Spectral multipliers on exponentially locally doubling metric measure spaces

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Abstract Let (X, ρ) be a geodesic space endowed with a positive Borel measure μ which satisfies an exponentially locally doubling condition. Assume that L is a nonnegative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel obeys a local Gaussian upper bound. In this paper, we prove that if Φ is a bounded even holomorphic function in a suitable strip of the complex plane, and satisfies the Mihlin-type condition of appropriate order at infinity, then the operator $\Phi(\sqrt{L})$ extends to an operator bounded on $L^p(X, d\mu)$ for $1 < p < \infty$ and of weak type $(1, 1)$. This partially extends some existing results concerning spherical multipliers on symmetric spaces of noncompact type and spectral multipliers on Riemannian manifolds with bounded geometry.

Keywords Spectral multipliers · Metric measure spaces · Heat kernel · Functional calculus

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1 Introduction and statement of main result

Let K be a tempered distribution on $\mathbb{R}^n$ and let $m = \widehat{K}$ be its Fourier transform. The classical Hörmander–Mihlin multiplier theorem asserts that $Tf = f * K$ is a bounded operator on $L^p(\mathbb{R}^n)$ provided that m satisfies certain symbol-type estimates, e.g.

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^i |\nabla^i m(\xi)| < +\infty \quad \left(i = 0, 1, \dots, \left[\left| \frac{1}{p} - \frac{1}{2} \right| n \right] + 1 \right).$$

This theorem has been generalized to various geometrical contexts, including symmetric spaces of noncompact type and Riemannian manifolds with bounded geometry. In the setting of noncompact symmetric spaces, the pioneer work was done by Clerc and Stein. In

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their important paper [7], they obtained a result concerning the L^p -continuity of spherical multipliers on a noncompact symmetric space G/K when G is a complex semisimple Lie group. Moreover, they were the first to point out the necessity of having the multiplier m holomorphic on a strip. Subsequently, similar results were obtained by Stanton and Tomas [28] when G has real rank one, and by Anker and Lohoué [4] when G is a normal real form. For related results on general noncompact symmetric spaces, we refer to [2, 3, 19, 20, 25] and the references therein.

The study of spectral multipliers on Riemannian manifolds with bounded geometry was begun by Cheeger et al. [6]. To recall their result, we need some notions. Let M be a Riemannian manifold. We say that M has C^∞ bounded geometry, if the injectivity radius of M is positive and the Riemann curvature tensor is bounded in the C^∞ topology. Obviously, if M has C^∞ bounded geometry, then $\text{Ric}(M) \geq -(n-1)K^2$ for some $K > 0$, and hence by the Bishop comparison theorem (see [5, §11.10] or [16, §2.1]) we have

$$\text{vol}(B(x, r)) \leq Cr^n e^{(n-1)Kr}$$

for all $r \in [1, \infty)$ and all $x \in X$, where $\text{vol}(B(x, r))$ denotes the Riemannian volume of the geodesic ball with centre x and radius r . For each $W \in \mathbb{R}_+$, denote by Ω_W the strip $\{z \in \mathbb{C} : \text{Im } z \in (-W, W)\}$. Cheeger et al. [6, Theorem 3.4] proved that if (i) M has C^∞ bounded geometry (which implies that $\text{Ric}(M) \geq -(n-1)K^2$ for some $K > 0$), and (ii) Φ is a bounded even holomorphic function in Ω_W for some $W > (n-1)K$, satisfying the Mihlin-type estimates

$$|\Phi^{(j)}(\zeta)| \leq C(1 + |\zeta|)^{-j} \quad (1.1)$$

for all $\zeta \in \Omega_W$ and all $j \in \{0, 1, 2, \dots, J\}$, where J is a sufficiently large integer depending on the dimension n of M , then the operator $\Phi(\sqrt{-\Delta})$ is bounded on $L^p(M)$ for $1 < p < \infty$. Here, Δ denotes the Laplace–Beltrami operator on M . Subsequently, Taylor [32, Theorem 1.6] improved this result. He proved the L^p -boundedness of $\Phi(\sqrt{-\Delta - b})$, where b is the bottom of the spectrum of $-\Delta$, under the assumption that Φ is holomorphic in a strip Ω_W of width $W \geq |1/p - 1/2|(n-1)K$. Note that the proofs of these results require control only of a finite number of covariant derivatives of the Riemann tensor, but this number is of the same order of magnitude of the dimension of M .

Recently, Mauceri, Meda and Vallarino [24] produced an important advance in the study of spectral multipliers on Riemannian manifolds. They proved that $\Phi(\sqrt{-\Delta - b})$ is bounded on $L^p(M)$ and is of weak type $(1, 1)$ provided that the following conditions are satisfied: (i) M is a Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius; (ii) there exist constants $\rho > 1/2$ and C such that

$$\|\mathcal{H}^t\|_{1,\infty} \leq Ce^{-bt}t^{-\rho} \quad \forall t \in [1, \infty), \quad (1.2)$$

where $\{\mathcal{H}^t\}$ denotes the heat semigroup generated by $-\Delta$, and $\|\mathcal{H}^t\|_{1,\infty}$ denotes the $L^1(M) \rightarrow L^\infty(M)$ norm of the operator $\mathcal{H}^t$; (iii) Φ is a bounded even holomorphic function in Ω_W with $W \geq (n-1)K/2$ and satisfies the Mihlin-type estimates (1.1). Note that by using Stein's interpolation theorem, the width of the strip in the last condition can be improved to $|1/p - 1/2|(n-1)K$ (see [32, p. 778]). Condition (ii) is known to be satisfied by non-amenable unimodular Lie groups with a left invariant Riemannian metric and noncompact Riemannian symmetric spaces. The result of [24] was subsequently sharpened by Taylor [33], who removed the condition (ii).

The purpose of the current paper is to (partially) generalize the above-mentioned results to the setting of metric measure spaces. More precisely, we consider the following setting. Let X be a set endowed with a metric ρ . We assume that (X, ρ) is a *geodesic space*, that

is, for every pair of points $x, y \in X$ there exists a geodesic $\sigma : [0, 1] \rightarrow X$ joining x to y . Assume further that (X, ρ) is endowed with a positive Borel measure μ such that the following *exponentially locally doubling condition* is valid:

$$0 < \mu(B(x, \lambda r)) \leq A\lambda^d e^{\tau(\lambda-1)r} \mu(B(x, r)) < \infty \quad (1.3)$$

for all $x \in X$, $r > 0$ and $\lambda \geq 1$, where $B(x, r)$ is the open ball centred at x with radius r , and $A \geq 1$, $d > 0$ and $\tau > 0$ are constants. Consider a nonnegative self-adjoint operator L acting on $L^2(X, d\mu)$. We assume that the associated semigroup $P_t = e^{-tL}$ consists of integral operators with (heat) kernel $p(t, x, y)$ obeying the *local Gaussian upper bound*:

$$|p(t, x, y)| \leq \frac{C^*}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \exp\left(-\frac{c^* \rho^2(x, y)}{t} + \kappa t\right) \quad (1.4)$$

for all $x, y \in M$ and $t > 0$, where C^* , c^* are positive constants and $0 < \kappa < 1$.

Given a bounded Borel measurable function Φ on $[0, \infty)$, one can define the operator $\Phi(\sqrt{L})$ by the spectral theorem, according to the prescription

$$\Phi(\sqrt{L}) = \int_0^\infty \Phi(\sqrt{\lambda}) dE_\lambda,$$

where dE_λ is the projection valued measure associated with L . The purpose of this paper is to study the action of the operator $\Phi(\sqrt{L})$ on the spaces $L^p(X, d\mu)$. Our main result can be stated as follows.

Theorem 1.1 Suppose $W \geq \frac{\tau}{4\sqrt{c^*}}$ and Φ is a bounded even holomorphic function in $\Omega_W := \{\zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < W\}$, satisfying the Mihlin-type condition:

$$|\Phi^{(j)}(\zeta)| \leq C(1 + |\zeta|)^{-j} \quad (1.5)$$

for all $\zeta \in \Omega_W$ and all $j \in \{0, 1, 2, \dots, J\}$, where $J := [d/2] + 4$. Then, $\Phi(\sqrt{L})$ is bounded on $L^p(X, d\mu)$ for $1 < p < \infty$, and is of weak type $(1, 1)$.

Remark 1.2 By using Stein's interpolation theorem, the condition $W \geq \frac{\tau}{4\sqrt{c^*}}$ about the width of the strip can be improved to $W \geq \left|\frac{1}{p} - \frac{1}{2}\right| \frac{\tau}{2\sqrt{c^*}}$; see [32, p. 778].

By the Bishop comparison theorem (see [5, §11.10] or [16, §2.1]), all Riemannian manifolds with Ricci curvature bounded from below satisfy the exponentially locally doubling condition (1.3), and it is well known from Li and Yau [22, Corollary 3.1] (see also [26, §4.2]) that the heat kernel associated with the Laplace–Beltrami operator on such a manifold satisfies the local Gaussian upper estimate (1.4). Hence, Theorem 1.1 can be applied to arbitrary complete connected Riemannian manifolds with Ricci curvature bounded from below, without assuming the positivity of the injectivity radius. Thus, our result partially extends the existing results concerning spectral multipliers on Riemannian manifolds with bounded geometry. More generally, our result can be applied to the so-called $RCD^*(K, N)$ spaces introduced by Erbar et al. [14] and Ambrosio et al. [1]. For more details and explanations of the applications of our result, see Sect. 5.

It should be mentioned that we only assume that the heat kernel of the operator L obeys an upper bound, while the arguments in [24, 32, 33] heavily rely on the regularity of the heat kernel and the differential structure of Riemannian manifolds.

It is also worth pointing out that when (X, ρ, μ) is a doubling metric measure space and L is a nonnegative self-adjoint operator whose heat kernel satisfies global Gaussian estimates, the spectral multipliers associated with L has been studied in [11–13] and others.

