Global well-posedness of solutions to the Cauchy problem of convective Cahn–Hilliard equation

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Abstract In this paper, we consider the global well-posedness of solutions for the Cauchy problem of N -dimensional Cahn–Hilliard equation with convective term. We first construct the local smooth solutions; then, by combining some a priori estimates, continuity argument, the local smooth solutions are extended step by step to all $t > 0$ provided that the initial data are suitably small and the smooth nonlinear functions satisfy certain local growth conditions at some fixed point $\bar{u} \in \mathbb{R}$. In addition, for the N -dimensional Cahn–Hilliard equation without convective term, we also establish the similar result.

Keywords Global well-posedness · Convective Cahn–Hilliard equation · Local solution · Cauchy problem

Mathematics Subject Classification 35K55 · 35A01 · 35B45

1 Introduction

The Cahn–Hilliard equation

$$\begin{cases} \partial_t u = \Delta \mu, \\ \mu := -\gamma \Delta u + \varphi(u), \end{cases} \quad (1.1)$$

arises in the study of phase separation on cooling binary solutions such as glasses, alloys and polymer mixtures [7, 25, 26]. Here, $u(x, t)$ is the relative concentration of the two phases, $\Phi(u) = \int_0^u \varphi(s) ds$ is a double-well potential with two equal minima at $u^- \leq u^+$ corresponding to the two pure phases, M is the diffusion mobility. During the past years, there are many classical papers related to the study of Cahn–Hilliard equation, see, e.g., Elliott and

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Zheng [16], Dlotko [12], Temam [30], Gilardi, Miranville and Schimperna [18], Yin [33], Dai and Du [8] and the reference therein.

It is particularly worth mentioning here that Cahn–Hilliard equation is only a phenomenological model. Hence, various modifications of it have been proposed in order to capture the dynamical picture of the phase transition phenomena better: to name only a few, viscous Cahn–Hilliard equation [5, 13], Cahn–Hilliard–Gurtin equation [4, 22], Cahn–Hilliard–Cook equation [3], Cahn–Hilliard–Brinkman equation [6], Cahn–Hilliard–Hele–Shaw systems [31] and so on.

Recently, many people got interested in the convective Cahn–Hilliard equation

$$\begin{cases} \partial_t u = \Delta u + \nabla \cdot g(u), \\ \mu := -\gamma \Delta u + f(u), \end{cases} \quad (1.2)$$

which arises naturally as a continuous model for the formation of facets and corners in crystal growth [19, 32]. Here, $u(x, t)$ denotes the slope of the interface. The convective term $\nabla \cdot g(u)$ [19] stems from the effect of kinetic that provides an independent flux of the order parameter, similar to the effect of an external field in spinodal decomposition of a driven system. In [34], Zaks et al. investigated the bifurcations of stationary periodic solutions of a convective Cahn–Hilliard equation. Eden and Kalantarov [14, 15] established some results on the existence of a compact attractor for the convective Cahn–Hilliard equation with periodic boundary conditions in one space dimension and three space dimension. Very recently, Della Porta and Grasselli [9] considered the initial-boundary value problem of convective nonlocal Cahn–Hilliard equation as dynamical systems and showed that they have bounded absorbing sets and global attractors. In [35], the existence of optimal solutions and optimality condition for 2D convective Cahn–Hilliard equation were studied by Zhao and Liu. Rocca and Sprekels [28] studied a distributed control problem for a 3D convective nonlocal Cahn–Hilliard-type system involving a degenerate mobility and a singular potential.

The goal of this paper is to investigate the global well-posedness for the Cauchy problem of N -dimensional convective Cahn–Hilliard equation

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta f(u) - \nabla \cdot g(u) = 0, \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad t \geq 0. \end{cases} \quad (1.3)$$

As far as we know, there's no reference concerning this aspect of the convective Cahn–Hilliard equation till now. We will prove the existence and uniqueness of global smooth solutions for Cauchy problem (1.1) by using Hoff and Smoller's method [20, 21, 29] with a slight modification. We show that if the smooth nonlinear functions $f(u)$ and $g(u)$ satisfy certain local growth condition at $u = \bar{u}$ for some $\bar{u} \in \mathbb{R}$ and the $L^{\frac{N(l-1)}{3}}(\mathbb{R})$ -norm of the initial data is sufficiently small, then there exists a unique global smooth solution $u(t, x)$ to the Cauchy problem (1.3). More precisely, the result can be stated as follows.

Theorem 1.1 *Suppose that $r > 0$ and $l > 1$ are two given positive constants, $L = \max\{5, N\}$ and $\bar{u} \in \mathbb{R}$ is some fixed constant. Assume that $u_0(x) - \bar{u} \in L^{\frac{N(l-1)}{3}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|u_0(x) - \bar{u}\|_{L^\infty} \leq r$ and $\|u_0(x) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}$ sufficiently small.*

The nonlinear functions $f(u), g(u) \in C^L(\bar{B}(\bar{u}, 2r), \mathbb{R})$ satisfying $f(u) = O(1)|u - \bar{u}|^p$ and $g(u) = O(1)|u - \bar{u}|^l$ as $u \rightarrow \bar{u}$, where $p = \frac{2l+1}{3}$. Then, there exists a unique global smooth solution $u(t, x) \in L^\infty(0, \infty; L^{\frac{N(l-1)}{3}}(\mathbb{R}^N))$ for the Cauchy problem (1.1), which satisfies

$$\|u(t, x) - \bar{u}\|_{L^\infty} \leq 2r, \quad t \geq 0. \quad (1.4)$$

Remark 1.1 For the details on the fractional Laplacian operator $(-\Delta)^\gamma$ ($\gamma \in \mathbb{R}^+$) and fractional Sobolev spaces, the reader may refer to [1, 2, 10, 23].

Remark 1.2 The solution $u(t, x)$ is said to be the global smooth solution to the convective Cahn–Hilliard equation if $u(t, x) \in C^{1,4}([0, \infty) \times \mathbb{R})$.

It is worth pointing that Liu et. al. [24] studied the global well-posedness of Cauchy problem for Cahn–Hilliard equation in $\mathbb{R}^N$:

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta f(u) = 0, \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad t \geq 0. \end{cases} \quad (1.5)$$

The authors supposed that $f(u) = O(1)|u|^{1+\frac{2}{N}}$ and $\|u - \bar{u}\|_{L^1}$ sufficiently small, obtained the global well-posedness and decay rate of solutions. In Theorem 1.1, we consider the convective Cahn–Hilliard equation with two nonlinear terms $\Delta f(u)$ and $\nabla \cdot g(u)$. We give a fewer restriction on the nonlinear functions $f(u)$ and $g(u)$, only assume $f(u) = O(1)|u - \bar{u}|^{\frac{2l+1}{3}}$, $g(u) = O(1)|u - \bar{u}|^l$ with $l > 1$ and $\|u - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}$ sufficiently small, proved the global well-posedness of solutions. In fact, our method can also be generalized to the classical Cahn–Hilliard equation without convective term.

