Hardy spaces for semigroups with Gaussian bounds

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Abstract Let $T_t = e^{-tL}$ be a semigroup of self-adjoint linear operators acting on $L^2(X, \mu)$, where (X, d, μ) is a space of homogeneous type. We assume that T_t has an integral kernel $T_t(x, y)$ which satisfies the upper and lower Gaussian bounds:

$$\frac{C_1}{\mu(B(x, \sqrt{t}))} \exp(-c_1 d(x, y)^2/t) \leq T_t(x, y) \leq \frac{C_2}{\mu(B(x, \sqrt{t}))} \exp(-c_2 d(x, y)^2/t).$$

By definition, f belongs to $H^1(L)$ if $\|f\|_{H^1(L)} = \|\sup_{t>0} |T_t f(x)|\|_{L^1(X, \mu)} < \infty$. We prove that there is a function $\omega(x)$, $0 < c \leq \omega(x) \leq C$, such that $H^1(L)$ admits an atomic decomposition with atoms satisfying: $\text{supp } a \subset B$, $\|a\|_{L^\infty} \leq \mu(B)^{-1}$, and the weighted cancelation condition $\int a(x)\omega(x)d\mu(x) = 0$.

Keywords Hardy space · Maximal function · Atomic decomposition · Gaussian bounds · Hölder estimates

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1 Introduction

Let (X, d) be a metric space equipped with a nonnegative Borel measure μ . We shall assume that $\mu(X) = \infty$ and $0 < \mu(B(x, r)) < \infty$, $r > 0$, where $B(x, r) = \{y \in X \mid d(x, y) \leq r\}$ denotes the closed ball centered at x and radius r . Suppose (X, d, μ) is a space of homoge-

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neous type in the sense of Coifman and Weiss [8], which means that the doubling condition holds, namely: There is $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for $r > 0$ and $x \in X$. It is well known that the doubling condition implies that there are $q > 0$ and $C_d > 0$ such that

$$\mu(B(x, sr)) \leq C_d s^q \mu(B(x, r)) \quad \text{for } x \in X, r > 0, \text{ and } s \geq 1. \quad (1.1)$$

Suppose that L is a nonnegative densely defined self-adjoint linear operator on $L^2(X, \mu)$. Let $T_t = e^{-tL}$, $t > 0$, denote the semigroup of linear operators generated by $-L$. We impose that there exists $T_t(x, y)$, such that

$$T_t f(x) = \int_X T_t(x, y) f(y) d\mu(y). \quad (1.2)$$

Clearly, $T_t(x, y) = T_t(y, x)$ for $t > 0$ and a.e. $x, y \in X$. Moreover, we assume the following lower and upper Gaussian bounds, that is, there are constants $c_1 \geq c_2 > 0$ and $C_0 > 0$ such that

$$\frac{C_0^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{c_1 d(x, y)^2}{t}\right) \leq T_t(x, y) \leq \frac{C_0}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{c_2 d(x, y)^2}{t}\right) \quad (1.3)$$

for $t > 0$ and a.e. $x, y \in X$. It is well known that (1.3) implies that for every $n \in \mathbb{N}$ we have

$$\left| \frac{\partial^n}{\partial t^n} T_t(x, y) \right| \leq \frac{C_n t^{-n}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{c'_n d(x, y)^2}{t}\right), \quad (1.4)$$

for $t > 0$ and a.e. $x, y \in X$ with some $C_n, c'_n > 0$. For this fact see, e.g., [20, (7.1)], [10, 27].

The Hardy space $H^1(L)$ related to L is defined by means of the maximal function of the semigroup T_t , namely

$$H^1(L) := \left\{ f \in L^1(X, \mu) \mid \|f\|_{H^1(L)} := \left\| \sup_{t>0} |T_t f| \right\|_{L^1(X, \mu)} < \infty \right\}.$$

On the other hand, we define atomic Hardy spaces as follows. Suppose we have a space X with a doubling measure σ and a quasi-metric ρ . We call a function a an (ρ, σ) -atom if there exists a ball $B = B_\rho(x_0, r) := \{x \in X \mid \rho(x, x_0) \leq r\}$ such that: $\text{supp } a \subseteq B$, $\|a\|_\infty \leq \sigma(B)^{-1}$, and

$$\int_B a(x) d\sigma(x) = 0.$$

By definition, a function $f \in L^1(X, \sigma)$ belongs to $H_{\text{at}}^1(\rho, \sigma)$, if there exist (ρ, σ) -atoms a_k and complex numbers λ_k , such that

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \quad \text{and} \quad \sum_{k=1}^{\infty} |\lambda_k| < \infty.$$

If such sequences exist, we define the norm $\|f\|_{H_{\text{at}}^1(\rho, \sigma)}$ to be the infimum of $\sum_{k=1}^{\infty} |\lambda_k|$ in the above presentations of f . Notice that in this paragraph we have changed the notation. This is because in the article we will use different metrics and measures.

For $x \in X$ let $\Phi_x : (0, \infty) \rightarrow (0, \infty)$ be the nondecreasing function defined by

$$\Phi_x(t) = \mu(B(x, t)). \quad (1.5)$$

The first main result of this paper is the following.

Theorem 1 Suppose that (X, d, μ) is a space of homogeneous type and assume that for each $x \in X$ the function Φ_x is a bijection on $(0, \infty)$. Let a nonnegative self-adjoint linear operator L on $L^2(X, \mu)$ be given such that the semigroup $T_t = e^{-tL}$ satisfies (1.3). Then there exist a constant $C > 0$ and a function ω on X , $0 < C^{-1} \leq \omega(x) \leq C$, such that the spaces $H^1(L)$ and $H_{\text{at}}^1(d, \omega\mu)$ coincide and the corresponding norms are equivalent,

$$C^{-1} \|f\|_{H_{\text{at}}^1(d, \omega\mu)} \leq \|f\|_{H^1(L)} \leq C \|f\|_{H_{\text{at}}^1(d, \omega\mu)}.$$

Moreover, ω is L -harmonic, that is $T_t\omega = \omega$ for $t > 0$.

Let us notice that the assumption on Φ_x implies that $\mu(X) = \infty$ and μ is nonatomic. This will be used later on. By definition, we call a semigroup *conservative* if

$$\int_X T_t(x, y) d\mu(y) = 1 \quad (1.6)$$

for $t > 0$ and $x \in X$.

Theorem 2 Suppose that (X, d, μ) is a space of homogeneous type and assume that for each $x \in X$ the function Φ_x is a bijection on $(0, \infty)$. Let a nonnegative self-adjoint linear operator L on $L^2(X, \mu)$ be given such that the semigroup $T_t = e^{-tL}$ satisfies (1.3) and (1.6). Then, the spaces $H^1(L)$ and $H_{\text{at}}^1(d, \mu)$ coincide and the corresponding norms are equivalent, i.e., there exists $C > 0$ such that

$$C^{-1} \|f\|_{H_{\text{at}}^1(d, \mu)} \leq \|f\|_{H^1(L)} \leq C \|f\|_{H_{\text{at}}^1(d, \mu)},$$

It appears that Theorem 2 is equivalent to Theorem 1, see Sect. 3. Let us also emphasize that we do not require any regularity conditions on the kernels $T_t(x, y)$. However, it turns out that (1.3) implies Hölder-type estimates on $T_t(x, y)$, which are crucial for this paper. This will be discussed in Sect. 4 (see Theorem 5 and Corollary 14). In fact, Sect. 4 gives a short, independent, and self-contained proof of Hölder-type estimates of the heat kernel that satisfies (1.3), which can be interesting in its own right. Furthermore, although Theorems 1 and 2 are stated for the heat semigroup, we would like to emphasize that the same theorems can be proved for the Hardy space $H^1(\sqrt{L})$ associated with the Poisson semigroup $e^{-t\sqrt{L}}$ as well, see Theorem 4 and Remark 21.

The theory of the classical Hardy spaces on the Euclidean spaces $\mathbb{R}^n$ has its origin in studying holomorphic functions of one variable in the upper half-plane. The reader is referred to the original works of Stein and Weiss [32], Burkholder et al. [6], Fefferman and Stein [15]. Very important contribution to the theory is the atomic decomposition of the elements of the H^p spaces proved by Coifman [7] in the one-dimensional case and then by Latter [23] for $H^p(\mathbb{R}^n)$. The theory was then extended to the space of homogeneous type (see, e.g., [8, 24, 33]). For more information concerning the classical Hardy spaces, their characterizations and historical comments we refer the reader to Stein [31]. A very general approach to the theory of Hardy spaces associated with semigroups of linear operators satisfying the Davies–Gaffney estimates was introduced by Hofmann et al. [20] (see also [1, 30]). Let us point out that the classical Hardy spaces can be thought as those associated with the classical heat semigroup $e^{t\Delta}$. Finally we want to remark that the present paper takes motivation from [13, 14], where the authors studied H^1 spaces associated with Schrödinger operators $-\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$, with Green bounded potentials $V \geq 0$.

In what follows C and c denote different constants that may depend on C_d, q, C_0, c_1, c_2 . By $U \lesssim V$ we understand $U \leq CV$ and $U \simeq V$ means $C^{-1}V \leq U \leq CV$.

The paper is organized as follows. In Sect. 2 we prove that estimate (1.3) implies the existence of an L -harmonic function ω such that $0 < C^{-1} \leq \omega(x) \leq C$. The equivalence of Theorems 1 and 2 is a consequence of the Doob transform, see Sect. 3. Then a proof of Theorem 2 is given in a few steps. First, in Sect. 4 we prove Hölder-type estimates for a conservative semigroup. Then, in Sect. 5 we introduce a new quasi-metric $\tilde{d}$ and study its properties. In Sect. 6 we apply a theorem of Uchiyama on the space $(X, \tilde{d}, \mu)$ to complete the proof of Theorem 2. Finally, in Sect. 7 we provide some examples of semigroups that satisfy assumptions of Theorem 1.

