On the Lipschitz regularity and asymptotic behaviour of the free boundary for classes of minima of inhomogeneous two-phase Alt–Caffarelli functionals in Orlicz spaces

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Abstract In this paper, we study classes of minimizers of inhomogeneous two-phase Alt–Caffarelli functionals of the type

$$\mathcal{J}_G(u, \Omega) = \int_{\Omega} \left[G(|\nabla u|) + f_1(x)H_1(u^+) + f_2(x)H_2(u^-) + Q(h_1, h_2)(u)(x) \right] dx,$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, where G , H_1 and H_2 are power-like N -functions, $f_1, f_2 \in L^q(\Omega)$ for suitable $n \leq q \leq \infty$, and $h_1, h_2 \in L^\infty(\Omega)$. Hölder and non-degeneracy estimates for minima are obtained and in the particular case where such minimizers are weak solutions of non-singular PDEs we provide log-Lipschitz type estimates. In the sequel, since the Alt–Caffarelli–Friedman monotonicity formula is missing in our context and there is a mistake in a proof of Lipschitz continuity for minimizers in Zheng et al. (Monatsh Math 172:441–475, 2013), we extend the results of Braga et al. (Ann Inst H Poincaré Anal Non Linéaire 31(4):823–850, 2014) establishing the Lipschitz regularity for more general class of minima under the additional condition of small Lebesgue density on one of the phases along the free boundary. We finish this paper with a result that establishes density estimates from below for the positive and negative phase on points inside the contact set between the free boundaries in the case where minimizers are not Lipschitz. Such estimates allow us to provide a preliminary full description of the free boundary for any minima even if the Lipschitz regularity (as optimal regularity) is unknown.

Keywords Free boundary problems · Lipschitz regularity · Minimizers · Orlicz spaces · Degenerate/singular equations

Mathematics Subject Classification 35J60 · 35J70 · 35J75 · 35R35

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1 Introduction

In the beginning of the 80s W. Alt and L. Caffarelli investigated in [2] the following problem

$$\min_{u \in K_\varphi} \left\{ \int_{\Omega} [|\nabla u|^2 + Q(x)\chi_{\{u>0\}}] dx \right\}, \quad K_\varphi := \left\{ u \in H^1(\Omega) : u - \varphi \in H_0^1(\Omega) \right\}. \quad (1.1)$$

They proved that solutions of the above minimization problem are nonnegative Lipschitz functions and weak solutions of the one-phase Free Boundary Problem (FBP):

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap \Omega, \\ |\nabla u^+| = Q & \text{on } F^+(u) := \partial\{u > 0\} \cap \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

for $0 \leq \varphi \in H^1(\Omega) \cap L^\infty(\partial\Omega)$ and $0 < \lambda \leq Q(x) \leq \Lambda$. Particularly, (1.2) ensures that Lipschitz is the best regularity for minima. Still from (1.2) another interesting issue stands out in these work. More precisely, what is the regularity of the (a priori unknown) interface $F^+(u)$? In this way the authors show that if Q is sufficiently regular, then $F^+(u)$ (so-called free boundary of u) is a $C^{1,\alpha}$ -regular surface near “flat” free boundary points.

Few years later, Alt, Caffarelli together with A. Friedman extended the problem above in the papers [3,4]. In [3] they study a similar functional as in (1.1) related to suitable quasilinear equations (still in the one-phase scenario), and in [4] consider a minimization problem where minimizers may change of sign (originating so two-phase FBPs). Although in [3,4] the authors have followed a similar way as in [2] for the two-phase problem, a new ingredient needed to be developed: the well-known Alt–Caffarelli–Friedman monotonicity formula. With such powerful tool, they showed that the gradient of minima are bounded in the neighbourhood of the free boundary and hence the local Lipschitz regularity of minimizers and the regularity of the free boundary have been achieved.

Only from 2005 the (one-phase) Alt–Caffarelli minimum problem was extended for singular/degenerate type operators (see, for instance [9,18]). In these papers, the energy is substituted, respectively, by p -energy, $\int_{\Omega} |\nabla u|^p dx$, $1 < p < \infty$, and G -energy, $\int_{\Omega} G(|\nabla u|) dx$, where G is a suitable N -function.¹ By a new approach using indirect arguments and compactness results, similar theorems about the optimal regularity of minima and properties of the free boundary were recovered. As in [2], the minimizers in [9,18] are also weak solutions of a FBP as (1.2).

For two-phase setting, the situation is much more delicate and less understood in the general context. To the best of our knowledge, up to the present time and except in those cases where a variant of the ACF monotonicity formula exists, for example [4,7], two-phase minimization problems have only partial results. In fact, for singular/degenerate operators see [6,11] (homogeneous case), and [15] (inhomogeneous case for non-singular p -Laplace operator, i.e., $p \geq 2$). However, in 2013, in order to answer in a complete form the two main issues in a two-phase scenario (in the absence of the ACF monotonicity formula) for non-singular type operators² in the Orlicz context, the authors in [20] studied minimizers of

$$\mathcal{J}(u) = \int_{\Omega} [G(|\nabla u|) + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}} + fu] dx, \quad (1.3)$$

where $G'(t)/t$ is nondecreasing, $f \in L^\infty(\Omega)$ and $0 \leq \lambda_- < \lambda_+ < \infty$. In these work, the statement of Theorem 1.3 ensures that minimizers of (1.3) are locally Lipschitz continuous.

¹ A definition for N -functions is presented in the next page.

² For non-singular operators see Sect. 8.

But, unfortunately, there exists a mistake in a proof of this result. More precisely, the interior $C^{1,\alpha}$ regularity of the solutions to $\operatorname{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0$, where $g = G'$, in bounded domains $\Omega \subset \mathbb{R}^n$, was used incorrectly. In fact, the authors consider a ball $B_r \Subset \Omega$ and the following Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\frac{g(|\nabla h|)}{|\nabla h|}\nabla h\right) = 0 & \text{in } B_r, \\ h = u_0 & \text{on } \partial B_r, \end{cases}$$

where u_0 is a minimizer to (1.3). Since, a priori, u_0 is only log-Lipschitz continuous and h is not defined in $\Omega \setminus \overline{B_r}$ it is impossible to conclude that ∇h is bounded in B_r . We would need a Lipschitz estimate up to the boundary for h with a prescribed log-Lipschitz boundary data. However, we observe that if $G(t) = \frac{1}{2}t^2$ and $e_1 = (1, 0, \dots, 0)$,

$$\begin{cases} \Delta h = 0 & \text{in } B_1(e_1), \\ h(x) = |x| \cdot |\log |x|| & \text{on } \partial B_1(e_1) \end{cases}$$

gives us a simple counterexample. In fact, $\lim_{t \rightarrow 0^+} |\nabla h(te_1)| = +\infty$.

In view of all the considerations above, the objective of the present manuscript is to extend one more time this theory in the two-phase setting to the context of Orlicz–Sobolev spaces. More precisely, for $\Omega \subset \mathbb{R}^n$, $n \geq 2$, a bounded domain with Lipschitz boundary, $G, H_1, H_2 : [0, \infty) \rightarrow [0, \infty)$ suitable N -functions, $f_1, f_2 \in L^q(\Omega)$, $n \leq q \leq \infty$ and $h_1, h_2 \in L^\infty(\Omega)$, we investigate the optimal regularity and geometric properties for local and global minimizers of classes of functionals of the type

$$\mathcal{J}_G(u, \Omega) = \int_{\Omega} \left[G(|\nabla u|) + f_1(x)H_1(u^+) + f_2(x)H_2(u^-) + Q(h_1, h_2)(u)(x) \right] dx, \quad (1.4)$$

where

$$Q(h_1, h_2)(u)(x) := h_2(x) \cdot \chi_{\{u>0\}} + h_1(x) \cdot \chi_{\{u<0\}} + \min(h_1(x), h_2(x)) \cdot \chi_{\{u=0\}}.$$

In addition, we observe that fixed a boundary datum $\varphi \in W^{1,G}(\Omega) \cap L^\infty(\partial\Omega)$, if G , H_i and h_i are sufficiently regular, at least formally, the Euler–Lagrange equations for such minimizers are general FBPs of the type

$$\begin{cases} \mathcal{L}_g u = f_1 H_1'(u^+) + f_2 H_2'(u^-) & \text{in } (\{u > 0\} \cup \{u \leq 0\}^0) \cap \Omega, \\ \Gamma_G(|\nabla u^+|) - \Gamma_G(|\nabla u^-|) = h_2 - h_1 & \text{on } F^\pm(u) := (\partial\{u > 0\} \cup \partial\{u < 0\}) \cap \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

where

$$\mathcal{L}_g u := \operatorname{div}\left(\frac{g(|\nabla u|)}{|\nabla u|} \cdot \nabla u\right), \quad g(t) = G'(t) \quad \text{and} \quad \Gamma_G(t) = g(t)t - G(t).$$

Here, we remember that G is a N -function if $G(t) = \int_0^t g(s)ds$, where $g : [0, \infty) \rightarrow \mathbb{R}$ is a positive nondecreasing function such that $g(0) = 0$, $\lim_{t \rightarrow \infty} g(t) = \infty$ and g is right continuous, that is, if $t \geq 0$ then $\lim_{s \rightarrow t^+} g(s) = g(t)$. Basics properties and results in Orlicz Spaces theory can be found in [1].

This paper is an extension of the results obtained in [6], as we now explain. Here, we consider functionals such that the Euler–Lagrange equations related are inhomogeneous. We obtain non-degeneracy for minimizers in the neighbourhood of the free boundary and log-Lipschitz type estimates for minima of non-singular operators (see definition in Sect. 8). We observe that such results were not presented in [6]. We also extend the compactness result

about N -functions (Theorem 6.1 of [6]). For this, we weaken the Morrey β -control, namely, if $g = G'$ and $Q_g(t) = \frac{tg'(t)}{g(t)}$,

$$\int_t^{t+\kappa} |Q'_g(t)| dt \leq C(\delta, g_0, \beta) \left(\frac{\kappa}{t}\right)^\beta$$

for more general modulus of continuity for Q_g . For instance, a Dini type control,

$$\int_0^{L-l} \frac{\omega_g^{l,L}(t)}{t} dt \leq C(\delta, g_0) \cdot \xi_1\left(\frac{L}{l}\right) \cdot \xi_2\left(\frac{L-l}{l}\right),$$

for a modulus of continuity $\omega_g^{l,L}(t)$ of Q_g and some functions $\xi_i : (0, \infty) \rightarrow (0, \infty)$ (for more details see Definition 2.1). We point out that the Morrey β -control is a key ingredient in [6] to prove the Lipschitz continuity of minimizers under universal smallness of the density of the negative phase. We emphasize that the main contribution in this paper is to prove the main results of [6] for such Dini type modulus of continuity for Q_g , precisely, the Lipschitz regularity of minima under small density on one of the phases and universal asymptotic density estimates from below for the sets $\{u < 0\}$ and $\{u > 0\}$ for some possible minimizer $u \notin C_{loc}^{0,1}$. Finally, even in the homogeneous case, we observe that our main results are slightly more general than those obtained in [6] (see Theorems 2.2, 2.3 and 2.4 in the next section).

Let us now discuss the precise conditions that will be assumed throughout the remainder of the paper. For singular/degenerate elliptic operators, we will assume the same conditions as in [17]. In this work, G. Lieberman studies the regularity theory for equations of the type $\mathcal{L}_g u = B(x, u, \nabla u)$. The conditions we refer to are:

• **Primitive Condition:**

$$G'(t) = g(t), \text{ where } g \in C^0([0, +\infty)) \cap C^1((0, +\infty)); \quad (\text{PC})$$

• **Quotient Condition:** for $0 < \delta \leq g_0$ fixed constants,

$$0 < \delta \leq Q_g(t) := \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall t > 0. \quad (\text{QC})$$

In accordance with [18], Lieberman's condition (PC) and (QC) assure that $\mathcal{L}_g u = B(x, u, \nabla u)$ is equivalent to a uniformly elliptic equation (in non-divergence form) with (positive) ellipticity constants $\lambda = \min\{1, \delta\}$ and $\Lambda = \max\{1, g_0\}$ in the sets $\{|\nabla u| \geq c\}$, for any $c > 0$ (Remark 7.1, [18]). Such conditions do not imply any kind of homogeneity for the function G . Moreover, it allows different behaviours for g when $|\nabla u|$ is close to zero or infinity. This lack of homogeneity will result in further difficulties in obtaining certain estimates. It is also worth emphasizing the large number of examples of functions satisfying (PC) and (QC), namely, $g(t) = t^p$ with $\delta = g_0 = p$, $p > 0$; $g(t) = at^p + bt^{p'}$ where $a, b, p, p' > 0$ with $\delta = \min\{p, p'\}$ and $g_0 = \max\{p, p'\}$ and $g(t) = t^p \log(at + b)$ for $p, a, b > 0$ where in this case $\delta = p$ and $g_0 = p + 1$. Many others examples are presented in [6, 18, 20]. We also observe that FBP as (1.5) appear in many applications, in particular, in the study of the flow of two liquids in models of jets and cavities.

For the N -functions $H_i : [0, \infty) \rightarrow [0, \infty)$, we assume that $H_i \in C^1[0, \infty)$ and there exist constants $0 \leq p_i \leq \rho_i < \infty$ and $\gamma_i > 0$ such that

$$H_i(t) \leq \gamma_i \cdot \max \left\{ t^{1+p_i}, t^{1+\rho_i} \right\}, \quad \forall t \geq 0, \quad i = 1, 2. \quad (\text{HC})$$

As mentioned above, in this paper, we deal with the inhomogeneous case for two-phase Alt–Caffarelli minimum type problem. Similarly as in [6], instead of focusing on any particular case, we would like to obtain universal estimates for the largest class of functionals that share the same universal and specific parameters. Once established the existence and bounds for minimizers of (1.4), we focus on appropriate classes of local minimizers to obtain regularity results, non-degeneracy estimates, suitable conditions for Lipschitz regularity and some geometry of the free boundaries. Therefore, we define a preliminary class of N -functions

$$\mathcal{G}(\delta, g_0) := \left\{ G : [0, \infty) \rightarrow [0, \infty); \quad G \text{ is } N\text{-function satisfying (PC) and (QC)} \right\}. \quad (1.6)$$

This paper is organized as follows: In Sect. 2, we establish classes of the N -functions and functionals that we will systematically use along the paper. In this section, we also present our main results. Section 3 is dedicated to preliminary results in Orlicz–Sobolev spaces developed in [6, 17, 18]. Section 4 deals with the existence and L^∞ estimates for global minimizers. Section 5 is devoted to the uniform $C_{loc}^{0,\alpha}$ estimates for the class of local minimizers of functionals related to non-degenerate N -functions G , i.e. functionals where G satisfies the following condition: there exists $\varepsilon_0 > 0$ such that $G(1) \geq \varepsilon_0$, $\forall G \in \mathcal{G}(\delta, g_0)$. In Sect. 6, we provide a proof for the local Lipschitz regularity for one-phase minimizers. In Sect. 7, we show the non-degeneracy properties for minima. In Sect. 8, we prove the uniform log-Lipschitz regularity for minimizers in classes where $\delta \geq 1$, i.e. the minima are weak solutions of non-singular Euler–Lagrange equations. In Sect. 9, we address compactness and scaling results. The proof of our main result (Theorem 2.2) is discussed in Sect. 10. In Sect. 11, we discuss two applications of the Theorem 2.2: The Lipschitz regularity for two-phase minimizers under small density of one of the phases along the free boundary and density estimates from below for the positive and negative phase on points inside the contact set between the free boundaries in the case where a minimizer is not Lipschitz. We finish the section establish a preliminary description of the free boundary for any minima even in the case where Lipschitz regularity fails to be the optimal one.