Let us describe the main idea of the proof of Theorem 1.1. Following [32], we shall decompose the Schwartz kernel of the operator $\Phi(\sqrt{L})$ as the sum of a kernel with support near the diagonal in $X \times X$ and of a kernel supported off the diagonal. To do this, we need to use the fact that in our setting the operator L satisfies the finite speed propagation property, which will be proved in Lemma 2.5. To treat the part near the diagonal, we will use a localized variant of the method from [10, 11, 27]. Note that this method is originated from an idea of Hebisch [17]. To treat the part off the diagonal we will use similar estimates as in [24] and [33]. However, since the geometrical and analytical assumptions of our setting are much weaker, some essential refinements will be needed.

We close this section by making some conventions. Throughout this paper, we use the notation $L^p(X) := L^p(X, d\mu)$, $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. We denote by $\|T\|_{p \rightarrow q}$ the norm of a bounded operator $T : L^p(X) \rightarrow L^q(X)$. We also use the “variable constant convention”, that is, we denote by C and c constants that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. $a \sim b$ means that the variables a and b are comparable, that is, there exists a positive constant C such that $C^{-1}b \leq a \leq Cb$.

2 Preliminaries

In this section, we collect some facts and technical results that will be needed in the sequel. We start by noting that (1.3) readily implies that for any $R_0 > 0$ there exists a constant $C = C(R_0)$ such that

$$\mu(B(x, R)) \leq C \left(\frac{R}{r}\right)^d \mu(B(x, r)) \quad (2.1)$$

for all $x \in X$ and all r, R such that $0 < r \leq R \leq R_0$. Since $B(x, r) \subset B(y, \rho(x, y) + r)$, (1.3) also yields that

$$\mu(B(x, r)) \leq C \left(1 + \frac{\rho(x, y)}{r}\right)^d e^{\tau\rho(x, y)} \mu(B(y, r)) \quad (2.2)$$

for all $x, y \in X$ and $r > 0$. In particular, for any $R_0 > 0$ there exists a constant $C = C(R_0)$ such that if $\rho(x, y) \leq R_0$ then

$$\mu(B(x, r)) \leq C \left(1 + \frac{\rho(x, y)}{r}\right)^d \mu(B(y, r)). \quad (2.3)$$

It is also obvious that (1.4) implies that for any $t_0 > 0$ there exists a constant $C = C(t_0)$ such that

$$|p(t, x, y)| \leq \frac{C}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \exp\left(-\frac{c\rho^2(x, y)}{t}\right) \quad (2.4)$$

for all $x, y \in M$ and $t \in (0, t_0]$. The following lemma gives an estimate of the L^2 -norm of $p(t, \cdot, y)$ for all $t \in (0, \infty)$.

Lemma 2.1 For any $t_0 > 0$, there exists a constant $C = C(t_0)$ such that

$$\int_X |p(t, x, y)|^2 d\mu(x) \leq \begin{cases} C\mu(B(y, \sqrt{t}))^{-1}, & 0 < t \leq t_0 \\ Ce^{2\kappa t}\mu(B(y, \sqrt{2t}))^{-1}, & t \geq t_0. \end{cases}$$

Proof First, we consider the case where $0 < t \leq t_0$. In this case, by (2.4), (2.2) and (1.3), we have

$$\begin{aligned} & \int_X |p(t, x, y)|^2 d\mu(x) \\ & \leq \int_X \frac{C}{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))} \exp\left(-\frac{c\rho^2(x, y)}{t}\right) d\mu(x) \\ & \leq \frac{C}{\mu(B(y, \sqrt{t}))^2} \int_X \left(1 + \frac{\rho(x, y)}{\sqrt{t}}\right)^d \exp\left(\tau\rho(x, y) - \frac{c\rho^2(x, y)}{t}\right) d\mu(x) \\ & \leq \frac{C}{\mu(B(y, \sqrt{t}))^2} \int_X \exp\left(-\frac{c'\rho^2(x, y)}{t}\right) d\mu(x) \\ & = \frac{C}{\mu(B(y, \sqrt{t}))^2} \sum_{j=0}^{\infty} \int_{\{x: j\sqrt{t} \leq \rho(x, y) \leq (j+1)\sqrt{t}\}} \exp\left(-\frac{c'\rho^2(x, y)}{t}\right) d\mu(x) \\ & \leq \frac{C}{\mu(B(y, \sqrt{t}))^2} \sum_{j=0}^{\infty} \mu(B(y, (j+1)\sqrt{t})) e^{-c'j^2} \\ & \leq \frac{C}{\mu(B(y, \sqrt{t}))} \sum_{j=0}^{\infty} (j+1)^d e^{\tau j\sqrt{t}} e^{-c'j^2} \\ & \leq \frac{C}{\mu(B(y, \sqrt{t}))} \sum_{j=0}^{\infty} e^{\tau j\sqrt{t}} e^{-c''j^2} \\ & = \frac{C}{\mu(B(y, \sqrt{t}))} \sum_{j=0}^{\infty} e^{-c''[j - (\tau\sqrt{t})/(2c'')]^2} e^{(\tau^2 t)/(4c'')} \\ & \leq C\mu(B(y, \sqrt{t}))^{-1}. \end{aligned}$$

Next, we consider the case where $t \geq t_0$. In this case, it follows from (1.4) that

$$\begin{aligned} \int_X |p(t, x, y)|^2 d\mu(x) &= \int_X p(t, x, y) \overline{p(t, x, y)} d\mu(x) = \int_X p(t, x, y) p(t, y, x) d\mu(x) \\ &= p(2t, y, y) \leq Ce^{2\kappa t} \mu(B(y, \sqrt{2t}))^{-1} \leq Ce^{2\kappa t} \mu(B(y, \sqrt{t}))^{-1}. \end{aligned}$$

The proof of the lemma is complete. $\square$

A nonnegative self-adjoint operator L is said to satisfy the *Davies–Gaffney estimate*, if there exists a positive constant c_1 such that

$$|\langle e^{-tL} f_1, f_2 \rangle| \leq \exp\left(-\frac{c_1 r^2}{t}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}, \quad t > 0, \quad (2.5)$$

for all open sets $U_j \subset X$ and $f_j \in L^2(X)$ with $\text{supp } f_j \subset U_j$, $j = 1, 2$, where $r = \rho(U_1, U_2)$. A nonnegative self-adjoint operator L is said to satisfy the *finite speed propagation*

property (for the solution of the corresponding wave equation), if there exists a positive constant c_2 such that

$$\langle \cos(t\sqrt{L})f_1, f_2 \rangle = 0, \quad 0 < c_2 t < r, \quad (2.6)$$

for all open sets $U_i \in X$ and all $f_i \in L^2(X)$ with $\text{supp } f_i \subset U_i$, $i = 1, 2$, where $r = \rho(U_1, U_2)$. The following result concerns the relationship between the Davies–Gaffney estimate and the finite speed propagation property. This has been proved in [8, Theorem 3.4] in the setting of general metric measure spaces and general nonnegative self-adjoint operators.

Lemma 2.2 *A nonnegative self-adjoint operator L satisfies (2.5) if and only if it satisfies (2.6) with $c_2 = \frac{1}{2\sqrt{c_1}}$.*

The following two lemmas are consequences of the Phragmén–Lindelöf theorem. These are [8, Proposition 2.2] and [9, Lemma 9], respectively.

Lemma 2.3 *Let F be an analytic function in $\mathbb{C}_+$. Assume that there exist constants $B, K, \gamma > 0$ and $\sigma \geq 0$ such that*

$$|F(z)| \leq B$$

for all $z \in \mathbb{C}_+$, and

$$|F(t)| \leq K e^{\sigma t} e^{-\gamma/t}$$

for all $t \in \mathbb{R}_+$. Then

$$|F(z)| \leq B \exp\left(-\operatorname{Re} \frac{\gamma}{z}\right)$$

for all $z \in \mathbb{C}_+$.

Lemma 2.4 *Let F be a holomorphic function in $\mathbb{C}_+$. Assume that there exist constants $a, b > 0$, $\beta \geq 0$ and $\alpha \in (0, 1]$ such that*

$$|F(z)| \leq a(\operatorname{Re} z)^{-\beta}$$

for all $z \in \mathbb{C}_+$, and

$$|F(t)| \leq a t^{-\beta} e^{-bt^{-\alpha}}$$

for all $t \in \mathbb{R}_+$. Then

$$|F(z)| \leq a 2^\beta (\operatorname{Re} z)^{-\beta} \exp\left(-\frac{b\alpha}{2} \frac{\operatorname{Re} z}{|z|^{\alpha+1}}\right)$$

for all $z \in \mathbb{C}_+$.

It is well known that if (X, ρ, μ) satisfies the global doubling condition and L satisfies the global Gaussian upper bound estimate, then L satisfies the finite speed propagation property. In the next lemma, we show that in our setting the operator L also satisfies the finite speed propagation property.

Lemma 2.5 *Assume that (X, ρ, μ) is a metric measure space satisfying (1.3) and L is a nonnegative self-adjoint operator on $L^2(X)$ with heat kernel satisfying (1.4). Then*

$$\langle \cos(t\sqrt{L})f_1, f_2 \rangle = 0, \quad 0 < \frac{1}{2\sqrt{c^*}}t < r, \quad (2.7)$$

for all open sets $U_i \in X$ and all $f_i \in L^2(X)$ with $\text{supp } f_i \subset U_i$, $i = 1, 2$, where $r = \rho(U_1, U_2)$ and c^* is a constant from (1.4). Informally, we can write (2.7) as

$$\text{supp } K_{\cos(t\sqrt{L})}(\cdot, \cdot) \subset \left\{ (x, y) \in X \times X : \rho(x, y) \leq \frac{1}{2\sqrt{c^*}}t \right\}. \quad (2.8)$$

Proof By Lemma 2.2, the assertion (2.7) is equivalent to that

$$|\langle e^{-tL} f_1, f_2 \rangle| \leq \exp\left(-\frac{c^* r^2}{t}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}, \quad t \in \mathbb{R}_+, \quad (2.9)$$

for all open sets $U_j \subset X$ and $f_j \in L^2(X)$ with $\text{supp } f_j \subset U_j$, $j = 1, 2$, where $r = \rho(U_1, U_2)$. Hence, the proof will be complete once (2.9) is proved. As shown in [8, Lemma 3.1] it suffices to prove (2.9) only in the case when U_1 and U_2 are balls. Let $B_j = B(x_j, r_j)$ and $f_j \in L^2(X)$ with $\text{supp } f_j \subset B_j$, $j = 1, 2$. Write $r := \rho(B_1, B_2)$.