2 Gaussian estimates and bounded harmonic functions

In this section we assume that the semigroup T_t satisfies (1.3). Clearly,

$$C^{-1} \leq \int_X T_t(x, y) d\mu(y) \leq C$$

for all $t > 0$ and a.e. $x \in X$. For a positive integer n define

$$\omega_n(x) = \frac{1}{n} \int_0^n \int_X T_s(x, y) d\mu(y) ds.$$

Then,

$$C^{-1} \leq \omega_n(x) \leq C.$$

Recall that a metric space with the doubling condition is separable, then so is $L^1(X, \mu)$. Using the Banach–Alaoglu theorem for $L^\infty(X, \mu)$ there exists a subsequence n_k and $\omega \in L^\infty(X, \mu)$, such that $\omega_{n_k} \rightarrow \omega$ in $*$ -weak topology. Obviously,

$$C^{-1} \leq \omega(y) \leq C.$$

Our goal is to prove that $T_t \omega(x) = \omega(x)$ for $t > 0$. To this end, we write

$$\begin{aligned} \int_X T_t(x, y) \omega(y) d\mu(y) &= \lim_{k \rightarrow \infty} \int_X T_t(x, y) \omega_{n_k}(y) d\mu(y) \\ &= \lim_{k \rightarrow \infty} \frac{1}{n_k} \int_0^{n_k} \int_X T_{t+s}(x, z) d\mu(z) ds \\ &= \lim_{k \rightarrow \infty} \omega_{n_k}(x) + \lim_{k \rightarrow \infty} \frac{1}{n_k} \int_{n_k}^{n_k+t} \int_X T_s(x, z) d\mu(z) ds \\ &\quad - \lim_{k \rightarrow \infty} \frac{1}{n_k} \int_0^t \int_X T_s(x, z) d\mu(z) ds. \end{aligned}$$

Since the last two limits tend to zero, as $k \rightarrow \infty$, we obtain

$$\int_X T_t(x, y) \omega(y) d\mu(y) = \lim_{k \rightarrow \infty} \omega_{n_k}(x). \quad (2.1)$$

From (2.1) we get that $\lim_{k \rightarrow \infty} \omega_{n_k}(x)$ exists for a.e. $x \in X$ and the limit has to be $\omega(x)$. Moreover, $T_t \omega(x) = \omega(x)$. Thus we have proved the following proposition.

Proposition 3 *Assume that a semigroup T_t satisfies (1.3). Then there exists a function ω and $C > 0$ such that $0 < C^{-1} \leq \omega(x) \leq C$ and $T_t \omega(x) = \omega(x)$ for every $t > 0$.*

3 Doob transform

In this section we work in a slightly more general scheme. Let (X, μ) be a σ -finite measure space and L be a self-adjoint operator on $L^2(X, \mu)$. We assume that the strongly continuous semigroup $T_t = \exp(-tL)$ admits a nonnegative integral kernel $T_t(x, y)$, so that

$$T_t f(x) = \int_X T_t(x, y) f(y) d\mu(y).$$

Moreover, assume that there exists ω satisfying $0 < C^{-1} \leq \omega(x) \leq C$, that is L -harmonic. Namely, for every $t > 0$ one has

$$\int_X T_t(x, y) \omega(y) d\mu(y) = \omega(x)$$

for a.e. $x \in X$. Obviously, this implies that $\sup_{x \in X, t > 0} \int T_t(x, y) d\mu(y) \leq C$.

Define a new measure $d\nu(x) = \omega^2(x) d\mu(x)$ and a new kernel

$$K_t(x, y) = \frac{T_t(x, y)}{\omega(x)\omega(y)}. \quad (3.1)$$

The semigroup K_t given by

$$K_t f(x) = \int_X K_t(x, y) f(y) d\nu(y)$$

is a strongly continuous semigroup of self-adjoint integral operators on $L^2(X, \nu)$. The mapping $L^2(X, \mu) \ni f \mapsto \omega^{-1}f \in L^2(X, \nu)$ is an isometric isomorphism with the inverse $L^2(X, \nu) \ni g \mapsto \omega g \in L^2(X, \mu)$. Clearly,

$$K_t g(x) = \omega(x)^{-1} T_t(\omega g)(x).$$

Moreover, the positive integral kernel $K_t(x, y)$ is conservative, that is,

$$\int_X K_t(x, y) d\nu(y) = \omega(x)^{-1} T_t \omega(x) = 1.$$

Thus the above change of measure and operators, which is called Doob's transform (see, e.g., [16]), conjugates the semigroup T_t with the conservative semigroup K_t .

It is clear that the operators K_t are contractions on $L^1(X, \nu)$. Consequently, K_t is a strongly continuous semigroup of linear operators on $L^1(X, \nu)$. To see this, it suffices to show that $\lim_{t \rightarrow 0} \|K_t \chi_A - \chi_A\|_{L^1(X, \nu)} = 0$ for any measurable set A of finite measure. We have that

$$\lim_{t \rightarrow 0} \int_A |K_t \chi_A - \chi_A| d\nu \leq \lim_{t \rightarrow 0} \|K_t \chi_A - \chi_A\|_{L^2(X, \nu)} \nu(A)^{1/2} = 0.$$

On the other hand $\int_X K_t \chi_A(x) d\nu(x) = \nu(A) = \|\chi_A\|_{L^1(X, \nu)}$. Hence,

$$\begin{aligned} \int_{A^c} |K_t \chi_A(x)| d\nu(x) &= \int_{A^c} K_t \chi_A(x) d\nu(x) = \int_X K_t \chi_A(x) d\nu(x) - \int_A K_t \chi_A(x) d\nu(x) \\ &= \int_A \chi_A(x) d\nu(x) - \int_A K_t \chi_A(x) d\nu(x) \rightarrow 0 \text{ as } t \rightarrow 0, \end{aligned}$$

which completes the proof of the strong continuity of K_t on $L^1(X, \nu)$.

Further, we easily see that the semigroup T_t is strongly continuous on $L^1(X, \mu)$. Indeed, if $f \in L^1(X, \mu)$ then $g = \omega^{-1}f \in L^1(X, \nu)$ and

$$\begin{aligned} \|T_t f - f\|_{L^1(X, \mu)} &= \int_X |K_t g(x) - g(x)| \omega(x)^{-1} d\nu(x) \\ &\lesssim \int_X |K_t g(x) - g(x)| d\nu(x) \rightarrow 0 \text{ as } t \rightarrow 0. \end{aligned}$$

Now we discuss the equivalence of Theorems 1 and 2. Let T_t and K_t be the semigroups related by (3.1) with generators $-L$ and $-R$, respectively. It easily follows from the Doob transform that $f \in H^1(L)$ if and only if $\omega^{-1}f \in H^1(R)$ and $\|f\|_{H^1(L)} \simeq \|\omega^{-1}f\|_{H^1(R)}$. In other words

$$H^1(L) \ni f \mapsto \omega^{-1}f \in H^1(R)$$

is an isomorphism of the spaces.

Assume that the space $H^1(R)$ admits an atomic decomposition with atoms that satisfy the cancelation condition with respect to the measure ν , that is every $g \in H^1(R)$ can be written as $g = \sum \lambda_j a_j$ with $\sum_j |\lambda_j| \simeq \|g\|_{H^1(R)}$ and a_j are atoms with the property $\int a_j d\nu = 0$. Then every $f \in H^1(L)$ admits atomic decomposition $f = \sum \lambda_j b_j$ with atoms b_j that satisfy cancelation condition $\int b_j d\mu = 0$.

4 Hölder-type estimates on the semigroups

In this section we consider a conservative semigroup T_t having an integral kernel $T_t(x, y)$ that satisfies upper and lower Gaussian bounds (1.3). Let

$$P_t(x, y) = \pi^{-1/2} \int_0^\infty e^{-u} T_{t^2/(4u)}(x, y) \frac{du}{\sqrt{u}} \quad (4.1)$$

be the kernels of the subordinate semigroup $P_t = e^{-t\sqrt{L}}$.

Theorem 4 *Let L be a nonnegative self-adjoint linear operator on $L^2(X, \mu)$ such that the semigroup $T_t = e^{-tL}$ satisfies (1.3) and (1.6). Then there is a constant $\alpha > 0$ such that*

$$|P_t(x, y) - P_t(x, z)| \lesssim \left(\frac{d(y, z)}{t} \right)^\alpha P_t(x, y) \quad (4.2)$$

whenever $d(y, z) \leq t$.

Theorem 5 *Under the assumptions of Theorem 4 there are constants $\alpha, c > 0$ such that*

$$|T_t(x, y) - T_t(x, z)| \lesssim \mu(B(x, \sqrt{t}))^{-1} \left(\frac{d(y, z)}{\sqrt{t}} \right)^\alpha \exp \left(-\frac{cd(x, y)^2}{t} \right) \quad (4.3)$$

whenever $d(y, z) \leq \sqrt{t}$.

Hölder regularity of semigroups satisfying Gaussian bounds was considered in various settings by many authors. We refer the reader to Grigor'yan [16], Hebisch and Saloff-Coste [19], Saloff-Coste [28], Gyrya and Saloff-Coste [17], Bernicot, Coulhon, and Frey [2], and references therein. Here we present a short alternative proof of (4.3). To this end we shall first prove some auxiliary propositions and then Theorem 4. Finally, at the end of this section, we shall make use of functional calculi deducing Theorem 5 from Theorem 4.