2 Some definitions and main results

We start this section remember the definition of modulus of continuity. A modulus of continuity is a nondecreasing continuous function $\omega : [0, \infty) \rightarrow [0, \infty)$ where $\omega(0) = 0$ and $\omega(t) > 0$, $\forall t > 0$. Now, for $G \in \mathcal{G}(\delta, g_0)$, Q_g as in (QC) and for any $0 < l < L < \infty$, we define

$$\omega_g^{l,L}(t) := \sup \left\{ |Q_g(x) - Q_g(y)| : l \leq x, y \leq L \text{ and } |x - y| \leq t \right\}. \quad (2.1)$$

Definition 2.1 (*Modulus of continuity for Q_g*) We say that a N -function $G \in \mathcal{G}(\delta, g_0)$ belongs to:

- (i) $\mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0)$ if, for the nondecreasing functions $\xi_1, \xi_2 : (0, \infty) \rightarrow (0, \infty)$ with $\lim_{t \rightarrow 0^+} \xi_2(t) = 0$ and any $0 < l < L < \infty$, the following Dini type control holds

$$\int_0^{L-l} \frac{\omega_g^{l,L}(t)}{t} dt \leq C_1^*(\delta, g_0) \cdot \xi_1\left(\frac{L}{l}\right) \cdot \xi_2\left(\frac{L-l}{l}\right). \quad (\text{DTC})$$

- (ii) $\mathcal{G}_{\beta \cdot H}(\delta, g_0)$, for $0 < \beta \leq 1$, if $g \in W_{loc}^{2,1}((0, +\infty))$ and, for any $t > 0$ and $\kappa > 0$ the following β -Hölder type control holds

$$\int_t^{t+\kappa} |Q'_g(s)| ds \leq C_2^*(\delta, g_0, \beta) \cdot \left(\frac{\kappa}{t}\right)^\beta. \quad (\beta\text{-HTC})$$

- (iii) $\mathcal{G}_{Lip}(\delta, g_0)$ if $g \in W_{loc}^{2,1}((0, +\infty))$ and holds the Lipschitz type control

$$0 \leq \frac{t^2 |g''(t)|}{g(t)} \leq C_3^*(\delta, g_0), \quad \text{for a.e. } t > 0. \quad (\text{LTC})$$

In Sect. 9, we verify that for every $\beta \in (0, 1]$

$$\mathcal{G}_{Lip}(\delta, g_0) \subsetneq \mathcal{G}_{\beta \cdot H}(\delta, g_0) \subsetneq \bigcup_{\gamma \in (0, 1]} \mathcal{G}_{\gamma \cdot H}(\delta, g_0) \subsetneq \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0) \subseteq \mathcal{G}(\delta, g_0).$$

In this moment, we observe that there may exist sequences of N -functions in $\mathcal{G}(\delta, g_0)$ that converge to zero in any compact subset of $[0, \infty)$. Indeed, consider, for example, $G_N(t) = \frac{1}{N}t^3$. On the other hand, the graph of (one-phase local) minimizers of the functionals $\mathcal{J}_G$, when $f_1 = f_2 \equiv 0$, hits the free boundary with an angle comparable to negative powers of $G(1)$. In fact, if $G(1)$ is small enough

$$C_4^* \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+g_0}} \leq |\nabla u| \leq C_5^* \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+\delta}} \quad \text{along } F^+(u) = \partial\{u > 0\} \cap \Omega,$$

for some universal and positive constants C_4^*, C_5^* (see Section 6 in [6]). In particular, if u_k is a minimizer for $\mathcal{J}_{G_k}$ with $G_k(1) \searrow 0$, then $|\nabla u_k| \nearrow +\infty$. Thus, universal estimates in classes of functionals can only hold for non-degenerate subclasses of $\mathcal{G}(\delta, g_0)$, i.e., subclasses where $G(1)$ is bounded from below away from zero, say $G(1) \geq \varepsilon_0 > 0$. In order to ensure that N -functions belong to non-degenerate classes, we consider the following definition (see [6]):

Definition 2.2 (*Non-degenerate subclasses*) For $\varepsilon_0 > 0$ we define

$$\begin{aligned} \mathcal{G}(\delta, g_0, \varepsilon_0) &= \left\{ G \in \mathcal{G}(\delta, g_0) : G(1) \geq \varepsilon_0 \right\} \quad \text{and} \\ \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0) &= \left\{ G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0) : G(1) \geq \varepsilon_0 \right\}. \end{aligned}$$

Similarly we define $\mathcal{G}_{\beta \cdot H}(\delta, g_0, \varepsilon_0)$ and $\mathcal{G}_{Lip}(\delta, g_0, \varepsilon_0)$.

Now we define the main classes of functions in this paper.

Definition 2.3 We say that a function u belongs to class $\mathcal{S}(\Omega)$ if u is a minimizer of a functional

$$\mathcal{J}_G(u, \Omega) = \int_{\Omega} \left[G(|\nabla u|) + f_1(x)H_1(u^+) + f_2(x)H_2(u^-) + Q(h_1, h_2)(u)(x) \right] dx, \quad (2.2)$$

for some $G \in \mathcal{G}(\delta, g_0)$ and $H_i \in C^1([0, \infty))$ satisfying (HC) for nonnegative constants γ_i, p_i, ρ_i , where $p_i \leq \rho_i < \delta$ and λ_i and μ_i , positive constants such that

$$\|f_i\|_{L^q(\Omega)} \leq \lambda_i, \quad 0 \leq h_i \leq \mu_i.$$

In fact, $\mathcal{S}(\Omega) := \mathcal{S}(\Omega, \delta, g_0, p_i, \rho_i, \gamma_i, \lambda_i, \mu_i)$.

If $G \in \mathcal{G}(\delta, g_0, \varepsilon_0)$, we say that $u \in \mathcal{S}_{\varepsilon_0}(\Omega) =: \mathcal{S}(\Omega, \delta, g_0, p_i, \rho_i, \gamma_i, \lambda_i, \mu_i, \varepsilon_0)$.

For $M > 0$ we define

$$\mathcal{S}_{\varepsilon_0}(\Omega, M) := \left\{ u \in \mathcal{S}_{\varepsilon_0}(\Omega) : \|u\|_{L^\infty(\Omega)} \leq M \right\}.$$

We say that $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$, for the positive constant Λ_i^M , if $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M)$ and

$$|H'_i(M)| \leq \Lambda_i^M(\gamma_i, p_i, \rho_i, M). \quad (2.3)$$

In the case where $G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0)$ or $G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0)$, we define the classes

$$\mathcal{S}^{\xi_1, \xi_2}(\Omega), \quad \mathcal{S}_{\varepsilon_0}^{\xi_1, \xi_2}(\Omega), \quad \mathcal{S}_{\varepsilon_0}^{\xi_1, \xi_2}(\Omega, M), \quad \mathcal{S}_{\varepsilon_0}^{\xi_1, \xi_2}(\Omega, M, \Lambda_i^M).$$

Similar definitions holds for β -Hölder control, and Lipschitz control. We pointed out that $\|u\|_{L^\infty(\Omega)} \leq M$ is a natural condition because of the Theorem 4.1. We also observe that the condition (2.3) is not artificial. In fact, if we change the condition (HC) to

$$0 \leq H'_i(t) \leq \gamma_i \cdot \max \{t^{p_i}, t^{\rho_i}\},$$

both conditions (HC) and (2.3) are satisfied.

We are ready to present our main results. Suppose initially that $q > \frac{1+\delta}{\delta-\rho^*}$, where $\rho^* = \max\{\rho_1, \rho_2\}$.

Theorem 2.1 *Let $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$. Assume that $\delta \geq 1$ and $q \geq n$. Then, $\nabla u \in BMO_{loc}(\Omega)$. More precisely, for any $\Omega' \Subset \Omega$ there exists a constant $C = C(n, \delta, g_0, q, \gamma_i, \lambda_i, \mu_i, M, \Lambda_i^M, |\Omega|) > 0$ such that*

$$\|\nabla u\|_{BMO(\Omega')} \leq \frac{C}{\varepsilon_0} \left(\text{dist}(\Omega', \partial\Omega)^{-n} + \text{diam}(\Omega)^n \cdot \varepsilon_0^{-1/\delta} \right)^{\frac{1}{1+\delta}}. \quad (2.4)$$

Moreover, u is locally Log-Lipschitz and the following estimate holds

$$|u(x) - u(y)| \leq C(n) \|\nabla u\|_{BMO(\Omega')} |x - y| \log |x - y|. \quad (2.5)$$

In particular, $u \in C_{loc}^{0, \alpha}(\Omega)$ for all $\alpha \in (0, 1)$.

For the next results, we consider Θ_u^- and Θ_u^+ the density functions of the negative and positive sets along the free boundaries for u , that is:

$$\Theta_u^-(x_0, r) = \frac{|\{u < 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \quad \text{and} \quad \Theta_u^+(x_0, r) = \frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|}$$

where $x_0 \in F^\pm(u) = F^+(u) \cup F^-(u) := (\partial\{u > 0\} \cap \Omega) \cup (\partial\{u < 0\} \cap \Omega)$.

Theorem 2.2 *Assume that $q = \infty$. There exists a constant $c_0 = c_0(n, \delta, g_0, \varepsilon_0, p_i, \rho_i, \gamma_i, \lambda_i, \mu_i, M, \xi_i) \in (0, 1)$ such that the estimate*

$$|u(x)| \leq \frac{2M}{c_0} \cdot |x - x_0|, \quad \forall x \in B_1(x_0) \quad (2.6)$$

holds for any $u \in \mathcal{S}_{\varepsilon_0}^{\xi_1, \xi_2}(B_1(x_0), M)$ provided

$$\min \{ \Theta_u^+(x_0, r), \Theta_u^-(x_0, r) \} \leq c_0, \quad \forall r \in (0, 1).$$

Corollary 2.1 Assume that $q = \infty$. There exist positive constants $c_0^* = c_0^*(n, \delta, g_0, p_i, \rho_i, \gamma_i, M, \xi_i)$ and $r_0^* = r_0^*(\delta, g_0, \varepsilon_0, \lambda_i, \mu_i)$ such that the estimate

$$|u(x)| \leq \frac{2M}{c_0^* \cdot r_0} \cdot |x - x_0|, \quad \forall x \in B_{r_0}(x_0) \quad (2.7)$$

holds for any $u \in S_{\varepsilon_0}^{\xi_1, \xi_2}(B_{r_0}(x_0), M)$ provided $0 < r_0 \leq r_0^*$ and

$$\min \left\{ \Theta_u^+(x_0, r), \Theta_u^-(x_0, r) \right\} \leq c_0^*, \quad \forall r \in (0, r_0).$$

Theorem 2.3 Let $u \in S_{\varepsilon_0}^{\xi_1, \xi_2}(\Omega, M, \Lambda_i^M)$ with $q = \infty$ and $\sigma \in (0, 1]$. Assume that for any

$$x_0 \in \left\{ x \in F^\pm(u) : \max \left\{ \text{dist}(x, F^+(u)), \text{dist}(x, F^-(u)) \right\} < \sigma \cdot r_0^* \right\}$$

the following condition is satisfied

$$\min \left\{ \Theta_u^-(x_0, r), \Theta_u^+(x_0, r) \right\} \leq c_0^*,$$

for all $0 < r < \min \left\{ \sigma \cdot r_0^*, \text{dist}(x_0, \partial\Omega) \right\}$, where c_0^* and r_0^* are as in Corollary 2.1. Then, $u \in C_{loc}^{0,1}(\Omega)$. More precisely, for any $\Omega' \Subset \Omega$ there exists $C = C(n, \delta, g_0, \varepsilon_0, p_i, \rho_i, \gamma_i, \lambda_i, \mu_i, M, \Lambda_i^M, \xi_i, \text{diam}(\Omega)) > 0$ such that, for $r_\sigma^* := \min \left\{ \sigma \cdot r_0^*, \text{dist}(\Omega', \partial\Omega) \right\}$,

$$\|\nabla u\|_{L^\infty(\Omega')} \leq \frac{C}{r_\sigma^*}.$$

At last, consider the following notation:

$$[u]_{C^{0,1}(B_\rho(x))}(x) = \sup_{y \in B_\rho(x), y \neq x} \frac{|u(y) - u(x)|}{|y - x|},$$

$$[u]_{C^{0,1}(B_\rho(x))} = \sup_{z, w \in B_\rho(x), z \neq w} \frac{|u(z) - u(w)|}{|z - w|}.$$

Theorem 2.4 Let $u \in S_{\varepsilon_0}^{\xi_1, \xi_2}(\Omega, M)$ and $\Omega' \Subset \Omega$. Assume that $q = \infty$. Consider the sets

$$S_p(\Omega') := \left\{ x \in \overline{\Omega'} : [u]_{C^{0,1}(B_r(x))}(x) = \infty \text{ for all } 0 < r < \text{dist}(\Omega', \partial\Omega) \right\}$$

and

$$S_l(\Omega') := \left\{ x \in \overline{\Omega'} : [u]_{C^{0,1}(B_r(x))} = \infty \text{ for all } 0 < r < \text{dist}(\Omega', \partial\Omega) \right\}.$$

Then, $S_p(\Omega') \subset S_l(\Omega') \subset F^+(u) \cap F^-(u)$ and, for $d = \text{dist}(\Omega', \partial\Omega)$,

$$\limsup_{F^\pm(u) \times (0, d) \ni (x, r) \rightarrow (x_0, 0)} \left(\min \left\{ \Theta_u^+(x, r), \Theta_u^-(x, r) \right\} \right) \geq c_0^*, \quad \text{for every } x_0 \in S_l(\Omega'). \quad (2.8)$$

Still,

$$\limsup_{r \rightarrow 0} \left(\min \left\{ \Theta_u^+(x_0, r), \Theta_u^-(x_0, r) \right\} \right) \geq c_0^*, \quad \text{for every } x_0 \in S_p(\Omega'). \quad (2.9)$$

Here c_0^* is as in Corollary 2.1.

3 Preliminary results

In this section, we present some background results that will be used throughout the paper. They are part of the theory of Orlicz–Sobolev spaces and of the regularity theory of singular/degenerate elliptic equations of the type $\mathcal{L}_g u = B(x, u, \nabla u)$. Some proofs can be found in [6, 17, 18]. Here, we use freely the definitions, results and properties of the N -functions obtained in Section 2 of [18].

Initially, we observe that the conditions (PC) and (QC) imply the properties below.

Lemma 3.1 *Let $G \in \mathcal{G}(\delta, g_0)$. Then, for all $t, s > 0$:*

- (g – 1) $\min\{s^\delta, s^{g_0}\}g(t) \leq g(st) \leq \max\{s^\delta, s^{g_0}\}g(t)$;
- (g – 2) $\frac{tg(t)}{1+g_0} \leq G(t) \leq tg(t)$;
- (G – 1) G is convex and $C^2(0, \infty)$;
- (G – 2) $\frac{1}{1+g_0} \min\{s^{1+\delta}, s^{1+g_0}\}G(t) \leq G(st) \leq (1+g_0) \max\{s^{1+\delta}, s^{1+g_0}\}G(t)$;
- (G – 3) $G(a+b) \leq 2^{g_0}(1+g_0)(G(a)+G(b)), \forall a, b > 0$.

Proof See Lemma 1.1 of [17], Lemma 2.1 and Remark 2.2 of [18]. $\square$

The next lemma is useful to prove the L^∞ estimates for minimizers (see Theorem 4.1).

Lemma 3.2 *Let $u \in W^{1,p}(\Omega)$, $1 < p \leq n$, such that $\sup_{\partial\Omega} u < \infty$. For $k \geq k_0 > \sup_{\partial\Omega} u$ assume that*

$$\int_{\Omega_k} |\nabla u|^p dx \leq C \cdot k^\alpha |\Omega_k|^{1-\frac{p}{n}+\varepsilon}, \quad \Omega_k := \{u > k\},$$

where $C, \varepsilon > 0$ and $0 \leq \alpha \leq p + \varepsilon$. Then, there exists a positive constant C^* depending on n, α, p, C, k_0 and $\|u\|_{L^1(\Omega_{k_0})}$ such that

$$\sup_{\Omega} u \leq C^*.$$

Proof See Lemma 5.2, Chapter 2 in [12]. $\square$

The following result is a version of the Morrey's Lemma in the Orlicz–Sobolev setting. This Lemma performs a fundamental role to obtain the $C^{0,\alpha}$ estimates for minimizers.

Lemma 3.3 (Morrey's type Lemma) *Let $u \in W_{loc}^{1,1}(\Omega) \cap L^1(\Omega)$, $G \in \mathcal{G}(\delta, g_0)$ and $0 < \alpha \leq 1$. Suppose $\Omega' \subset\subset \Omega$ such that*

$$\int_{B_r(x_0)} G(|\nabla u|) dx \leq Lr^{\alpha-1} \text{ for every } x_0 \in \Omega' \text{ with } 0 < r \leq R_0 \leq \text{dist}(\Omega', \partial\Omega).$$

Then, there exist $C_1 = C_1(\alpha, n, g_0) > 0$ and $C_2 = C_2(\alpha, n, g_0, R_0) > 0$ such that

$$\begin{aligned} |u(x) - u(y)| &\leq \left(C_1 \cdot \max \left\{ \frac{L}{G(1)}, 1 \right\} \right) \cdot |x - y|^\alpha, \text{ for } x, y \in \Omega' \text{ with } |x - y| \leq \frac{R_0}{2}, \\ \|u\|_{L^\infty(\Omega')} &\leq C_2 \left(L + \|u\|_{L^1(\Omega)} \right), \\ [u]_{\alpha, \Omega'} &\leq \max \left(C_1 \cdot \max \left\{ \frac{L}{G(1)}, 1 \right\}, \frac{2^{\alpha+1} \cdot \|u\|_{L^\infty(\Omega')}}{R_0^\alpha} \right). \end{aligned} \quad (3.1)$$

Proof See Appendix in [6]. $\square$

We recall still that a function $v \in W^{1,G}(\Omega)$ is a weak solution for the equation $\mathcal{L}_g v = B(x, v, \nabla v)$ in Ω if $v \in W^{1,G}(\Omega)$ and for all $\xi \in C_0^\infty(\Omega)$ we have

$$-\int_{\Omega} g(|\nabla v|) \frac{\nabla v}{|\nabla v|} \nabla \xi \, dx + \int_{\Omega} B(x, v, \nabla v) \xi \, dx = 0.$$

We state a version of Harnack inequality for $B(x, v, \nabla v) := f(x)$. The proof of this result can be found in [17] (see, for instance, Corollary 1.4 and Sections 3 and 4).

Lemma 3.4 (Harnack inequality) *Let $G \in \mathcal{G}(\delta, g_0)$ and u be a nonnegative weak solution to $\mathcal{L}_g u = f$ in B_r , where $g = G'$ and $f \in L^q(B_r)$, $q > n$. Then, there exists a constant $C = C(n, \delta, g_0, q) > 0$ such that*

$$\sup_{B_{r/2}} u \leq C \left(\inf_{B_{r/2}} u + r g^{-1} \left(r^{1-\frac{n}{q}} \|f\|_{L^q(B_r)} \right) \right).$$

The next Theorem compiles some estimates on the regularity theory of weak solutions to elliptic equations in Orlicz context (see, for instance, Theorem 1.7 and Lemma 5.1 in [17]). We present a special version only with the desired estimates.