Since L is a positive self-adjoint operator, the semigroup $\{e^{-zL} : z \in \mathbb{C}_+\}$ is holomorphic in $\mathbb{C}_+$ and $\|e^{-zL}\|_{2 \rightarrow 2} \leq 1$. Therefore, the function $F(z) := \langle e^{-zL} f_1, f_2 \rangle$ is holomorphic in $\mathbb{C}_+$ and

$$|F(z)| \leq \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)} \quad \forall z \in \mathbb{C}_+. \quad (2.10)$$

To estimate $|F(t)|$ for $t \in \mathbb{R}_+$, we divide our argument into two cases.

Case 1 $0 < t \leq 1$. In this case, using (1.4) we infer

$$\begin{aligned} |\langle e^{-tL} f_1, f_2 \rangle| &\leq \int_{B_1} \int_{B_2} |p(t, x, y)| |f_1(x)| |f_2(y)| d\mu(x) d\mu(y) \\ &\leq C^* e^{-\frac{c^* r^2}{t}} \int_{B_1} \frac{f_1(x)}{\mu(B(x, \sqrt{t}))^{1/2}} d\mu(x) \int_{B_2} \frac{f_2(y)}{\mu(B(y, \sqrt{t}))^{1/2}} d\mu(y) \\ &\leq C^* e^{-\frac{c^* r^2}{t}} \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)} \\ &\quad \times \left(\int_{B_1} \mu(B(x, \sqrt{t}))^{-1} d\mu(x) \right)^{1/2} \left(\int_{B_2} \mu(B(y, \sqrt{t}))^{-1} d\mu(y) \right)^{1/2}. \end{aligned}$$

Let $\varepsilon \in (0, c^* r^2)$. If $\sqrt{t} \leq r_1$ and $x \in B_1 = B(x_1, r_1)$, then $B(x_1, r_1) \subset B(x, 2r_1)$, and hence using (1.3) we obtain

$$\begin{aligned} \mu(B(x_1, r_1)) &\leq \mu(B(x, 2r_1)) \leq A \left(\frac{2r_1}{\sqrt{t}} \right)^d e^{\tau(\frac{2r_1}{\sqrt{t}}-1)\sqrt{t}} \mu(B(x, \sqrt{t})) \\ &\leq A \left(\frac{2r_1}{\sqrt{t}} \right)^d e^{2\tau r_1} \mu(B(x, \sqrt{t})) \leq C e^{\frac{\varepsilon}{t}} \mu(B(x, \sqrt{t})), \end{aligned}$$

where the last constant C depends on $\varepsilon, r_1, d, \tau, A$, but not on t . From this, we infer

$$\int_{B_1} \mu(B(x, \sqrt{t}))^{-1} d\mu(x) \leq C e^{\frac{\varepsilon}{t}}. \quad (2.11)$$

If $\sqrt{t} > r_1$ and $x \in B_1 = B(x_1, r_1)$, then $B(x_1, r_1) \subset B(x, 2\sqrt{t})$ and hence

$$\mu(B(x_1, r_1)) \leq \mu(B(x, 2\sqrt{t})) \leq A 2^d e^{\tau\sqrt{t}} \mu(B(x, \sqrt{t})) \leq C \mu(B(x, \sqrt{t})),$$

implying $\int_{B_1} \mu(B(x, \sqrt{t}))^{-1} d\mu(x) \leq C \leq C e^{\frac{\varepsilon}{t}}$. Therefore, the estimate (2.11) holds for all $t \in (0, 1]$.

The same argument as above also shows that for any $\varepsilon \in (0, c^*r^2)$ there exists a positive constant C depending on $\varepsilon, r_2, d, \tau, A$ such that $\int_{B_2} \mu(B(y, \sqrt{t}))^{-1} d\mu(y) \leq Ce^{\frac{\varepsilon}{t}}$ for all $t \in (0, 1]$.

Combining the above estimates together, we conclude that for any $\varepsilon \in (0, c^*r^2)$ there exists a positive constant K_1 depending on $\varepsilon, r_1, r_2, d, \tau, A$ such that for all $t \in (0, 1]$,

$$|F(t)| = |\langle e^{-tL} f_1, f_2 \rangle| \leq K_1 e^{-\frac{c^*r^2}{t}} e^{\frac{\varepsilon}{t}} \leq K_1 e^{(c^*r^2 - \varepsilon)t} e^{-\frac{c^*r^2 - \varepsilon}{t}}. \quad (2.12)$$

Case 2 $t > 1$. In this case, let $B := \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}$, then clearly

$$|F(t)| \leq B \leq B e^{(c^*r^2 - \varepsilon)t} e^{-\frac{c^*r^2 - \varepsilon}{t}}. \quad (2.13)$$

Now let $K := \max\{K_1, B\}$, $\sigma := c^*r^2 - \varepsilon$ and $\gamma := c^*r^2 - \varepsilon$. Then, combining (2.12) and (2.13) we see that for all $t \in \mathbb{R}_+$ there holds $|F(t)| \leq K e^{\sigma t} e^{-\frac{\gamma}{t}}$. Finally, from this and (2.10), employing Lemma 2.3 we conclude that for all $\varepsilon \in (0, c^*r^2)$ and all $t \in \mathbb{R}_+$,

$$|F(t)| \leq \exp\left(-\frac{c^*r^2 - \varepsilon}{t}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}.$$

Letting $\varepsilon \rightarrow 0$ yields (2.9) and completes the proof. $\square$

3 Functional calculus induced by the heat kernel

In this section, we present functional calculus of operators of the form $F(\sqrt{L})$ associated with even functions $F \in W_s^\infty(\mathbb{R})$, which is a local version of the corresponding result from [11].

We first give an estimate for the heat kernel $p(z, x, y)$ with complex time $z \in \mathbb{C}_+$ such that $0 < \operatorname{Re} z \leq \lambda_0$.

Lemma 3.1 *For any $\lambda_0 > 0$, there exist constants $C = C(\lambda_0)$ and $c = c(\lambda_0)$ such that for all $z \in \mathbb{C}_+$ with $0 < \operatorname{Re} z \leq \lambda_0$ and all $x, y \in X$,*

$$|p(z, x, y)| \leq \frac{C}{\sqrt{\mu(B(x, \sqrt{\operatorname{Re} z}))\mu(B(y, \sqrt{\operatorname{Re} z}))}} \exp\left(-\frac{c\rho^2(x, y)\operatorname{Re} z}{|z|^2}\right). \quad (3.1)$$

Proof To save space, we denote

$$A(x, y; s) := \frac{1}{\sqrt{\mu(B(x, s))\mu(B(y, s))}}$$

for all $x, y \in X$ and $s > 0$. Fix $\lambda_0 > 0$ and take arbitrary $\lambda \in (0, \lambda_0]$.

If $z \in \mathbb{C}$ such that $\operatorname{Re} z \in (0, \lambda]$, then by (2.4) and (2.1) we have

$$\begin{aligned}
 |p(z, x, y)| &= \left| \int_X p(z/2, x, w) p(z/2, w, y) d\mu(w) \right| \\
 &\leq \left(\int_X |p(z/2, x, w)|^2 d\mu(w) \right)^{1/2} \left(\int_X |p(z/2, w, y)|^2 d\mu(w) \right)^{1/2} \\
 &= \left(\int_X p(z/2, x, w) p(\bar{z}/2, w, x) d\mu(w) \right)^{1/2} \\
 &\quad \times \left(\int_X p(z/2, w, y) p(\bar{z}/2, y, w) d\mu(w) \right)^{1/2} \\
 &= \sqrt{p(\operatorname{Re} z, x, x) p(\operatorname{Re} z, y, y)} \\
 &\leq CA(x, y; \sqrt{\operatorname{Re} z}) \\
 &\leq CA(x, y; \sqrt{\lambda}) \left(\frac{\lambda}{\operatorname{Re} z} \right)^{d/2}.
 \end{aligned} \tag{3.2}$$

If $z \in \mathbb{C}$ such that $\operatorname{Re} z \in (\lambda, \infty)$, then by (1.4) we have

$$\begin{aligned}
 |p(z, x, y)| &\leq \sqrt{p(\operatorname{Re} z, x, x) p(\operatorname{Re} z, y, y)} \\
 &\leq CA(x, y; \sqrt{\operatorname{Re} z}) e^{\kappa \operatorname{Re} z} \\
 &\leq CA(x, y; \sqrt{\lambda}) \left(\frac{\lambda}{\operatorname{Re} z} \right)^{d/2} e^{\frac{(\kappa+1)\operatorname{Re} z}{\lambda}},
 \end{aligned} \tag{3.3}$$

where for the last inequality we used the fundamental inequality $\left(\frac{\operatorname{Re} z}{\lambda}\right)^{d/2} \leq Ce^{\frac{\operatorname{Re} z}{\lambda}}$. Combing (3.2) and (3.3) we see that for all $z \in \mathbb{C}_+$,

$$|p(z, x, y)| \leq CA(x, y; \sqrt{\lambda}) \left(\frac{\lambda}{\operatorname{Re} z} \right)^{d/2} e^{\frac{(\kappa+1)\operatorname{Re} z}{\lambda}}. \tag{3.4}$$

If $t \in (0, \lambda]$, then it follows from (2.4) and (2.1) that

$$\begin{aligned}
 |p(t, x, y)| &\leq CA(x, y; \sqrt{t}) \exp\left(-\frac{c\rho^2(x, y)}{t}\right) \\
 &\leq CA(x, y; \sqrt{\lambda}) \exp\left(-\frac{c\rho^2(x, y)}{t}\right) \left(\frac{\lambda}{t}\right)^{d/2}.
 \end{aligned} \tag{3.5}$$