Theorem 1.2 Suppose that $r > 0$ and $l > 1$ are two given positive constants, $L = \max\{5, N\}$ and $\bar{u} \in \mathbb{R}$ is some fixed constant. Assume that $u_0(x) - \bar{u} \in L^{\frac{N(l-1)}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|u_0(x) - \bar{u}\|_{L^\infty} \leq r$ and $\|u_0(x) - \bar{u}\|_{L^{\frac{N(l-1)}{2}}}$ sufficiently small. The nonlinear function $f(u) \in C^L(\bar{B}(\bar{u}, 2r), \mathbb{R})$ satisfying $f(u) = O(1)|u - \bar{u}|^l$ as $u \rightarrow \bar{u}$. Then, there exists a unique global smooth solution $u(t, x) \in L^\infty(0, \infty; L^{\frac{N(l-1)}{2}}(\mathbb{R}^N))$ for the Cauchy problem (1.5), which satisfies

$$\|u(t, x) - \bar{u}\|_{L^\infty} \leq 2r, \quad t \geq 0. \quad (1.6)$$

Remark 1.3 We assume that $f(u) = O(1)|u - \bar{u}|^l$ ($l > 1$) as $u \rightarrow \bar{u}$ in Theorem 1.2. When $l = 1 + \frac{2}{N}$, we recover the result in [24].

The rest of this paper is organized as follows. In the next section, we introduce some preliminary results. The proof of Theorem 1.1 is postponed in Sect. 3. We also give a brief proof of Theorem 1.2 in the final section of this paper.

Throughout this article, we use C to denote the generic constant that can take different values in different places. In addition, $L^p(\mathbb{R}^N)$ ($1 \leq p \leq \infty$) represents the N -dimensional Lebesgue space with norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^3} |u(x, t)|^p dx \right)^{\frac{1}{p}}, \quad \|u\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^3} |u(x, t)|.$$

Moreover, for each $k \in \mathbb{Z}^+$, $D^k u$ is used to denote the set of all k th order derivatives of $u(t, x)$ with respect to x and $|D^k u|^2 = \sum_{|\alpha|=k} \left| \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \right|^2$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is some multi-index and $|\alpha| = \sum_{j=1}^N \alpha_j$ stands for the length of α .

2 Preliminaries

The following two lemmas will be used in this section.

Lemma 2.1 ([27]) If $1 \leq p \leq r \leq q \leq \infty$ and $u \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, then $u \in L^r(\mathbb{R}^N)$ with

$$\|u\|_{L^r} \leq \|u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha}, \quad \text{where } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

Lemma 2.2 (Strauss's inequality [29]) Suppose that $M(t)$ is a nonnegative continuous function of t . $M(t)$ satisfies

$$M(t) \leq d_1 + d_2 M(t)^r$$

in some interval containing 0, where d_1 and d_2 are positive constants and $r > 1$. If $M(0) \leq d_1$ and

$$d_1 d_2 < (1 - r^{-1}) r^{-(r-1)^{-1}},$$

then in the same interval

$$M(t) \leq \frac{d_1}{1 - r^{-1}}.$$

In the proof of lemmas and theorems, we frequently employ the Gagliardo–Nirenberg inequality:

Lemma 2.3 (Gagliardo–Nirenberg inequality [17]) Let $u \in L^q(\mathbb{R}^n)$, $\nabla^m u \in L^r(\mathbb{R}^n)$, $1 \leq q, r \leq \infty$. Then, there exists a positive constant $C = C(n, m, j, a, q, r)$, such that

$$\|\nabla^j u\|_{L^p} \leq C \|\nabla^m u\|_{L^r}^a \|u\|_{L^q}^{1-a},$$

where

$$\frac{1}{p} = \frac{j}{n} + a \left(\frac{1}{r} - \frac{m}{n} \right) + (1-a) \frac{1}{q}, \quad 1 \leq p \leq \infty, \quad 0 \leq j \leq m, \quad \frac{j}{m} \leq a \leq 1.$$

We give the $L^p(\mathbb{R}^3, \mathbb{R})$ -estimate on the fundamental solution to the convective Cahn–Hilliard equations.

Lemma 2.4 Suppose that $k(t) = \mathcal{F}^{-1}(e^{-|\xi|^4 t})$, where $\xi, x \in \mathbb{R}^N$ and $t > 0$. We have

$$\|k(t)\|_{L^p(\mathbb{R}^N)} \leq \frac{c_p}{c_q} t^{-\frac{N}{4} \left(\frac{1}{q} - \frac{1}{p} \right)} \|k(t)\|_{L^q(\mathbb{R}^N)}, \quad (2.1)$$

$$\|D^j k(t)\|_{L^p(\mathbb{R}^N)} \leq \frac{c_{p,j}}{c_q} t^{-\frac{N}{4} \left(\frac{1}{q} - \frac{1}{p} \right) - \frac{j}{4}} \|k(t)\|_{L^q(\mathbb{R}^N)}, \quad j = 1, 2, \dots, \quad (2.2)$$

where c_p, c_q and $c_{p,j}$ are positive constants with $c_1 = 1$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transformation with respect to ξ .

Proof Let $\xi = \eta t^{-\frac{1}{4}}$. Hence

$$k(t, x) = \int_{\mathbb{R}^N} e^{-|\xi|^4 t} e^{ix \cdot \xi} d\xi = t^{-\frac{N}{4}} \int_{\mathbb{R}^N} e^{-|\eta|^4} e^{ix \cdot \eta t^{-\frac{1}{4}}} d\eta.$$

Let $G(y) = \int_{\mathbb{R}^N} e^{-|\eta|^4} e^{iy \cdot \eta} d\eta$. Clearly, $G(y)$ is a rapidly decreasing function. Then

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |k(t, x)|^p dx \right)^{\frac{1}{p}} &= t^{-\frac{N}{4}} \left(\int_{\mathbb{R}^N} |G(xt^{-\frac{1}{4}})|^p dx \right)^{\frac{1}{p}} \\ &= t^{-\frac{N}{4}} t^{\frac{N}{4p}} \left(\int_{\mathbb{R}^N} |G(z)|^p dz \right)^{\frac{1}{p}} = c_p t^{-\frac{N}{4} \left(1 - \frac{1}{p} \right)}. \end{aligned}$$

Similarly, we have

$$\left(\int_{\mathbb{R}^N} |k(t, x)|^q dx \right)^{\frac{1}{q}} = c_q t^{-\frac{N}{4}(1-\frac{1}{p})}.$$

Combining the above two equalities together gives (2.1). In addition,

$$D^k k(t, x) = t^{-\frac{N+k}{\alpha}} D_y^k G(x t^{-\frac{1}{\alpha}}).$$

By simple calculations, we obtain (2.2). The proof is completed. $\square$

3 Proof of Theorem 1.1

In order to prove the global existence and uniqueness of smooth solutions for the Cauchy problem of convective Cahn–Hilliard equation, we first state the local existence result.