Proposition 6 For $x, y \in X$ and $t > 0$ we have

$$P_t(x, y) \simeq \frac{1}{\mu(B(x, t + d(x, y)))} \frac{t}{t + d(x, y)}. \quad (4.4)$$

Proof Consider first the case $d(x, y) \leq t$. Then $d(x, y) + t \simeq t$. The upper bound follows by (1.3) and (1.1). Indeed,

$$\begin{aligned} P_t(x, y) &\lesssim \int_0^{1/4} \mu\left(B\left(x, \frac{t}{2\sqrt{u}}\right)\right)^{-1} \frac{du}{\sqrt{u}} + \mu(B(x, t))^{-1} \int_{1/4}^\infty \frac{e^{-u} \mu(B(x, t))}{\mu\left(B\left(x, \frac{t}{2\sqrt{u}}\right)\right)} \frac{du}{\sqrt{u}} \\ &\lesssim \mu(B(x, t))^{-1} \int_0^{1/4} \frac{du}{\sqrt{u}} + \mu(B(x, t))^{-1} \int_{1/4}^\infty e^{-u} (2\sqrt{u})^q \frac{du}{\sqrt{u}} \\ &\lesssim \mu(B(x, t))^{-1}. \end{aligned}$$

Also, (1.3) implies lower bounds, since

$$P_t(x, y) \gtrsim \int_{1/4}^1 \frac{e^{-u} \exp(-4c_1 u)}{\mu\left(B\left(x, \frac{t}{2\sqrt{u}}\right)\right)} \frac{du}{\sqrt{u}} \gtrsim \mu(B(x, t))^{-1}.$$

Let us now turn to the case $d(x, y) \geq t$. Then $d(x, y) + t \simeq d(x, y)$. Using (1.3), we have

$$P_t(x, y) \lesssim \int_0^\infty \frac{e^{-u} \exp(-4c_2 u d(x, y)^2 / t^2)}{\mu\left(B\left(x, \frac{t}{2\sqrt{u}}\right)\right)} \frac{du}{\sqrt{u}} = \int_0^{t^2/d(x, y)^2} + \int_{t^2/d(x, y)^2}^\infty = (J_1) + (J_2). \quad (4.5)$$

Moreover, by (1.1),

$$\begin{aligned} (J_1) &\simeq \int_0^{t^2/d(x, y)^2} \frac{1}{\mu\left(B\left(x, \frac{t}{2\sqrt{u}}\right)\right)} \frac{du}{\sqrt{u}} \\ &\lesssim \int_0^{t^2/d(x, y)^2} \frac{1}{\mu\left(B\left(x, \frac{d(x, y)}{2}\right)\right)} \frac{du}{\sqrt{u}} \simeq \frac{1}{\mu(B(x, d(x, y)))} \frac{t}{d(x, y)} \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} (J_2) &\lesssim \frac{1}{\mu(B(x, d(x, y)))} \int_{t^2/d(x, y)^2}^\infty \exp(-4c_2 u d(x, y)^2 / t^2) \left(\frac{2d(x, y)\sqrt{u}}{t}\right)^q \frac{du}{\sqrt{u}} \\ &\simeq \frac{1}{\mu(B(x, d(x, y)))} \frac{t}{d(x, y)}. \end{aligned} \quad (4.7)$$

Estimates (4.5)–(4.7) give upper estimate. For the lower estimate, recall that $d(x, y) \geq t$ and observe that

$$\begin{aligned}
P_t(x, y) &\gtrsim \int_{t^2/d(x,y)^2}^{2t^2/d(x,y)^2} \frac{e^{-u} \exp(-4c_1 u d(x, y)^2/t^2)}{\mu\left(B\left(x, \frac{t}{2\sqrt{u}}\right)\right)} \frac{du}{\sqrt{u}} \\
&\simeq \frac{1}{\mu(B(x, d(x, y)/2))} \int_{t^2/d(x,y)^2}^{2t^2/d(x,y)^2} \frac{du}{\sqrt{u}} \\
&\simeq \frac{1}{\mu(B(x, d(x, y)))} \frac{t}{d(x, y)}.
\end{aligned}$$

□

Corollary 7 *There is a constant $C > 0$ such that if $d(y, z) \leq t$, then*

$$C^{-1} \leq \frac{P_t(x, y)}{P_t(x, z)} \leq C. \quad (4.8)$$

Proof The corollary is a simple consequence of (4.4). For the proof one may consider two cases: $d(x, y) \leq 2t$ (then $d(x, z) \leq 3t$) and $d(x, y) > 2t$ (then $d(x, y) \simeq d(x, z)$). □

Proposition 8 *There exists a constant $\gamma \in (0, 1)$ such that the following statement holds: If there are $y, z, t_0, a_1, b_1 > 0$ given such that $d(y, z) < t_0$ and for all $x \in X$ we have*

$$a_1 \leq \frac{P_{t_0}(x, y)}{P_{t_0}(x, z)} \leq b_1, \quad (4.9)$$

then there is a subinterval $[a_2, b_2] \subseteq [a_1, b_1]$ such that $b_2 - a_2 = \gamma(b_1 - a_1)$ and for all $t \geq 2t_0$ and all $x \in X$ one has

$$a_2 \leq \frac{P_t(x, y)}{P_t(x, z)} \leq b_2.$$

Proof The proof, which takes some ideas from [21], is an adapted version of the proof of [13, Prop. 3.1]. For the reader convenience we present the details. Let $m = (a_1 + b_1)/2$ and $\theta = (b_1 - a_1)/(a_1 + b_1) \in (0, 1)$, so that $a_1 = (1 - \theta)m$ and $b_1 = (1 + \theta)m$. Define

$$\Omega_+ = \left\{ x \in X \mid m \leq \frac{P_{t_0}(x, y)}{P_{t_0}(x, z)} \leq b_1 \right\}, \quad \Omega_- = X \setminus \Omega_+.$$

Obviously, either $\mu(\Omega_- \cap B(z, t_0)) \geq \mu(B(z, t_0))/2$ or $\mu(\Omega_+ \cap B(z, t_0)) \geq \mu(B(z, t_0))/2$. Here we shall assume that the latter holds. The proof in the opposite case is similar. Denote $B = B(z, t_0)$, $s = t - t_0 \geq t_0$. For $x \in X$, we have

$$\begin{aligned}
P_t(x, y) &\geq m(1 - \theta) \int_{\Omega_-} P_s(x, w) P_{t_0}(w, z) d\mu(w) + m \int_{\Omega_+} P_s(x, w) P_{t_0}(w, z) d\mu(w) \\
&= m(1 - \theta) P_t(x, z) + m\theta \int_{\Omega_+} P_s(x, w) P_{t_0}(w, z) d\mu(w) \\
&\geq m(1 - \theta) P_t(x, z) + m\theta \int_{\Omega_+ \cap B} P_s(x, w) P_{t_0}(w, z) d\mu(w) \\
&\geq m(1 - \theta) P_t(x, z) + m\theta \frac{\mu(B)}{2} \inf_{w \in B} P_s(x, w) \inf_{w \in B} P_{t_0}(w, z) = (J).
\end{aligned}$$

Notice that $d(w, z) \leq t_0 \leq s$ and, by Corollary 7,

$$\inf_{w \in B} P_s(x, w) \simeq P_s(x, z).$$

Since $t \simeq s$, Proposition 6 implies

$$P_s(x, z) \simeq P_t(x, z)$$

and

$$\inf_{w \in B} P_{t_0}(w, z) \simeq \mu(B)^{-1}.$$

Therefore

$$(J) \geq m((1 - \theta) + \kappa\theta) P_t(x, z) = m(1 - \theta(1 - \kappa)) P_t(x, z), \quad (4.10)$$

where $\kappa \in (0, 2)$. Moreover, from (4.9) and the semigroup property we easily get

$$P_t(x, y) \leq b_1 \int P_s(x, w) P_{t_0}(w, z) d\mu(w) \leq b_1 P_t(x, z). \quad (4.11)$$

Defining $\gamma = 1 - \kappa/2 \in (0, 1)$, $b_2 = b_1$ and $a_2 = m((1 - \theta) + \kappa\theta)$, we have

$$b_2 - a_2 = 2m\theta(1 - \kappa/2) = (b_1 - a_1)(1 - \kappa/2) = \gamma(b_1 - a_1).$$

Now (4.10) together with (4.11) gives

$$a_2 \leq \frac{P_t(x, y)}{P_t(x, z)} \leq b_2 \quad \text{for } x \in X.$$

□

Proof of Theorem 4 Having Corollary 7 and Proposition 8 proved, we follow arguments of [13] to obtain the theorem. By Corollary 7 there are $b_1 > a_1 > 0$ such that for $y, z \in X$ and $t > 0$ satisfying $d(y, z) < t$ we have

$$a_1 \leq \frac{P_t(x, y)}{P_t(x, z)} \leq b_1$$

for all $x \in X$. From Proposition 8 we deduce that there exists $\omega(y, z)$ such that

$$\lim_{t \rightarrow \infty} \frac{P_t(x, y)}{P_t(x, z)} = \omega(y, z) \quad \text{uniformly in } x \in X. \quad (4.12)$$

It follows from (4.1) that $\int_X P_t(x, y) d\mu(y) = 1$. Recall that $P_t(x, y) = P_t(y, x)$. Using (4.12),

$$1 = \int P_t(y, x) d\mu(x) = \int \frac{P_t(y, x)}{P_t(z, x)} P_t(z, x) d\mu(x) \xrightarrow{t \rightarrow \infty} \omega(y, z).$$

Thus $\omega(y, z) = 1$.