If $t \in (\lambda, \infty)$, then by (1.4) we have

$$\begin{aligned}
 |p(t, x, y)| &\leq CA(x, y; \sqrt{t}) \exp\left(-\frac{c\rho^2(x, y)}{t} + \kappa t\right) \\
 &\leq CA(x, y; \sqrt{\lambda}) \exp\left(-\frac{c\rho^2(x, y)}{t}\right) \left(\frac{\lambda}{t}\right)^{d/2} e^{\frac{(\kappa+1)t}{\lambda}},
 \end{aligned} \tag{3.6}$$

where for the last inequality we used the fundamental inequality $\left(\frac{t}{\lambda}\right)^{d/2} \leq Ce^{\frac{t}{\lambda}}$. Combing (3.5) and (3.6) we see that for all $t \in \mathbb{R}_+$,

$$|p(t, x, y)| \leq CA(x, y; \sqrt{\lambda}) \exp\left(-\frac{c\rho^2(x, y)}{t}\right) \left(\frac{\lambda}{t}\right)^{d/2} e^{\frac{(\kappa+1)t}{\lambda}}. \tag{3.7}$$

Now, define $F(z) = p(z, x, y)e^{-\frac{(\kappa+1)z}{\lambda}}$, $z \in \mathbb{C}_+$. Then, (3.4) implies that

$$|F(z)| \leq CA(x, y; \sqrt{\lambda})\lambda^{d/2}(\operatorname{Re} z)^{-d/2}$$

for all $z \in \mathbb{C}_+$, and (3.7) implies that

$$|F(t)| \leq CA(x, y; \sqrt{\lambda})\lambda^{d/2}t^{-d/2} \exp\left(-\frac{c\rho^2(x, y)}{t}\right)$$

for all $t \in \mathbb{R}_+$. Hence, by Lemma 2.4 we have

$$|F(z)| \leq CA(x, y; \sqrt{\lambda})\lambda^{d/2}(\operatorname{Re} z)^{-d/2} \exp\left(-\frac{c\rho^2(x, y) \operatorname{Re} z}{|z|^2}\right)$$

for all $z \in \mathbb{C}_+$, where the value of the constant c may be different from those before. From this, it follows that

$$|p(z, x, y)| \leq CA(x, y; \sqrt{\lambda}) \left(\frac{\lambda}{\operatorname{Re} z}\right)^{d/2} \exp\left(-\frac{c\rho^2(x, y) \operatorname{Re} z}{|z|^2} + \frac{(\kappa+1) \operatorname{Re} z}{\lambda}\right).$$

This estimate holds for all $\lambda \in (0, \lambda_0]$ and all $z \in \mathbb{C}_+$. By letting $0 < \operatorname{Re} z \leq \lambda_0$ and then taking $\lambda = \operatorname{Re} z$ in this estimate, we obtain (3.1), as desired. $\square$

Using the local doubling condition (2.1) and proceeding as in the proof of [11, Lemma 4.4], we can get the following lemma.

Lemma 3.2 *For any $R_0 > 0$, there exists a constant $C = C(R_0)$ such that for all $R, r \in (0, R_0]$ and $y \in X$,*

$$\int_{B(y, R_0) \setminus B(y, r)} (1 + R^{-1}\rho(x, y))^{-s} d\mu(x) \leq C\mu(B(y, R))(1 + rR^{-1})^{d-s}.$$

Using Lemmas 3.1 and 2.4, we can get the following lemma, which is a local version of [11, Lemma 4.1].

Lemma 3.3 *For any $R_0 > 0$, there exists a constant $C = C(R_0)$ such that for all $R \in (0, R_0]$, $\xi \in \mathbb{R}$ and $y \in X$,*

$$\int_{B(y, R_0)} |p(R^2(1 + i\xi), x, y)|^2 \rho(x, y)^s d\mu(x) \leq C\mu(B(y, R))^{-1} R^s (1 + |\xi|)^s.$$

Now, we are ready to state a result concerning functional calculus of operators of the form $F(\sqrt{L})$, which is a local version of [11, Lemma 4.3 (i)].

Lemma 3.4 *Suppose $R_0 > 0$ and $s > 0$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(R_0, s, \varepsilon)$ such that for all $R \in (0, R_0]$ and all even functions $F \in W_{(s/2)+\varepsilon}^\infty(\mathbb{R})$ with $\operatorname{supp} F \subset [-R^{-1}, R^{-1}]$,*

$$\int_{B(y, R_0)} |K_{F(\sqrt{L})}(x, y)|^2 (1 + R^{-1}\rho(x, y))^s d\mu(x) \leq C \frac{\|\delta_{R^{-1}} F\|_{W_{(s/2)+\varepsilon}^\infty(\mathbb{R})}^2}{\mu(B(y, R))}. \quad (3.8)$$

where $K_{F(\sqrt{L})}$ is the Schwartz kernel of the operator $F(\sqrt{L})$, $\delta_R F(\lambda) := F(R\lambda)$, and $\|F\|_{W_s^\infty(\mathbb{R})} := \|(I - \frac{d^2}{d\lambda^2})^{s/2} F\|_{L^\infty(\mathbb{R})}$.

Proof Following the proof of [11, Lemma 2.2] and using Lemma 2.1, one can show that if (X, ρ, μ) is a metric measure space satisfying (1.3) and L is a nonnegative self-adjoint operator on $L^2(X)$ with heat kernel $p(t, x, y)$ satisfying (1.4), then L satisfies the following version of Plancherel type estimate: for any $R_0 > 0$, there exists a constant $C = C(R_0)$ such that for all $R \in (0, R_0]$ and all even functions F on $\mathbb{R}$ with $\text{supp } F \subset [-R^{-1}, R^{-1}]$,

$$\int_X |K_{F(\sqrt{L})}(x, y)|^2 d\mu(x) \leq C \mu(B(y, R))^{-1} \|F\|_{L^\infty(\mathbb{R})}^2.$$

Hence, the proof of the lemma is same as that of [11, Lemma 4.3 (i)], except that one needs to replace [11, Lemma 2.1], [11, Lemma 4.1] in the proof of [11, Lemma 4.3 (i)] by Lemmas 3.2 and 3.3, respectively. $\square$

4 Proof of Theorem 1.1

Let φ be a smooth even function on $\mathbb{R}$ such that $\varphi(t) = 1$ for $|t| \leq \sqrt{c^*}$, and $\varphi(t) = 0$ for $|t| \geq 2\sqrt{c^*}$, where c^* is a constant from (1.4). Following an idea of Taylor [32], we decompose the operator $\Phi(\sqrt{L})$ as

$$\Phi(\sqrt{L}) = \Phi^\sharp(\sqrt{L}) + \Phi^b(\sqrt{L}), \quad (4.1)$$

where

$$\Phi^\sharp(\sqrt{L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}(t) \varphi(t) \cos t\sqrt{L} dt \quad (4.2)$$

and

$$\Phi^b(\sqrt{L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}(t) (1 - \varphi(t)) \cos t\sqrt{L} dt, \quad (4.3)$$

In the rest of this section, we show that $\Phi^\sharp(\sqrt{L})$ is bounded on $L^p(X)$ for $p \in (1, \infty)$ and is of weak type $(1, 1)$, and $\Phi^b(\sqrt{L})$ is bounded on $L^p(X)$ for $p \in [1, \infty]$. Combining these results, we obtain the conclusion of Theorem 1.1.

4.1 The operator $\Phi^\sharp(\sqrt{L})$

In this subsection, based on the estimates developed in Sects. 2 and 3, we use a localized variant of the method from [10, 11] to show that the operator $\Phi^\sharp(\sqrt{L})$ is bounded on $L^p(X)$ for $p \in (1, \infty)$ and is of weak type $(1, 1)$. Note that this method is originated from an idea of Hebisch [17].

From Lemma 2.5, we see that the Schwartz kernel of $\Phi^\sharp(\sqrt{L})$ is supported in $\{(x, y) \in X \times X : \rho(x, y) \leq 1\}$. Hence, to prove that $\Phi^\sharp(\sqrt{L})$ is of weak type $(1, 1)$, by an argument of partition of the unity as used in the proof of [24, Theorem 3.1], it suffices to show that there exists a constant C such that for all $x_0 \in X$ and all $f \in L^1(X)$ with $\text{supp } f \subset B(x_0, 1)$,

$$\mu\left(\left\{x \in B(x_0, 2) : \Phi^\sharp(\sqrt{L})f(x) > \alpha\right\}\right) < \frac{C\|f\|_{L^1(X)}}{\alpha}. \quad (4.4)$$

To prove the latter estimate, we fix arbitrary $x_0 \in X$. For any $x \in B(x_0, 1)$ and any $r > 0$, we write $\tilde{B}(x, r) := B(x, r) \cap B(x_0, 1)$ and call it a ball of the set $B(x_0, 1)$. Note that for all $x \in B(x_0, 1)$ and all $r \geq 2$, we have $\tilde{B}(x, r) = B(x_0, 1)$. Hence, we only need to consider the balls $\tilde{B}(x, r)$ with $0 < r \leq 2$. Using the hypothesis that (X, ρ) is a geodesic length space

and (2.1), it is not difficult to show that there exists a constant C (independent of x_0) such that

$$\mu(\tilde{B}(x, r)) \geq C\mu(B(x, r)) \quad (4.5)$$

for all $x \in B(x_0, 1)$ and all $r \in (0, 2]$. Thus, endowed with the distance and the measure inherited from X , the set $B(x_0, 1)$ is a space of homogeneous type, and the constant appearing in the volume doubling condition is independent of x_0 .