Theorem 3.1 *Let $G \in \mathcal{G}(\delta, g_0)$ and v be a weak solution to $\mathcal{L}_g v = f$ in Ω , where $g = G'$ and $f \in L^q(\Omega)$, $q > n$. Then v is $C_{loc}^{1,\alpha}(\Omega)$ for some constant $\alpha(n, g_0, \delta, q) \in (0, 1)$. Moreover, for any $\Omega' \Subset \Omega$, there exists a constant $C_0 > 0$, depending only on n, δ, g_0, q and $\text{dist}(\Omega', \partial\Omega)$ such that*

$$\|v\|_{C^{1,\alpha}(\Omega')} \leq C_0 \left(\|v\|_{L^\infty(\Omega)} + g^{-1}(\|f\|_{L^q(\Omega)}) \right). \quad (3.2)$$

Also, there exists a constant $C_1 = C_1(n, \delta, g_0, q) > 0$ such that for all $B_r \subset \Omega$,

$$\sup_{B_{r/2}} |\nabla v| \leq C_1 \left(\frac{\|v\|_{L^\infty(B_r)}}{r} + g^{-1} \left(r^{1-n/q} \|f\|_{L^q(B_r)} \right) \right). \quad (3.3)$$

Further, if $f \equiv 0$, for every $\epsilon \in (0, n)$, there exists $C_2 = C_2(n, \epsilon, \delta, g_0) > 0$ such that,

$$\int_{B_{r/2}} G(|\nabla v|) \, dx \leq C_2 \sup_{B_r} |v| \cdot r^\epsilon. \quad (3.4)$$

Proof We prove (3.2). By standard covering procedure, it is sufficient to prove that

$$\|v\|_{C^{1,\alpha}(B_{\frac{r}{2}})}^* \leq C \left(\|v\|_{L^\infty(B_r)} + r g \left(r^{1-n/q} \|f\|_{L^q(B_r)} \right) \right),$$

where $\|v\|_{C^{1,\alpha}(B_{\frac{r}{2}})}^* := \|v\|_{L^\infty(B_{\frac{r}{2}})} + r \|\nabla v\|_{L^\infty(B_{\frac{r}{2}})} + r^{1+\alpha} [\nabla v]_{C^{0,\alpha}(B_{\frac{r}{2}})}$ and $[\nabla v]_{C^{0,\alpha}(B_{\frac{r}{2}})}$ is the semi-norm Hölder of ∇v . By scaling of the type $v_r(x) = v(rx)/r$, we can assume that $r = 1$. Now, for $\varepsilon > 0$, consider

$$\kappa_0 := \|u\|_{L^\infty(B_1)} + g^{-1}(\|f\|_{L^q(B_1)}) + \varepsilon \quad \text{and} \quad w(x) := \kappa_0^{-1} \cdot v(x), \quad \text{for } x \in B_1.$$

This way, it is easy to check that $\mathcal{L}_{g^*} w = \frac{f(x)}{g(\kappa_0)}$ in B_1 in the distributional sense, where $g^*(t) = \frac{g(\kappa_0 \cdot t)}{g(\kappa_0)}$. Now, since g^* satisfies (PC) and (QC), $g^*(1) = 1$, $\|w\|_{L^\infty(B_1)} \leq 1$ and $\|f_0\|_{L^q(B_1)} \leq$

1, we have by Theorem 1.7 in [17] (see still comments in pages 347–350) that, for some $\alpha = \alpha(n, \delta, g_0, q) \in (0, 1)$ and $C = C(n, \delta, g_0, q) > 0$,

$$w \in C^{1,\alpha}(B_{1/2}) \quad \text{and} \quad \kappa_0^{-1} \cdot \|v\|_{C^{1,\alpha}(B_{1/2})} = \|w\|_{C^{1,\alpha}(B_{1/2})} \leq C.$$

This implies the desired estimate taking $\varepsilon \rightarrow 0$. The proof of (3.3) follows immediately by (3.2), and (3.4) can be founded in Lemma 5.1 [17] (see also Remark 3.2 in [6]). $\square$

4 Existence theory and boundedness for minimizers of $\mathcal{J}_G$.

In this section, we establish existence and boundedness of (global) minimizers for the functional $\mathcal{J}_G$. The procedure to determine the existence of minimizers is standard, so we only present some steps. Before, we need of a definition.

Definition 4.1 We say $u \in W^{1,G}(\Omega)$ is a minimizer for $\mathcal{J}_G$ over K_φ if

$$\mathcal{J}_G(u, \Omega) = \min_{v \in K_\varphi} \mathcal{J}_G(v, \Omega),$$

where for $\varphi \in W^{1,G}(\Omega) \cap L^\infty(\partial\Omega)$ with $\int_\Omega G(|\nabla\varphi|)dx < \infty$ and

$$K_\varphi = \left\{ v \in L^1(\Omega) : \int_\Omega G(|\nabla v|)dx < \infty, \quad v = \varphi \text{ on } \partial\Omega \right\}.$$

Minimizers over K_φ are also called global minimizers.

In order to make shorter the mathematical estimates in this section, we consider the following simplifications

$$[v]_G := \int_\Omega G(|v|)dx \quad \text{and} \quad [v]_p := \|v\|_{L^p(\Omega)},$$

for any N -function G and $1 \leq p \leq \infty$.

Proposition 4.1 Let $G \in \mathcal{G}(\delta, g_0)$ and $\mathcal{J}_G(\cdot, \Omega)$ a functional of the type (2.2). Assume that $H(|\varphi|) \in L^{q'}(\Omega)$ where $1/q + 1/q' = 1$. Then, there exists, at least, one (global) minimizer for the optimization problem

$$\min_{v \in K_\varphi} \mathcal{J}_G(v, \Omega).$$

Remark 4.1 In the proof of the above proposition we need that for any $\psi \in W^{1,G}(\Omega)$ the condition $H_i(|\psi|)^{q'} \in L^G(\Omega)$ to be satisfied. This condition is always true if $\delta > n - 1$. When $0 < \delta \leq n - 1$, it is sufficient to assume that $(1 + \rho_i)q' < 1 + \delta$, $i = 1, 2$. We observe that this happens if we impose that $\delta > \rho^* = \max\{\rho_1, \rho_2\}$ and $q > \frac{1+\delta}{\delta-\rho^*}$. From now on, we assume that this conditions holds.

Proof of Proposition 4.1 First we note that

$$\min_{v \in K_\varphi} \mathcal{J}_G(v, \Omega) \leq [\nabla\varphi]_G + \sum \left([f_i]_q [H_i(|\varphi|)]_{q'} + [h_i]_\infty |\Omega| \right). \quad (4.1)$$

Now, we claim that there exists $C = C(\varphi) > 0$ such that $\mathcal{J}_G(v, \Omega) \geq -C$ for any $v \in K_\varphi$. In fact, by Poincaré type inequality (Lemma 2.4 of [18]), (G - 2) and (G - 3)

$$C(g_0, d)[v - \varphi]_G \leq [\nabla v - \nabla\varphi]_G \leq C(g_0)([\nabla v] + [\nabla\varphi]_G) \quad \text{and} \quad c(g_0)[v]_G - [\varphi]_G \leq [v - \varphi]_G, \quad (4.2)$$

where $d = \text{diam}(\Omega)$. Combination of the inequalities in (4.2) give us

$$-C(g_0, d)([\varphi]_G + [\nabla\varphi]_G) \leq [\nabla v]_G - c[v]_G, \quad (4.3)$$

where $c = c(g_0, d) < 1$. For $P_i(t) = t^{(1+\rho_i)q'}$ we have that $G \circ P_i^{-1}$ is a N -function. Thus, for $0 < \kappa < 1$, follows by (G - 2) together with Young's inequality that,

$$[f_1 H_1(v^+)]_1 \leq C_1 \cdot [f]_q \left(\kappa^{-(1+\rho_1)} [\kappa v]_{(1+\rho_1)q'}^{1+\rho_1} + |\Omega|^{1/q'} \right) \leq \frac{c}{4} [v]_G + \tilde{C}_1, \quad (4.4)$$

where $C_1 = 2^{1+\rho_1+1/q'} \gamma_1$ and $\kappa = [4(1 + g_0)c^{-1}]^{-\frac{1}{(1+\delta)q'}}$. Similarly,

$$[f_2 H_2(v^-)]_1 \leq \frac{c}{4} [v]_G + \tilde{C}_2. \quad (4.5)$$

Combining the estimates (4.3), (4.4) and (4.5) the claim is proved. Now, the proof follows similarly as in Theorem 3.1 of [18] considering a minimizing sequence $\{u_j\} \subset K_\varphi$ for $\mathcal{J}_G(v, \Omega)$. $\square$

Remark 4.2 It is possible to show the existence of minima if $q = \frac{1+\delta}{\delta-\rho^*}$ provided $[f]_q$ is small enough.

The next Lemma provides a universal control on the L^1 -norm of minima. We combine these estimate with Lemma 3.2 to obtain L^∞ estimates for minimizers.

Lemma 4.1 (L^1 -estimate) *Let u a minimizer for $\mathcal{J}_G$ in K_φ . Then,*

$$[u]_1 \leq C = C(n, \delta, g_0, q, f_i, h_i, \gamma_i, \rho, \varphi, \Omega).$$

Proof By Young and Poincaré type inequalities, (G - 2) and (G - 3)

$$[u]_1 \leq [u]_G + \tilde{G}(1)|\Omega| \quad \text{and} \quad [u]_G \leq C(\delta, g_0, d) ([\nabla u]_G + [\varphi]_G + [\nabla\varphi]_G), \quad (4.6)$$

where $\tilde{G}$ is the N -complementary function of G and $d = \text{diam}(\Omega)$. By minimality, we have

$$[\nabla u]_G \leq \mathcal{J}_G(\varphi) + [f_1 H_1(u_j^+)]_1 + [f_2 H_2(u_j^-)]_1 + C_1. \quad (4.7)$$

Similarly as in the proof of the Proposition 4.1, we obtain that $[\nabla u]_G \leq C$ and by (4.6) the proof is now complete. $\square$

Below we establish the L^∞ estimates for (global) minimizers.

Theorem 4.1 (L^∞ -estimate) *Let $G \in \mathcal{G}(\delta, g_0)$, $\mathcal{J}_G$ a functional as in (2.2) and u be a minimizer for $\mathcal{J}_G$ in K_φ . Then, $u \in L^\infty(\Omega)$. In fact,*

$$[u]_\infty \leq C = C(n, \delta, g_0, q, f_i, h_i, \gamma_i, \rho_i, \varphi, \Omega). \quad (4.8)$$

Proof We prove upper boundedness of u . If $\delta > n - 1$, since $W^{1,1+\delta}(\Omega) \hookrightarrow W^{1,G}(\Omega)$, the existence of the universal constant C is an immediate consequence of the Rellich-Kondrachov embedding Theorem. Thus, we can assume that $0 < \delta \leq n - 1$. Let $k_0 = \min \{k \in \mathbb{N} : k \geq \sup_{\partial\Omega} \varphi\}$. For each $k \geq k_0$, we define a truncation function $u_k : \Omega \rightarrow \mathbb{R}^n$ by

$$u_k = \begin{cases} u, & \text{if } u \leq k, \\ k, & \text{if } u > k. \end{cases}$$

Obviously $u_k \in K_\varphi$ and, up to sets of measure zero,

$$\{u_k > 0\} = \{u > 0\}, \quad \{u_k < 0\} = \{u < 0\}, \quad \{u_k = 0\} = \{u = 0\}.$$

Setting $\Omega_k := \{u > k\}$ we get by definition of u_k , minimality of u , (HC), (G - 3) and Hölder inequality that

$$[\nabla u]_G \leq C_1 \left[[(u - k)]_{(1+\rho_1)q'}^{1+\rho_1} + k^{1+\rho_1} |\Omega_k|^{1/q'} \right]. \quad (4.9)$$

Now, for any $(1 + \rho_1)q' < \delta_0 < 1 + \delta$ we have by Sobolev's embedding and Theorem 2.2 in [18] that $u \in W^{1,\delta_0}(\Omega)$. So, applying Hölder, Gagliardo-Nirenberg-Sobolev and Young inequalities in the left side in the above estimate we get

$$[(u - k)]_{(1+\rho_1)q'}^{1+\rho_1} \leq C_2 |\Omega_k|^{(1-\tilde{\beta})/q'} \cdot [\nabla u]_{\delta_0}^{1+\rho_1} \leq C_3 |\Omega_k|^{1/q'} + \frac{1}{2C_1} \cdot [\nabla u]_{\delta_0}^{\delta_0}, \quad (4.10)$$

where $\tilde{\beta} = \frac{(1+\rho_1)(n-\delta_0)}{n\delta_0}$. Hence, combining (4.9) and (4.10), and one more time by Young inequality, we obtain

$$[\nabla u]_{\delta_0}^{\delta_0} - \tilde{G}_{\delta_0}(1) |\Omega|^{1/q} |\Omega_k|^{1/q'} \leq \frac{1}{2} [\nabla u]_{\delta_0}^{\delta_0} + C_4 k^{1+\rho_1} |\Omega_k|^{1/q'}, \quad (4.11)$$

where $\tilde{G}_{\delta_0}$ is the N -complementary function of $G \circ P^{-1}$, with $P(t) = t^{\delta_0}$. By (4.11) we conclude that

$$\int_{\Omega_k} |\nabla u|^{\delta_0} dx \leq C_5 \cdot k^{1+\rho_1} |\Omega_k|^{1/q'} = C_5 k^{1+\rho_1} |\Omega_k|^{1-\frac{\delta_0}{n}+\varepsilon},$$

where $\varepsilon = \frac{1}{q'} + \frac{\delta_0}{n} - 1$. Since $\sup_{\partial\Omega} u \leq [\varphi]_\infty$ the upper boundedness of u is consequence of Lemma 4.1 together with Lemma 3.2. By similar procedure we prove that $\inf_{\Omega} u \geq -C$. This completes the proof. $\square$

Remark 4.3 An immediate consequence of the Theorem above is a uniform control of the $W^{1,G}$ -norm of the minimizers depending only on universal constants and intrinsic parameters.

Remark 4.4 If $\varphi \geq 0$ and $f_2 \geq 0$, then $u \geq 0$. In particular u minimizes the functional,

$$E_G(v, \Omega) = \int_{\Omega} [G(|\nabla v|) + f_1(x)H_1(v^+) + h_1(x)\chi_{\{v>0\}}] dx.$$

In fact, for $\tilde{Q}(h_1, h_2)(v) = (h_1 - h_2)\chi_{\{v>0\}} - (h_1 - h_2)^- \chi_{\{v=0\}}$, we observe that u minimize

$$\tilde{E}_G(v) := \int_{\Omega} [G(|\nabla u|) + f_1(x)H_1(u^+) + f_2(x)H_2(u^-) + \tilde{Q}(h_1, h_2)(u)(x)] dx.$$

Now, let $u_\varepsilon(x) = u(x) + \varepsilon u^-$ for $0 < \varepsilon < 1$. By the minimality of u with respect to $\tilde{E}_G$, we obtain

$$\int_{\{u<0\}} \{G(|\nabla u|) + f_2[H_2(u^-) - H_2(u_\varepsilon^-)]\} dx \leq (1 + g_0)(1 - \varepsilon)^{1+\delta} \int_{\{u<0\}} G(|\nabla u|) dx.$$

Since H_2 is increasing follows for $\varepsilon \ll 1$ and by boundary condition that $u \geq 0$.

5 The uniform Hölder regularity for local minimizers in $\mathcal{S}_{\varepsilon_0}(\Omega, M)$

For the purpose of the Hölder, log-Lipschitz and Lipschitz regularity for minimizers, we do not need to consider global minimizers, or else, functions for which the G -energy is finite in the whole domain. We can consider only local minimizers as in [10]. Follows the precise definition.

Definition 5.1 A function $u \in W_{loc}^{1,G}(\Omega)$ is said to be a local minimizer for the functional $\mathcal{J}_G(\cdot, \Omega)$ if for any $W^{1,G}$ -perturbation compactly supported, i.e. $\psi \in W_c^{1,G}(\Omega)$, we have

$$\mathcal{J}_G(u, \Omega) \leq \mathcal{J}_G(u + \psi, \Omega).$$

It follows that u is a local minimizer for $\mathcal{J}_G(\cdot, \Omega)$ if and only if $\forall D \subset\subset \Omega$ with Lipschitz boundary

$$\mathcal{J}_G(u, D) = \min_{v \in K_u} \mathcal{J}_G(v, D), \quad K_u = \left\{ v \in W^{1,G}(D) : v = u \text{ on } \partial D \right\}.$$

It is easy to check that global minimizers are also local minimizers. By simplicity, we call local minimizers only minimizers.

We now aim towards the uniform (local) Hölder regularity for minimizers in the class $\mathcal{S}_{\varepsilon_0}(\Omega, M)$. Here, we assume that $q > n$ if $\delta > n - 1$ or $q > \max\{\frac{1+\delta}{\delta-\rho^*}, n\}$ when $0 < \delta \leq n - 1$. The proof follows the same spirit of the proof of the Theorem 4.1 of [18] and Theorem 5.1 in [6]. As mentioned in [6], the non-degeneracy condition (bound from below on $G(1)$) is crucial for the uniform Hölder estimate.