Assume that $d(y, z) < t$. Let $n \in \mathbb{N}$ be such that $d(y, z) \leq t2^{-n} < 2d(y, z)$. Set $t_0 = t2^{-n}$. Clearly, $d(y, z) \leq t_0$ and

$$a_1 \leq \frac{P_{t_0}(x, y)}{P_{t_0}(x, z)} \leq b_1.$$

Observe that $n \simeq \log(t/d(y, z))$. Applying Proposition 8 n -times we arrive at

$$\left| \frac{P_t(x, y)}{P_t(x, z)} - 1 \right| \lesssim \gamma^n \lesssim \gamma^{c \log(t/d(y, z))} \lesssim \left(\frac{d(y, z)}{t} \right)^\alpha,$$

with $\alpha > 0$ and the proof of Theorem 4 is finished. □

Finally, we devote the remaining part of this section for deducing Theorem 5 from Theorem 4. This is done by using a functional calculi. First, we need some preparatory facts. Recall that q is a fixed constant satisfying (1.1). By $W^{2,\sigma}(\mathbb{R})$ we denote the Sobolev space with the norm

$$\|f\|_{W^{2,\sigma}(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^\sigma |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Let $\int_0^\infty \xi dE_{\sqrt{L}}(\xi)$ be the spectral resolution for $\sqrt{L}$. For a bounded function m on $[0, \infty)$ the formula

$$m(\sqrt{L}) = \int_0^\infty m(\xi) dE_{\sqrt{L}}(\xi)$$

defines a bounded linear operator on $L^2(X, \mu)$.

Further we shall use the following lemma, whose proof based on finite speed propagation of the wave equation (see [9]) can be found in [11].

Lemma 9 [11, Lemma 4.8] *Let $\kappa > 1/2$, $\beta > 0$. Then there exists a constant $C > 0$ such that for every even function $m \in W^{2,\beta/2+\kappa}(\mathbb{R})$ and every $g \in L^2(X, \mu)$, $\text{supp } g \subset B(y_0, r)$, we have*

$$\int_{d(x,y_0) > 2r} \left| m(2^{-j}\sqrt{L})g(x) \right|^2 \left(\frac{d(x, y_0)}{r} \right)^\beta d\mu(x) \leq C(r2^j)^{-\beta} \|m\|_{W^{2,\beta/2+\kappa}(\mathbb{R})}^2 \|g\|_{L^2(X,\mu)}^2$$

for $j \in \mathbb{Z}$.

Proposition 10 *Let $\beta > q$ and $\kappa > 1/2$. There is a constant $C > 0$ such that for every $F \in W^{2,\beta/2+\kappa}(\mathbb{R})$ with $\text{supp } F \subset (1/2, 2)$ the integral kernels $F(2^{-j}\sqrt{L})(x, y)$ of the operators $F(2^{-j}\sqrt{L})$ satisfy*

$$\int_X |F(2^{-j}\sqrt{L})(x, y)| d\mu(x) \leq C \|F\|_{W^{2,\beta/2+\kappa}(\mathbb{R})} \quad \text{for } j \in \mathbb{Z}.$$

Proof For $y \in X$ and $j \in \mathbb{Z}$ set $U_0 = B(y, 2^{-j})$, $U_k = B(y, 2^{k-j}) \setminus B(y, 2^{k-j-1})$, $k = 1, 2, \dots$. Define $g_{j,k,y}(x) = T_{2^{-2j}}(x, y)\chi_{U_k}(x)$, $k = 0, 1, 2, \dots$. Then, using (1.1) and (1.3), we have

$$\|g_{j,y,k}\|_{L^2(X,\mu)} \leq a_k \mu(U_0)^{-1/2}, \quad (4.13)$$

where $a_k = C_0 \sqrt{C_q 2^{kq}} \exp(-c_2 2^{2k-2})$ is a rapidly decreasing sequence.

Let $m(\xi)$ be the even extension of $e^{\xi^2} F(\xi)$. Obviously, $\|m\|_{W^{2,\beta/2+\kappa}(\mathbb{R})} \simeq \|F\|_{W^{2,\beta/2+\kappa}(\mathbb{R})}$. Then, $F(2^{-j}\sqrt{L}) = m(2^{-j}\sqrt{L})T_{2^{-2j}}$, and, consequently,

$$F(2^{-j}\sqrt{L})(x, y) = \sum_{k=0}^{\infty} m(2^{-j}\sqrt{L})g_{j,y,k}(x). \quad (4.14)$$

By the Cauchy–Schwartz inequality, (1.1), and (4.13) we get

$$\begin{aligned} \|m(2^{-j}\sqrt{L})g_{j,y,k}\|_{L^1(B(y, 2^{k-j+1}), \mu)} &\leq \mu(B(y, 2^{k-j+1}))^{1/2} \|m(2^{-j}\sqrt{L})g_{j,y,k}\|_{L^2(X,\mu)} \\ &\lesssim \left(\frac{\mu(B(y, 2^{k-j+1}))}{\mu(B(y, 2^{-j}))} \right)^{1/2} \mu(B(y, 2^{-j}))^{1/2} \|m\|_{L^\infty(\mathbb{R})} \|g_{j,y,k}\|_{L^2(X,\mu)} \\ &\lesssim 2^{kq} a_k \|m\|_{W^{2,\beta/2+\kappa}(\mathbb{R})}. \end{aligned} \quad (4.15)$$

We turn to estimate $\|m(2^{-j}\sqrt{L})g_{j,y,k}\|_{L^1(B(y,2^{k-j+1})^c,\mu)}$. Utilizing the Cauchy–Schwarz inequality and Lemma 9, we obtain,

$$\begin{aligned} & \int_{d(x,y)>2^{k-j+1}} \left| m(2^{-j}\sqrt{L})g_{j,y,k}(x) \right| d\mu(x) \\ & \leq \left[\int_{d(x,y)>2^{k-j+1}} \left| m(2^{-j}\sqrt{L})g_{j,y,k}(x) \right|^2 \left(\frac{d(x,y)}{2^{k-j}} \right)^\beta d\mu(x) \right]^{1/2} \\ & \quad \times \left[\int_{d(x,y)>2^{k-j+1}} \left(\frac{d(x,y)}{2^{k-j}} \right)^{-\beta} d\mu(x) \right]^{1/2} \\ & \lesssim 2^{-k\beta/2} \|m\|_{W^{2,\beta/2+\kappa}(\mathbb{R})} \|g_{j,y,k}\|_{L^2(X,\mu)} \left[\int_{d(x,y)>2^{k-j+1}} \left(\frac{d(x,y)}{2^{k-j}} \right)^{-\beta} d\mu(x) \right]^{1/2}. \end{aligned} \quad (4.16)$$

Recall that $\beta > q$ and hence is not difficult to check that (4.1) leads to

$$\int_{d(x,y)>2^{k-j+1}} \left(\frac{d(x,y)}{2^{k-j}} \right)^{-\beta} d\mu(x) \lesssim 2^{kq} \mu(U_0). \quad (4.17)$$

Thus, by (4.13), (4.16), and (4.17), we get

$$\int_{d(x,y)>2^{k-j+1}} \left| m(2^{-j}\sqrt{L})g_{j,y,k}(x) \right| d\mu(x) \lesssim 2^{-k\beta/2} 2^{kq/2} \|m\|_{W^{2,\beta/2+\kappa}(\mathbb{R})} a_k,$$

which, combined with (4.15) and (4.14), completes the proof of the proposition. $\square$

For $t > 0$ set

$$\psi_t(\xi) = \exp\left(\sqrt{t}\xi - t\xi^2\right).$$

Lemma 11 *The operators $\psi_t(\sqrt{L})$ have integral kernels $\Psi_t(x, y)$ that satisfy*

$$\sup_{x \in X, t > 0} \int |\Psi_t(x, y)| d\mu(y) \leq C.$$

Proof Denoting $\theta_t(\xi) = \exp(-t\xi^2) (\exp(\sqrt{t}\xi) - 1)$, where $t, \xi > 0$, we have

$$\psi_t(\sqrt{L}) = T_t + \theta_t(\sqrt{L}).$$

Clearly, by (1.3), $\sup_{t,x>0} \int T_t(x, y) d\mu(y) \leq C$. Thus we concentrate our attention on $\theta_t(\sqrt{L})$.