Given $f \in L^1(X) \cap L^2(X)$ with $\text{supp } f \subset B(x_0, 1)$, we regard f as a function on $B(x_0, 1)$, then there exists a constant c such that for any $\alpha > \|f\|_{L^1(X)}/\mu(B(x_0, 1))$, one has a Calderón-Zygmund decomposition

$$f = g + b = g + \sum_j b_j,$$

so that $\text{supp } g, \text{supp } b_j \subset B(x_0, 1)$ and

- (a) $|g(x)| \leq c\alpha$ for almost all $x \in X$;
- (b) There exists a sequence of balls $\tilde{B}(x_j, r_j) \subset B(x_0, 1)$ ($0 < r_j \leq 2$) so that $\text{supp } b_j \subset \tilde{B}(x_j, r_j)$ and

$$\int |b_j(x)| d\mu(x) \leq c\alpha\mu(\tilde{B}(x_j, r_j));$$

- (c) $\sum_j \mu(\tilde{B}(x_j, r_j)) \leq \frac{c}{\alpha} \int |f(x)| d\mu(x)$;

- (d) Each point of $B(x_0, 1)$ is contained in at most a finite number N of the balls $\tilde{B}(x_j, r_j)$.

Note that properties b) and c) also imply that $\|b\|_{L^1(X)} \leq c\|f\|_{L^1(X)}$ and hence that $\|g\|_{L^1(X)} \leq (1+c)\|f\|_{L^1(X)}$.

We have

$$\begin{aligned} & \mu\left(\left\{x \in B(x_0, 2) : \Phi^\sharp(\sqrt{L})f(x) > \alpha\right\}\right) \\ & \leq \mu\left(\left\{x \in B(x_0, 2) : \Phi^\sharp(\sqrt{L})g(x) > \frac{\alpha}{2}\right\}\right) \\ & \quad + \mu\left(\left\{x \in B(x_0, 2) : \Phi^\sharp(\sqrt{L})b(x) > \frac{\alpha}{2}\right\}\right). \end{aligned} \quad (4.6)$$

Since the operator $\Phi^\sharp(\sqrt{L})$ is bounded on $L^2(X)$ and $|g(x)| \leq c\alpha$, we have

$$\begin{aligned} & \mu\left(\left\{x \in B(x_0, 2) : \Phi^\sharp(\sqrt{L})g(x) > \frac{\alpha}{2}\right\}\right) \leq 4\alpha^{-2}\|\Phi^\sharp(\sqrt{L})g\|_{L^2(X)}^2 \\ & \leq C\alpha^{-2}\|g\|_{L^2(X)}^2 \leq C\alpha^{-1}\|g\|_{L^1(X)} \leq C\alpha^{-1}\|f\|_{L^1(X)}. \end{aligned} \quad (4.7)$$

Now, we consider the “bad” part $b(x)$. Let us decompose

$$\Phi^\sharp(\sqrt{L})b = \sum_j \Phi^\sharp(\sqrt{L})e^{-r_j^2 L} b_j + \sum_j \Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})b_j.$$

In order to analyse $\Phi^\sharp(\sqrt{L})e^{-r_j^2 L} b_j$, we first estimate the function $e^{-r_j^2 L} b_j$. Define

$$q(t, x, y) := \frac{C^*}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \exp\left(-\frac{c^*\rho^2(x, y)}{t}\right)$$

for $t > 0$ and $x, y \in X$. Observe that for any $t_0 > 0$ there exists a constant $C = C(t_0)$ such that for all $t \in (0, t_0]$ and $w \in X$,

$$\sup_{y \in B(w, \sqrt{t})} q(t, x, y) \leq C \inf_{y \in B(w, \sqrt{t})} q(2t, x, y).$$

Using this fact and (2.4), we infer

$$\begin{aligned} |e^{-r_j^2 L} b_j(x)| &\leq \int q(r_j^2, x, y) |b_j(y)| d\mu(y) \leq \|b_j\|_{L^1(X)} \sup_{y \in \tilde{B}(x_j, r_j)} q(r_j^2, x, y) \\ &\leq C\alpha \mu(\tilde{B}(x_j, r_j)) \inf_{y \in \tilde{B}(x_j, r_j)} q(2r_j^2, x, y) \leq C\alpha \int q(2r_j^2, x, y) \chi_{\tilde{B}(x_j, r_j)}(y) d\mu(y). \end{aligned}$$

Hence, for any $h \in L^2(X)$, we have

$$\begin{aligned} |\langle h, e^{-r_j^2 L} b_j \rangle| &\leq C\alpha \iint |h(x)| q(2r_j^2, x, y) \chi_{\tilde{B}(x_j, r_j)}(y) d\mu(y) d\mu(x) \\ &\leq C\alpha \int \left(\int |h(x)| q(2r_j^2, x, y) d\mu(x) \right) \chi_{\tilde{B}(x_j, r_j)}(y) d\mu(y). \end{aligned}$$

By the Cauchy–Schwartz inequality and the proof of Lemma 2.1, we have

$$\begin{aligned} \int |h(x)| q(2r_j^2, x, y) d\mu(x) &\leq \|h\|_{L^2(X)} \left(\int |q(2r_j^2, x, y)|^2 d\mu(x) \right)^{1/2} \\ &\leq C \|h\|_{L^2(X)} \mu(B(y, \sqrt{2}r_j))^{-1/2} \leq C \|h\|_{L^2(X)} \mu(B(y, r_j))^{-1/2}. \end{aligned}$$

Therefore,

$$|\langle h, e^{-r_j^2 L} b_j \rangle| \leq C\alpha \|h\|_{L^2(X)} \int \frac{\chi_{\tilde{B}(x_j, r_j)}(y)}{\mu(B(y, r_j))^{1/2}} d\mu(y) \leq C\alpha \mu(\tilde{B}(x_j, r_j))^{1/2} \|h\|_{L^2(X)},$$

where we used that $\mu(\tilde{B}(x_j, r_j)) \leq C\mu(B(y, r_i))$ whenever $y \in \tilde{B}(x_j, r_j)$. This implies

$$\|e^{-r_j^2 L} b_j\|_{L^2(X)} \leq C\alpha \mu(B(x_j, r_j))^{1/2}.$$

Hence, by the L^2 -boundedness of $\Phi^\sharp(\sqrt{L})$, we have

$$\begin{aligned} \left\| \sum_j \Phi^\sharp(\sqrt{L}) e^{-r_j^2 L} b_j \right\|_{L^2(X)} &\leq C \left\| \sum_j e^{-r_j^2 L} b_j \right\|_{L^2(X)} \leq C \sum_j \|e^{-r_j^2 L} b_j\|_{L^2(X)} \\ &\leq C\alpha \sum_j \mu(\tilde{B}(x_j, r_j))^{1/2} \leq C\alpha \left(\sum_j \mu(\tilde{B}(x_j, r_j)) \right)^{1/2} \leq C\alpha^{1/2} \|f\|_{L^1(X)}^{1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} &\mu \left(\left\{ x \in B(x_0, 2) : \left| \sum_j \Phi^\sharp(\sqrt{L}) e^{-r_j^2 L} b_j(x) \right| > \frac{\alpha}{4} \right\} \right) \\ &\leq 16\alpha^{-2} \int_{B(x_0, 2)} \left| \sum_j \Phi^\sharp(\sqrt{L}) e^{-r_j^2 L} b_j(x) \right|^2 d\mu(x) \\ &\leq 16\alpha^{-2} \left\| \sum_j \Phi^\sharp(\sqrt{L}) e^{-r_j^2 L} b_j \right\|_{L^2(X, d\mu)}^2 \leq \frac{C\|f\|_{L^1(X)}}{\alpha}. \end{aligned} \quad (4.8)$$

On the other hand,

$$\begin{aligned} & \mu \left(\left\{ x \in B(x_0, 2) : \left| \sum_j \Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})b_j(x) \right| > \frac{\alpha}{4} \right\} \right) \\ & \leq \sum_j \mu(B(x_j, 2r_j)) + \sum_j \frac{4}{\alpha} \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} |\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})b_j(x)| d\mu(x). \end{aligned}$$

Using (2.1), (4.5), and property (c) of the decomposition, we have

$$\sum_j \mu(B(x_j, 2r_j)) \leq C \sum_j \mu(B(x_j, r_j)) \leq C \sum_j \mu(\tilde{B}(x_j, r_j)) \leq \frac{C \|f\|_{L^1(X)}}{\alpha}.$$

By Fubini's Theorem, we have

$$\begin{aligned} & \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} |\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})b_j(x)| d\mu(x) \\ & \leq \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} \left| \int_{\tilde{B}(x_j, r_j)} K_{\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})}(x, y) b_j(y) d\mu(y) \right| d\mu(x) \\ & \leq \int_{\tilde{B}(x_j, r_j)} |b_j(y)| \left(\int_{B(x_0, 2) \setminus B(x_j, 2r_j)} |K_{\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})}(x, y)| d\mu(x) \right) d\mu(y). \end{aligned}$$

To proceed, we claim that

$$\sup_j \sup_{y \in \tilde{B}(x_j, r_j)} \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} |K_{\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})}(x, y)| d\mu(x) < \infty. \quad (4.9)$$

Assume the claim for the moment. Then, it follows from the preceding estimates and properties (b) and (c) of the decomposition that

$$\begin{aligned} & \mu \left(\left\{ x \in B(z, 2) : \left| \sum_j \Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})b_j(x) \right| > \frac{\alpha}{4} \right\} \right) \\ & \leq \sum_j \mu(B(x_j, 2r_j)) + \sum_j \frac{4}{\alpha} \int_{\tilde{B}(x_j, r_j)} |b_j(y)| d\mu(y) \leq \frac{C \|f\|_{L^1(X)}}{\alpha}, \end{aligned}$$

which along with (4.8) yields

$$\mu \left(\left\{ x \in B(x_0, 2) : \Phi^\sharp(\sqrt{L})b(x) > \frac{\alpha}{2} \right\} \right) \leq \frac{C \|f\|_{L^1(X)}}{\alpha}. \quad (4.10)$$

Combining (4.6), (4.7) and (4.10) yields (4.4).