Lemma 3.1 *Suppose that $\max_{u \in \bar{B}(\bar{u}, 2r)} \sum_{k=1}^L (|D^k f(u)| + |D^k g(u)|) = b$ and the conditions listed in Theorem 1.1 are satisfied. Then, there exists a unique local solution $u(x, t)$ on the strip $\Pi_{t_1} := \{(x, t) : x \in \mathbb{R}^N, 0 < t \leq t_1\}$ such that*

$$\|u(x, t) - \bar{u}\|_{L^\infty} \leq 2r, \quad 0 \leq t \leq t_1, \quad (3.1)$$

where $t_1 = \min \left\{ 1, \frac{1}{64b^2c_{1,2}^2}, \left(\frac{3}{16bc_{1,1}} \right)^{\frac{4}{3}} \right\}$. In addition, assume that C_k is a continuous increasing function of $t - s'_k$, then for each $0 < s'_1 < s'_2 < \dots < s'_N < t \leq t_1$, we have

$$\|D^k u(t, x)\|_{L^\infty} \leq (t - s'_k)^{-\frac{k}{4}} C_k(r, s'_k - s'_1, t - s'_k), \quad k = 1, 2, \dots, L. \quad (3.2)$$

Proof The smooth solution of Cauchy problem (1.3) should satisfy the following integro-differential equation

$$\begin{aligned} u(t, x) = & \int k(t, x - y) u_0 dy - \int_0^t ds \int \Delta k(x - y, t - s) f(u(s, y)) dy \\ & - \int_0^t ds \int \nabla \cdot k(x - y, t - s) g(u(s, y)) dy. \end{aligned} \quad (3.3)$$

We first prove that there is a sufficiently small $t_1 > 0$ such that the integro-differential equation (3.3) admits a unique continuous solution $u(t, x)$ on the strip Π_{t_1} . Then, if we can show that the solution obtained above is a smooth solution, such a $u(t, x)$ is indeed a local smooth solution to problem (1.1).

Assume $T(t)u = k(t, x) * u(t, x)$. Hence, (3.3) is equivalent to

$$u(t, x) = T(t)u_0 - \int_0^t \Delta T(t - s) f(u(s)) ds - \int_0^t \nabla \cdot T(t - s) g(u(s)) ds. \quad (3.4)$$

It is easy to see that $T(t)1 = 1$. In order to prove that there exists a local smooth solution for (3.3), we just need to show that there exists a local smooth solution for

$$\begin{aligned} u(t, x) - \bar{u} = & T(t)(u_0 - \bar{u}) - \int_0^t \Delta T(t - s) f(u(s)) ds \\ & - \int_0^t \nabla \cdot T(t - s) g(u(s)) ds. \end{aligned} \quad (3.5)$$

By using the successive approximations method [11, 24]: Set $u^{(0)}(t, x) = u_0(x)$ and for $n \geq 1$, we define

$$\begin{aligned} u^{(n)}(t, x) - \bar{u} &= T(t)(u^{(0)} - \bar{u}) - \int_0^t \Delta T(t-s)f(u^{(n-1)}(s))ds \\ &\quad - \int_0^t \nabla \cdot T(t-s)g(u^{(n-1)}(s))ds. \end{aligned} \quad (3.6)$$

Obviously, $u^{(n)}(t, x)$ is well defined on $[0, \infty) \times \mathbb{R}$ for each $n \geq 0$.

Set $v^{(n)}(t, x) = u^{(n)}(t, x) - \bar{u}$ and $\mathbb{Q}\chi\mathbb{Q} = \sup_{(t,x) \in \Pi_{t_1}} |\chi(t, x)|$. By induction, we prove that if $t_1 = \min \left\{ 1, \frac{1}{64b^2c_{1,2}^2}, \left(\frac{3}{16bc_{1,1}} \right)^{\frac{4}{3}} \right\}$, then

$$\mathbb{Q}v^{(n)}\mathbb{Q} \leq 2r. \quad (3.7)$$

From the assumption we imposed on the initial data, we obtain (3.7) holds for $n = 0$. As to the case $n = 1$, since Hausdorff–Young’s inequality together with the fact that $u_0(x) \in \tilde{B}(\bar{u}, r)$ implies that

$$\left| \int k(t, x-y)(u_0(y) - \bar{u})dy \right| \leq \|u_0 - \bar{u}\|_{L^\infty} \leq r, \quad (3.8)$$

we have from (3.6) and (3.8) that

$$\begin{aligned} \mathbb{Q}v^{(1)}(t, x)\mathbb{Q} &\leq r + bc_{1,2} \int_0^t (t-s)^{-\frac{1}{2}} \mathbb{Q}v^{(0)}\mathbb{Q}ds + bc_{1,1} \int_0^t (t-s)^{-\frac{1}{4}} \mathbb{Q}v^{(0)}\mathbb{Q}ds \\ &\leq r + 4rbc_{1,2}(t_1)^{\frac{1}{2}} + \frac{8}{3}rbc_{1,1}(t_1)^{\frac{3}{4}} \leq 2r. \end{aligned} \quad (3.9)$$

Hence, (3.7) is right for $n = 1$. In addition, assume that (3.7) holds for $n \leq m-1$ for some positive integer $m \geq 1$. We now show that (3.7) also holds for $n = m$. Applying Hausdorff–Young’s inequality and (3.6), we deduce that

$$\begin{aligned} \mathbb{Q}v^{(m)}\mathbb{Q} &\leq r + b \int_0^t \|\Delta k(t-s)\|_{L^1} \mathbb{Q}v^{(m-1)}\mathbb{Q}ds + b \int_0^t \|\nabla k(t-s)\|_{L^1} \mathbb{Q}v^{(m-1)}\mathbb{Q}ds \\ &\leq r + bc_{1,2} \mathbb{Q}v^{(m-1)}\mathbb{Q} \int_0^t (t-s)^{-\frac{1}{2}} ds + bc_{1,1} \mathbb{Q}v^{(m-1)}\mathbb{Q} \int_0^t (t-s)^{-\frac{1}{4}} ds \\ &\leq r + 4rbc_{1,2}(t_1)^{\frac{1}{2}} + \frac{8}{3}rbc_{1,1}(t_1)^{\frac{3}{4}} \leq 2r. \end{aligned} \quad (3.10)$$

Therefore, (3.7) holds for $n = m$ and by induction, (3.7) is true for any $n \geq 0$.