Let $\eta \in C_c^\infty(1/2, 2)$ be a partition of unity such that

$$\theta_t(\xi) = \sum_{j \in \mathbb{Z}} \theta_t(\xi) \eta(2^{-j}\xi) = \sum_{j \in \mathbb{Z}} \theta_{t,j}(\xi).$$

Denote $\widetilde{\theta}_{t,j}(\xi) = \theta_{t,j}(2^j\xi) = \eta(\xi)\theta_1(\sqrt{t}2^j\xi)$. One can easily verify that for $n \in \mathbb{N} \cup \{0\}$ there are constants $C_n, c_n > 0$ such that

$$\left\| \frac{d^n}{d\xi^n} \widetilde{\theta}_{t,j} \right\|_{L^\infty(\mathbb{R})} \leq C_n \begin{cases} \sqrt{t} 2^j & \text{for } j \leq -\log_2 \sqrt{t} \\ \exp(-c_n t 2^{2j}) & \text{for } j \geq -\log_2 \sqrt{t} \end{cases}.$$

In other words, for arbitrary N ,

$$\sum_{j \in \mathbb{Z}} \|\tilde{\theta}_{t,j}\|_{W^{2,N}(\mathbb{R})} \leq C(N).$$

Using Proposition 10 (with a fixed $N > q/2 + 1$) we get that the integral kernels $\Theta_{t,j}(x, y)$ of the operators $\theta_{t,j}(\sqrt{L}) = \tilde{\theta}_{t,j}(2^{-j}\sqrt{L})$ satisfy

$$\int |\Theta_{t,j}(x, y)| d\mu(y) \lesssim \|\tilde{\theta}_{t,j}\|_{W^{2,N}(\mathbb{R})}.$$

Therefore, $\Theta_t(x, y) = \sum_{j \in \mathbb{Z}} \Theta_{t,j}(x, y)$ is the integral kernel of $\theta_t(\sqrt{L})$ and it satisfies

$$\sup_{t,x>0} \int |\Theta_t(x, y)| d\mu(y) \leq C.$$

□

Proof of Theorem 5 By the spectral theorem $T_t = \psi_t(\sqrt{L})P_{\sqrt{t}}$ and

$$T_t(x, y) = \int \Psi_t(x, w) P_{\sqrt{t}}(w, y) d\mu(w).$$

Using Theorem 4 together with Lemma 11 and Proposition 6, for $\sqrt{t} \geq d(y, z)$, we have

$$\begin{aligned} |T_t(x, y) - T_t(x, z)| &= \left| \int \Psi_t(x, w) (P_{\sqrt{t}}(w, y) - P_{\sqrt{t}}(w, z)) d\mu(w) \right| \\ &\lesssim \left(\frac{d(y, z)}{\sqrt{t}} \right)^\alpha \int |\Psi_t(x, w)| P_{\sqrt{t}}(w, y) d\mu(w) \\ &\lesssim \mu(B(y, \sqrt{t}))^{-1} \left(\frac{d(y, z)}{\sqrt{t}} \right)^\alpha. \end{aligned} \quad (4.18)$$

We claim that for $d(y, z) \leq \sqrt{t}$ one has

$$|T_t(x, y) - T_t(x, z)| \lesssim \mu(B(y, \sqrt{t}))^{-1} \left(\frac{d(y, z)}{\sqrt{t}} \right)^{\alpha/2} \exp\left(-\frac{cd(x, y)^2}{t}\right). \quad (4.19)$$

To prove the claim we consider two cases.

Case 1: $2d(y, z) \geq d(x, y)$. Recall that $\sqrt{t} \geq d(y, z)$; thus, $d(x, y) \leq 2\sqrt{t}$ and (4.19) follows directly from (4.18).

Case 2: $2d(y, z) \leq d(x, y)$. In this case $d(x, y) \simeq d(x, z)$, so by (1.3) we obtain

$$\begin{aligned} |T_t(x, y) - T_t(x, z)| &\lesssim \frac{\exp\left(-\frac{c_2 d(x, y)^2}{t}\right)}{\mu(B(y, \sqrt{t}))} + \frac{\exp\left(-\frac{c_2 d(x, z)^2}{t}\right)}{\mu(B(z, \sqrt{t}))} \\ &\lesssim \mu(B(y, \sqrt{t}))^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right), \end{aligned} \quad (4.20)$$

where in the last inequality we have utilized that $\mu(B(z, \sqrt{t})) \simeq \mu(B(y, \sqrt{t}))$, since $d(y, z) < \sqrt{t}$. By taking the geometric mean of (4.18) and (4.20) we obtain (4.19). To finish the proof observe that (4.19) implies (4.3). Indeed,

$$\mu(B(x, \sqrt{t})) = \mu(B(y, \sqrt{t})) \frac{\mu(B(x, \sqrt{t}))}{\mu(B(y, \sqrt{t}))} \leq \mu(B(y, \sqrt{t})) \frac{\mu(B(y, \sqrt{t} + d(x, y)))}{\mu(B(y, \sqrt{t}))}$$

$$\leq \mu(B(y, \sqrt{t})) \left(1 + \frac{d(x, y)}{\sqrt{t}}\right)^q$$

and using the exponent factor we can replace $\mu(B(y, \sqrt{t}))$ by $\mu(B(x, \sqrt{t}))$. $\square$

Remark 12 Let us remark that Lemma 11, which is crucial in our proof of Theorem 5, can be proved by applying functional calculus of Hebisch [18]. Thus, Theorem 5 can be also obtained without using the finite propagation speed of the solution of the wave equation

$$(\partial_t^2 + L)u(t, x) = 0, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = 0.$$

As a consequence of (1.4) and Theorem 5 we get what follows.

Corollary 13 *The function $T_t(x, y)$ is continuous on $(0, \infty) \times X \times X$.*

As a direct consequence of Theorem 5 and Doob transform (see (3.1)) we get the following corollary. Notice that in Corollary 14 we do not assume that $T_t(x, y)$ is conservative.

Corollary 14 *Assume that the semigroup T_t satisfies (1.3). Then there are constants $\alpha, c > 0$ such that*

$$\left| \frac{T_t(x, y)}{\omega(x)\omega(y)} - \frac{T_t(x, z)}{\omega(x)\omega(z)} \right| \lesssim \mu(B(x, \sqrt{t}))^{-1} \left(\frac{d(y, z)}{\sqrt{t}} \right)^\alpha \exp \left(-\frac{cd(x, y)^2}{t} \right)$$

whenever $d(y, z) \leq \sqrt{t}$.

5 Measures and distances

To prove Theorem 2 we introduce a new quasi-metric $\tilde{d}$ on X , which is related to d and μ . To this end, set

$$\tilde{d}(x, y) = \inf \mu(B),$$

where the infimum is taken over all closed balls B containing x and y (see, e.g., [8, 25]). Denote

$$\tilde{B}(x, r) = \{y \in X \mid \tilde{d}(x, y) \leq r\}.$$

In the lemma below we state some properties of $\tilde{d}$, which are known among specialists, and which we shall need latter on. Since their proofs are very short and it is difficult for us to indicate one reference which contains all of them, we provide the details for the convenience of the reader.

Lemma 15 *The function $\tilde{d}$ has the following properties: 0*

(a) *there exists $C_b > 0$ such that for $x, y \in X$ we have*

$$C_b^{-1} \mu(B(x, d(x, y))) \leq \tilde{d}(x, y) \leq \mu(B(x, d(x, y))). \quad (5.1)$$

(b) *$\tilde{d}$ is a quasi-metric, namely there exists A_1 such that*

$$\tilde{d}(x, y) \leq A_1 (\tilde{d}(x, z) + \tilde{d}(z, y)).$$

Moreover, if the measure μ has no atoms and $\mu(X) = \infty$, then:

(c) *the measure μ is regular with respect to $\tilde{d}$, namely for $x \in X$ and $r > 0$,*

$$\mu(\tilde{B}(x, r)) \simeq r;$$

(d) for $x \in X$ and $r > 0$ there exists $R > 0$ such that

$$\tilde{B}(x, r) \subset B(x, R) \text{ and } \mu(B(x, R)) \lesssim \mu(\tilde{B}(x, r));$$

(e) for $x \in X$ and $R > 0$ there exists $r > 0$ such that

$$B(x, R) \subset \tilde{B}(x, r) \text{ and } \mu(\tilde{B}(x, r)) \lesssim \mu(B(x, R)).$$

Proof (a) Set $R = d(x, y)$. Clearly, $\tilde{d}(x, y) \leq \mu(B(x, R))$, as x and y belong to $B(x, R)$. On the other hand, if x and y belong to a ball $B = B(z, r)$, then $R \leq 2r$; hence, $B(x, R) \subset B(z, 3r)$ and $\mu(B(x, R)) \leq \mu(B(z, 3r)) \simeq \mu(B(z, r))$. By taking the infimum over all balls B containing both x and y , we conclude that $\mu(B(x, R)) \leq C\tilde{d}(x, y)$.

(b) For every $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$. Assume that $r = d(x, z) \geq d(z, y)$. Then $x, y \in B(z, r)$. By using (a), we deduce that

$$\tilde{d}(x, y) \leq \mu(B(z, r)) \simeq \tilde{d}(z, x) \leq \tilde{d}(x, z) + \tilde{d}(z, y).$$

(c) Given $x \in X$, by our additional assumptions, the function $(0, \infty) \ni r \mapsto \mu(B(x, r))$ is increasing and

$$\begin{cases} \mu(B(x, r)) \searrow 0 & \text{as } r \searrow 0, \\ \mu(B(x, r)) \nearrow +\infty & \text{as } r \nearrow +\infty. \end{cases}$$

Let $x \in X$ and $r > 0$. For every $y \in \tilde{B}(x, r)$, we have $\mu(B(x, d(x, y))) \simeq \tilde{d}(x, y) \leq r$. Hence

$$R = \sup \{d(x, y) \mid y \in \tilde{B}(x, r)\} < +\infty.$$

Let $y \in \tilde{B}(x, r)$ such that $d(x, y) \geq \frac{R}{2}$. Then

$$\tilde{B}(x, r) \subset B(x, R) \subset B(x, 2d(x, y)). \quad (5.2)$$

Hence

$$\mu(\tilde{B}(x, r)) \leq \mu(B(x, R)) \leq \mu(B(x, 2d(x, y))) \simeq \mu(B(x, d(x, y))) \simeq \tilde{d}(x, y) \leq r. \quad (5.3)$$

On the other hand,

$$T = \inf \{t > 0 \mid \mu(B(x, t)) \geq r\} > 0.$$

As $\mu(B(x, T/2)) < r$, we have $\tilde{d}(x, y) < r$, for every $y \in B(x, T/2)$; hence, $B(x, T/2) \subset \tilde{B}(x, r)$. Consequently,

$$r \leq \mu(B(x, 2T)) \simeq \mu(B(x, T/2)) \leq \mu(\tilde{B}(x, r)),$$

which together with (5.3) completes the proof of (c).