It remains to prove the claim. Choose $\omega_0 \in C^\infty(\mathbb{R})$ such that $\text{supp } \omega_0 \in [-1, 1]$, $0 \leq \omega_0 \leq 1$, and $\omega_0(\lambda) = 1$ for $\lambda \in [-1/2, 1/2]$. Let $\omega(\lambda) := \omega_0(\lambda) - \omega_0(2\lambda)$. Then $\text{supp } \omega \subset [-1, -1/4] \cup [1/4, 1]$. Set $\omega_k(\lambda) := \omega(2^{-k}\lambda)$, $k = 1, 2, \dots$. Clearly, we have $\sum_{k=0}^\infty \omega_k(\lambda) = 1$ for $\lambda \in \mathbb{R}$. Hence,

$$\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L}) = \sum_{k=1}^\infty \omega_k(\sqrt{L})\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L}).$$

Note that if $y \in \tilde{B}(x_j, r_j)$ then $B(y, r_j) \subset B(x_j, 2r_j)$ and $B(x_0, 2) \subset B(y, 5)$. Hence for all j and all $y \in \tilde{B}(x_j, r_j)$, we have

$$\begin{aligned} & \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} \left| K_{\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})}(x, y) \right| d\mu(x) \\ & \leq \sum_{k=0}^{\infty} \int_{B(y, 5) \setminus B(y, r_j)} \left| K_{\omega_k(\sqrt{L})\Phi^\sharp(\sqrt{L})(I - e^{-r_j^2 L})}(x, y) \right| d\mu(x) \\ & \leq \sum_{k=0}^{\infty} \left(\int_{B(y, 5)} \left| K_{\omega_k(\sqrt{L})\Phi^\sharp(\sqrt{L})\Psi_{r_j}(\sqrt{L})}(x, y) \right|^2 (1 + 2^k \rho(x, y))^{2s} d\mu(x) \right)^{1/2} \\ & \quad \times \left(\int_{B(y, 5) \setminus B(y, r_j)} (1 + 2^k \rho(x, y))^{-2s} d\mu(x) \right)^{1/2}, \end{aligned}$$

where s is a number such that $d/2 < s < [d/2] + 1$, and Ψ_{r_j} is a function on $\mathbb{R}$ given by $\Psi_{r_j}(\lambda) := 1 - e^{-r_j^2 \lambda^2}$, $\lambda \in \mathbb{R}$. Now, let J' be an even integer such that $s < J' \leq J$. Since $\text{supp}[\omega_0 \Phi^\sharp \Psi_{r_j}] \subset [-1, 1]$ and $\text{supp}[\omega_k \Phi^\sharp \Psi_{r_j}] \subset [-2^k, -2^{k-2}] \cup [2^{k-2}, 2^k]$ for $k = 0, 1, 2, \dots$, it follows from Lemma 3.4 that

$$\begin{aligned} & \int_{B(y, 5)} \left| K_{\omega_k(\sqrt{L})\Phi^\sharp(\sqrt{L})\Psi_{r_j}(\sqrt{L})}(x, y) \right|^2 (1 + 2^k \rho(x, y))^{2s} d\mu(x) \\ & \leq C \mu(B(y, 2^{-k}))^{-1} \|\delta_{2^k} [\omega_k \Phi^\sharp \Psi_{r_j}]\|_{W_{J'}^\infty(\mathbb{R})}^2 \end{aligned}$$

for $k = 0, 1, 2, \dots$. On the other hand, by Lemma 3.2 we have

$$\int_{B(y, 5) \setminus B(y, r_j)} (1 + 2^k \rho(x, y))^{-2s} d\mu(x) \leq \mu(B(y, 2^{-k})) (1 + 2^k r_j)^{d-2s}.$$

Therefore,

$$\begin{aligned} & \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} \left| K_{\Phi^\sharp(\sqrt{L})\Psi_{r_j}(\sqrt{L})}(x, y) \right| d\mu(x) \\ & \leq C \sum_{k=0}^{\infty} (1 + 2^k r)^{d/2-s} \|\delta_{2^k} [\omega_k \Phi^\sharp \Psi_{r_j}]\|_{W_{J'}^\infty(\mathbb{R})}. \end{aligned} \quad (4.11)$$

Since J' is a nonnegative even integer, we have $\|F\|_{W_{J'}^\infty(\mathbb{R})} \sim \sum_{v=0}^{J'} \left\| \frac{d^v}{d\lambda^v} F \right\|_{L^\infty(\mathbb{R})}$; see, for instance [29, Ch. V, Section 6.6]. Using this fact and the properties of ω_k , we infer

$$\begin{aligned} & \|\delta_{2^k} [\omega_k \Phi^\sharp \Psi_{r_j}]\|_{W_{J'}^\infty(\mathbb{R})} \\ & \leq \begin{cases} C \sum_{v_1+v_2=J'} \sup_{\lambda \in [-1, 1]} \left| \frac{d^{v_1}}{d\lambda^{v_1}} [\delta_{2^k} \Phi^\sharp](\lambda) \right| \left| \frac{d^{v_2}}{d\lambda^{v_2}} [\delta_{2^k} \Psi_{r_j}](\lambda) \right|, & k = 0, \\ C \sum_{v_1+v_2=J'} \sup_{1/4 \leq |\lambda| \leq 1} \left| \frac{d^{v_1}}{d\lambda^{v_1}} [\delta_{2^k} \Phi^\sharp](\lambda) \right| \left| \frac{d^{v_2}}{d\lambda^{v_2}} [\delta_{2^k} \Psi_{r_j}](\lambda) \right|, & k = 1, 2, \dots \end{cases} \end{aligned}$$

It is straightforward to check that if Φ satisfies the Mihlin-type condition (1.5) then $\Phi^\sharp$ also satisfies (1.5) with the constant C depending on the choice of φ . From this, we infer that there exists a constant C such that

$$\left| \frac{d^{v_1}}{d\lambda^{v_1}} [\delta_{2^k} \Phi^\sharp](\lambda) \right| \leq \begin{cases} C & \text{for } k = 0, v_1 \in \{0, 1, \dots, J'\} \text{ and } \lambda \in [-1, 1], \\ C & \text{for } k \geq 1, v_1 \in \{0, 1, \dots, J'\} \text{ and } 1/4 \leq |\lambda| \leq 1. \end{cases}$$

On the other hand, the properties of the function Ψ_{r_j} imply that there exists a constant C such that

$$\left| \frac{d^{v_2}}{d\lambda^{v_2}} [\delta_{2^k} \Psi_{r_j}](\lambda) \right| \leq \begin{cases} Cr_j & \text{for } k = 0, v_2 \in \{0, 1, \dots, J'\} \text{ and } \lambda \in [-1, 1], \\ C \frac{2^k r_j}{1+2^k r_j} & \text{for } k \geq 1, v_2 \in \{0, 1, \dots, J'\} \text{ and } 1/4 \leq |\lambda| \leq 1. \end{cases}$$

Inserting all these estimates into (4.11), we see that for all j and all $y \in \tilde{B}(x_j, r_j)$,

$$\begin{aligned} & \int_{B(x_0, 2) \setminus B(x_j, 2r_j)} \left| K_{\Phi^\sharp(\sqrt{L})\Psi_{r_j}(\sqrt{L})}(x, y) \right| d\mu(x) \\ & \leq C \sum_{k=0}^{\infty} (1 + 2^k r_j)^{d/2-s} \frac{2^k r_j}{1 + 2^k r_j} \leq C, \end{aligned}$$

which verifies the claim. Hence, we have proved that $\Phi^\sharp(\sqrt{L})$ is of weak type $(1, 1)$.

Interpolation of the weak type $(1, 1)$ estimate with the obvious L^2 -continuity yields that $\Phi^\sharp(\sqrt{L})$ is bounded on $L^p(X)$ for all $p \in (1, 2]$. Then, by duality, we get that $\Phi^\sharp(\sqrt{L})$ is bounded on $L^p(X)$ for all $p \in (1, \infty)$.

4.2 The operator $\Phi^b(\sqrt{L})$

In this subsection, we prove that the operator $\Phi^b(\sqrt{L})$ is bounded on $L^p(X)$ for all $p \in [1, \infty]$. The main idea comes from [32, 33].

We will need the following lemma.

Lemma 4.1 *Suppose $k > d/4$. Then, the integral kernel $K_{(I+L)^{-k}}(x, y)$ of the operator $(I + L)^{-k}$ satisfies*

$$\|K_{(I+L)^{-k}}(\cdot, y)\|_{L^2(X)} \leq C\mu(B(y, 1))^{-1/2}.$$

Proof By the formula

$$\lambda^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty e^{-s\lambda} s^{k-1} ds, \quad \lambda \in [0, \infty),$$

we have

$$(I + L)^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty e^{-s(I+L)} s^{k-1} ds = \frac{1}{(k-1)!} \int_0^\infty e^{-s} s^{k-1} e^{-sL} ds.$$

From this, Lemma 2.1 and (2.1), it follows that

$$\begin{aligned}
 \|K_{(I+L)^{-k}}(\cdot, y)\|_{L^2(X)} &\leq \frac{1}{(k-1)!} \int_0^\infty e^{-s} s^{k-1} \|p(s, \cdot, y)\|_{L^2(X)} ds \\
 &= \frac{1}{(k-1)!} \int_0^1 e^{-s} s^{k-1} \|p(s, \cdot, y)\|_{L^2(X)} ds \\
 &\quad + \frac{1}{(k-1)!} \int_1^\infty e^{-s} s^{k-1} \|p(s, \cdot, y)\|_{L^2(X)} ds \\
 &\leq C \int_0^1 e^{-s} s^{k-1} \mu(B(y, \sqrt{s}))^{-1/2} ds \\
 &\quad + C \int_1^\infty e^{-s} s^{k-1} e^{\kappa s} \mu(B(y, \sqrt{2s}))^{-1/2} ds \\
 &\leq C \mu(B(y, 1))^{-1/2} \int_0^1 e^{-s} s^{k-1-d/4} ds \\
 &\quad + C \mu(B(y, 1))^{-1/2} \int_1^\infty e^{-(1-\kappa)s} s^{k-1} ds \\
 &\leq C \mu(B(y, 1))^{-1/2},
 \end{aligned}$$

where for the last inequality we used that $k > d/4$ and $0 < \kappa < 1$. The proof is complete. $\square$

To show that $\Phi^b(\sqrt{L})$ is bounded on $L^p(X)$ for $1 \leq p \leq \infty$, it suffices to prove the following proposition.