We also prove that $v^{(n)}(t, x)$ satisfies

$$\begin{aligned} \mathbb{Q}v^{(n)} - v^{(n-1)}\mathbb{Q} &\leq \frac{(C_1(t_1)^{\frac{1}{2}})^{n-1}}{\Gamma(\frac{n-1}{2} + \frac{1}{2})} M_1 + \frac{(C_2(t_1)^{\frac{3}{4}})^{n-1}}{\Gamma(\frac{n-1}{2} + \frac{1}{2})} M_2 \\ &\leq \frac{C_1^{n-1} M_1 + C_2^{n-1} M_2}{\Gamma(\frac{n-1}{2} + \frac{1}{2})}, \quad n \geq 1, \end{aligned} \quad (3.11)$$

where $M_1 = 2rbc_{1,2}\sqrt{\pi}$, $C_1 = bc_{1,2}\sqrt{\pi}$, $M_2 = \frac{3}{4}rbc_{1,1}\sqrt{\pi}$ and $C_2 = bc_{1,1}\sqrt{\pi}$. (3.11) can be established by induction. For $n = 1$, from (3.6) that

$$\begin{aligned} & \|v^{(1)}(t, x) - v^{(0)}(t, x)\| \\ & \leq \int_0^t \|\Delta k(t-s)\|_{L^1} \|f(u^{(0)}) - f(\bar{u})\| ds \\ & \quad + \int_0^t \|\nabla k(t-s)\|_{L^1} \|g(u^{(0)}) - g(\bar{u})\| ds \\ & \leq rbc_{1,2} \int_0^t (t-s)^{-\frac{1}{2}} ds + rbc_{1,1} \int_0^t (t-s)^{-\frac{1}{4}} ds \\ & \leq 2rbc_{1,2}(t_1)^{\frac{1}{2}} + \frac{4}{3}rbc_{1,1}(t_1)^{\frac{3}{4}} \leq \frac{M_1 + M_2}{\sqrt{\pi}}, \end{aligned}$$

which implies that the estimate (3.11) is right for $n = 1$. Suppose that (3.11) holds for $n \leq m-1$, where $m \geq 2$ is a positive integer. Then, we can get from (3.6) that

$$\begin{aligned} & \|v^{(m)}(t, x) - v^{(m-1)}(t, x)\| \\ & \leq \int_0^t \|\Delta k(t-s)\|_{L^1} \|f(u^{(m-1)}) - f(u^{(m-2)})\| ds \\ & \quad + \int_0^t \|\nabla k(t-s)\|_{L^1} \|g(u^{(m-1)}) - g(u^{(m-2)})\| ds \\ & \leq bc_{1,2} \int_0^t (t-s)^{-\frac{1}{2}} \frac{(C_1 s^{\frac{1}{2}})^{m-2}}{\Gamma(\frac{m-2}{2} + \frac{1}{2})} M_1 ds \\ & \quad + bc_{1,1} \int_0^t (t-s)^{-\frac{3}{4}} \frac{(C_2 s^{\frac{3}{4}})^{m-2}}{\Gamma(\frac{m-2}{2} + \frac{1}{2})} M_2 ds \\ & \leq bc_{1,2}\sqrt{\pi} \frac{\Gamma(\frac{m-2}{2} + 1)}{\Gamma(\frac{m-2}{2} + \frac{3}{2})} (t_1)^{\frac{m-1}{2}} \frac{(C_1)^{m-2} M_1}{\Gamma(\frac{m-2}{2} + \frac{1}{2})} \\ & \quad + bc_{1,1}\sqrt{\pi} \frac{\Gamma(\frac{m-2}{2} + 1)}{\Gamma(\frac{m-2}{2} + \frac{3}{2})} (t_1)^{\frac{3(m-1)}{4}} \frac{(C_2)^{m-2} M_2}{\Gamma(\frac{m-2}{2} + \frac{1}{2})} \\ & \leq \frac{(C_1)^{m-1} M_1 + (C_2)^{m-1} M_2}{\Gamma(\frac{m-2}{2} + \frac{1}{2})}, \end{aligned}$$

which implies that (3.11) is true for $n = m$. By induction again, we derive that the estimate (3.11) holds for any $n \geq 1$.

It is easy to see that $\sum_{n=0}^{\infty} \frac{(C_1)^{m-1} M_1 + (C_2)^{m-1} M_2}{\Gamma(\frac{m-2}{2} + \frac{1}{2})}$ is convergent. Then, follows from (3.11), we obtain $v^{(n)}(t, x)$ converges uniformly to $v(t, x) = u(t, x) - \bar{u}$ on the strip Π_{t_1} . It is clear that the unique limit $u(t, x)$ is a continuous solution of (3.3) on Π_{t_1} .

In order to show that such a $u(t, x)$ obtained above is indeed a smooth solution of problem (1.1) on Π_{t_1} , we also need to obtain the regularity of $u(t, x)$. To do so, we only need to derive the following estimates: For each $1 \leq k \leq L$, $n \geq 1$, there exists a C_k which is a continuous increasing function of $t - s'_k$ such that

$$\|D^k u^{(n)}(t)\|_{L^\infty} \leq (t - s'_k)^{-\frac{k}{4}} C_k(r, s'_k - s'_1, t - s'_k), \quad s'_k < t \leq t_1. \quad (3.12)$$

By (3.6) and the semigroup property of $T(t)$, we derive that

$$\begin{aligned} u^{(n)}(t, x) - \bar{u} &= T(t - \bar{t})(u^{(n)}(\bar{t}, x) - \bar{u}) - \int_{\bar{t}}^t \Delta T(t - s)f(u^{(n-1)}(s))ds \\ &\quad - \int_{\bar{t}}^t \nabla \cdot T(t - s)g(u^{(n-1)}(s))ds. \end{aligned} \quad (3.13)$$

For $k = 1$, we have

$$\begin{aligned} Du^{(n)}(t, x) &= DT(t - s'_1)(u^{(n)}(s'_1, x) - \bar{u}) - \int_{s'_1}^t D^3T(t - s)f(u^{(n-1)}(s))ds \\ &\quad - \int_{s'_1}^t D^2T(t - s)g(u^{(n-1)}(s))ds. \end{aligned}$$