(d) is a simple consequence of (5.2), (5.3), and (c).

(e) Set $r = \mu(B(x, R))$. If $y \in B(x, R)$, then $\tilde{d}(x, y) \leq r$ and, consequently, $B(x, R) \subset \tilde{B}(x, r)$. Clearly, by (c), $\mu(\tilde{B}(x, r)) \simeq r = \mu(B(x, R))$. $\square$

Let us recall that in Theorems 1 and 2 we assume that Φ_x is a bijection on $(0, \infty)$. This obviously implies that $\mu(X) = \infty$ and that μ is nonatomic. As a consequence of (d) and (e) of Lemma 15 we obtain the following corollary.

Corollary 16 *Suppose that μ has no atoms and $\mu(X) = \infty$. Then, the atomic Hardy spaces $H_{\text{at}}^1(d, \mu)$ and $H_{\text{at}}^1(\tilde{d}, \mu)$ coincide and the corresponding atomic norms are equivalent.*

We finish this section by Lemma 17, which is used latter on. Define $A_2 := C_b(C_d 2^q)^3$, where C_d , q , and C_b are as in (1.1) and (5.1).

Lemma 17 Suppose that we have a space of homogeneous type (X, d, μ) such that the function Φ_x defined in (1.5) is a bijection on $(0, \infty)$. Assume that $x \in X$, $r, t > 0$ are related by $r = \mu(B(x, \sqrt{t}))$ and satisfy: $\sqrt{t} \leq d(y, z)$, $A_2 \tilde{d}(y, z) < r$. Then

$$\sqrt{t} \leq d(x, y) \leq 2d(x, z).$$

Proof Suppose, toward a contradiction, that $d(x, y) < \sqrt{t}$. From (5.1) we get

$$\begin{aligned} r &= \mu(B(x, \sqrt{t})) \leq \mu(B(y, 2\sqrt{t})) \leq C_d 2^q \mu(B(y, \sqrt{t})) \\ &\leq C_d 2^q \mu(B(y, d(y, z))) \leq C_d 2^q C_b \tilde{d}(y, z) < r, \end{aligned}$$

so the first inequality is proved.

Similarly, assume $d(x, z) < d(x, y)/2$. Then $d(x, y)/2 \leq d(y, z) \leq 2d(x, y)$. Thus, using (5.1),

$$\begin{aligned} \tilde{d}(y, z) &\geq C_b^{-1} \mu(B(y, d(y, z))) \geq C_b^{-1} (C_d 2^q)^{-1} \mu(B(y, d(x, y))) \\ &\geq C_b^{-1} (C_d 2^q)^{-2} \mu(B(x, d(x, y))) \geq C_b^{-1} (C_d 2^q)^{-3} \mu(B(x, \sqrt{t})) = A_2^{-1} r \end{aligned}$$

and we come to a contradiction. $\square$

6 Proof of Theorem 2

In order to prove Theorem 2 we shall use a result of Uchiyama [33], which we state below in Theorem 18. Denote by $(X, \tilde{d}, \mu)$ the space X equipped with a quasi-metric $\tilde{d}$ and a nonnegative measure μ , where $\mu(X) = \infty$. Assume moreover that

$$\mu(\tilde{B}(x, r)) \simeq r, \quad (6.1)$$

where $x \in X$, $r > 0$ and $\tilde{B}(x, r) \subseteq X$ is a ball in the quasi-metric $\tilde{d}$. Let A_1 be a constant in the quasi-triangle inequality, i.e.,

$$\tilde{d}(x, y) \leq A_1 (\tilde{d}(x, z) + \tilde{d}(z, y)), \quad x, y, z \in X. \quad (6.2)$$

Additionally, we impose that there exist constants $\gamma_1, \gamma_2, \gamma_3 > 0$, $A \geq A_1$ and a continuous function $\tilde{T}(r, x, y)$ of variables $x, y \in X$ and $r > 0$ such that

$$\tilde{T}(r, x, x) \gtrsim r^{-1}, \quad (U_1)$$

$$0 \leq \tilde{T}(r, x, y) \lesssim r^{-1} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\gamma_1}, \quad (U_2)$$

$$\text{if } \tilde{d}(y, z) < (r + \tilde{d}(x, y))/(4A), \text{ then} \quad (U_3)$$

$$|\tilde{T}(r, x, y) - \tilde{T}(r, x, z)| \lesssim r^{-1} \left(\frac{\tilde{d}(y, z)}{r}\right)^{\gamma_2} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\gamma_3},$$

for all $x, y, z \in X$ and $r > 0$.

As in [33], we consider the maximal function

$$f^{(+)}(x) = \sup_{r>0} \left| \int_X \tilde{T}(r, x, y) f(y) d\mu(y) \right|$$

and the Hardy space $H^1(X, \tilde{T}) = \left\{ f \in L^1(X, \mu) \mid \|f\|_{H^1(X, \tilde{T})} := \|f^{(+)}\|_{L^1(X, \mu)} < \infty \right\}$.

Recall that the atomic space $H_{\text{at}}^1(\tilde{d}, \mu)$ is defined in Sect. 1.

Theorem 18 ([33], Corollary 1') Suppose that $(X, \tilde{d}, \mu, \tilde{T})$ satisfy (6.1), (6.2), (U1), (U2), and (U3). Then the spaces $H^1(X, \tilde{T})$ and $H_{\text{at}}^1(\tilde{d}, \mu)$ coincide and

$$\|f\|_{H^1(X, \tilde{T})} \simeq \|f\|_{H_{\text{at}}^1(\tilde{d}, \mu)}.$$

Assume that the kernel $T_t(x, y)$ satisfies (1.3) and the semigroup T_t is conservative. Recall that in Sect. 4 we proved Hölder-type estimate (4.3) for $T_t(x, y)$. Define $\tilde{T}(r, x, y)$ by

$$\tilde{T}(r, x, y) := T_t(x, y),$$

where $t = t(x, r)$ is such that

$$\mu(B(x, \sqrt{t})) = r. \quad (6.3)$$

In what follows $t, r > 0$ and $x \in X$ are always related by (6.3). Let us notice that from Corollary 13 and by the assumption that Φ_x is a continuous bijection on $(0, \infty)$ we have that $\tilde{T}$ is a continuous function on $(0, \infty) \times X \times X$.

Theorem 19 Suppose that $T_t(x, y)$ satisfies upper and lower Gaussian bounds (1.3) and the semigroup T_t is conservative. Then the kernel $\tilde{T}(r, x, y)$ satisfies (U1), (U2), and (U3).

Proof On-diagonal lower estimate (U1) is an immediate consequence of lower Gaussian bound (1.3).

For every fixed $\delta > 0$, the upper estimate

$$\tilde{T}(r, x, y) \lesssim r^{-1} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\delta} \quad (6.4)$$

follows from the upper estimates for $T_t(x, y)$, more precisely by combining

$$\tilde{T}(r, x, y) \lesssim r^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right) \quad (6.5)$$

with

$$\begin{aligned} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{1+\delta} &\leq \left(1 + \frac{\mu(B(x, d(x, y)))}{\mu(B(x, \sqrt{t}))}\right)^{1+\delta} \\ &\lesssim \left(1 + \frac{d(x, y)}{\sqrt{t}}\right)^{q(1+\delta)} \lesssim \exp\left(\frac{cd(x, y)^2}{t}\right). \end{aligned} \quad (6.6)$$

The latter estimate holds for any $\delta > 0$. Thus (U2) is proved with any $\gamma_1 > 0$. To finish the proof we need Hölder-type estimate (U3). This is proved in Proposition 20 below. $\square$

Proposition 20 Let α be a constant as in Theorem 5. There exists $A \geq A_1$ such that for $\delta > 0$ we have

$$|\tilde{T}(r, x, y) - \tilde{T}(r, x, z)| \lesssim r^{-1} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\delta} \left(\frac{\tilde{d}(y, z)}{r}\right)^{\frac{\alpha}{q}} \quad (6.7)$$

if $\tilde{d}(y, z) \leq (r + \tilde{d}(x, y))/(4A)$.

Proof Set $A = \max(A_1, A_2)$, see (6.2) and Lemma 17. Assuming that $\tilde{d}(y, z) \leq (r + \tilde{d}(x, y))/(4A)$ let us begin with some observations.

Firstly, we claim that it suffices to prove (6.7) for $\tilde{d}(y, z) < r/(2A)$. Indeed, if $\tilde{d}(y, z) > r/(2A)$, then $\tilde{d}(y, z) \leq \tilde{d}(x, y)/(2A)$ and, consequently,

$$\tilde{d}(x, y) \leq A_1 \tilde{d}(x, z) + A_1 \tilde{d}(z, y) \leq A_1 \tilde{d}(x, z) + \tilde{d}(x, y)/2.$$

So, $\tilde{d}(x, y) \lesssim \tilde{d}(x, z)$ and (6.7) follows from (6.4) by using the triangle inequality. From now on we assume that $\tilde{d}(y, z) < r/(2A)$.