Proposition 4.2 *Then, the Schwartz kernel $K_{\Phi^b(\sqrt{L})}(x, y)$ of $\Phi^b(\sqrt{L})$ satisfies*

$$\sup_{y \in X} \int_X |K_{\Phi^b(\sqrt{L})}(x, y)| d\mu(x) < +\infty, \quad (4.12)$$

$$\sup_{x \in X} \int_X |K_{\Phi^b(\sqrt{L})}(x, y)| d\mu(y) < +\infty. \quad (4.13)$$

Proof The idea comes from [32, 33]. Set $A_j = \{x : j \leq \rho(x, y) \leq j+1\}$, $j = 0, 1, 2, \dots$. Then,

$$\begin{aligned}
 \int_X |K_{\Phi^b(\sqrt{L})}(x, y)| d\mu(x) &= \sum_{j=0}^\infty \int_{A_j} |K_{\Phi^b(\sqrt{L})}(x, y)| d\mu(x) \\
 &\leq \sum_{j=0}^\infty \mu(B(y, j+1))^{1/2} \left(\int_{X \setminus B(y, j)} |K_{\Phi^b(\sqrt{L})}(x, y)|^2 d\mu(x) \right)^{1/2}.
 \end{aligned} \quad (4.14)$$

We claim that there exists a constant C such that for all $R \geq 0$,

$$\begin{aligned}
 &\int_{X \setminus B(y, R+1)} |K_{\Phi^b(\sqrt{L})}(x, y)|^2 d\mu(x) \\
 &\leq C(R+1/2)^{-J} e^{-2\sqrt{c^*}WR} \mu(B(y, 1))^{-1/2}.
 \end{aligned} \quad (4.15)$$

Assuming the claim for the moment. Then, for all $j \geq 1$,

$$\begin{aligned} & \left(\int_{X \setminus B(y, j)} |K_{\Phi^b(\sqrt{L})}(x, y)|^2 d\mu(x) \right)^{1/2} \\ & \leq C(j-1/2)^{-J} e^{-2\sqrt{c^*}W(j-1)} \mu(B(y, 1))^{-1/2}. \end{aligned} \quad (4.16)$$

For $j = 0$, we have $B(y, 0) = \emptyset$. Hence, by the spectral theorem and Lemma 4.1, there holds

$$\begin{aligned} & \left(\int_{X \setminus B(y, 0)} |K_{\Phi^b(\sqrt{L})}(x, y)|^2 d\mu(x) \right)^{1/2} = \|K_{\Phi^b(\sqrt{L})}(\cdot, y)\|_{L^2(X)} \\ & \leq \|\Phi^b(\sqrt{L})(I + L)^{m_0}\|_{2 \rightarrow 2} \|K_{(I+L)^{-m_0}}(\cdot, y)\|_{L^2(X)} \\ & \leq C \|K_{(I+L)^{-m_0}}(\cdot, y)\|_{L^2(X)} \\ & \leq C \mu(B(y, 1))^{-1/2}, \end{aligned} \quad (4.17)$$

where $m_0 := [d/4] + 1$. Now, from (4.14), (4.16), (4.17) and (1.3) it follows that

$$\begin{aligned} & \int_X |K_{\Phi^b(\sqrt{L})}(x, y)| d\mu(x) \\ & = \mu(B(y, 1))^{1/2} \left(\int_{X \setminus B(y, 0)} |K_{\Phi^b(\sqrt{L})}(x, y)|^2 d\mu(x) \right)^{1/2} \\ & \quad + \sum_{j=1}^{\infty} \mu(B(y, j+1))^{1/2} \left(\int_{X \setminus B(y, j)} |K_{\Phi^b(\sqrt{L})}(x, y)|^2 d\mu(x) \right)^{1/2} \\ & \leq C \left(1 + \sum_{j=1}^{\infty} (j-1/2)^{-J} e^{-2\sqrt{c^*}W(j-1)} \frac{\mu(B(y, j+1))^{1/2}}{\mu(B(y, 1))^{1/2}} \right) \\ & \leq C \left(1 + \sum_{j=1}^{\infty} (j-1/2)^{-J} e^{-2\sqrt{c^*}W(j-1)} (j+1)^{d/2} e^{\tau j/2} \right) \\ & \leq C, \end{aligned}$$

where for the last inequality we used that $W \geq \frac{\tau}{4\sqrt{c^*}}$ and $J > d/2$. This verifies (4.12). Since $K_{\Phi^b(\sqrt{L})}(y, x) = \overline{K_{\Phi^b(\sqrt{L})}(x, y)}$, (4.13) is also valid.

It remains to prove the claim. Indeed, choose $\psi \in C^\infty(\mathbb{R})$ such that $\psi(t) = 0$ for $t \in (-\infty, \sqrt{c^*}]$, and $\psi(t) = 1$ for $t \in [2\sqrt{c^*}, \infty)$. For $R \geq 0$, define $\psi_R(t) = \psi(t - 2\sqrt{c^*}R) + \psi(-t - 2\sqrt{c^*}R)$. Note that ψ_R is an even function on $\mathbb{R}$, $\psi_R(t) = 0$ for $|t| \leq 2\sqrt{c^*}(R + \frac{1}{2})$, and $\psi_R(t) = 1$ for $|t| \geq 2\sqrt{c^*}(R + 1)$. Set

$$\Phi_R(\sqrt{L}) = \int_{-\infty}^{\infty} \widehat{\Phi}(t) \psi_R(t) \cos(t\sqrt{L}) dt.$$

Then, since

$$\text{supp} [\widehat{\Phi} \psi_R - \widehat{\Phi}(1 - \varphi)] \subset [-2\sqrt{c^*}(R + 1), 2\sqrt{c^*}(R + 1)],$$

it follows from Lemma 2.5 that

$$K_{\Phi^b(\sqrt{L})}(x, y) = K_{\Phi_R(\sqrt{L})}(x, y) \quad \text{if } \rho(x, y) \geq R + 1.$$

Consequently,

$$\begin{aligned} \|K_{\Phi^b(\sqrt{L})}(\cdot, y)\|_{L^2(X \setminus B(y, R+1))} &= \|K_{\Phi_R(\sqrt{L})}(\cdot, y)\|_{L^2(X \setminus B(y, R+1))} \\ &\leq \|K_{\Phi_R(\sqrt{L})}(\cdot, y)\|_{L^2(X)}. \end{aligned} \quad (4.18)$$

To estimate $\|K_{\Phi_R(\sqrt{L})}(\cdot, y)\|_{L^2(X)}$, we write

$$\begin{aligned} &\|K_{\Phi_R(\sqrt{L})}(\cdot, y)\|_{L^2(X)} \\ &= \|\Phi_R(\sqrt{L})(I + L)^{k_0}[K_{(I+L)^{-k_0}}(\cdot, y)]\|_{L^2(X)} \\ &\leq \|\Phi_R(\sqrt{L})(I + L)^{k_0}\|_{2 \rightarrow 2} \|K_{(I+L)^{-k_0}}(\cdot, y)\|_{L^2(X)}. \end{aligned} \quad (4.19)$$

where k_0 is an integer such that $k_0 > d/4$ and $2k_0 \leq J - 2$. (Such k_0 exists, since by the definition of J there exists at least one even integer k such that $d/2 < k \leq J - 2$.) Since Φ is holomorphic in Ω_W and satisfies the Mihlin-type condition (1.5), it follows from [18, Lemma 5.4] (or [24, Lemma 3.3]) that there exists a constant C such that

$$|\partial_t^k \widehat{\Phi}(t)| \leq C|t|^{-J} e^{-W|t|}. \quad (4.20)$$

for all $t \in \mathbb{R} \setminus \{0\}$ and all $k \in \{1, 2, \dots, J - 2\}$. On the other hand, from the construction of ψ_R we see that there exists a constant C (depending only on ψ) such that for all $R \geq 0$, $k \in \{0, 1, \dots, 2k_0\}$ and $t \in \mathbb{R}$,

$$|\partial_t^k \psi_R(t)| \leq C.$$

This together with (4.20) and the fact that $2k_0 \leq J - 2$ yields

$$\left| (I - \partial_t^2)^{k_0} [\widehat{\Phi}(t) \psi_R(t)] \right| \leq C|t|^{-J} e^{-W|t|}.$$

Using this and the spectral theorem, we have

$$\begin{aligned} &\|\Phi_R(\sqrt{L})(I + L)^{k_0}\|_{2 \rightarrow 2} \\ &= \frac{1}{\sqrt{2\pi}} \sup_{\lambda \in [0, \infty)} \left| (1 + \lambda)^{k_0} \int_{-\infty}^{\infty} \widehat{\Phi}(t) \psi_R(t) \cos t\sqrt{\lambda} dt \right| \\ &= \frac{1}{\sqrt{2\pi}} \sup_{\lambda \in [0, \infty)} \left| \int_{-\infty}^{\infty} \widehat{\Phi}(t) \psi_R(t) (I - \partial_t^2)^{k_0} \cos t\sqrt{\lambda} dt \right| \\ &= \frac{1}{\sqrt{2\pi}} \sup_{\lambda \in [0, \infty)} \left| \int_{-\infty}^{\infty} (I - \partial_t^2)^{k_0} [\widehat{\Phi}(t) \psi_R(t)] \cos t\sqrt{\lambda} dt \right| \\ &\leq C \int_{|t| \geq 2\sqrt{c^*}(R + \frac{1}{2})} \left| (I - \partial_t^2)^{k_0} [\widehat{\Phi}(t) \psi_R(t)] \right| dt \\ &\leq C \int_{|t| \geq 2\sqrt{c^*}(R + \frac{1}{2})} |t|^{-J} e^{-W|t|} dt \\ &\leq C(R + 1/2)^{-J} \int_{|t| \geq 2\sqrt{c^*}(R + \frac{1}{2})} e^{-W|t|} dt \\ &\leq C(R + 1/2)^{-J} e^{-2\sqrt{c^*}WR}. \end{aligned} \quad (4.21)$$

Combining (4.18), (4.19), (4.21) and Lemma 4.1 yields the claim. Hence, the proof of the proposition is complete. $\square$

5 Applications

In this section, we give some applications of Theorem 1.1.