Applying Hausdorff–Young’s inequality, we deduce that

$$\begin{aligned} \|Du^{(n)}(t)\| &\leq c_{1,1}(t - \bar{s}'_1)^{-\frac{1}{4}}r + bc_{1,5} \int_{a'_1}^t (t - s)^{-\frac{3}{4}}\|v^{(n-1)}\|ds \\ &\quad + bc_{1,2} \int_{s'_1}^t (t - s)^{-\frac{1}{2}}\|v^{(n-1)}\|ds \\ &\leq (t - \bar{s}'_1)^{-\frac{1}{4}}C_1(r, t - s'_1). \end{aligned}$$

For $k = 2$,

$$\begin{aligned} D^2u^{(n)}(t, x) &= D^2T(t - s'_2)(u^{(n)}(s'_2, x) - \bar{u}) - \int_{s'_2}^t D\Delta T(t - s)Df(u^{(n-1)}(s))ds \\ &\quad - \int_{s'_2}^t D^2T(t - s)\nabla \cdot g(u^{(n-1)}(s))ds. \end{aligned}$$

Then, applying Hausdorff–Young’s inequality and (3.12) with $k = 1$ that

$$\begin{aligned} \|D^2u^{(n)}\| &\leq c_{1,2}(t - s'_2)^{-\frac{1}{2}}\|u^{(n)}\| + bc_{1,3} \int_{s'_2}^t (t - s)^{-\frac{3}{4}}(s - s'_1)^{-\frac{1}{4}}C_1(r, s - s'_1)ds \\ &\quad + bc_{1,2} \int_{s'_2}^t (t - s)^{-\frac{1}{2}}(s - s'_1)^{-\frac{1}{4}}C_1(r, s - s'_1)ds \\ &\leq (t - s'_2)^{-\frac{1}{2}}C_2(r, s'_2 - s'_1, t - s'_2), \end{aligned}$$

which implies that (3.12) is true for $k = 2$ and $n \geq 1$. In addition, we suppose that (3.12) holds for $k \leq m - 1$ for some $3 \leq m \leq L$, i.e.,

$$\|D^k u^{(n)}\| \leq (t - s'_k)^{-\frac{k}{4}}C_k(r, s'_k - s'_1, t - s'_k), \dots, k = 1, 2, \dots, m - 1. \quad (3.14)$$

Combining (3.7), (3.12) and (3.14) together, applying Hausdorff–Young’s inequality, we derive that

$$\begin{aligned}
 & \|D^m u^{(n)}\| \\
 & \leq c_{1,m} (t - s'_m)^{-\frac{m}{4}} \|v^{(n)}\| + bc_{1,3} \int_{s'_m}^t (t-s)^{-\frac{3}{4}} \sum_{\sum_{i=1}^m i\beta_i = m} \Pi_{i=1}^{m-1} \|(D^i u^{(n-1)}(s))^{\beta_i}\| ds \\
 & \quad + bc_{1,2} \int_{s'_m}^t (t-s)^{-\frac{1}{2}} \sum_{\sum_{i=1}^m i\beta_i = m} \Pi_{i=1}^{m-1} \|(D^i u^{(n-1)}(s))^{\beta_i}\| ds \\
 & \leq 2rc_{1,m} (t - s'_m)^{-\frac{m}{4}} \\
 & \quad + bc_{1,3} \int_{s'_m}^t (t-s)^{-\frac{3}{4}} C_1(r, s - s'_1) \cdots C_{m-1}(r, s'_{m-1} - s'_1, s - s'_{m-1}) ds \\
 & \quad + bc_{1,2} \int_{s'_m}^t (t-s)^{-\frac{1}{2}} C_1(r, s - s'_1) \cdots C_{m-1}(r, s'_{m-1} - s'_1, s - s'_{m-1}) ds \\
 & \leq (t - s'_m)^{-\frac{m}{4}} C_m(r, s'_m - s'_1, t - s'_m),
 \end{aligned}$$

which implies that (3.12) is true for $k = m$ and $n \geq 1$. Consequently, by induction, (3.12) holds for $1 \leq k \leq L$, $n \geq 1$.

Having obtained (3.12), it is a routine matter to verify that for each $\varsigma > 0$, $D^k u^{(n)}(t, x)$ converges uniformly to $D^k u(t, x)$ on $[\varsigma, t_1] \times \mathbb{R}$ for $k = 1, 2, \dots, L-1$. Hence, we obtain $u(t, x) \in C^{1,4}([\varsigma, t_1] \times \mathbb{R})$ and since $\varsigma > 0$ can be chosen sufficiently small, we get $u(t, x) \in C^{1,4}((0, t_1] \times \mathbb{R})$. Applying the above regularity result, we can show that the solution $u(t, x)$ is indeed a smooth solution to problem (1.1) on Π_{t_1} and (3.2) is the direct consequence of (3.13). Therefore, we complete the proof. $\square$

In the following, we establish the certain $L^{\frac{N(l-1)}{3}}(\mathbb{R}^N, \mathbb{R})$ estimates on $u(t, x)$ on the time interval on which the smooth solutions exist.

Lemma 3.2 *Suppose that the solution $u(t, x)$ obtained in Lemma 3.1 has been extended up to time T ($T \geq t_1 > 0$) and the smooth properties and the a priori estimate (3.1) [and hence (3.2)] are kept unchanged. Then, for any $0 < s'_1 < \bar{s}'_1 < s'_2 < \bar{s}'_2 < \dots < s'_k < \bar{s}'_k < t \leq T$, we have*

$$\begin{aligned}
 & \|D^{\frac{3}{l-1}} u\|_{L^{\frac{N(l-1)}{3}}} \\
 & \leq C(t - \bar{s}'_k)^{-\frac{3}{4(l-1)}} (1 + \mathcal{M}_k(r, t - \bar{s}'_k, \bar{s}'_k - s'_k, t - \bar{s}'_k)) \|u(\bar{s}'_k) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}},
 \end{aligned} \tag{3.15}$$

where $k = 2, 3, \dots, N$ and $\mathcal{M}_k$ is a continuous increasing function of $t - \bar{s}'_k$.