Secondly, if $d(y, z) \leq \sqrt{t}$, then using (1.1) and (5.1),

$$\begin{aligned} \frac{d(y, z)}{\sqrt{t}} &\lesssim \left(\frac{\mu(B(y, d(y, z)))}{\mu(B(y, \sqrt{t}))} \right)^{1/q} \\ &\leq \left(\frac{\mu(B(y, d(y, z)))}{\mu(B(x, \sqrt{t}))} \right)^{1/q} \left(\frac{\mu(B(y, \sqrt{t} + d(x, y)))}{\mu(B(y, \sqrt{t}))} \right)^{1/q} \\ &\lesssim \left(\frac{\tilde{d}(y, z)}{r} \right)^{1/q} \left(1 + \frac{d(x, y)}{\sqrt{t}} \right). \end{aligned} \quad (6.8)$$

Thirdly, if $d(y, z) \geq \sqrt{t}$, then using (1.1) and (5.1),

$$\begin{aligned} 1 &\lesssim \frac{\tilde{d}(y, z)}{\mu(B(y, \sqrt{t}))} = \frac{\tilde{d}(y, z)}{r} \frac{\mu(B(x, \sqrt{t}))}{\mu(B(y, \sqrt{t}))} \\ &\lesssim \frac{\tilde{d}(y, z)}{r} \frac{\mu(B(y, \sqrt{t} + d(x, y)))}{\mu(B(y, \sqrt{t}))} \\ &\lesssim \frac{\tilde{d}(y, z)}{r} \left(1 + \frac{d(x, y)}{\sqrt{t}} \right)^q. \end{aligned} \quad (6.9)$$

Let us turn to the proof of (6.7).

Case 1: $d(y, z) \leq d(x, y)/2$. Then $d(x, y) \simeq d(x, z)$. Thus, according to (1.3) and (4.3) combined with (6.8) and (6.9) we obtain

$$\begin{aligned} |\tilde{T}(r, x, y) - \tilde{T}(r, x, z)| &\lesssim r^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right) \min\left(\frac{d(y, z)}{\sqrt{t}}, 1\right)^\alpha \\ &\lesssim r^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right) \left(\frac{\tilde{d}(y, z)}{r}\right)^{\alpha/q} \left(1 + \frac{d(x, y)}{\sqrt{t}}\right)^\alpha \\ &\lesssim r^{-1} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\delta} \left(\frac{\tilde{d}(y, z)}{r}\right)^{\alpha/q}, \end{aligned}$$

where in the last inequality we have used (6.6).

Case 2: $d(y, z) > d(x, y)/2$ and $\sqrt{t} > d(y, z)$. Then $\tilde{d}(x, y) \lesssim \mu(B(x, \sqrt{t})) = r$. Using (4.3) and (6.8),

$$|\tilde{T}(r, x, y) - \tilde{T}(r, x, z)| \lesssim r^{-1} \left(\frac{d(y, z)}{\sqrt{t}}\right)^\alpha \lesssim r^{-1} \left(\frac{\tilde{d}(y, z)}{r}\right)^{\alpha/q}.$$

Case 3: $d(y, z) > d(x, y)/2$ and $\sqrt{t} < d(y, z)$. Then, $\tilde{d}(y, z) \leq r/(2A) \leq r/(2A_2)$ and by Lemma 17 we have $2d(x, z) > d(x, y) \geq \sqrt{t}$. Hence, using (6.5) and (6.9),

$$\begin{aligned} |\tilde{T}(r, x, y) - \tilde{T}(r, x, z)| &\lesssim r^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right) \\ &\lesssim r^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right) \left(\frac{\tilde{d}(y, z)}{r}\right)^{\alpha/q} \left(1 + \frac{d(x, y)}{\sqrt{t}}\right)^\alpha \\ &\lesssim r^{-1} \left(\frac{\tilde{d}(y, z)}{r}\right)^{\alpha/q} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\delta}, \end{aligned}$$

where in the last inequality we have used (6.6). This finishes the proof of Proposition 20. $\square$

Proof of Theorem 2 Assuming (1.3) and (1.6) we obtain Hölder-type estimate (4.3). Recall once more that the assumption on Φ_x implies $\mu(X) = \infty$ and μ is nonatomic. Then we define a new quasi-metric $\tilde{d}$. By Corollary 16 we get that $H_{\text{at}}^1(d, \mu) = H_{\text{at}}^1(\tilde{d}, \mu)$. We apply Theorem 18 to the space $(X, \tilde{d}, \mu)$. The assumptions of Theorem 18 are verified in Theorem 19 and Proposition 20. In this way we get

$$\|f\|_{H^1(X, \tilde{T})} \simeq \|f\|_{H_{\text{at}}^1(\tilde{d}, \mu)}.$$

Using once again the assumption on Φ_x and the definition of $\tilde{T}$ we easily observe that

$$\|f\|_{H^1(X, \tilde{T})} = \|f\|_{H^1(L)},$$

which finishes the proof of Theorem 2. $\square$

Let us recall that Theorem 1 follows from Theorem 2. This is elaborated at the end of Sect. 3.

Remark 21 Under the assumptions of Theorem 1 one can prove, by the same methods, that the Hardy space

$$H^1(\sqrt{L}) = \left\{ f \in L^1(X, \mu) \mid \|f\|_{H^1(\sqrt{L})} := \left\| \sup_{t>0} |P_t f(x)| \right\|_{L^1(X, \mu)} < \infty \right\}$$

coincides with $H_{\text{at}}^1(d, \omega \mu)$ and the corresponding norms are equivalent. To this end, one uses: Proposition 6, Doob's transform, Theorem 4, and Theorem 18 applied to the kernel $\tilde{P}(r, x, y) = P_t(x, y)$, where $t = t(x, r)$ is defined by the relation $\mu(B(x, t)) = r$.

7 Examples

In this section we give examples of self-adjoint semigroups with the two-sided Gaussian bounds.

7.1 Laplace–Beltrami operators

Let (X, g) be a complete Riemannian manifold with the Riemannian distance $d(x, y)$ and the Riemannian measure μ satisfying the doubling property and the Poincaré inequality

$$\int_{B(x, r)} |f - f_B|^2 d\mu \leq Cr^2 \int_{B(x, 2r)} |\nabla f|^2 d\mu \quad \text{for all } r > 0,$$

where ∇ denotes the gradient on X . It is well known that the kernel of the heat semigroup generated by the Laplace–Beltrami operator satisfies two-sided Gaussian bounds (1.3) and Hölder estimates (4.3). For details and more information concerning the heat equation on Riemannian manifolds we refer the reader to [28] and references therein.

7.2 Schrödinger operators

On $X = \mathbb{R}^d$ with the Euclidean metric and the Lebesgue measure we consider the Schrödinger operator

$$L = -\Delta + V,$$

where Δ is the standard Laplacian and V is a locally integrable function.

It is well known (see, e.g., [29]) that for $V \geq 0$, $d \geq 3$, the semigroup $T_t = e^{-tL}$ admits kernels $T_t(x, y)$ with the upper and lower Gaussian bounds (1.3) if and only if V is a Green bounded potential, that is,

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^{2-d} V(y) dy < \infty. \quad (7.1)$$

Hardy spaces associated with Schrödinger operators on $\mathbb{R}^d$ satisfying (7.1) were studied in [13]. Actually, as we have already mentioned, this work motivated us to study the problem of H^1 spaces with the Gaussian bounds in the generality as in Theorems 1 and 2.

Our second example concerns Schrödinger operators $L = -\Delta + V$ with nonpositive potentials. For $d \geq 3$ fix $p_1, p_2 > 1$ satisfying $p_1 < d/2 < p_2$. Then there is a constant $c(p_1, p_2, d) > 0$ such that if $V \leq 0$ and

$$\|V\|_{L^{p_1}(\mathbb{R}^d)} + \|V\|_{L^{p_2}(\mathbb{R}^d)} \leq c(p_1, p_2, d),$$

then the integral kernel $T_t(x, y)$ of the semigroup $T_t = e^{-tL}$ exists and satisfies two-sided Gaussian bounds

$$T_t(x, y) \simeq t^{-d/2} \exp\left(-\frac{|x - y|^2}{4t}\right). \quad (7.2)$$

The result can be found in Zhang [35]. A slightly different proof of (7.2), based on bridges of the Gauss–Weierstrass semigroup, can be obtained by using Lemma 1.2 together with Proposition 2.2 of [4].

7.3 Bessel–Schrödinger operator

Let $\alpha > -1$ and consider $X = (0, \infty)$ and $d\mu(x) = x^\alpha dx$. Notice that the space (X, μ) with the Euclidean metric $d_e(x, y) = |x - y|$ is a space of homogeneous type. We consider the classical Bessel operator

$$\mathcal{B}f(x) = -f''(x) - \frac{\alpha}{x}f'(x),$$

which is self-adjoint positive on $L^2(X, \mu)$, and the associated semigroup of linear operators $S_t = e^{-t\mathcal{B}}$. It is well known that S_t is conservative and has the integral kernel

$$S_t(x, y) = (2t)^{-1} \exp\left(-\frac{x^2 + y^2}{4t}\right) I_{(\alpha-1)/2}\left(\frac{xy}{2t}\right) (xy)^{-(\alpha-1)/2}, \quad (7.3)$$

see, e.g., [5, Chapter 6]. Here $I_{(\alpha-1)/2}$ denotes the modified Bessel function of the first kind, see, e.g., [34]. The kernel $S_t(x, y)$ satisfies two-sided Gaussian estimates

$$\frac{C_0^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{c_1|x - y|^2}{t}\right) \leq S_t(x, y) \leq \frac{C_0}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{c_2|x - y|^2}{t}\right) \quad (7.4)$$

For a short proof of (7.4) see [12, proof of Lemma 4.2]. Therefore, using Theorem 2 we obtain atomic characterization of $H^1(\mathcal{B})$ that was previously proved in [3].