5.1 Riemannian manifolds with Ricci curvature bounded from below

Let M be a complete connected n -dimensional Riemannian manifold with

$$\operatorname{Ric}(M) \geq -(n-1)K^2 \quad (5.1)$$

for some $K > 0$. Then, by the Bishop comparison theorem (see [5, §11.10] or [16, §2.1]), we have

$$\operatorname{vol}(B(x, r)) \leq Cr^n e^{(n-1)Kr}$$

for all $x \in M$ and $r > 0$. Denote by Δ the Laplace–Beltrami operator on M , and let b be the bottom of the spectrum of $-\Delta$. If one assumes further that M has positive injectivity radius, i.e.

$$\operatorname{inj}(M) \geq a > 0, \quad (5.2)$$

then the main result of [33] yields the following

Theorem A ([33, Theorem 1.1]) *Assume (5.1) and (5.2). Let $W \geq \frac{(n-1)K}{2}$ and let Φ be a bounded even holomorphic function in $\Omega_W := \{\zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < W\}$, satisfying the Mihlin-type condition:*

$$|\Phi^{(j)}(\zeta)| \leq C(1 + |\zeta|)^{-j}$$

for all $\zeta \in \Omega_W$ and all $j \in \{0, 1, 2, \dots, J\}$, where J is a sufficiently large integer. Then, $\Phi(\sqrt{-\Delta - b})$ is bounded on $L^p(M)$ for $1 < p < \infty$, and is of weak type $(1, 1)$.

Next, we will see that, if we only assume (5.1) but not (5.2), we still have a spectral multiplier theorem on M . Indeed, since $\operatorname{Ric}(M) \geq -(n-1)K^2$, it follows from the Bishop comparison theorem (see [5, §11.10] or [16, §2.1]) that

$$0 < \operatorname{Vol}(B(x, \lambda r)) \leq A\lambda^n e^{(n-1)K(\lambda-1)r} \operatorname{Vol}(B(x, r)) < \infty$$

for all $x \in X$, $r > 0$ and $\lambda \geq 1$. On the other hand, under the assumption (5.1), it is well known from Li and Yau [22, Corollary 3.1] (see also [26, §4.2]) that the heat kernel associated with the Laplace–Beltrami operator satisfies the following estimate: For any $\varepsilon \in (0, 1)$,

$$|p(t, x, y)| \leq \frac{C(\varepsilon, n)}{\sqrt{\operatorname{vol}(B(x, \sqrt{t})) \operatorname{vol}(B(y, \sqrt{t}))}} \exp\left(-\frac{\rho(x, y)^2}{(4 + \varepsilon)t} + C_1 \varepsilon K^2 t\right), \quad (5.3)$$

where C_1 depends only on n . Hence, by applying Theorem 1.1, we have the following

Theorem 5.1 (Corollary of Theorem 1.1) *Assume (5.1). Let ε be an arbitrary positive number and let $W \geq \frac{(n-1)K}{2} + \varepsilon$. Let Φ be a bounded even holomorphic function in Ω_W , satisfying the Mihlin-type condition:*

$$|\Phi^{(j)}(\zeta)| \leq C(1 + |\zeta|)^{-j}$$

for all $\zeta \in \Omega_W$ and all $j \in \{0, 1, 2, \dots, J\}$, where $J := [n/2] + 4$. Then, $\Phi(\sqrt{-\Delta})$ is bounded on $L^p(M)$ for $1 < p < \infty$ and is of weak type $(1, 1)$.

Note that there is considerable discrepancy between Theorem A and Theorem 5.1. The first is that the width of the strip appearing in Theorem A is required to be no less than $(n-1)K/2$, while that appearing in Theorem 5.1 is required to be strictly larger than $(n-1)K/2$. The second is that Theorem A obtains L^p -boundedness of the operator $\Phi(\sqrt{-\Delta - b})$, while

Theorem 5.1 only obtains the L^p -boundedness of $\Phi(\sqrt{-\Delta})$. In the setting where the positivity of the injectivity radius is not assumed, we have not been able to obtain the L^p -boundedness of $\Phi(\sqrt{-\Delta - b})$. The reason is that in such a setting the estimate (1.2) may not be valid, even for $\rho = 0$. Although we have not been able to obtain the L^p -boundedness of $\Phi(\sqrt{-\Delta - b})$ on general Riemannian manifolds with Ricci curvature bounded from below, in certain limited cases our main result improves those of Taylor [33] and Mauceri et al. [24]. Indeed, from [15, Corollary 5.5] we see that, if M is a complete n -dimensional Riemannian manifold with $\text{Ric}(M) \geq -(n-1)K^2$, then the bottom of the spectrum of $-\Delta$ satisfies

$$b \leq \frac{(n-1)^2 K^2}{4}.$$

As a consequence, we have the following implication:

$$\text{Ric}(M) > -\frac{4}{n-1} \implies b < 1. \quad (5.4)$$

On the other hand, if we denote by $p(t, x, y)$ and $p^b(t, x, y)$ the heat kernels of $-\Delta$ and $-\Delta - b$ respectively, then

$$p^b(t, x, y) = e^{bt} p(t, x, y).$$

From this and (5.3), it follows that for any $\varepsilon \in (0, 1)$,

$$|p^b(t, x, y)| \leq \frac{C(\varepsilon, n)}{\sqrt{\text{Vol}(B(x, \sqrt{t}))\text{Vol}(B(y, \sqrt{t}))}} \exp\left(-\frac{\rho(x, y)^2}{(4+\varepsilon)t} + (C_1 \varepsilon K^2 + b)t\right), \quad (5.5)$$

where C_1 depends only on n . Note that if $b < 1$, then we can choose sufficiently small ε such that

$$0 < C_1 \varepsilon K^2 + b < 1. \quad (5.6)$$

Having (5.4), (5.5) and (5.6), our main result (Theorem 1.1) yields the following

Theorem 5.2 (Corollary of Theorem 1.1) *Assume that M is a complete connected n -dimensional Riemannian manifold with $\text{Ric}(M) > -\frac{4}{n-1}$. Let $W \geq 1$ and let Φ be a bounded even holomorphic function in Ω_W , satisfying the Mihlin-type condition:*

$$|\Phi^{(j)}(\zeta)| \leq C(1 + |\zeta|)^{-j}$$

for all $\zeta \in \Omega_W$ and all $j \in \{0, 1, 2, \dots, J\}$, where $J := [n/2] + 4$. Then, $\Phi(\sqrt{-\Delta - b})$ is bounded on $L^p(M)$ for $1 < p < \infty$ and is of weak type $(1, 1)$.

5.2 Schrödinger operators on Riemannian manifolds

We emphasize again that our main result requires only a local Gaussian upper bound of the heat kernel of the operator, but does not require any regularity (such as Hölder continuity or gradient estimate) of the heat kernel. Thus, our result have a wide range of applications. For example, it can be applied to the associated Schrödinger operators. To be more precise, let M be a complete connected n -dimensional Riemannian manifold satisfying (5.1). Denote by Δ the Laplace–Beltrami operator on M . Let V be a nonnegative locally integrable function on M . We consider the Schrödinger operator $H = -\Delta + V$ defined by the quadratic form

technique. Denote by $p(t, x, y)$ and $p'(t, x, y)$ the heat kernels of $-\Delta$ and H , respectively. Then as a consequence of the Trotter product formula,

$$p'(t, x, y) \leq p(t, x, y)$$

for all $t > 0$ and $x, y \in M$. Since $p(t, x, y)$ satisfies (5.3), so does $p'(t, x, y)$. Hence, by using our main result (Theorem 1.1), we have the following

Theorem 5.3 (Corollary of Theorem 1.1) *Assume (5.1). Let ε be arbitrary positive number. Let $W \geq \frac{(n-1)K}{2} + \varepsilon$ and let Φ be a bounded even holomorphic function in Ω_W , satisfying the Mihlin-type condition:*

$$|\Phi^{(j)}(\zeta)| \leq C(1 + |\zeta|)^{-j}$$

for all $\zeta \in \Omega_W$ and all $j \in \{0, 1, 2, \dots, J\}$, where $J := [n/2] + 4$. Then, $\Phi(\sqrt{H})$ is bounded on $L^p(M)$ for $1 < p < \infty$, and is of weak type $(1, 1)$.

5.3 Spectral multiplier theorem on $RCD^*(K, N)$ spaces

Recently, on metric measure spaces, Erbar et al. [14] and Ambrosio et al. [1] introduced the Riemannian curvature-dimension condition, denoted by $RCD^*(K, N)$, which is a generalization of the “Ricci curvature lower bound” for the nonsmooth setting and is a strengthening of the curvature-dimension condition introduced by Lott and Villani [23] and Sturm [30, 31]. A metric measure space is called a $RCD^*(K, N)$ space if it satisfies the $RCD^*(K, N)$ condition and certain other conditions; see [14] for more details.

Now assume that (X, ρ, μ) is a $RCD^*(K, N)$ space. Then, from [14, Proposition 3.6] we see that (X, ρ, μ) satisfies the exponentially locally doubling condition (1.3). In addition, it is proved by Jiang et al. [21, Theorem 1.2] that the heat kernel associated with the Laplacian on (X, ρ, μ) obeys the local Gaussian upper bound (1.4). Therefore, as an application of Theorem 1.1, one can easily formulate a spectral multiplier theorem on (X, ρ, μ) . We omit the details here.

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