Theorem 5.1 Let $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M)$ and $\alpha \in (0, 1 - \frac{n}{q})$. Then u is uniformly in $C_{loc}^{0,\alpha}(\Omega)$. More precisely, for any $\Omega' \Subset \Omega$, there exists a positive constant $\bar{C} = \bar{C}(\delta, g_0, p_i, \rho_i, \gamma_i, \Lambda_i, \lambda_i, \mu_i, q, M)$ such that

$$[u]_{C^{0,\alpha}(\Omega')} \leq C := \max \left(C_1 \cdot \max \left\{ \frac{\bar{C}}{\varepsilon_0}, 1 \right\}, \frac{2^{\alpha+1} \cdot M}{\min \left\{ \left(\frac{1}{2} \right)^{1+1/\varepsilon}, \text{dist}(\Omega', \partial\Omega) \right\}^\alpha} \right),$$

where $\varepsilon = \varepsilon(n, q, \alpha) \in (0, 1)$ and $C_1 = C_1(\delta, g_0, n) > 0$. In fact,

$$C = C(n, \delta, g_0, p_i, \rho_i, \gamma_i, \Lambda_i, \lambda_i, \mu_i, q, \varepsilon_0, M, \alpha, \text{dist}(\Omega', \partial\Omega)).$$

Proof Since $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M)$, u is a local minimizer of a functional of the type

$$\mathcal{J}_G(v, \Omega) = \int_{\Omega} [G(|\nabla v|) + f_1(x)H_1(v^+) + f_2(x)H_2(v^-) + Q(h_1, h_2)(v)(x)] dx$$

satisfying the conditions of the Definition 2.3. Fix $\alpha \in (0, 1 - n/q)$, $\Omega' \Subset \Omega$ and $y \in \Omega'$. Now, consider $r \in (0, 1)$ such that $B_r(y) \subset \Omega$. By simplify the notation, assume $y = 0$. Let w be the solution to the Dirichlet problem

$$\mathcal{L}_g w = 0 \text{ in } B_r \quad \text{and} \quad w - u \in W_0^{1,G}(B_r).$$

From Theorem 2.3 of [18] we have

$$\begin{aligned} \int_{B_r} [G(|\nabla u|) - G(|\nabla w|)] dx &\geq C_1(g_0, \delta) \left(\int_{A_2} G(|\nabla u - \nabla w|) dx \right. \\ &\quad \left. + \int_{A_1} F(|\nabla u|) |\nabla u - \nabla w|^2 dx \right), \end{aligned} \quad (5.1)$$

where

$$A_1 = \left\{ x \in B_r : |\nabla u - \nabla w| \leq 2|\nabla u| \right\}, \quad A_2 = \left\{ x \in B_r : |\nabla u - \nabla w| > 2|\nabla u| \right\}$$

and $F(t) = g(t)/t$. On the other hand, by Maximum Principle (Lemma 2.8 of [18]) we have $\|w\|_{L^\infty(B_r)} = \|u\|_{L^\infty(B_r)} \leq M$. Hence, follows by the minimality of u and Hölder inequality that

$$\int_{B_r} [G(|\nabla u|) - G(|\nabla w|)] dx \leq C_2 \cdot r^{n/q'}, \quad (5.2)$$

where $C_2 = C_2(n, \lambda_i, \mu_i, \Lambda_i, \gamma_i, M, |\Omega|, q) > 0$. Combination of (5.1) and (5.2) reveals that

$$\max \left\{ \int_{A_2} G(|\nabla u - \nabla w|) dx, \int_{A_1} F(|\nabla u|) |\nabla u - \nabla w|^2 dx \right\} \leq C_3 \cdot r^{n/q'}. \quad (5.3)$$

Let $\varepsilon > 0$ and suppose that $r^\varepsilon \leq 1/2$. By Hölder inequality, (g - 2) and (5.3) we get

$$\int_{A_1 \cap B_{r+1+\varepsilon}} G(|\nabla u - \nabla w|) dx \leq C_4 \cdot r^{n/2} \left(\int_{B_{r+1+\varepsilon}} G(|\nabla u|) dx \right)^{1/2}. \quad (5.4)$$

Combining now (5.3) and (5.4) we arrive at

$$\int_{B_{r+1+\varepsilon}} G(|\nabla u - \nabla w|) dx \leq C_5 \left[r^{n/q'} + r^{n/2} \left(\int_{B_{r+1+\varepsilon}} G(|\nabla u|) dx \right)^{1/2} \right]. \quad (5.5)$$

Now, it follows by (3.4) in Theorem 3.1, Maximum Principle and (G - 3) that for any $\bar{\beta} \in (0, n/q')$

$$\int_{B_{r+1+\varepsilon}} G(|\nabla u|) dx \leq C_6 \left[r^{\bar{\beta}} + r^{\bar{\beta}/2} \left(\int_{B_{r+1+\varepsilon}} G(|\nabla u|) dx \right)^{1/2} \right], \quad (5.6)$$

where $C_6 = C_6(\delta, g_0, n, \lambda_i, \mu_i, \Lambda_i, \gamma_i, M, |\Omega|, q, \bar{\beta}) > 0$. Hence as in Theorem 4.1 of [18], we obtain

$$\int_{B_{r+1+\varepsilon}} G(|\nabla u|) dx \leq [(C_6 + 1)^{1/2} + C_6^{1/2}]^2 C_6 \cdot r^{\bar{\beta}}. \quad (5.7)$$

Taking $\bar{\beta} := (1 + \varepsilon)(n - (1 - \alpha))$, we choose $\varepsilon > 0$ so that $0 < \bar{\beta} < n/q'$. Finally, by setting $\varrho_0 := \min \left\{ (\frac{1}{2})^{1+1/\varepsilon}, \text{dist}(\Omega', \partial\Omega) \right\}$ and, for $0 < \varrho < \varrho_0$, $r = \varrho^{1/(1+\varepsilon)}$, we have that $r^\varepsilon \leq 1/2$ and therefore by (5.7),

$$\int_{B_\varrho} G(|\nabla u|) dx \leq \bar{C} \varrho^{n+\alpha-1}.$$

By Lemma 3.3 the prove of Theorem is finished. $\square$

By above result, we have that $\{u \neq 0\}$ is an open set. The following Corollary establishes the Euler–Lagrange equations for which minimizers are weak (sub/super) solutions.

Corollary 5.1 *Let u a global (local) minimizer for $\mathcal{J}_G$ in K_φ . Then*

$$\mathcal{L}_g u = f_1(x) H'_1(u^+) + f_2(x) H'_2(u^-) \quad \text{in } \{u \neq 0\},$$

in the weak distributional sense. If $h_1 \geq h_2$, we have

$$\mathcal{L}_g u \geq f_1(x) H'_1(u) \chi_{\{u>0\}} + f_2(x) H'_2(-u) \chi_{\{u<0\}} \quad \text{in } \Omega.$$

Similarly, if $h_2 \geq h_1$, then

$$\mathcal{L}_g u \leq f_1(x)H'_1(u)\chi_{\{u>0\}} + f_2(x)H'_2(-u)\chi_{\{u<0\}} \quad \text{in } \Omega.$$

(ii) In this case where $f_2, \varphi \geq 0$, we have

$$\mathcal{L}_g u - f_1(x)H'_1(u)\chi_{\{u>0\}} \geq 0 \quad \text{in } \Omega.$$

Proof The prove follows standard procedure as in [11] or [18]. $\square$

6 Lipschitz regularity for one-phase minimizers in $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$

In this section, we address the local Lipschitz continuity for one-phase minimizers. We observe that with the restriction of the sign of u , the sharp regularity of minima holds in general class $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$ when $q = \infty$. By Remark 4.4, the restriction of the sign of u is obtained when we assume $f_2, \varphi \geq 0$. The proof this fact follows a combination of similar strategies founded in Theorem 3.1 of [16] and Lemma 4.2 of [18]. For the records, we state and prove this result.

Theorem 6.1 *Let $0 \leq u \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$. Suppose that $q = \infty$. Then, for any $\Omega' \Subset \Omega$, there exists $C_1 = C_1(n, \delta, g_0, q, \varepsilon_0, \gamma_1, \lambda_1, \mu_1, \rho_1, \Lambda_1^M, \Omega', \Omega) > 0$ such that*

$$u(x) \leq \frac{C_1}{\text{dist}(\Omega', \partial\Omega)} \cdot \text{dist}(x, \partial\{u > 0\}), \quad \forall x \in \Omega'. \quad (6.1)$$

In particular, $u \in C_{loc}^{0,1}(\Omega)$ and for some $C_2 = C_2(C_1) > 0$ holds the following estimate

$$\|\nabla u\|_{L^\infty(\Omega')} \leq \frac{C_2}{\text{dist}(\Omega', \partial\Omega)}. \quad (6.2)$$

Proof Assume initially that $\text{dist}(\Omega', \partial\Omega) = 1$. Now, suppose by contradiction, that (6.1) does not hold. In this case, there exist a sequence of functions $u_k \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$ and a sequence of points $x_k \in \Omega' \cap \{u_k > 0\}$ such that

$$\frac{u_k(x_k)}{d_k} \geq k, \quad \text{where } d_k := \text{dist}(x_k, \partial\{u_k > 0\}). \quad (6.3)$$

Since $\{u_k\}$ is uniformly bounded we have $d_k \rightarrow 0$. Observe now that, for each $k \in \mathbb{N}$, we can found $y_k \in \partial\{u_k > 0\}$ such that

$$d_k = |x_k - y_k|.$$

By definition of $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$, there exist G_k, H_k, f_k and h_k satisfying the conditions in Definition 2.3 such that u_k is a minimizer of

$$\mathcal{J}_{G_k}(u, \Omega) = \int_{\Omega} [G_k(|\nabla u|) + f_k(x)H_k(u^+) + h_k(x)\chi_{\{u>0\}}]dx.$$

Follows by Corollary 5.1 that

$$\mathcal{L}_{g_k} u_k = f_k H'_k(u_k) \quad \text{in } \{u_k > 0\},$$

where $g_k = G'_k$. Thus, by Harnack's inequality, we obtain

$$u_k(x_k) \leq \sup_{\bar{B}_{\frac{3}{4}d_k}(x_k)} u_k \leq C_1 \left(\inf_{\bar{B}_{\frac{3}{4}d_k}(x_k)} u_k + C_2 \cdot d_k \right), \quad (6.4)$$

where C_1 and C_2 depends on $n, \delta, g_0, \mu_1, \lambda_1, \rho_1, \gamma_1, M$ and Λ_1^M .

Since $\overline{B}_{\frac{3}{4}d_k}(x_k) \cap \overline{B}_{\frac{1}{4}d_k}(x_k) \neq \emptyset$ we conclude by (6.4) that

$$\sup_{\overline{B}_{\frac{1}{4}d_k}(y_k)} u_k \geq cu_k(x_k) - C_2 d_k. \quad (6.5)$$

In sequel, consider the set

$$\Omega_k := \left\{ z \in B_{d_k}(y_k) : \text{dist}(z, \partial\{u_k > 0\}) \leq \frac{1}{3} \text{dist}(z, \partial B_{d_k}(y_k)) \right\}.$$

Since $y_k \in \partial\{u_k > 0\}$, we have $B_{\frac{d_k}{4}}(y_k) \subseteq \Omega_k$ and hence

$$\begin{aligned} A_k &:= \sup_{z \in \Omega_k} \text{dist}(z, \partial B_{d_k}(y_k)) \cdot u_k(z) \\ &\geq \inf_{B_{\frac{d_k}{4}}(y_k)} \text{dist}(z, \partial B_{d_k}(y_k)) \cdot \sup_{B_{\frac{d_k}{4}}(y_k)} u_k(z) \\ &= \frac{3}{4} d_k \cdot \sup_{B_{\frac{d_k}{4}}(y_k)} u_k(z). \end{aligned} \quad (6.6)$$

On the other hand, there exists $z_k \in \Omega_k$ such that

$$A_k = \text{dist}(z_k, \partial B_{d_k}(y_k)) \cdot u_k(z_k). \quad (6.7)$$

Combination of (6.6) and (6.7) lead us to conclude that

$$u_k(z_k) \geq \frac{3}{4} \sup_{B_{\frac{d_k}{4}}(y_k)} u_k(z) \quad (6.8)$$

and by (6.5) and (6.8) we arrive at

$$u_k(z_k) \geq \frac{3}{4} (cu_k(x_k) - C_2 d_k). \quad (6.9)$$

Let $w_k \in \partial\{u_k > 0\}$ such that

$$|z_k - w_k| = \text{dist}(z_k, \partial\{u_k > 0\}).$$

Now, as $z_k \in \Omega_k$ we get

$$\delta_k := |z_k - w_k| \leq \frac{1}{3} (d_k - |z_k - y_k|) \quad (6.10)$$

and as $\delta_k \leq |z_k - y_k|$,

$$\delta_k \leq \frac{1}{3} (d_k - \delta_k) \implies \delta_k \leq \frac{1}{4} d_k.$$

By above estimate and (6.9)

$$\frac{u_k(z_k)}{\delta_k} \geq \frac{3}{4} \frac{4}{d_k} (cu_k(x_k) - C_2 d_k) = 3 \left(c \cdot \frac{u_k(x_k)}{d_k} - C_2 \right). \quad (6.11)$$

This implies, by (6.3), $\frac{u_k(z_k)}{\delta_k} \longrightarrow +\infty$.

Once $\text{dist}(x, \partial\{u_k > 0\}) \leq \frac{\delta_k}{2}$, $\forall x \in B_{\frac{\delta_k}{2}}(w_k)$,

$$\begin{aligned} \text{dist}(x, \partial B_{d_k}(y_k)) &\geq \text{dist}(z_k, \partial B_{d_k}(y_k)) - |z_k - x| \\ &\geq \text{dist}(z_k, \partial B_{d_k}(y_k)) - (|z_k - w_k| + |x - w_k|) \\ &\geq \text{dist}(z_k, \partial B_{d_k}(y_k)) - \frac{3}{2}\delta_k \\ &\geq \frac{1}{2}\text{dist}(z_k, \partial B_{d_k}(y_k)), \end{aligned} \quad (6.12)$$

and hence

$$\text{dist}(x, \partial\{u_k > 0\}) \leq \frac{1}{6}\text{dist}(z_k, \partial B_{d_k}(y_k)) \leq \frac{1}{3}\text{dist}(x, \partial B_{d_k}(y_k)),$$

i.e. $B_{\frac{\delta_k}{2}}(w_k) \subseteq \Omega_k$. Also, by estimate (6.12), we have that

$$u_k(z_k) = \frac{A_k}{\text{dist}(z_k, \partial B_{d_k}(y_k))} \geq \frac{\text{dist}(x, \partial B_{d_k}(y_k)) \cdot u_k(x)}{\text{dist}(z_k, \partial B_{d_k}(y_k))} \geq \frac{1}{2}u_k(x), \quad \forall x \in B_{\frac{\delta_k}{2}}(w_k).$$

Thus,

$$\sup_{B_{\frac{\delta_k}{2}}(w_k)} u_k \leq 2u_k(z_k). \quad (6.13)$$

As $B_{\delta_k}(z_k) \subset \{u > 0\}$, Harnack inequality secures that

$$\inf_{\overline{B}_{\frac{3}{4}\delta_k}(z_k)} u_k \geq \tilde{c}u_k(z_k) - \tilde{C}\delta_k.$$

Thus,

$$\sup_{\overline{B}_{\frac{1}{4}\delta_k}(w_k)} u_k \geq \tilde{c}u_k(z_k) - \tilde{C}\delta_k. \quad (6.14)$$

Let now $\xi_k : B_1 \rightarrow \mathbb{R}$ the following family of auxiliary functions:

$$\xi_k(x) = \frac{u_k(w_k + \frac{\delta_k}{2}x)}{u_k(z_k)}, \quad x \in B_1.$$

Clearly $\xi_k(0) = 0$ and, by (6.13) and (6.14), for $k \gg 1$,

$$\max_{B_1} \xi_k \leq 2 \quad \text{and} \quad \max_{B_{\frac{1}{2}}} \xi_k \geq \frac{3\tilde{c}}{4} > 0. \quad (6.15)$$

Define still the following family of N -functions

$$\overline{G}_k(t) := \frac{G_k(\sigma_k t)}{\sigma_k g_k(\sigma_k)},$$

where $\sigma_k = \frac{2u_k(z_k)}{\delta_k}$. Observe that, for k sufficient large, $\sigma_k > c \cdot k \rightarrow +\infty$. By Proposition 6.1 of [6] we know that $\{\overline{G}_k\}_{k \geq 1} \subset \mathcal{G}(\delta, g_0, (1 + g_0)^{-1})$ and $\overline{G}_k \leq 1$. Also, by simple computation, we observe that ξ_k is a local minimizer of

$$\mathcal{F}_k(\psi) = \int_{B_1} [\overline{G}_k(|\nabla \psi|) + \overline{f}_k \overline{H}_k(\psi^+) + \overline{h}_k \chi_{\{\psi > 0\}}] dx,$$

where

$$\overline{H}_k(v) = H_k(u_k(z_k) \cdot v), \quad \overline{f}_k(x) = \frac{f_k(w_k + \frac{\delta_k}{2}x)}{\sigma_k g_k(\sigma_k)}, \quad \overline{h}_k(x) = \frac{h_k(w_k + \frac{\delta_k}{2}x)}{\sigma_k g_k(\sigma_k)}.$$

Now, we estimate

$$\|\overline{h}_k\|_{L^\infty(B_1)} \leq C \cdot \frac{1+g_0}{\varepsilon_0} \cdot \frac{\mu_1}{k^{1+\delta}} \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (6.16)$$

$$\|\overline{f}_k(x)\overline{H}_k(v)\|_{L^\infty(B_1)} \leq C \cdot \frac{1+g_0}{\varepsilon_0} \cdot \frac{\lambda_1}{k^{1+\delta}} \cdot \gamma_1 (1 + M \cdot \|v\|_{L^\infty(B_1)})^{1+\rho_1} \rightarrow 0, \quad (6.17)$$

for any bounded function v .