In this subsection we consider perturbations of $\mathcal{B}$ of the form

$$L = \mathcal{B} + V,$$

where a potential V is nonnegative and locally integrable. More precisely, on $L^2(X, \mu)$ we define the quadratic form

$$Q(f, g) = \int_X f'(x) \overline{g'(x)} d\mu(x) + \int_X V(x) f(x) \overline{g(x)} d\mu(x),$$

with the domain

$$\text{Dom}(Q) = \text{cl} \left\{ f \in C_c^1[0, \infty) \mid f'(0^+) = 0 \right\} \cap \left\{ f \in L^2(X, \mu) \mid \sqrt{V}f \in L^2(X, \mu) \right\},$$

where $\text{cl}(A)$ stands for the closure of the set A in the norm $\|f\|_{L^2(X, \mu)} + \|f'\|_{L^2(X, \mu)}$. The form Q is positive and closed. Thus, it corresponds to the unique self-adjoint operator L on $L^2(X, \mu)$ with the domain

$$\text{Dom}(L) = \left\{ f \in \text{Dom}(Q) \mid \left(\exists h \in L^2(X, \mu) \right) \left(\forall g \in \text{Dom}(Q) \right) Q(f, g) = \int h \overline{g} d\mu \right\}.$$

By definition, $Lf = h$ when f, h are related as above.

Let $T_t = \exp(-tL)$ be the semigroup generated by $-L$. The Feynman–Kac formula states that

$$T_t f(x) = E^x \left(\exp \left(- \int_0^t V(b_s) ds \right) f(b_t) \right), \quad (7.5)$$

where b_s is the Bessel process on $(0, \infty)$ associated with S_t . Using (7.5) one gets that the semigroup T_t has form (1.2), where

$$0 \leq T_t(x, y) \leq S_t(x, y). \quad (7.6)$$

Therefore the upper Gaussian estimates for $T_t(x, y)$ follows simply from (7.4) for any locally integrable $V \geq 0$. On the other hand, the relation between $S_t(x, y)$ and $T_t(x, y)$ is given by the perturbation formula

$$S_t(x, y) = T_t(x, y) + \int_0^t \int_X S_{t-s}(x, z) V(z) T_s(z, y) d\mu(z) ds \quad (7.7)$$

From now on we consider only $\alpha > 1$. We are interested in proving the lower Gaussian estimates, but this can be done only for some potentials V . For other potentials Hardy spaces may have a local character, (see, e.g., [22]). Let

$$\Gamma(x, y) = \int_0^\infty S_t(x, y) dt$$

be the formal kernel of B^{-1} . In addition to $V \in L_{loc}^1(X)$ and $V \geq 0$, we shall need one more assumption, a version of the global Kato condition, cf. (7.1),

$$\|V\|_{\text{Kato}} := \sup_{x \in X} \int_X \Gamma(x, y) V(y) d\mu(y) < \infty. \quad (7.8)$$

Formally, we can rephrase this as $B^{-1}V \in L^\infty(X, \mu)$. Let us point out that

$$\Gamma(x, y) \simeq (x + y)^{-\alpha+1}.$$

This can be easily obtained from (7.3) and well-known asymptotics for the Bessel function $I_{(\alpha-1)/2}$, see also [26, Sect. 2].

In Lemmas 22 and 23 we prove that under assumption (7.8) lower Gaussian estimates (1.3) hold for $T_t(x, y)$. The estimates are proved in a similar way as in the case of the Schrödinger operator on the Euclidean space. For the convenience of the reader we provide the details.

Lemma 22 Suppose that $\|V\|_{\text{Kato}} < \infty$. If $|x - y| \leq \sqrt{t}$, then

$$T_t(x, y) \geq C_l^{-1} \mu(B(x, \sqrt{t}))^{-1}.$$

Proof First we shall prove Lemma 22 with an additional assumption that $\|V\|_{\text{Kato}} \leq \varepsilon$ for a fixed small $\varepsilon > 0$. By (7.7) and (7.6),

$$\begin{aligned} S_t(x, y) - T_t(x, y) &= \int_0^t \int_X S_{t-s}(x, z) V(z) T_s(z, y) d\mu(z) ds = \int_0^{t/2} \dots ds + \int_{t/2}^t \dots ds \\ &\lesssim \mu(B(x, \sqrt{t}))^{-1} \int_0^{t/2} \int_X V(z) T_s(z, y) d\mu(z) ds \\ &\quad + \mu(B(y, \sqrt{t}))^{-1} \int_{t/2}^t \int_X S_{t-s}(x, z) V(z) d\mu(z) ds \\ &\lesssim \mu(B(x, \sqrt{t}))^{-1} \|V\|_{\text{Kato}}. \end{aligned}$$

By choosing proper $\varepsilon > 0$ we deduce the thesis from lower estimate (7.4) for $S_t(x, y)$.

Now assume that the norm $\|V\|_{\text{Kato}} < \infty$ is arbitrary. Fix $q > 1$, such that $\|V\|_{\text{Kato}} = q\varepsilon$. Set $V_q(x) = V(x)/q$, $\|V_q\|_{\text{Kato}} = \varepsilon$, and let T_t^q be the semigroup related to $L_q = \mathcal{B} + V_q$. Using (7.5) and Hölder's inequality,

$$\begin{aligned} T_t \left(\frac{\chi_{B(y,r)}(\cdot)}{\mu(B(y,r))} \right) (x) &= E^x \left(\left(\exp \left(- \int_0^t \frac{V(b_s)}{q} ds \right) \right)^q \frac{\chi_{B(y,r)}(b_t)}{\mu(B(y,r))} \right) \\ &\geq \left[E^x \left(\exp \left(- \int_0^t \frac{V(b_s)}{q} ds \right) \right)^q \frac{\chi_{B(y,r)}(b_t)}{\mu(B(y,r))} \right]^q \left[E^x \left(\frac{\chi_{B(y,r)}(b_t)}{\mu(B(y,r))} \right) \right]^{-\frac{q}{q'}} \\ &= \left[T_t^q \left(\frac{\chi_{B(y,r)}(\cdot)}{\mu(B(y,r))} \right) (x) \right]^q \cdot \left[S_t \left(\frac{\chi_{B(y,r)}(\cdot)}{\mu(B(y,r))} \right) (x) \right]^{-q/q'} \end{aligned} \quad (7.9)$$

Let us notice that

$$T_t(x, y) = \lim_{r \rightarrow 0} \mu(B(y, r))^{-1} \int_{B(y,r)} T_t(x, z) d\mu(z)$$

for a.e. (x, y) . By letting $r \rightarrow 0$ in (7.9), using (7.4) and the first part of the proof, for a.e. (x, y) we obtain that

$$T_t(x, y) \geq T_t^q(x, y)^q S_t(x, y)^{-q/q'} \gtrsim \mu(B(x, \sqrt{t}))^{-1}.$$

□

Lemma 23 Suppose that $\|V\|_{\text{Kato}} < \infty$. Then

$$T_t(x, y) \gtrsim \mu(B(x, \sqrt{t}))^{-1} \exp \left(- \frac{c|x - y|^2}{t} \right).$$

Proof Assume that $|x - y|^2/t \geq 1$ and set $m = \lceil 4|x - y|^2/t \rceil \geq 4$. For $i = 0, \dots, m$, let $x_i = x + i(y - x)/m$, so that $x_0 = x$, $x_m = y$, and $|x_{i+1} - x_i| = |x - y|/m$. Denote $B_i = B(x_i, \sqrt{t}/(4\sqrt{m}))$ and observe that

$$|y_{i+1} - y_i| \leq |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \leq \frac{\sqrt{t}}{4\sqrt{m}} + \frac{\sqrt{t}}{2\sqrt{m}} + \frac{\sqrt{t}}{4\sqrt{m}} = \frac{\sqrt{t}}{\sqrt{m}}$$

for $y_i \in B_i$ and $y_{i+1} \in B_{i+1}$. Now we use the semigroup property, Lemma 22, and the doubling property of μ to obtain

$$\begin{aligned} T_t(x, y) &= \int_X \dots \int_X T_{t/m}(x, y_1) T_{t/m}(y_1, y_2) \dots T_{t/m}(y_{m-1}, y) d\mu(y_1) \dots d\mu(y_{m-1}) \\ &\geq c_1^{m-1} \int_{B_1} \dots \int_{B_{m-1}} \mu(B(x, \sqrt{t/m}))^{-1} \dots \mu(B(y_{m-1}, \sqrt{t/m}))^{-1} d\mu(y_1) \dots d\mu(y_{m-1}) \\ &\geq c_2^{m-1} \mu(B(x, \sqrt{t/m}))^{-1} \frac{\mu(B_1) \dots \mu(B_{m-1})}{\mu(B(x_1, \sqrt{t/m})) \dots \mu(B(x_{m-1}, \sqrt{t/m}))} \\ &\geq c_3^m \mu(B(x, \sqrt{t}))^{-1} = \mu(B(x, \sqrt{t}))^{-1} e^{-m \ln c_3^{-1}} \gtrsim \mu(B(x, \sqrt{t}))^{-1} e^{-\frac{c|x-y|^2}{t}}. \end{aligned}$$

Notice that c_1, c_2, c_3 , and c in this estimate depend only on the constant C_l from Lemma 22 and the doubling constant of μ . $\square$

Obviously, the space (X, d_e, μ) satisfies the assumptions of Theorem 1. Since we have the two-sided Gaussian estimates for T_t (see Lemma 22, (7.6), and (7.4)) we obtain the following corollary.

Corollary 24 *Suppose that $\alpha > 1$ and $V \geq 0$ satisfies (7.8). Then there exists ω such that $0 < C^{-1} \leq \omega(x) \leq C$, $T_t \omega = \omega$ for $t > 0$ and $H^1(L) = H_{\text{at}}^1(d_e, \omega \mu)$. Moreover,*

$$\|f\|_{H^1(L)} \simeq \|f\|_{H_{\text{at}}^1(d_e, \omega \mu)}.$$

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