Centralizers on a super-reflexive Schatten ideal

Jesús Suárez de la Fuente¹

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Abstract We give a simple proof that there is no strictly singular bicentralizer on a super-reflexive Schatten ideal. This result applies, in particular, to the p -Schatten class for $1 < p < \infty$.

Keywords Schatten · Strictly singular · Bicentralizer

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1 Main

Let H be a separable Hilbert space and denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on H . We will follow the deep work of Kalton [4], the interested reader may consider also [2, 3], to introduce the following definitions. Let E be a Köthe sequence space which is symmetric, then we define the corresponding Schatten ideal $\mathcal{C}_E$ to be the algebra of all operators $A : H \rightarrow H$ whose singular values $s_n(A)$ satisfy that $(s_n(A)) \in E$. We endow $\mathcal{C}_E$ with the norm $\|A\|_E = \|(s_n(A))\|_E$. In this notation, the usual p -Schatten class is written as $\mathcal{C}_{\ell_p}$. Let us recall that a map $\Omega : \mathcal{C}_E \rightarrow \mathcal{B}(H)$ is a *bicentralizer* provided for some constant $k = k(\Omega)$ we have

$$\|\Omega(VAW) - V\Omega(A)W\|_E \leq k\|V\|\|A\|_E\|W\|,$$

for $A \in \mathcal{C}_E$ and $V, W \in \mathcal{B}(H)$. We will also say that Ω is *strictly singular* if its restriction to every infinite-dimensional closed subspace is never bounded.

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✉ Jesús Suárez de la Fuente
jesus@unex.es

¹ Departamento de Matemáticas, Avda. de Elvas s/n, Badajoz, Spain

Regarding the L_p -spaces, the following remarkable result for centralizers was proved by Cabello Sánchez [1, Proposition 1].

Proposition 1 *There is no strictly singular centralizer on L_p for $0 < p < \infty$.*

The proof of the proposition above contains several simplifications that are of independent interest. However, the core of the argument is based on a result of Kalton contained in the aforementioned paper [4], namely that every centralizer on L_p arises as a derivation from some interpolation process for $p > 1$. The same work of Kalton contains a similar statement for the Schatten class [4, Theorem 8.3]. The aim of this short note is to observe that using this last result one may prove a claim equivalent to Proposition 1 but for the Schatten class $\mathcal{C}_{\ell_p}$, and even overstep the limits of the p -class.

Proposition 2 *There is no strictly singular bicentralizer on a super-reflexive Schatten ideal.*

Proof Let Ω denote a bicentralizer on $\mathcal{C}_E$. Then there exists Hermitian bicentralizers Ω_1, Ω_2 and a constant $K > 0$ so that if $A \in \mathcal{C}_E$ then

$$\|\Omega(A) - \Omega_1(A) - i\Omega_2(A)\|_E \leq K\|A\|_E, \quad (1)$$

see [4, Lemma 8.1]. Let us find a copy of ℓ_2 where Ω_1 and Ω_2 are bounded and the same will follow, by (1), for Ω . Thus, pick Ω_1 , our first Hermitian bicentralizer, and apply [4, Theorem 8.3] to find symmetric Köthe sequence spaces E_0, E_1 such that $(\mathcal{C}_{E_0}, \mathcal{C}_{E_1})_{\frac{1}{2}} = \mathcal{C}_{\tilde{E}}$ (where $\tilde{E}$ and E have equivalent norms) and Ω_1 is *equivalent* to the bicentralizer induced by the previous interpolation formula, say $\tilde{\Omega}_1$. We write this last claim in a formal language as there is some constant K_1 such that

$$\|\Omega_1(A) - \tilde{\Omega}_1(A)\|_E \leq K_1\|A\|_E, \quad (2)$$

for $A \in \mathcal{C}_E$. Now, let us give the copy of ℓ_2 that makes the trick: Pick $(e_n)_{n=1}^\infty$ an orthonormal basis in H . For this basis, consider the span of the corresponding first column matrix, or in other words the natural embedding

$$J : \ell_2 \longrightarrow \mathcal{C}_E$$

$$a = (a_n)_{n=1}^\infty \longrightarrow A = \sum_{n=1}^\infty a_n (\cdot | e_1) e_n.$$

Writing

$$A = \left(\sum_{n=1}^\infty a_n e_n \right) (\cdot | e_1) = \|(a_n)\|_{\ell_2} \left(\sum_{n=1}^\infty \frac{a_n}{\|(a_n)\|_{\ell_2}} e_n \right) (\cdot | e_1),$$

one may easily deduce that $s_1(A) = \|(a_n)\|_{\ell_2}$ and $s_n(A) = 0$ for $n \geq 2$. Therefore, the operator J is well defined and the image spans a copy of ℓ_2 . Observe that E does not play any special role here so that the same holds for E_0 and E_1 . Indeed,

$$\|(s_1(A), 0, 0, \dots)\|_E = \|(s_1(A), 0, 0, \dots)\|_{E_0} = \|(s_1(A), 0, 0, \dots)\|_{E_1} = s_1(A). \quad (3)$$

Thus, for a fixed $A = J(a)$, defining the constant function $F_A(z) = A$ where z runs on the complex unit strip, the previous comments show that F_A takes values in $\mathcal{C}_{E_0} \cap \mathcal{C}_{E_1}$. This gives that F_A is a well-defined map in the space of functions of Calderón $\mathcal{F}(\mathcal{C}_{E_0}, \mathcal{C}_{E_1})$ (see [5] for a clear exposition). And moreover, by (3),

$$\|F_A\|_{\mathcal{F}(\mathcal{C}_{E_0}, \mathcal{C}_{E_1})} = \|A\|_E.$$

Hence, F_A is a C -bounded selection of A for the quotient map $\delta_{1/2} : \mathcal{F}(\mathcal{C}_{E_0}, \mathcal{C}_{E_1}) \rightarrow C_{\tilde{E}}$, where C depends only on the equivalence constant between E and $\tilde{E}$. Since F_A is constant, its derivative must be zero, that in particular means $\tilde{\Omega}_1(A) = 0$ (see [5, Page 1159] for a quick description of this phenomenon) and the same must hold for every $A \in J(\ell_2)$. Thus, by (2), one has that Ω_1 is bounded when restricted to $J(\ell_2)$, that is

$$\|\Omega_1(A)\|_E \leq K_1 \|A\|_E,$$

for every $A \in J(\ell_2)$. The same arguments work to find a constant K_2 satisfying $\|\Omega_2(A)\|_E \leq K_2 \|A\|_E$ for every $A \in J(\ell_2)$. Using K_1, K_2 and the inequality (1), one may easily check that Ω is bounded on $J(\ell_2)$ which finishes the proof. $\square$

Corollary 1 *There is no strictly singular bicentralizer on C_{ℓ_p} for $1 < p < \infty$.*

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