Let us take $v_k \in W^{1,G}(B_{5/8})$ such that

$$\mathcal{L}_k v_k = 0 \quad \text{in } B_{5/8} \quad \text{and} \quad v_k = \xi_k \quad \text{in } \partial B_{5/8}, \quad (6.18)$$

where $\mathcal{L}_k$ is the operator related to $\overline{G}_k$.

By stability estimates (5.5) and (5.7) in Theorem 5.1, for $\varepsilon = 0$ and $r = \frac{5}{8}$, Maximum Principle, (6.17) and (6.18) we obtain,

$$\int_{B_{\frac{5}{8}}} \overline{G}_k(|\nabla v_k - \nabla \xi_k|) dx \leq \tilde{C} \cdot k^{-\frac{1+\delta}{2}}. \quad (6.19)$$

Since, for each k , holds

$$\frac{1}{(g_0 + 1)^2} \min\{t^{1+\delta}, t^{1+g_0}\} \leq \overline{G}_k(t) \leq (g_0 + 1) \max\{t^{1+\delta}, t^{1+g_0}\} \quad \forall t \geq 0, \quad (6.20)$$

by setting

$$B_{\frac{5}{8}}^- := B_{\frac{5}{8}} \cap \{|\nabla v_k - \nabla \xi_k| < 1\} \text{ and } B_{\frac{5}{8}}^+ := B_{\frac{5}{8}} \cap \{|\nabla v_k - \nabla w_k| \geq 1\},$$

estimate (6.19) together with (6.20) give us

$$C \cdot k^{-\frac{1+\delta}{2}} \geq (1+g_0)^{-2} \left(\int_{B_{\frac{5}{8}}^-} |\nabla v_k - \nabla \xi_k|^{g_0+1} dx + \int_{B_{\frac{5}{8}}^+} |\nabla v_k - \nabla w_k|^{\delta+1} dx \right).$$

Applying Hölder inequality in the above first integral we have

$$\begin{aligned} C \cdot k^{-\frac{1+\delta}{2}} &\geq (1+g_0)^{-2} |B_{\frac{5}{8}}^-|^{\frac{\delta-g_0}{1+\delta}} \left(\int_{B_{\frac{5}{8}}^-} |\nabla v_k - \nabla \xi_k|^{\delta+1} dx \right)^{(1+g_0)/(1+\delta)} \\ &\quad + (1+g_0)^{-2} \int_{B_{\frac{5}{8}}^+} |\nabla v_k - \nabla \xi_k|^{\delta+1} dx. \end{aligned}$$

Hence,

$$\tilde{C} \geq k^{\frac{1}{2(1+g_0)}} \int_{B_{\frac{5}{8}}} |\nabla v_k - \nabla \xi_k|^{1+\delta} dx.$$

By Poincaré inequality, we conclude that

$$h_k := v_k - \xi_k \longrightarrow 0 \quad \text{strongly in } W_0^{1,\delta+1}(B_{\frac{5}{8}}). \quad (6.21)$$

Using one more time Theorem 5.1 and since interior Hölder estimates for v_k is a consequence of Harnack inequality we conclude that there exists $\alpha \in (0, 1)$ and constant $C' = C'(\delta, g_0, n, \rho_0, \rho, \varepsilon_0, \gamma, \Lambda_0, \mu_0, \mu, \alpha) > 0$ such that

$$\|v_k\|_{C^\alpha(B_{5/9})}, \|\xi_k\|_{C^\alpha(B_{5/9})} \leq C'.$$

This imply that there exist functions v_∞, ξ_∞ such that

$$\xi_k \rightarrow \xi_\infty \quad \text{uniformly in } B_{5/9} \quad \text{and} \quad v_k \rightarrow v_\infty \quad \text{uniformly in } B_{5/9}.$$

So, from (6.21) we arrive at $v_\infty = \xi_\infty$. Once more by Harnack inequality

$$\sup_{B_{1/2}} v_k \leq C_0 \inf_{B_{1/2}} v_k, \quad (6.22)$$

where C_0 is a universal constant. Then, passing the limit in the second estimate of (6.15) and in (6.22), we have

$$0 < \frac{3\tilde{c}}{4C_0} \leq \frac{1}{C_0} \sup_{B_{1/2}} v_\infty \leq \inf_{B_{1/2}} v_\infty \leq \xi_\infty(0) = 0.$$

But this is a contradiction. Thus,

$$u(x) \leq C_1 \cdot \text{dist}(x, \partial\{u > 0\}), \quad \forall x \in \Omega'.$$

Suppose now that $0 < d := \text{dist}(\Omega', \partial\Omega) \neq 1$. If $d > 1$ we define

$$\Omega^* := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1\}.$$

Notice that Ω^* is an open set that depends only on Ω and satisfies $\Omega' \subseteq \Omega^*$ and $\text{dist}(\Omega^*, \partial\Omega) = 1$. Using the previous case for Ω^* , there exists $C_1^* = C_1^*(n, \delta, g_0, q, \varepsilon_0, \gamma_1, \lambda_1, \mu_1, \rho_1, \Lambda_1^M, \Omega) > 0$ such that

$$u(x) \leq C_1^* \cdot \text{dist}(x, \partial\{u > 0\}) \leq \frac{C_1^* \cdot \text{diam}(\Omega)}{\text{dist}(\Omega', \partial\Omega)} \cdot \text{dist}(x, \partial\{u > 0\}), \quad \forall x \in \Omega'.$$

Hence, put $C_1 = C_1^* \cdot \text{diam}(\Omega)$. Finally assume that $d \in (0, 1)$ and consider the following rescaling

$$v(x) := u(dx), \quad x \in \Omega_d := \{d^{-1}y : y \in \Omega\}.$$

We have $\text{dist}(\Omega'_d, \partial\Omega_d) = 1$. In addition, we observe that $v \in \mathcal{S}_{\varepsilon_0}(\Omega_d, M, \Lambda_i^M)$. In fact, if u is a minimizer of

$$\mathcal{J}_G(w, \Omega) = \int_{\Omega} [G(|\nabla w|) + f(x)H(w^+) + h(x)\chi_{\{w>0\}}] dx,$$

satisfying the conditions of Definition 2.3, then, v is a minimizer of

$$\mathcal{J}_{G_d^*}(\xi, \Omega_d) = \int_{\Omega_d} [G_{d^{-1}}^*(|\nabla \xi|) + f(dx)H(\xi^+) + h(dx)\chi_{\{\xi>0\}}] dx,$$

where $G_{d^{-1}}^*(t) = G(d^{-1}t)$, and by Proposition 6.1 of [6], $G_{d^{-1}}^* \in \mathcal{G}(\delta, g_0, \varepsilon_0)$. Therefore,

$$v(x) \leq C_1 \cdot \text{dist}(x, \partial\{v > 0\}).$$

If $\partial\{u > 0\}_d := \{d^{-1}z : z \in \partial\{u > 0\}\}$, we can translate this back in terms of u and to conclude that

$$u(y) \leq C_1 \cdot \text{dist}(d^{-1}y, \partial\{u > 0\}_d) = \frac{C_1}{d} \cdot \text{dist}(y, \partial\{u > 0\}).$$

This proves (6.1). Estimate (6.2) follows essentially by (6.1) and local gradient estimates (3.3). $\square$

7 Non-degeneracy of minimizers in the neighbourhood of the free boundary

In this current section, we show that minimizers in a suitable class of $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$ grow at most linearly away from the free boundary. One more time, we assume that $q > n$. Now, we define a subclass of $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$.

Definition 7.1 Let $E_0 > 0$ and $\mu_i \geq \theta_i > 0$, $i = 1, 2$. A minimizer $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M, E_0, \theta_i)$ if

- (i) $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$, i.e. u is a minimizer of a functional as (2.2) satisfying the conditions of Definition 2.3;
- (ii) $0 < \varepsilon_0 \leq G(1) \leq E_0$ and;
- (iii) $0 < \theta_i \leq h_i \leq \mu_i$, $i = 1, 2$.

Theorem 7.1 (Nondegeneracy of minimizers) *Let $u \in \mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M, E_0, \theta_i)$ and $\Omega' \Subset \Omega$. Then, there exists a positive constant $c = c(n, \delta, g_0, q, \lambda_1, \Lambda_1^M, \theta_1, E_0, \Omega', \Omega)$ such that*

$$u(x_0) \geq c \cdot \text{dist}(x_0, F^+(u)), \quad (7.1)$$

for any point $x_0 \in \{u > 0\} \cap \Omega'$. If $x_0 \in \{u < 0\} \cap \Omega'$ then,

$$u^-(x_0) \geq \tilde{c} \cdot \text{dist}(x_0, F^-(u)), \quad (7.2)$$

for some $\tilde{c} = \tilde{c}(n, \delta, g_0, q, \lambda_2, \Lambda_2^M, \theta_2, E_0, \Omega', \Omega) > 0$. Here $F^-(u) := \partial\{u < 0\} \cap \Omega$.

Proof We prove (7.1). The proof of (7.2) follows the same guidelines. First, notice that it suffices to show such estimate for point $x_0 \in \{u > 0\} \cap \Omega'$ that are close enough to the free boundary, i.e. satisfying

$$0 < \text{dist}(x_0, F^+(u)) < \min\{d, \text{dist}(\Omega', \partial\Omega)\}$$

for some $d = d(n, \delta, g_0, q, \lambda_1, \Lambda_1^M, \theta_1, \varepsilon_0, E_0) > 0$. Let $d_0 := \text{dist}(x_0, \partial\{u > 0\})$. Clearly, we can assume that $0 < d_0 \leq 1$. We know that there exist functional $\mathcal{J}_G$ as in (2.2) such that u is a minimizer of $\mathcal{J}_G$. Defining the rescaled function

$$w(x) := \frac{u(x_0 + d_0 x)}{d_0}$$

it is enough to show that $w(0) \geq c_0$. Now, we observe that w is a local minimizer for the functional of the type

$$\tilde{\mathcal{J}}_G(v, B_1) = \int_{B_1} [G(|\nabla v|) + \tilde{f}_1(x)\tilde{H}_i(v^+) + \tilde{f}_2(x)\tilde{H}_i(v^-) + Q(\tilde{h}_1, \tilde{h}_2)] dx,$$

where $\tilde{f}_i(x) = f_i(x_0 + d_0x)$, $\tilde{H}_i(t) = H_i(d_0 \cdot t)$ and $\tilde{h}_i(x) = h_i(x_0 + d_0x)$. It follows by Corollary 5.1 that

$$\mathcal{L}_g w = d_0 \cdot f_1(x) H'_1(d_0 \cdot w) \quad \text{in } B_1 \subset \{w > 0\}.$$

By Lemma 2.2 of [18] we have that $g^{-1}(1) \leq 1/\min\{\varepsilon_0^{1/\delta}, \varepsilon_0^{1/g_0}\}$ and using one more time Lemma 2.2 of [18] and Harnack inequality (Lemma 3.4) we arrive that, $\forall x \in B_{\frac{7}{8}}$,

$$w(x) \leq C_0(w(0) + \Upsilon_0) \quad \text{where} \quad \Upsilon_0 := d_0^{\frac{q-n}{qg_0}} \frac{(\Lambda_1^M \cdot \lambda_1 + 1)^{1/\delta}}{\min\{\varepsilon_0^{1/\delta}, \varepsilon_0^{1/g_0}\}}.$$

Now, considering $\psi \in C^\infty(B_1(0))$ such that

$$\psi \equiv 1 \quad \text{in } B_1 \setminus B_{1/2}, \quad \psi \equiv 0 \quad \text{in } B_{1/4}, \quad |\nabla \psi| \leq \tilde{C} \quad \text{in } B_1,$$

we observe that

$$\tilde{\psi} := \begin{cases} \min\{w, C_0(w(0) + \Upsilon_0)\psi\} & \text{in } B_{\frac{7}{8}}, \\ w & \text{in } B_1 \setminus B_{\frac{7}{8}} \end{cases}$$

belongs to $K_w = \{v \in W^{1,G}(B_1) : v = w \text{ on } \partial B_1\}$ and $0 \leq \tilde{\psi} \leq w$. By minimality of w , definition of ψ , mean value theorem, Hölder and Young inequalities we have

$$\begin{aligned} & \int_{\{\tilde{\psi} < w\}} (G(|\nabla \tilde{\psi}|) - G(|\nabla w|)) dx \\ & \geq -\Lambda_1^M d_0 \int_{\{\tilde{\psi} < w\}} |\tilde{f}_1| |w - \tilde{\psi}| dx + \theta_1 \int_{\{\tilde{\psi} < w\}} (\chi_{\{w>0\}} - \chi_{\{\tilde{\psi}>0\}}) dx \\ & \geq -C_1 \cdot d_0^{1-\frac{n}{q}} \left(\int_{\{\tilde{\psi} < w\}} |w - \tilde{\psi}|^{1+\delta} dx \right)^{\frac{1}{1+\delta}} + \theta_1 |B_{1/4}|. \\ & \geq -C_1 \cdot d_0^{1-\frac{n}{q}} \left[C_0^{1+\delta} (w(0) + \Upsilon_0)^{1+\delta} + \frac{\delta}{1+\delta} \right] + \theta_1 |B_{1/4}|, \end{aligned}$$

where $C_1 = \Lambda_1^M \lambda_1 |B_1|^{\frac{\delta}{1+\delta} - \frac{1}{q}}$. On the other hand,

$$\int_{\{\tilde{\psi} < w\}} (G(|\nabla \tilde{\psi}|) - G(|\nabla w|)) dx \leq \int_{B_1} G(C_0(w(0) + \Upsilon_0)|\nabla \psi|) dx \leq C_2 \cdot (w(0)^\tau + \Upsilon_0^\tau),$$

where $C_2 = (1 + g_0)[2C_0(\tilde{C} + 1)]^{1+g_0} E_0 |B_1|$ and $\tau = 1 + \delta$ or $\tau = 1 + g_0$. If $\tau = 1 + \delta$ follows by above estimates that

$$C_4 w^{1+\delta}(0) \leq \theta_1 |B_{1/4}| - \frac{C_1 \delta}{1+\delta} d_0^{1-\frac{n}{q}} - C_4 \Upsilon_0^{1+\delta}, \quad (7.3)$$

for $C_4 = 2^{1+g_0} C_0^{1+\delta} C_1 + C_2$. If $\tau = 1 + g_0$, then using again Young inequality,

$$C_4 w^{1+g_0}(0) \leq \theta_1 |B_{1/4}| - C_1 \left(\frac{C_0^{1+\delta}(g_0 - \delta)}{1 + g_0} + \frac{\delta}{1 + \delta} \right) d_0^{1-\frac{n}{q}} - C_4 \Upsilon_0^{1+g_0}. \quad (7.4)$$

Now, by above estimates, we choose d_0 such that

$$\frac{\theta_1}{8} |B_{1/4}| \leq \max \left\{ C_1 \left(\frac{C_0^{1+\delta}(g_0 - \delta)}{1 + g_0} + \frac{\delta}{1 + \delta} \right) d_0^{1-\frac{n}{q}}, C_4 \Upsilon_0^{1+\delta} \right\} \leq \frac{\theta_1}{4} |B_{1/4}|.$$

Thus, by (7.3) and (7.4)

$$w(0) \geq \max \left\{ \left(\frac{\theta_1}{2C_4} |B_{1/4}| \right)^{\frac{1}{1+\delta}}, \left(\frac{\theta_1}{2C_4} |B_{1/4}| \right)^{\frac{1}{1+g_0}} \right\}.$$

□

In the presence of Lipschitz regularity, we obtain the strong non-degeneracy for minimizers. Before we provide the following key step.

Lemma 7.1 *Let u be a locally Lipschitz function in $S_{\varepsilon_0}(\Omega, M, \Lambda_i^M, E_0, \theta_i)$. Then, for any $\Omega' \subset \subset \Omega$, there exist constants $r_0, \delta_0 > 0$ and $\gamma_0 \in (0, 1]$ such that if $x_0 \in \{u > 0\} \cap \Omega'$ and $\text{dist}(x_0, F^+(u)) = r \leq r_0$, the following estimate is satisfied*

$$u(x_1) \geq (1 + \delta_0)u(x_0) \quad (7.5)$$

for some $x_1 \in \overline{B_r}(x_0) \setminus B_{\gamma_0 r}(x_0)$.

