = 0.$$

Therefore,

$$f(s) \geq 0 \Leftrightarrow 0 \leq s \leq |x-y|^2. \tag{46}$$

If $|x|^2 - 2x\cdot y \neq 0$, it follows that $f(s)$ is a quadratic polynomial, and always has two roots,

$$s_1 = |x-y|^2, \qquad s_2 = \frac{|x|^2|x-y|^2}{|x|^2 - 2x\cdot y}.$$

Now, let us divide into the following three cases to consider.

Case 1: For $|x|^2 - 2x\cdot y < 0$, then $s_2 < 0 < s_1$, which implies that

$$f(s) \geq 0 \Leftrightarrow 0 \leq s \leq s_1. \tag{47}$$

Case 2: For $|x|^2 - 2x\cdot y > 0$ and $x\cdot y \geq 0$, then $0 < s_1 \leq s_2$, it is obviously to obtain that

$$f(s) \geq 0 \Leftrightarrow 0 \leq s \leq s_1 \quad \text{or} \quad s \geq s_2. \tag{48}$$

Case 3: For $|x|^2 - 2x\cdot y > 0$ and $x\cdot y < 0$, then $0 < s_2 \leq s_1$. As before, we have

$$f(s) \geq 0 \Leftrightarrow 0 \leq s \leq s_2 \quad \text{or} \quad s \geq s_1. \tag{49}$$

Combining (46), (47), (48) and (49), we finish the proof of this proposition. □

Proof of Corollary 1.4 If $x\cdot y \geq 0$, it is easy to see that

$$\frac{|x|^2|x-y|^2}{|x|^2 + (-2x\cdot y)^+} = |y-x|^2.$$

It follows that λ satisfies (14) whether $x\cdot y$ is bigger or smaller than $|x|^2/2$.

If $x \cdot y < 0$, by a direct calculation, we have

$$\frac{|x|^2|x-y|^2}{|x|^2+(-2x\cdot y)^+} \geq |x|^2.$$

This also implies that λ satisfies (14). Therefore, (13) has been established. □

4 Appendix

The first one is called the trace inequality.

Proposition 4.1 [12, Proposition 2.1] *If $U \in C_c^2(\mathbb{R}_+^{n+1})$, then there exists a positive constant C depending only on n and σ such that*

$$\left(\int_{\mathbb{R}^n} |U(\cdot,0)|^{\frac{2n}{n-2\sigma}} \mathrm{d}x\right)^{\frac{n-2\sigma}{2n}} \leq C\left(\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma}|\nabla U|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \tag{50}$$

The next one is on a maximum principle for positive supersolutions with an isolated singularity.

Proposition 4.2 [12, Proposition 3.1] *Suppose that $U \in C^2(\mathcal{B}_R^+ \cup \partial'\mathcal{B}_R^+ \setminus \{0\})$ and $U > 0$ in $\mathcal{B}_R^+ \cup \partial'\mathcal{B}_R^+ \setminus \{0\}$ is a solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) \leq 0 & \text{in } \mathcal{B}_R^+, \\ \dfrac{\partial U}{\partial \nu^\sigma} \geq 0 & \text{on } \partial'\mathcal{B}_R^+ \setminus \{0\}, \end{cases}$$

then

$$\liminf_{X\to 0} U(X) > 0.$$

We also recall the standard maximum principle.

Proposition 4.3 [12, Lemma 2.5] *Suppose that $U \in C^2(D) \cap C^1(\overline{D})$ is a solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) \leq 0 & \text{in } D, \\ \dfrac{\partial U}{\partial \nu^\sigma} \geq 0 & \text{on } \partial' D, \end{cases}$$

where $D \subset \mathbb{R}_+^{n+1}$ is an open domain. If $U \geq 0$ on $\partial'' D$, then $U \geq 0$ in D.

The last one is the Harnack inequality, and Tan-Xiong [26] provide more details for the Harnack inequality.

Proposition 4.4 [12, Proposition 2.6] *Suppose that $U \in C^2(\mathcal{B}_{2R}^+) \cap C^1(\overline{\mathcal{B}_{2R}^+})$ is a nonnegative solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_{2R}^+, \\ \dfrac{\partial U}{\partial \nu^\sigma} = a(x)U(x,0) & \text{on } \partial'\mathcal{B}_{2R}^+. \end{cases}$$

If $a \in L^q(B_{2R})$ for some $q > \frac{n}{2\sigma}$, then we have

$$\sup_{\mathcal{B}_R^+} U \leq C \inf_{\mathcal{B}_R^+} U,$$

where the positive constant C depends only on n, σ, R and $\|a\|_{L^q(B_{2R})}$.

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