Proof Note that

$$\begin{aligned}
 D^{\frac{3}{l-1}} u(t, x) &= D^{\frac{3}{l-1}} T(t - \bar{s}'_k) (u(\bar{s}'_k, x) - \bar{u}) - \int_{\bar{s}'_k}^t D^{\frac{3}{l-1}+1} T(t-s) Df(u(s)) ds \\
 &\quad - \int_{\bar{s}'_k}^t D^{\frac{3}{l-1}} T(t-s) Dg(u(s)) ds.
 \end{aligned}$$

Then,

$$\begin{aligned}
 & \|D^{\frac{3}{l-1}} u(t)\|_{L^{\frac{N(l-1)}{3}}} \\
 & \leq C(t - \bar{s}'_k)^{-\frac{3}{4(l-1)}} \|u(\bar{s}'_k, x) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} + bC \int_{\bar{s}'_k}^t (t-s)^{-\frac{l+2}{4(l-1)}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} ds \\
 & \quad + bC \int_{\bar{s}'_k}^t (t-s)^{-\frac{3}{4(l-1)}} \|u(s)\|_{L^{\frac{N(l-1)}{3}}} ds \\
 & \leq C(t - \bar{s}'_k)^{-\frac{3}{4(l-1)}} (1 + \mathcal{M}_k(r, t - \bar{s}'_k, \bar{s}'_k - s'_k, t - \bar{s}'_k)) \|u(\bar{s}'_k) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}.
 \end{aligned}$$

The proof of Lemma 3.2 is completed. $\square$

The following lemma plays an important role in extending the local solution step by step to a global one.

Lemma 3.3 *Suppose that the assumptions listed in Lemma 3.2 are satisfied, then there exists a positive constant $C_1(r)$ depending only on r such that*

$$\|u - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} + t^{\frac{3}{4l}} \|u - \bar{u}\|_{L^{\frac{Nl(l-1)}{3}}} \leq C_1(r) \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}, \quad 0 \leq t \leq T, \quad (3.16)$$

where the constant $C_1(r)$ is independent of T .

Proof Let

$$X = \left\{ u(t, x) : u(t, x) - \bar{u} \in C\left(0, T; L^{\frac{N(l-1)}{3}}\right), t^{\frac{3}{4l}}(u(t, x) - \bar{u}) \in C\left(0, T; L^{\frac{Nl(l-1)}{3}}\right) \right\},$$

with its norm defined by

$$\|u\|_X = \sup_{[0, T]} \{ \|u\|_{L^{\frac{N(l-1)}{3}}} + t^{\frac{3}{4l}} \|u\|_{L^{\frac{Nl(l-1)}{3}}} \}.$$

It then follows from (3.5) that

$$\begin{aligned}
 \|u - \bar{u}\|_X & \leq \|T(t)(u_0 - \bar{u})\|_X + \left\| \int_0^t D^2 T(t-s) f(u(s)) ds \right\|_X \\
 & \quad + \left\| \int_0^t DT(t-s) g(u(s)) ds \right\|_X = I_1 + I_2 + I_3.
 \end{aligned}$$

Based on Hausdorff–Young’s inequality, we deduce that

$$\begin{aligned}
 I_1 & = \sup_{[0, T]} \{ \|T(t)(u_0 - \bar{u})\|_{L^{\frac{N(l-1)}{3}}} + t^{\frac{3}{4l}} \|T(t)(u_0 - \bar{u})\|_{L^{\frac{Nl(l-1)}{3}}} \} \\
 & \leq \sup_{[0, T]} \{ \|k(t)\|_{L^1} \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} + t^{\frac{3}{4l}} \|k(t)\|_{L^{\frac{Nl(l-1)}{3}}} \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} \} \\
 & \leq \sup_{[0, T]} \left\{ \left(1 + c_l t^{\frac{3}{4l} - \frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{Nl(l-1)} \right)} \right) \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} \right\} \\
 & \leq (1 + c_l) \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}.
 \end{aligned}$$

For I_3 , by employing a similar argument, we derive that

$$\begin{aligned} I_3 &\leq \sup_{[0,T]} \left\{ \int_0^t \|DT(t-s)g(u(s))\|_{L^{\frac{N(l-1)}{3}}} ds + t^{\frac{3}{4l}} \int_0^t \|DT(t-s)g(u(s))\|_{L^{\frac{N(l-1)}{3}}} ds \right\} \\ &\leq C_2 \sup_{[0,T]} \left(\int_0^t (t-s)^{-\frac{1}{4}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^l \right. \\ &\quad \left. + t^{\frac{3}{4l}} \int_0^t (t-s)^{-\frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{N(l-1)} \right) - \frac{1}{4}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^l ds \right) \\ &\leq C_2 \sup_{[0,T]} \left(\int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} ds + t^{\frac{3}{4l}} \int_0^t (t-s)^{-\frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{N(l-1)} \right) - \frac{1}{4}} s^{-\frac{3}{4}} ds \right) \\ &\quad \|u(s) - \bar{u}\|_X^l \leq C_3 \|u - \bar{u}\|_X^l. \end{aligned}$$

As to I_2 , note that $p = \frac{2l+1}{3}$, we have

$$\begin{aligned} I_2 &\leq \sup_{[0,T]} \left\{ \int_0^t \|D^2T(t-s)f(u(s))\|_{L^{\frac{N(l-1)}{3}}} ds + t^{\frac{3}{4l}} \int_0^t \|D^2T(t-s)f(u(s))\|_{L^{\frac{N(l-1)}{3}}} ds \right\} \\ &\leq C_4 \sup_{[0,T]} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|u(s) - \bar{u}\|_{L^{\frac{Np(l-1)}{3}}}^p \right. \\ &\quad \left. + t^{\frac{3}{4l}} \int_0^t (t-s)^{-\frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{N(l-1)} \right) - \frac{1}{2}} \|u(s) - \bar{u}\|_{L^{\frac{Np(l-1)}{3}}}^p ds \right) \\ &\leq C_4 \sup_{[0,T]} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^{\frac{l-p}{l-1}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^{\frac{pl-l}{l-1}} \right. \\ &\quad \left. + t^{\frac{3}{4l}} \int_0^t (t-s)^{-\frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{N(l-1)} \right) - \frac{1}{2}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^{\frac{l-p}{l-1}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^{\frac{pl-l}{l-1}} ds \right) \\ &\leq C_4 \sup_{[0,T]} \left(\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + t^{\frac{3}{4l}} \int_0^t (t-s)^{-\frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{N(l-1)} \right) - \frac{1}{2}} s^{-\frac{1}{2}} ds \right) \\ &\quad \times \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^{\frac{l-p}{l-1}} \|u(s) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}^{\frac{pl-l}{l-1}} \\ &\leq C_4 \sup_{[0,T]} \left(\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + t^{\frac{3}{4l}} \int_0^t (t-s)^{-\frac{1}{4} \left(\frac{3}{N(l-1)} - \frac{3}{N(l-1)} \right) - \frac{1}{2}} s^{-\frac{1}{2}} ds \right) \\ &\quad \|u(s) - \bar{u}\|_X^l \leq C_5 \|u - \bar{u}\|_X^l. \end{aligned}$$

Summing up, we immediately conclude

$$\|u - \bar{u}\|_X \leq (1 + c_l) \|u - \bar{u}\|_{L^{\frac{N(l-1)}{3}}} + C \|u - \bar{u}\|_X^l.$$

If we suppose that $\|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}$ is sufficiently small, applying Strauss's inequality, we can get (3.16) immediately. The proof is completed. $\square$

Now, we prove Theorem 1.1.