Remark 7.1 We point out that if $f_1 \equiv 0$ we can take $\gamma_0 = 1$ and hence $x_1 \in \partial B_r(x_0)$. The constants r_0, δ_0 and γ_0 depend on $n, \delta, g_0, q, \lambda_1, \Lambda_1^M, \theta_1, E_0, \Omega', \Omega$ and $\|\nabla u\|_{L^\infty(\Omega')}$ where

$$\Omega'' := \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{2} \text{dist}(\Omega', \partial\Omega) \right\}.$$

Similar estimate as in (7.5) holds if $x_0 \in \{u < 0\} \cap \Omega'$. Precisely,

$$u^-(x_1) \geq (1 + \tilde{\delta}_0)u^-(x_0),$$

where $\text{dist}(x_0, F^-(u)) = r \leq \tilde{r}_0$ and $x_1 \in \overline{B_r}(x_0) \setminus B_{\tilde{\gamma}_0 r}(x_0)$ for some constants $\tilde{\delta}_0, \tilde{r}_0 > 0$ and $\tilde{\gamma}_0 \in (0, 1]$ depend on $n, \delta, g_0, q, \lambda_2, \Lambda_2^M, \theta_2, E_0, \Omega', \Omega$ and $\|\nabla u\|_{L^\infty(\Omega')}$.

Proof of Lemma 7.1 Taking into account the linear growth for $u \in S_{\varepsilon_0}(\Omega, M, \Lambda_i^M, E_0, \theta_i)$, this result can be proven as a variant of Proposition 4.3 in [19] since the crucial argument is based on an iteration performed via weak Harnack inequality also available for our operator $\mathcal{L}_g$ (see Theorem 1.3 and Section 4 in [17]).

Theorem 7.2 (Strong non-degeneracy estimate) *Let u be a locally Lipschitz function in $S_{\varepsilon_0}(\Omega, M, \Lambda_i^M, E_0, \theta_i)$. Then, for any $\Omega' \subset \subset \Omega$ there exists a positive constant c_0 such that, for any $0 < r < \text{dist}(\Omega', \partial\Omega)$,*

$$\sup_{B_r(x_0)} u^+ \geq c_0 \cdot r, \quad (7.6)$$

provided $x_0 \in \overline{\{u > 0\}} \cap \Omega'$. If we have $x_0 \in \overline{\{u < 0\}} \cap \Omega'$, then there exists $\tilde{c}_0 > 0$ such that

$$\sup_{B_r(x_0)} u^- \geq \tilde{c}_0 \cdot r. \quad (7.7)$$

The constants c_0 and $\tilde{c}_0$ depend on $n, \delta, g_0, q, \lambda_i, \Lambda_i^M, \theta_i, E_0, \Omega', \Omega$ and $\|\nabla u\|_{L^\infty(\Omega')}$.

Proof The proof follows the same lines of Theorem 4.1 of [19]. □

8 Log-Lipschitz regularity in the class $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$ - non-singular operators

It is a well-known fact among specialists of this research area that the Lipschitz continuity of minimizers plays a important role on the development of the study of the free boundary. But from previous considerations made in Sect. 1, such regularity may be unknown in more general scenarios. However, in [15], the authors were able to show asymptotically optimal result, i.e. local log-Lipschitz regularity. Motivated by this fact, we investigate suitable conditions to the asymptotic optimal regularity for local minimizers in the class $\mathcal{S}_{\varepsilon_0}(\Omega, M, \Lambda_i^M)$. We observe that in [20] (Orlicz–Sobolev context), this result has been proven for local minima of (1.3) since $\frac{g(t)}{t}$ is nondecreasing. We notice that such restriction hide precisely that the $\mathcal{L}_g$ -operator is, in certain sense, non-singular. This is justified by the following definition and next lemma.

Definition 8.1 Let $G \in \mathcal{G}(\delta, g_0)$ and $G' = g$. We say that the $\mathcal{L}_g$ -operator is:

- (i) (strongly) singular operator when $g_0 < 1$;
- (ii) weakly singular operator if $0 < \delta < g_0 = 1$;
- (iii) non-singular operator when $\delta \geq 1$;

By simplicity, we call (strongly) singular operators only singular operators. We observe still that if $\delta > 1$ then $\mathcal{L}_g$ is degenerate. The above lemma provides a criterion for new concepts. In the cases where $0 < \delta < 1 < g_0$ the $\mathcal{L}_g$ -operator does not enjoy a character purely singular or degenerate.

Lemma 8.1 Let $G \in \mathcal{G}(\delta, g_0)$ and $G' = g$. Then,

- (i) $\mathcal{L}_g$ is singular if $\frac{g(t)}{t}$ is decreasing for $t > 0$.
- (ii) $\mathcal{L}_g$ is weakly singular if and only if $\mathcal{L}_g$ is not singular, $\frac{g(t)}{t}$ is nonincreasing for $t > 0$ and G is not a power of t .
- (iii) $\mathcal{L}_g$ is non-singular if and only if $\frac{g(t)}{t}$ is nondecreasing for $t > 0$.

Proof (i) In fact,

$$\mathcal{L}_g \text{ is singular} \implies \frac{tg'(t)}{g(t)} < 1 \implies \left(\frac{g(t)}{t}\right)' < 0 \implies \frac{g(t)}{t} \text{ is decreasing for } t > 0.$$

(ii) Suppose $\mathcal{L}_g$ is weakly singular. By definition $\mathcal{L}_g$ is not singular. Still, $\left(\frac{g(t)}{t}\right)' \leq 0$ and hence $\frac{g(t)}{t}$ is nonincreasing for $t > 0$. Moreover, since $\delta < 1$, $\frac{tg'(t)}{g(t)}$ can not be a constant. This implies that G is not a power of t . Conversely, assume that $\mathcal{L}_g$ is not singular, $\frac{g(t)}{t}$ is nonincreasing for $t > 0$ and G is not a power of t . This implies that $g_0 \geq 1$, $\frac{tg'(t)}{g(t)} \leq 1$ and $0 < \delta < g_0$. This shows that $\mathcal{L}_g$ is a weakly singular operator.

(iii) Indeed, for $t > 0$ we have that $\left(\frac{g(t)}{t}\right)' \geq 0$ if and only if $\delta \geq 1$. □

The next lemma is simple, but an important fact for non-singular $\mathcal{L}_g$ operators.

Lemma 8.2 Let u a minimizer of a functional of the type $\mathcal{J}_G$ with $G \in \mathcal{G}(\delta, g_0)$ where $\delta \geq 1$. Assume $B_r(x_0) \subset \Omega$ and w be the solution to

$$\mathcal{L}_g w = 0 \quad \text{in } B_r(x_0) \quad \text{and} \quad w = u \quad \text{on } \partial B_r(x_0).$$

Then,

$$\int_{B_r(x_0)} (G(|\nabla u|) - G(|\nabla w|)) dx \geq C(\delta, g_0) \int_{B_r(x_0)} G(|\nabla u - \nabla w|) dx.$$

Proof By notation of Theorem 5.1 we have

$$\begin{aligned} \int_{A_1} F(|\nabla u|) |\nabla u - \nabla v|^2 dx &\geq 2^{1-g_0} \int_{A_1} F(|\nabla u - \nabla v|) |\nabla u - \nabla v|^2 dx \\ &\geq 2^{1-g_0} \int_{A_1} G(|\nabla u - \nabla v|) dx. \end{aligned}$$

Now, the proof follows by combination of the above estimate and (5.1) in Theorem 5.1. $\square$

Remark 8.1 We note that if $\delta < 1$, then

$$\int_{A_1} F(|\nabla u|) |\nabla u - \nabla v|^2 dx \leq 2^{1-g_0} \int_{A_1} G(|\nabla u - \nabla v|) dx.$$

The next lemma was proved in [20] (see Lemma 4.1).

Lemma 8.3 Let $u, w \in W^{1,G}(B_R)$ where $\mathcal{L}_g w = 0$ in B_R in the distributional sense. Then, there exist constants $C > 0$ and $\sigma \in (0, 1)$ depending only on n, δ, g_0 such that, for all $0 < r \leq R$ holds

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) dx \leq C \left[\left(\frac{r}{R} \right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) dx + \int_{B_R} G(|\nabla u - \nabla w|) dx \right],$$

where, for any function $\psi \in L^1(B_R)$ and $0 < s \leq R$,

$$(\psi)_s := \int_{B_s} |\psi(x)| dx.$$

We now have all the ingredients to show that the gradient of local minimizers are locally BMO for non-singular $\mathcal{L}_g$ operators. In the sequel, we assume that $q \geq n$ and the assumptions in the Remark 4.1.

Proof of Theorem 2.1 Let $y \in \Omega'$ and $0 < r < R = \text{dist}(\Omega', \partial\Omega)$. By simplicity assume that $y = 0$. Consider $w : B_R \rightarrow \mathbb{R}$ the solution to

$$\mathcal{L}_g w = 0 \quad \text{in } B_R \quad \text{and} \quad u = w \quad \text{on } \partial B_R.$$

By Lemma 8.2 and using the minimality of u we have that

$$\begin{aligned} \int_{B_r} G(|\nabla u - \nabla w|) dx &\leq \frac{1}{C_1} \left[C_2 R^n + \int_{B_R} f_1(H_1(u^+) - H_1(w^+)) dx \right. \\ &\quad \left. + \int_{B_R} f_2(H_2(u^-) - H_2(w^-)) dx \right], \end{aligned}$$

where $C_1 = C_1(\delta, g_0)$ and $C_2 = C_2(n, \mu_1, \mu_2)$. By maximum principle we know that $\sup_{B_R} |w| \leq M$. Thus, by mean value theorem

$$\begin{aligned} &\int_{B_R} f_1(H_1(u^+) - H_1(w^+)) dx + \int_{B_R} f_2(H_2(u^-) - H_2(w^-)) dx \\ &\leq (\Lambda_1^M + \Lambda_2^M) \cdot \int_{B_R} (|f_1| + |f_2|) |u - w| dx. \end{aligned}$$

Now, by Hölder, Poincaré and Young inequalities we arrive at

$$\begin{aligned} \int_{B_R} (|f_1| + |f_2|)|u - w|dx &\leq (||f_1||_{L^q(B_R)} + ||f_2||_{L^q(B_R)})|B_R|^{\frac{\delta}{1+\delta}-\frac{1}{q}}||u - w||_{L^{1+\delta}(B_R)} \\ &\leq C(n)(\lambda_1 + \lambda_2)|\Omega|^{\frac{q-n}{qn}}|R|^{\frac{n\delta}{1+\delta}}||\nabla u - \nabla w||_{L^{1+\delta}(B_R)} \\ &\leq \frac{C_4}{c^{\frac{1+\delta}{\delta}}} \cdot R^n + c^{1+\delta} \int_{B_R} |\nabla u - \nabla w|^{1+\delta} dx, \end{aligned}$$

where $c > 0$ will be chosen a posteriori. Now, by (G - 2)

$$\begin{aligned} \int_{B_R} |\nabla u - \nabla w|^{1+\delta} dx &\leq \int_{\{|\nabla u - \nabla w| < 1\}} |\nabla u - \nabla w|^{1+\delta} dx + \int_{\{|\nabla u - \nabla w| \geq 1\}} |\nabla u - \nabla w|^{1+\delta} dx \\ &\leq C(n)R^n + \frac{(1 + g_0)}{\varepsilon_0} \int_{B_R} G(|\nabla u - \nabla w|) dx. \end{aligned}$$

Combining the above estimates, we obtain

$$\int_{B_r} G(|\nabla u - \nabla w|) dx \leq C_5 \cdot \varepsilon_0^{-1/\delta} \cdot R^n + \frac{1}{2} \int_{B_r} G(|\nabla u - \nabla w|) dx,$$

where $c = \left[\frac{C_1 \varepsilon_0}{2(1+g_0)C_3} \right]^{\frac{1}{1+\delta}}$. Now, by Lemma 8.3, we have

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) dx \leq C \left[\left(\frac{r}{R} \right)^{n+\sigma} \int_{B_r} G(|\nabla u - (\nabla u)_R|) dx + 2C_5 \cdot \varepsilon_0^{-1/\delta} \cdot R^n \right].$$

Using Lemma 2.7 of [15] and Remark 4.3 follows that

$$\begin{aligned} \int_{B_r} G(|\nabla u - (\nabla u)_r|) dx &\leq C_6 \left(R^{-n} \int_{B_R} G(|\nabla u - (\nabla u)_R|) dx + \varepsilon_0^{-1/\delta} \cdot R^n \right) r^n \\ &\leq C_7 \left(R^{-n} + \text{diam}(\Omega)^n \cdot \varepsilon_0^{-1/\delta} \right) r^n, \end{aligned}$$

where $C_7 = C_7(n, \delta, g_0, q, M, \lambda_i, \mu_i, \Lambda_{i,M}, |\Omega|) \geq 1$. Finally, by Jensen inequality and (G - 2) we conclude that

$$\int_{B_r} |\nabla u - (\nabla u)_r| dx \leq \frac{C_8}{\varepsilon_0} \cdot \left(\text{dist}(\Omega', \partial\Omega)^{-n} + \text{diam}(\Omega)^n \cdot \varepsilon_0^{-1/\delta} \right)^{\frac{1}{1+\delta}},$$

where $C_8 = (1 + g_0)C_7^{\frac{1}{1+\delta}}$. This proves (2.4). The estimate (2.5) follows immediately by Lemma 1 of [5]. $\square$

9 Compactness result and Invariance of Dini type control under suitable Scalings

In the proof of Theorem 2.2 (main result), the use of a compactness approach is fundamental. We point out that in our context for classes, we need two compactness results, more precisely, compactness for minimizers and N -functions. The Hölder regularity for minimizers is sufficient for our purposes. In the case of N -functions, we need that the limit of N -functions considered here continue satisfying the Lieberman conditions (PC) and (QC).

Unfortunately, the classes $\mathcal{G}(\delta, g_0)$ and $\mathcal{G}(\delta, g_0, \varepsilon_0)$ are weak to provide the compactness desired. For instance, consider the following family $\{G_N\}$ of N -functions:

$$G_N(t) = \begin{cases} \frac{3}{4}t^2 & \text{if } 0 \leq t \leq 2 \\ \frac{(4+5N)}{12}t^3 - \frac{(5+10N)}{4}t^2 + (4+5N)t - \frac{(8+10N)}{3} & \text{if } 2 \leq t \leq 2 + \frac{1}{N} \\ \frac{1}{3}t^3 - \frac{(5+4N)}{4N}t + \frac{7}{3} + \frac{5}{2N} + \frac{5}{12N^2} & \text{if } 2 + \frac{1}{N} \leq t. \end{cases}$$

We see that $G_N \in \mathcal{G}(1, 7, 3/4)$ for all $N \in \mathbb{N}$ and $G_N \rightarrow G_\infty$ uniformly in the compact sets of $[0, +\infty)$ where

$$G_\infty(t) = \begin{cases} \frac{3}{4}t^2 & \text{if } 0 \leq t \leq 2 \\ \frac{1}{3}t^3 - t + \frac{7}{3} & \text{for } t \geq 2. \end{cases}$$

Since $G_\infty \in C^{1,1}([0, \infty)) \setminus C^2[0, \infty)$ follows that $G_\infty \notin \mathcal{G}(\delta, g_0)$ for whatever $0 < \delta \leq g_0$.

In view of this, we need a suitable uniform modulus of continuity for $\mathcal{Q}_g$. In this section, we show that the Dini type control meets our interests before we prove a lemma about Dini continuity and uniform convergence.

Lemma 9.1 *Let $\{f_N\}$ a locally bounded sequence of continuous functions defined in the same domain. Assume that there exist modulus of continuity $\{\omega_N\}$ such that, for some positive constant D_1 ,*

$$|f_N(x) - f_N(y)| \leq D_1 \cdot \omega_N(|x - y|), \quad \forall N \in \mathbb{N}$$

and, for some $L, D_2 > 0$,

$$\int_0^L \frac{\omega_N(t)}{t} dt \leq D_2, \quad \forall N \in \mathbb{N}.$$

Then, up to subsequence, $\{f_N\}$ converges locally uniformly for a function f .

Proof By Arzelà-Ascoli theorem, it is sufficient to prove that $\{\omega_N\}$ is equicontinuous. Suppose that this does not occurs. Then, there exists a subsequence $\{\omega_{N_j}\}$ and $t_{N_j} \rightarrow 0$ such that, for some $\varepsilon > 0$, $\omega_{N_j}(t_{N_j}) \geq \varepsilon$, $\forall j \in \mathbb{N}$. Clearly we can assume that $0 < t_{N_j} < L$. In particular,

$$D_2 \geq \int_{t_{N_j}}^L \frac{\omega_{N_j}(t)}{t} dt \geq \varepsilon \int_{t_{N_j}}^L \frac{1}{t} dt = \varepsilon (\ln(L) - \ln(t_{N_j})).$$

This is a contradiction for $t_{N_j} \ll 1$. □

Theorem 9.1 (Compactness for N -functions) *Let*

$$\vartheta(\varepsilon_0, E_0) := \left\{ G \in \mathcal{G}_{\xi_i, D}(\delta, g_0, \varepsilon_0) : G(1) \leq E_0 \right\}.$$

If $\{G_N\} \subset \vartheta(\varepsilon_0, E_0)$ then, up to subsequence, $\{G_N\}$ converges to a N -function $G_\infty \in \mathcal{G}(\delta, g_0)$ in C^2 topology on compact subsets of $(0, +\infty)$ and in the C^1 topology on compact subsets of $[0, +\infty)$.