Proof of Theorem 1.1 Let β be a sufficiently small positive constant. Choose $0 < s_1 < \bar{s}_1 < s_2 < \bar{s}_2 < s_3 < \dots < s_N < \bar{s}_N \leq T$ sufficiently small such that $\bar{s}_N \leq t_1$ and

$$t_1 - \bar{s}_1 = \bar{s}_i - s_i = s_i = \bar{s}_{i-1} = \bar{s}_1 = s_1 = \beta, \quad i = 2, 3, \dots, N. \quad (3.17)$$

By (3.15) and (3.16), we have

$$\begin{cases} \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq C_1(r) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}, & 0 \leq t \leq t_1, \\ \|u(t_1) - \bar{u}\|_{W^{\frac{3}{l-1}, \frac{N(q-1)}{3}}} \leq C_7(\beta, r, t_1) \sup_{[0, t_1]} \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}. \end{cases} \quad (3.18)$$

Assume that $\mathfrak{C}$ is the constant in Sobolev's embedding

$$\|u(t) - \bar{u}\|_{L^\infty} \leq \mathfrak{C} \|u(t) - \bar{u}\|_{W^{\frac{3}{l-1}, \frac{N(q-1)}{3}}},$$

if we choose $\|u_0 - \bar{u}\|_{L^1}$ sufficiently small such that

$$\mathfrak{C} C_1(r) C_7(\beta, r, t_1) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq \|u_0 - \bar{u}\|_{L^\infty}, \quad (3.19)$$

then

$$\begin{aligned} \|u(t_1) - \bar{u}\|_{L^\infty} &\leq \mathfrak{C} \|u(t_1) - \bar{u}\|_{W^{\frac{3}{l-1}, \frac{N(q-1)}{3}}} \leq \mathfrak{C} C_7(\beta, r, t_1) \sup_{[0, t_1]} \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \\ &\leq \mathfrak{C} C_1(r) C_7(\beta, r, t_1) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq \|u_0 - \bar{u}\|_{L^\infty} \leq r. \end{aligned}$$

Follows from Lemmas 3.1 and 3.3, the solution $u(t, x)$ can be extended up to $2t_1$ and satisfies

$$\begin{cases} \|u(t) - \bar{u}\|_{L^\infty} \leq 2r, & 0 \leq t \leq 2t_1, \\ \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq C_1(r) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}, & 0 \leq t \leq 2t_1. \end{cases} \quad (3.20)$$

Take $t = 2t_1$, $s'_1 = s_1 + t_1$ and $\bar{s}'_1 = \bar{s}_1 + t_1$ in (3.15) and (3.16), we derive that

$$\|u(2t_1) - \bar{u}\|_{W^{\frac{3}{l-1}, \frac{N(q-1)}{3}}} \leq C_7(\beta, r, t_1) \sup_{[0, 2t_1]} \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}. \quad (3.21)$$

Now, assume that $u(t, x)$ has been defined up to kt_1 for some $k \in \mathbb{Z}_+$ such that

$$\begin{cases} \|u(t) - \bar{u}\|_{L^\infty} \leq 2r, & 0 \leq t \leq kt_1, \\ \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq C_1(r) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}, & 0 \leq t \leq kt_1. \end{cases} \quad (3.22)$$

Take $t = kt_1$, $s'_1 = s_1 + (k-1)t_1$ and $\bar{s}'_1 = \bar{s}_1 + (k-1)t_1$ in (3.15) and (3.16), we derive that

$$\|u(kt_1) - \bar{u}\|_{W^{\frac{3}{l-1}, \frac{N(q-1)}{3}}} \leq C_7(\beta, r, t_1) \sup_{[0, kt_1]} \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}. \quad (3.23)$$

It then follows from (3.19), (3.22) and (3.23) that

$$\begin{aligned} \|u(kt_1) - \bar{u}\|_{L^\infty} &\leq \mathfrak{C} \|u(kt_1) - \bar{u}\|_{W^{\frac{3}{l-1}, \frac{N(q-1)}{3}}} \leq \mathfrak{C} C_7(\beta, r, t_1) \sup_{[0, t_1]} \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \\ &\leq \mathfrak{C} C_1(r) C_7(\beta, r, t_1) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq \|u_0 - \bar{u}\|_{L^\infty} \leq r. \end{aligned}$$

Follows from Lemmas 3.1 and 3.3 again, the solution $u(t, x)$ can be extended up to $(k+1)t_1$ and satisfies

$$\begin{cases} \|u(t) - \bar{u}\|_{L^\infty} \leq 2r, & 0 \leq t \leq (k+1)t_1, \\ \|u(t) - \bar{u}\|_{L^{\frac{N(q-1)}{3}}} \leq C_1(r) \|u_0 - \bar{u}\|_{L^{\frac{N(q-1)}{3}}}, & 0 \leq t \leq (k+1)t_1. \end{cases} \quad (3.24)$$

Proceeding inductively, we thus establish the existence of solution $u(x, t)$ in all $t > 0$. The proof is completed. $\square$

4 The brief proof of Theorem 1.2

In this section, we give a brief proof of Theorem 1.2.

Similar to the proof of Theorem 1.1, we first consider the existence and uniqueness of local smooth solutions for problem (1.5).

By using the same method as the proof of Lemma 3.1, we have

Lemma 4.1 *Suppose that $\max_{u \in \bar{B}(\bar{u}, 2r)} \sum_{k=1}^L |D^k f(u)| = b$ and the conditions listed in Theorem 1.2 are satisfied. Then, there exists a unique local solution $u(x, t)$ on the strip $\Pi_{t_1} := \{(x, t) : x \in \mathbb{R}^N, 0 < t \leq t_1\}$ such that*

$$\|u(x, t) - \bar{u}\|_{L^\infty} \leq 2r, \quad 0 \leq t \leq t_1, \quad (4.1)$$

where $t_1 = \min \left\{ 1, \frac{1}{64b^2 c_{1,2}^2} \right\}$. In addition, assume that C_k is a continuous increasing function of $t - s'_k$, then for each $0 < s'_1 < s'_2 < \dots < s'_l < t \leq t_1$, we have

$$\|D^k u(t, x)\|_{L^\infty} \leq (t - s'_k)^{-\frac{k}{4}} C_k(r, s'_k - s'_1, t - s'_k), \quad k = 1, 2, \dots, L. \quad (4.2)$$

It is worth pointing that the integro-differential equation of (1.5) is

$$u(t, x) = \int k(t, x - y) u_0 dy - \int_0^t ds \int \Delta k(x - y, t - s) f(u(s, y)) dy. \quad (4.3)$$

and the main difference between the proof of Lemma 3.1 and the proof of Lemma 4.1

is the choice of t_1 . In Lemma 4.1, we assume that $t_1 = \min \left\{ 1, \frac{1}{64b^2 c_{1,2}^2} \right\}$ whereas $t_1 =$

$\min \left\{ 1, \frac{1}{64b^2 c_{1,2}^2}, \left(\frac{3}{16bc_{1,1}} \right)^{\frac{4}{3}} \right\}$ in Lemma 3.1.