Proof Let $\{G_N\}$ be a sequence in $\vartheta(\varepsilon_0, E_0)$. By properties of G_N ,

$$\frac{\varepsilon_0}{(g_0 + 1)} \min\{t^{1+\delta}, t^{1+g_0}\} \leq G_N(t) \leq E_0(g_0 + 1) \max\{t^{1+\delta}, t^{1+g_0}\} \quad \forall t \geq 0, \forall N > 0, \quad (9.1)$$

$$\varepsilon_0 \min\{t^\delta, t^{g_0}\} \leq g_N(t) \leq (1 + g_0)E_0 \max\{t^\delta, t^{g_0}\} \quad \forall t \geq 0, \forall N > 0, \quad (9.2)$$

$$0 < g'_N(t) \leq g_0(1 + g_0)E_0 \max\{t^{\delta-1}, t^{g_0-1}\} \quad \forall t > 0, \forall N > 0, \quad (9.3)$$

where $g_N = G'_N$. Once G_N satisfies (DTC) follows that g'_N is locally Dini continuous with uniform Dini constant. Indeed, let $K \subseteq (0, +\infty)$ a closed interval. We have that, $g'_N(t) = Q_{g_N}(t) \cdot \frac{g_N(t)}{t}$. By the way, $\{g'_N\}$ is uniform bounded in K and if $\omega_N, \tilde{\omega}_N$ are the modulus of continuity of g'_N and $\frac{g_N(t)}{t}$ respectively, then by (9.2),

$$\omega_N(t) \leq C_1 \cdot \omega_{g'_N}^{l,L}(t) + g_0 \cdot \tilde{\omega}_N(t), \quad (9.4)$$

where $K = [l, L]$ and $C_1 = C_1(\delta, g_0, E_0, l, L) > 0$. We know that $\frac{g_N(t)}{t}$ is locally Lipschitz and holds

$$\tilde{\omega}_N(t) \leq \frac{(1 + g_0)C_1}{l} \cdot t, \quad 0 \leq t \leq L - l. \quad (9.5)$$

Thus, combination of (9.4), (9.5) and (DTC) we assure that

$$\int_0^{L-l} \frac{\omega_N(t)}{t} dt \leq C_1 \cdot C_1^* \cdot \xi_1 \left(\frac{L}{l} \right) \cdot \xi_2 \left(\frac{L-l}{l} \right) + \frac{(1 + g_0)C_1}{l} (L - l), \quad \forall N \in \mathbb{N}.$$

Since G_N is uniform bounded in $C^2(K)$ by (9.1), (9.2) and (9.3) and holds the above estimate, follows by Lemma 9.1 the existence of a C^2 -function $\overline{G}_\infty : (0, \infty) \rightarrow \mathbb{R}$ such that, up to subsequence, $G_N \rightarrow G_\infty$ in C^2 topology for any compact interval of $(0, \infty)$. Note that we can extend G_∞ to become $C^1[0, \infty)$. In fact, passing to the limit in (9.1), for $0 < t < 1$, we have $G_\infty(t) \leq L_0(1 + g_0) \cdot t^{1+\delta}$, which yields $\lim_{t \rightarrow 0^+} G_\infty(t) = 0$. By

$$0 \leq \lim_{t \rightarrow 0^+} \frac{G_\infty(t)}{t} \leq E_0(1 + g_0) \lim_{t \rightarrow 0^+} t^\delta = 0 \quad \text{and} \quad 0 < t < 1 \implies g_\infty(t) \leq (1 + g_0)E_0 t^\delta,$$

we see that $G_\infty \in C^1[0, \infty)$. Thus, follows that $G_\infty \in \mathcal{G}(\delta, g_0)$ and $G_N \rightarrow G_\infty$ in $C^1_{loc}[0, \infty)$. $\square$

Remark 9.1 By [6] we know that $\mathcal{G}_{Lip}(\delta, g_0) \subsetneq \mathcal{G}_{\beta \cdot H}(\delta, g_0) \subsetneq \mathcal{G}_{\epsilon \cdot H}(\delta, g_0)$ for any $0 < \epsilon < \beta < 1$. We can check that there exists a class $\mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0)$ such that $\mathcal{G}_{\beta \cdot H}(\delta, g_0) \subsetneq \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0)$ for any $\beta \in (0, 1]$. In fact, let $G \in \mathcal{G}_{\beta \cdot H}(\delta, g_0)$. Then, Q_g is locally Hölder continuous in $(0, \infty)$ and for some constant $C_\beta = C_\beta(\delta, g_0, \beta) > 0$ holds $[Q_g]_{C^{0,\beta}([l, L])} \leq \frac{C_\beta}{l^\beta}$. In particular, $\omega_g^{l,L}(t) \leq \frac{C_\beta}{l^\beta} t^\beta$ and hence

$$\int_0^{L-l} \frac{\omega_g^{l,L}(t)}{t} dt \leq \frac{C_\beta}{\beta} \cdot \left(\frac{L-l}{l} \right)^\beta.$$

Since $\beta \in (0, 1]$ is fixed, we prove that $G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0)$ for

$$C_1^* = \frac{C_\beta}{\beta}, \quad \xi_1 \equiv 1, \quad \xi_2(t) = t^\beta.$$

Finally, we observe that

$$\mathcal{G}_{\beta \cdot H}(\delta, g_0) \subsetneq \bigcup_{\gamma \in (0, 1]} \mathcal{G}_{\gamma \cdot H}(\delta, g_0) \subseteq \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0).$$

Example 9.1 In this example, we show that there exist a considerable number of N -functions in $\mathcal{G}(\delta, g_0, \varepsilon_0)$ which do not belong to any $\mathcal{G}_{\gamma \cdot H}(\delta, g_0, \varepsilon_0)$ but belongs to $\mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0)$. In particular, we show that, for suitable $0 < \delta \leq g_0$, and functions ξ_1, ξ_2 ,

$$\bigcup_{\gamma \in (0,1]} \mathcal{G}_{\gamma \cdot H}(\delta, g_0) \subsetneq \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0).$$

Indeed, let $\nu \in \mathbb{N}$ fixed arbitrarily and $g_\nu : [0, \infty) \rightarrow [0, \infty)$ the following function

$$g_\nu(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ \begin{cases} 1 & \text{if } t = 1, \\ 1 + [\ln(t-1)]^{-2\nu} & \text{if } 1 < t < 1 + e^{-(2\nu+1)} \end{cases} & \text{if } 1 \leq t < 1 + e^{-(2\nu+1)}, \\ t + (2\nu+1)^{-2\nu} - e^{-(2\nu+1)} & \text{if } 1 + e^{-(2\nu+1)} \leq t. \end{cases}$$

Clearly g_ν is continuous and increasing in $[0, \infty)$. Now, define for all $t \geq 0$,

$$g(t) = \int_0^t g_\nu(s) ds \quad \text{and} \quad G(t) = \int_0^t g(s) ds.$$

Simple but long computations show that $G \in \mathcal{G}(1, g_0, 1)$, where $g_0 = g_0(\nu) > 1$. Moreover, for any $\beta \in (0, 1]$, we have that

$$\lim_{t \rightarrow 1^+} \frac{|g_\nu(t) - g_\nu(0)|}{t^\beta} = \infty.$$

This proves that $G \notin \mathcal{G}_{\gamma \cdot H}(\delta, g_0, \varepsilon_0)$ whatever $\beta \in (0, 1)$. On the other hand, there exist functions ξ_1, ξ_2 such that $G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(1, g_0, 1)$. In fact, first we note that, if $0 < l < L \leq 1$, then $\omega_g^{l,L} \equiv 0$ satisfying (DTC) in this interval. Still, if $h(t) = \frac{t}{g(t)}$ and

$$\begin{aligned} \omega_v^{l,L}(t) &:= \sup \left\{ |g_\nu(x) - g_\nu(y)| : l \leq x, y \leq L \text{ and } |x - y| \leq t \right\}, \\ \omega_h^{l,L}(t) &:= \sup \left\{ |h(x) - h(y)| : l \leq x, y \leq L \text{ and } |x - y| \leq t \right\}, \end{aligned}$$

we have,

$$\omega_g^{l,L}(t) \leq \|h\|_{L^\infty[l,L]} \cdot \omega_v^{l,L}(t) + \|g_\nu\|_{L^\infty[l,L]} \cdot \omega_h^{l,L}(t). \quad (9.6)$$

Again by long computations, we can to check that g_ν is a Dini continuous function with modulus of continuity satisfying a condition of the type (DTC). Precisely, there exists a constant $C(\nu) > 0$, such that

$$\int_0^{L-l} \frac{\omega_v^{l,L}(t)}{t} dt \leq C(\nu) \cdot \xi \left(\frac{L-l}{l} \right), \quad (9.7)$$

where

$$\xi(t) := \begin{cases} (\ln t^{-1})^{1-2\nu} + t, & \text{if } L-l \in (0, e^{-(2\nu+1)}], l \in (0, 1], \\ \left[\ln \left(\frac{t^{-1}}{1+e^{-(2\nu+1)}} \right) \right]^{1-2\nu} + t, & \text{if } L-l \in (0, e^{-(2\nu+1)}], l \in (1, 1+e^{-(2\nu+1)}], \\ t, & \text{if } L-l \in (e^{-(2\nu+1)}, \infty) \text{ or } l \in (1+e^{-(2\nu+1)}, \infty). \end{cases}$$

Finally, using the definitions of g_ν and g , the local Lipschitz regularity of h , estimate (9.6) and one more time by long and tedious computations we conclude that there exists $C^*(\nu) > 0$, such that

$$\omega_g^{l,L}(t) \leq C^*(\nu) \cdot \left(1 + \frac{L}{l} \right) \left(\omega_v^{l,L}(t) + \frac{t}{l} \right).$$

This example now follows by direct computation of the Dini condition by inequality above using (9.7).

We finish this section discussing about scaling. The estimates for minimizers of functionals considered here involve universal constants, i.e., depending, for instance, of δ and g_0 . Many these estimates also depend on $G(1)$. In this case, it is very important that our minimizers admit rescaling that normalize the values $G(1)$. This fact is what allows us to run a compactness argument using the result above to perform a blow-up analysis in the proof of Theorem 2.2. For $s > 0$ consider the following scalings

$$G_s(t) := \frac{G(st)}{sG'(s)} \quad \text{and} \quad G_s^*(t) := G(st). \quad (9.8)$$

We observe that this rescalings preserve regularity. Also, if $G \in \mathcal{G}(\delta, g_0) \implies G_s \in \mathcal{G}(\delta, g_0, (1 + g_0)^{-1})$ and $G_s(1) \leq 1$. Still, $G \in \mathcal{G}(\delta, g_0, \varepsilon_0)$, $s \geq 1 \implies G_s^* \in \mathcal{G}(\delta, g_0, \varepsilon_0)$. In the next lemma, we show the invariance under this scaling of the Dini type control (DTC).

Lemma 9.2 (Scaling properties)

$$G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0) \implies G_s \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, (1 + g_0)^{-1}) \text{ and } G_s(1) \leq 1. \quad (\text{S-1})$$

$$G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0), \quad s \geq 1 \implies G_s^* \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0). \quad (\text{S-2})$$

Proof Let $G \in \mathcal{G}(\delta, g_0)$. We see that $Q_{g_s}(t) = Q_{g_s^*}(t) = Q_g(st)$, $\forall t > 0$. Using the Definition 2.7 we obtain $\omega_{g_s}^{l,L}(t) = \omega_{g_s^*}^{l,L}(t) \leq \omega_g^{sl,sL}(st)$. Thus,

$$\begin{aligned} \int_0^{L-l} \frac{\omega_{g_s}^{l,L}(t)}{t} dt &= \int_0^{L-l} \frac{\omega_{g_s^*}^{l,L}(t)}{t} dt \leq \int_0^{L-l} \frac{\omega_g^{sl,sL}(st)}{t} dt \\ &\leq \int_0^{s(L-l)} \frac{\omega_g^{sl,sL}(\zeta)}{\zeta} d\zeta \\ &\leq C_1^*(\delta, g_0) \cdot \xi_1 \left(\frac{L}{l} \right) \cdot \xi_2 \left(\frac{L-l}{l} \right). \end{aligned}$$

10 The proof of Theorem 2.2

In this moment, we discuss the proof of our main result. Before prove it, we enunciate a lemma presented in [6] that provides an estimate putting in perspective the Hölder and Lipschitz continuity character of functions. We observe that the method of the proof of Theorem 2.2 appears first in the papers [8, 14]. We follow the same spirit of the Theorem 2.1 of [6]. For this reason, we present only the major steps.

Lemma 10.1 (Hölder/Lipschitz continuity character) *Let $w : (0, 1] \rightarrow \mathbb{R}$ be a nonnegative and nondecreasing function such that $w(1) \leq L$ for some $L > 0$. Suppose $0 < \tau < 1$ and $0 < \alpha \leq 1$ are such that*

$$w(\tau^{k+1}) \leq \max_{0 \leq m \leq k} \left\{ L \cdot \tau^{\alpha(k+1)}, \tau^{\alpha(m+1)} \cdot w(\tau^{k-m}) \right\} \text{ for every } k \geq 0. \quad (10.1)$$

Then,

$$w(r) \leq \tau^{-\alpha} L r^{\alpha} \quad \text{for } 0 < r \leq 1.$$

In particular, if u is a bounded function in B_1 such that $\sup_{B_1} |u| \leq L$ and

$$\sup_{B_{\tau^{k+1}}} |u| \leq \max_{0 \leq m \leq k} \left\{ L \cdot \tau^{\alpha(k+1)}, \tau^{\alpha(m+1)} \cdot \sup_{B_{\tau^{k-m}}} |u| \right\} \quad \text{for every } k \geq 0, \quad (10.2)$$

we have the following estimate,

$$|u(x)| \leq \tau^{-\alpha} L |x|^{\alpha} \quad \text{for every } x \in B_1.$$

Proof of Theorem 2.2 Without loss of generality, we can assume that $x_0 = 0$. By Lemma 10.1, it is enough to show that there exists a constant $c_0 \in (0, 1)$ such that for any $u \in S_{\varepsilon_0}^{\xi_1, \xi_2}(B_1, M)$ satisfying

$$\min \{ \Theta_u^+(0, r), \Theta_u^-(0, r) \} \leq c_0, \quad \forall r \in (0, 1)$$

we have

$$\sup_{B_{2^{-(k+1)}}} |u(x)| \leq \max_{0 \leq m \leq k} \left\{ \frac{M}{c_0 \cdot 2^{k+1}}, \frac{S(k-m)}{2^{m+1}} \right\}, \quad \forall k \geq 0 \quad (10.3)$$

where $S(j) := \sup_{B_{2^{-j}}} |u|$. So, let us suppose by contradiction, that (10.3) does not hold.