The certain $L^{\frac{N(l-1)}{2}}(\mathbb{R}^N, \mathbb{R})$ estimates on $u(t, x)$ on the time interval can be established similar to Lemma 3.2.

Lemma 4.2 *Suppose that the solution $u(t, x)$ obtained in Lemma 4.1 has been extended up to time T ($T \geq t_1 > 0$) and the smooth properties and the a priori estimate (4.1) (and hence (4.2)) are kept unchanged. Then, for any $0 < s'_1 < \bar{s}'_1 < s'_2 < \bar{s}'_2 < \dots < s'_k < \bar{s}'_k < t \leq T$,*

$$\begin{aligned} & \|D^{\frac{2}{l-1}} u\|_{L^{\frac{N(l-1)}{2}}} \\ & \leq C(t - \bar{s}'_k)^{-\frac{1}{2(l-1)}} (1 + \mathcal{M}_k(r, t - \bar{s}'_k, \bar{s}'_k - s'_k, t - \bar{s}'_k)) \|u(\bar{s}'_k) - \bar{u}\|_{L^{\frac{N(l-1)}{3}}}. \end{aligned} \quad (4.4)$$

where $k = 2, 3, \dots, N$ and $\mathcal{M}_k$ is a continuous increasing function of $t - \bar{s}'_k$.

We also have the following lemma, which is concerned with the time-independent $L^{\frac{N(l-1)}{2}}(\mathbb{R}, \mathbb{R})$ a priori estimate on the solution $u(t, x)$ of problem (1.5).

Lemma 4.3 *Suppose that the assumptions listed in Lemma 4.2 are satisfied, then there exists a positive constant $C_2(r)$ depending only on r such that*

$$\|u - \bar{u}\|_{L^{\frac{N(l-1)}{2}}} + t^{\frac{1}{2l}} \|u - \bar{u}\|_{L^{\frac{N(l-1)}{2}}} \leq C_2(r) \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{2}}}, \quad 0 \leq t \leq T, \quad (4.5)$$

where the constant $C_1(r)$ is independent of T .

Proof Let

$$X = \left\{ u(t, x) : u(t, x) - \bar{u} \in C \left(0, T; L^{\frac{N(l-1)}{2}} \right), t^{\frac{1}{2l}} (u(t, x) - \bar{u}) \in C \left(0, T; L^{\frac{Nl(l-1)}{2}} \right) \right\},$$

with its norm defined by

$$\|u\|_X = \sup_{[0, T]} \left\{ \|u\|_{L^{\frac{N(l-1)}{2}}} + t^{\frac{1}{2l}} \|u\|_{L^{\frac{Nl(l-1)}{2}}} \right\}.$$

Then,

$$\|u - \bar{u}\|_X \leq \|T(t)(u_0 - \bar{u})\|_X + \left\| \int_0^t D^2 T(t-s) f(u(s)) ds \right\|_X = I_1 + I_2.$$

Note that

$$\begin{aligned} I_1 &= \sup_{[0, T]} \left\{ \|T(t)(u_0 - \bar{u})\|_{L^{\frac{N(l-1)}{2}}} + t^{\frac{1}{2l}} \|T(t)(u_0 - \bar{u})\|_{L^{\frac{Nl(l-1)}{2}}} \right\} \\ &\leq \sup_{[0, T]} \left\{ \|k(t)\|_{L^1} \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{2}}} + t^{\frac{1}{2l}} \|k(t)\|_{L^{\frac{Nl(l-1)}{2}}} \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{2}}} \right\} \\ &\leq \sup_{[0, T]} \left\{ \left(1 + c_l t^{\frac{1}{2l} - \frac{N}{4} \left(\frac{2}{N(l-1)} - \frac{2}{Nl(l-1)} \right)} \right) \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{2}}} \right\} \leq (1 + c_l) \|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{2}}}. \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq \sup_{[0, T]} \left\{ \int_0^t \|D^2 T(t-s) f(u(s))\|_{L^{\frac{N(l-1)}{2}}} ds + t^{\frac{1}{2l}} \int_0^t \|D^2 T(t-s) f(u(s))\|_{L^{\frac{Nl(l-1)}{2}}} ds \right\} \\ &\leq C_2 \sup_{[0, T]} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|u(s) - \bar{u}\|_{L^{\frac{Nl(l-1)}{2}}}^l \right. \\ &\quad \left. + t^{\frac{1}{2l}} \int_0^t (t-s)^{-\frac{N}{4} \left(\frac{2}{N(l-1)} - \frac{2}{Nl(l-1)} \right) - \frac{1}{2}} \|u(s) - \bar{u}\|_{L^{\frac{Nl(l-1)}{2}}}^l ds \right) \\ &\leq C_2 \sup_{[0, T]} \left(\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + t^{\frac{1}{2l}} \int_0^t (t-s)^{-\frac{N}{4} \left(\frac{2}{N(l-1)} - \frac{2}{Nl(l-1)} \right) - \frac{1}{2}} s^{-\frac{1}{2}} ds \right) \\ &\quad \|u(s) - \bar{u}\|_X^l \leq C_3 \|u - \bar{u}\|_X^l. \end{aligned}$$

Summing up, we immediately conclude

$$\|u - \bar{u}\|_X \leq (1 + c_l) \|u - \bar{u}\|_{L^{\frac{N(l-1)}{2}}} + C \|u - \bar{u}\|_X^l.$$

If we suppose that $\|u_0 - \bar{u}\|_{L^{\frac{N(l-1)}{2}}}$ is sufficiently small, applying Strauss's inequality, we can get (4.5) immediately. $\square$

The proof of Theorem 1.2 is in the following.

Proof of Theorem 1.2 With the above preparations in hand, proceeding inductively, we thus establish the existence of the solution $u(x, t)$ for problem (1.5) in all $t > 0$. Hence, the proof of Theorem 1.1 is completed. $\square$

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