Then, for each $j \in \mathbb{N}$, $j \geq 2$, we can find integers $k_j \geq 0$ and functions $u_j \in S_{\varepsilon_0}^{\xi_1, \xi_2}(B_1, M)$ such that

$$\min \{ \Theta_{u_j}^+(0, 2^{-k_j}), \Theta_{u_j}^-(0, 2^{-k_j}) \} \leq \frac{1}{j} \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad (10.4)$$

but

$$\sup_{B_{2^{-(k_j+1)}}} |u_j(x)| > \max_{0 \leq m \leq k_j} \left\{ \frac{j \cdot M}{2^{k_j+1}}, \frac{S_j(k_j - m)}{2^{m+1}} \right\}, \quad \text{where } S_j(k_j - m) = \sup_{B_{2^{-(k_j-m)}}} |u_j|. \quad (10.5)$$

Because of (10.4) we can assume that, up to subsequence of $\{k_j\}$,

$$\Theta_{u_j}^-(0, 2^{-k_j}) \leq \frac{1}{j} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (10.6)$$

Moreover, (10.5) implies that $k_j \geq \log_2 j - 1 \rightarrow \infty$. Now we consider the following family of auxiliary functions

$$v_j(x) := \frac{u_j(2^{-k_j}x)}{S_j(k_j + 1)}, \quad x \in B_{2^{k_j}}. \quad (10.7)$$

Since density is scaling invariant, (10.6) implies that $|\{v_j < 0\} \cap B_1| \leq \frac{1}{j} |B_1|$. We also know that u_j is a local minimizer of a functional

$$\mathcal{J}_{G_j}(u, \Omega) = \int_{\Omega} \left[G_j(|\nabla u|) + f_{1,j}(x) H_{1,j}(u^+) + f_{2,j}(x) H_{2,j}(u^-) + Q(h_{1,j}, h_{2,j})(u)(x) \right] dx, \quad (10.8)$$

satisfying the conditions of the Definition 2.3 and $G_j \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0)$. It easy to check that u_j minimizes a functional similar to (10.8) substituting the function $Q(h_{1,j}, h_{2,j})(u)(x)$ by

$$\tilde{Q}(h_{1,j}, h_{2,j})(u)(x) = (h_{2,j} - h_{1,j}) \chi_{\{u>0\}} - (h_{2,j} - h_{1,j})^- \chi_{\{u=0\}}.$$

Setting $g_j = G'_j$, we can consider the following family of normalized rescalings

$$\bar{G}_j(t) := G_{\sigma_j}(t) = \frac{G_j(\sigma_j t)}{\sigma_j g_j(\sigma_j)}, \quad \text{where } \sigma_j := 2^{k_j} \cdot S_j(k_j + 1). \quad (10.9)$$

By Lemma 9.2 we have $\bar{G}_j \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, (1 + g_0)^{-1})$ and $\bar{G}_j(1) \leq 1$. Again by (10.5) follows that $\sigma_j \geq \frac{j}{2} \rightarrow \infty$. Now, we note that v_j is a minimizer for the functional

$$\begin{aligned} \tilde{\mathcal{J}}_{\bar{G}_j}(w, \Omega) = & \int_{\Omega} \left[\bar{G}_j(|\nabla w|) + \bar{f}_{1,j}(x) \bar{H}_{1,j}(w^+) \right. \\ & \left. + \bar{f}_{2,j}(x) \bar{H}_{2,j}(w^-) + \tilde{Q}(\bar{h}_{1,j}, \bar{h}_{2,j})(w)(x) \right] dx, \end{aligned}$$

where

$$\bar{f}_{i,j}(x) = \frac{f_{i,j}(2^{-k_j}x)}{\sigma_j g_j(\sigma_j)}, \quad \bar{H}_{i,j}(w^{\pm}) = H_{i,j}(S_j(k_j + 1)w^{\pm}) \quad \text{and} \quad \bar{h}_{i,j}(x) = \frac{h_{i,j}(2^{-k_j}x)}{\sigma_j g_j(\sigma_j)}.$$

Once more by (10.5), $G_j(1) \geq \varepsilon_0$, $(G - 2)$ and since $\sigma_j \rightarrow \infty$, we conclude that

$$\|v_j\|_{L^\infty(B_8)} \leq 16, \quad \|\bar{f}_{i,j}(x) \bar{H}_{i,j}(w^{\pm})\|_{L^\infty(B_8)} \rightarrow 0, \quad \|\tilde{Q}(\bar{h}_{1,j}, \bar{h}_{2,j})(w)\|_{L^\infty(B_8)} \rightarrow 0, \quad (10.10)$$

for any bounded function $w : B_{2^{k_j}} \rightarrow \mathbb{R}$. This imply that for j large enough $v_j \in S_{(1+g_0)^{-1}}^{\xi_1, \xi_2}(B_8, 16)$. This way, the uniform Hölder estimate - Theorem 5.1 - implies that

$$\|v_j\|_{C^{0,\alpha}(B_4)} \leq 16 + C.$$

So, we conclude that there exists a $v_\infty \in C^{0,\alpha}(B_4)$ such that $v_j \rightarrow v_\infty$ uniformly in B_4 . Now, considering $\{w_j\}_{j \geq 1}$ solutions in $W^{1, \bar{G}_j}(B_4)$ to the following Dirichlet problem

$$\mathcal{L}_j w_j = 0 \quad \text{in } B_4 \quad \text{and} \quad w_j = v_j \quad \text{on } \partial B_4,$$

where the $\mathcal{L}_j$'s are given by

$$\mathcal{L}_j w := \operatorname{div} \left(\bar{g}_j(|\nabla w|) \frac{\nabla w}{|\nabla w|} \right), \quad \bar{g}_j = \bar{G}'_j,$$

we conclude similarly to the proof of Theorem 2.1 of [6] that

$$h_j := v_j - w_j \rightarrow 0 \quad \text{strongly in } W_0^{1, \delta+1}(B_4). \quad (10.11)$$

By Maximum Principle and (10.10) follow that $\|w_j\|_{L^\infty(B_4)} = \|v_j\|_{L^\infty(B_4)} \leq 16$.

Now, Theorem 3.1 guarantees that there exists $C = C(n, \delta, g_0) > 0$ such that $\|w_j\|_{C^{1,\alpha}(B_2)} \leq C$. Therefore, we can find $w_\infty \in C^{1,\alpha/2}(B_2)$ such that up to a subsequence

$$w_j \rightarrow w_\infty, \quad \nabla w_j \rightarrow \nabla w_\infty \quad \text{uniformly in } B_2.$$

We conclude this way that $v_\infty = w_\infty$ in B_2 by (10.11). Now, we can use our compactness result - Theorem 9.1 - to conclude that there exists a $G_\infty \in \mathcal{G}(\delta, g_0)$ such that, again up to a subsequence,

$$\bar{G}_j \rightarrow G_\infty \quad \text{and} \quad \bar{G}'_j \rightarrow G'_\infty \quad \text{uniformly in compact subsets of } [0, \infty),$$

and

$$\bar{G}''_j \rightarrow G''_\infty \quad \text{uniformly in compact subsets of } (0, \infty).$$

By estimates (10.10) and convergence of w_j and $\bar{G}_j$ follows that v_∞ is a minimizer of $I(v, B_1) := \int_{B_1} G_\infty(|\nabla v|)dx$. This implies that for $g_\infty := G'_\infty$, v_∞ is a weak solution to

$$\begin{aligned} \mathcal{L}_\infty v_\infty &:= \operatorname{div} \left(g_\infty(|\nabla v_\infty|) \frac{\nabla v_\infty}{|\nabla v_\infty|} \right) = 0 \quad \text{in } B_1, \\ \text{and } v_\infty(0) &= 0 \quad \text{and } 0 \leq v_\infty \leq 2 \quad \text{and } \sup_{B_{1/2}} v_\infty = 1. \end{aligned}$$

Therefore, by Harnack inequality (Lemma 3.4), we have for $C_0 = C_0(\delta, g_0, n) > 0$

$$1 = \sup_{B_{1/2}} v_\infty \leq C_0 \cdot v_\infty(0) = 0,$$

which is a contradiction. This way (10.3) holds. If instead of (10.6) we have $\Theta_{u_j}^+(0, 2^{-k_j}) \leq \frac{1}{j}$ the proof follows by slightly modifications of the previous case to consider the auxiliary function $V_j = -v_j$. This finish the proof of Theorem. $\square$

We are ready to prove the scaled version of Theorem 2.2, i.e. Corollary 2.1. In particular, it shows the dependence between the Lipschitz constant of minimizers and the scale where the density starts to become small enough. We use this result for to prove Lipschitz regularity and also density estimates from below in the next section.

Proof of Corollary 2.1 We can assume that $x_0 = 0$. For $u \in S_{\varepsilon_0}^{\xi_1, \xi_2}(B_r, M)$ follows that u is a minimizer of

$$\mathcal{J}_G(u, B_r) = \int_{B_r} \left[G(|\nabla u|) + f_1(x)H_1(u^+) + f_2(x)H_2(u^-) + Q(h_1, h_2)(u)(x) \right] dx,$$

satisfying the Definition 2.3 and $G \in \mathcal{G}_{(\xi_1, \xi_2) \cdot D}(\delta, g_0, \varepsilon_0)$. Define $v(x) = u(rx)$, $x \in B_1$. In this case, for

$$0 < r \leq r_0^* := \min \left\{ 1, \left(\frac{\varepsilon_0}{(1 + g_0) \cdot \max_{i=1,2} \{\lambda_i, \mu_i\}} \right)^{(1+\delta)^{-1}} \right\},$$

we can show that $v \in S_{(1+g_0)^{-1}}^{\xi_1, \xi_2}(B_1, M)$ with $\lambda_i = \mu_i = 1$ in this class. Indeed, by simple computations and using Lemma 9.2, we note that v is a local minimizer of

$$\begin{aligned} \mathcal{J}_{G_{r^{-1}}}(w, B_1) &= \int_{B_1} \left[G_{r^{-1}}(|\nabla w|) + \frac{r}{g(r^{-1})} (\tilde{f}_1(x)H_1(w^+) \right. \\ &\quad \left. + \tilde{f}_2(x)H_2(w^-) + Q(\tilde{h}_1, \tilde{h}_2)(w)(x)) \right] dx, \end{aligned}$$

$$\text{where } \tilde{f}_i(x) = f_i(rx), \quad \tilde{h}_i(x) = h_i(rx) \quad \text{for } x \in B_1 \text{ and } i = 1, 2,$$

and, by $(g - 2)$ and $(G - 2)$, we have for $r \leq r_0^*$ that

$$r^{-1}g(r^{-1}) \geq G(r^{-1}) \geq \frac{\varepsilon_0}{1 + g_0} r^{-(1+\delta)} \geq \max_{i=1,2} \{\lambda_i, \mu_i\}.$$

Now, applying Theorem 2.2 to v we obtain

$$|v(x)| \leq \frac{2M}{c_0^*} \cdot |x|, \quad \forall x \in B_1(0),$$

provided

$$\min \left\{ \Theta_v^+(x_0, \rho), \Theta_v^-(x_0, \rho) \right\} \leq c_0^*$$

for all $0 < \rho < 1$, where $c_0^* = c_0^*(n, \delta, g_0, \varepsilon_0, \wp_i, \rho_i, \gamma_i, M, \xi_i) > 0$. The result follows translating this back in terms of u . $\square$

11 Lipschitz regularity under a “small density criterion” and asymptotic behaviour of the free boundary

In this final section, we investigate the optimal regularity for local minimizers for functionals of the type (2.2) and the local behaviour of the free boundary in the case where the minimizers are only locally Hölder continuous both as a consequence of Theorem 2.2. We start with the proof of Theorem 2.3.

Proof of Theorem 2.3 Consider $\Omega' \Subset \Omega$ and $y_0 \in \Omega' \cap \{u \neq 0\}$. We can assume that $u(y_0) > 0$ since the case $u(y_0) < 0$ can be treated similarly. Define

$$d_y^+ := \text{dist}(y, \partial\{u > 0\}) \quad \text{and} \quad d_y^- := \text{dist}(y, \partial\{u < 0\}).$$

Assume first that $d_{y_0}^- \geq r_\sigma^*/2$. In this case we have $0 \leq u \in \mathcal{S}_{\varepsilon_0}(B_{r_\sigma^*/2}(y_0), M, \Lambda_i^M)$. By Theorem 6.1,

$$|\nabla u(y_0)| \leq \frac{4 \cdot C_2}{r_\sigma^*}.$$

Suppose now $d_{y_0}^- < r_\sigma^*/2$. Let $z_0 \in F^+(u)$ such that $d_{y_0}^+ = |y_0 - z_0|$. Since also $d_{y_0}^+ < r_\sigma^*/2$ we conclude that $d_{z_0}^- < r_\sigma^*$. Thus, Corollary 2.1 assure us

$$u(y_0) \leq \frac{2M}{c_0 \cdot r_\sigma^*} \cdot d_{y_0}^+.$$

By Corollary 5.1, $\mathcal{L}_g u = H'_1(u) f_1$ in $B_{d_{y_0}^+}(y_0)$, thus by Harnack inequality and above estimate, we obtain

$$u(x) \leq C_3 \left(\frac{2M}{c_0 \cdot r_\sigma^*} + g^{-1} \left(d_{y_0}^+ \Lambda_1^M \cdot \lambda_1 \right) \right) \cdot d_{y_0}^+, \quad \forall x \in B_{\frac{d_{y_0}^+}{2}}(y_0).$$

Now, by local gradient estimate (3.3),

$$|\nabla u(y_0)| \leq C_4 \left[C_3 \left(\frac{2M}{c_0 \cdot r_\sigma^*} + g^{-1} \left(d_{y_0}^+ \Lambda_1^M \cdot \lambda_1 \right) \right) + g^{-1} \left(d_{y_0}^+ \Lambda_1^M \cdot \lambda_1 \right) \right] \leq \frac{C_5}{r_\sigma^*}.$$

$\square$

In the sequel, we investigate the touching behaviour of the free boundaries $F^+(u)$ and $F^-(u)$ even though local minima u are only Hölder continuous. Notice that in the classical paper [4], suitable conditions imply that minimizers are globally subharmonic functions (see Theorem 2.3 in [4]) and this fact together with the maximum principle restrict the way the free boundaries $F^+(u)$, $F^-(u)$ may touch. Essentially $F^-(u)$ cannot separate from $F^+(u)$. In other words, the set $F^-(u) \setminus F^+(u)$ is empty. In this generality treated here, minimizers

are neither subsolutions nor supersolutions and the negative free boundary may, in principle, separate from the positive one, i.e.,

$$(F^+(u) \setminus F^-(u)) \cap \Omega \neq \emptyset.$$

A similar situation occurs even in the standard case involving the Laplace operator in a inhomogeneous setting as it appears in flame propagation problems in [13].

As a second application of Theorem 2.2, we can show that if Lipschitz regularity fails to be the optimal regularity for a minimizer $u \in S_{\varepsilon_0}^{\xi_1, \xi_2}(\Omega, M)$ say in a point, then this point is a contact point between the free boundaries and the positive and negative phase of u have a universal (upper) density from below at that point. In particular, in this contact point the free boundary $F^\pm(u)$ is cusp free. In the case where u is not locally Lipschitz around x_0 , then x_0 is also a contact point between the free boundaries and $F^\pm(u)$ is “asymptotically cusp free”. The proof follows the same guide lines of Proposition 2.1 of [6].

Proof of Theorem 2.4 Let $\Omega' \Subset \Omega$. It is easy to see that $S_p(\Omega') \subset S_l(\Omega')$. Now consider $x_0 \in S_l(\Omega')$. We can find two sequences $\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \subset \Omega'$ such that:

$$x_n \rightarrow x_0, y_n \rightarrow x_0 \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \rho_n := \frac{|u(x_n) - u(y_n)|}{|x_n - y_n|} > n.$$

Note that $\Omega = \{u > 0\} \cup \{u < 0\} \cup \{u = 0\}^\circ \cup F^\pm(u)$ and by interior $C^{1,\alpha}$ -regularity we conclude that

$$x_0 \notin \{u > 0\} \cup \{u < 0\} \cup \{u = 0\}^\circ.$$

Thus, $x_0 \in F^\pm(u)$. Let us suppose that $x_0 \in F^+(u) \setminus F^-(u)$. In this case, $x_0 \notin F^-(u) \cap \overline{\Omega'}$ which is a compact set. Hence, for $0 < \varepsilon = \min \left\{ \frac{3}{4} d_{x_0}^-, \text{dist}(\Omega', \partial\Omega) \right\}$, we have that $B_\varepsilon(x_0) \cap F^-(u) \cap \overline{\Omega'} = \emptyset$. Since $0 \leq u \in S_{\varepsilon_0}^{(\xi_1, \xi_2)-D}(B_\varepsilon(x_0), M)$, we have by one-phase case (Theorem 6.1) that

$$n < \frac{|u(x_n) - u(x_0)|}{|x_n - x_0|} \leq \|\nabla u\|_{L^\infty(B_{\varepsilon/2}(x_0))} \leq \frac{C}{\varepsilon} < \infty,$$

which is a contradiction. Similarly we treat the case where $x_0 \in F^-(u) \setminus F^+(u)$. Hence, $x_0 \in F^+(u) \cap F^-(u)$. Now we verify (2.8) and (2.9). Consider first $x_0 \in S_l(\Omega')$. By Theorem 2.3, we know that for any $r \in (0, r_1^*)$ there exists $x_r \in B_{\frac{3r}{4}}(x_0) \cap F^\pm(u)$ such that for some $\rho_r \in (0, r/4)$ we have

$$\min \left\{ \Theta_u^+(x_r, \rho_r), \Theta_u^-(x_r, \rho_r) \right\} > c_0^*.$$

This proves (2.8). Suppose now that $x_0 \in S_p(\Omega')$, we also see that for any $0 < r \leq r_1^*$ there exists $0 < \rho \leq r$ such that

$$\min \left\{ \Theta_u^+(x_0, \rho), \Theta_u^-(x_0, \rho) \right\} > c_0^*.$$

This follows by Corollary 2.1. Thus, (2.9) is proved. $\square$

Remark 11.1 By above Theorem we obtain that

$$F^\pm(u) = S_p(\Omega') \dot{\bigcup} (S_l(\Omega') \setminus S_p(\Omega')) \dot{\bigcup} (F^\pm(u) \setminus S_l(\Omega')).$$

In the set $\mathcal{S}_p(\Omega')$, the free boundary is cusp free and, in the set $\mathcal{S}_l(\Omega') \setminus \mathcal{S}_p(\Omega')$, asymptotically cusp free. Finally, we note that for any $z_0 \in F^\pm(u) \setminus \mathcal{S}_l(\Omega')$ there exists $r_0 \in (0, r_1^*)$ such that

$$[u]_{C^{0,1}(B_r(z_0))} \leq C^*, \quad r \in (0, r_0].$$

In this case, if $z_0 \in F^+(u)$, by Lipschitz continuity and strong non-degeneracy (see Theorem 4.1 of [16] or Theorem 5.1 of [18]) there exists a constant $c_1^* \in (0, 1)$ depending only on intrinsic parameters and C^* such that

$$(1 - c_1^*) \cdot r^n \geq |B_r(z_0) \cap \{u > 0\}| \geq c_1^* \cdot r^n, \quad r \in (0, r_0).$$

Furthermore, following a similar strategy of the Lemma 4.2 and Theorem 4.3 of [16], there exist positive constants $c_2^* \leq C_2^* < \infty$ such that

$$c_2^* \cdot r^{n-1} \leq |B_r(z_0) \cap F^+(u)| \leq C_2^* \cdot r^{n-1}.$$

The two above estimates follows similarly in the case where $z_0 \in F^-(u)$. This proves that the open set $F^\pm(u) \setminus \mathcal{S}_l(\Omega')$ (with respect to the topology induced by $F^\pm(u)$) is cusp free and has locally finite perimeter in each connected component.

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