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Energy estimates for the Cauchy problem of Klein–Gordon-type equations with non-effective and very fast oscillating time-dependent potential

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Abstract The aim of this paper is to prove some energy estimates for Klein–Gordon equations with time-dependent potential. If the potential is "non-effective" and has "very slow oscillations" in the time-dependent coefficient, then energy estimates are proved in Ebert et al. (in: Dubatovskaya and Rogosin (eds) AMADE 2012, Cambridge Scientific Publishers, Cambridge, 2014). In contrast, the main goal of the present paper is to generalize the previous results to potentials with "very fast oscillations" in the time-dependent coefficient; consequently, the positivity of the potential is not required anymore.

Keywords Energy estimates · Very fast oscillations · Klein–Gordon equation · Time-dependent potential

Mathematics Subject Classification 35L15 · 35B40 · 35L71

1 Introduction

In this paper we consider the following Cauchy problem for Klein–Gordon equations with time-dependent potential:

$$\begin{cases} u_{tt} - \Delta u + M(t)u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

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where $M = M(t)$ is real-valued. A large amount of work has been devoted to (1). In particular, we take up the recent works [1–3,6], which studied the asymptotic properties of the energy as t tends to infinity. For a positive continuous function $p = p(t)$ we introduce the following energy to the solution of (1):

$$E(u; p)(t) := \frac{1}{2}\left(\|\nabla u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2 + p(t)\|u(t,\cdot)\|_{L^2}^2\right). \tag{2}$$

In [1,2] the authors studied the following scale-invariant model for the coefficient in the potential

$$M(t) = \frac{\mu^2}{(1+t)^2} \tag{3}$$

with a positive number μ, and they showed the following energy estimate:

$$E(u; p)(t) \lesssim E(u; p)(0) \tag{4}$$

with

$$p(t) = \begin{cases} (1+t)^{-1} & \text{for } \mu > 1/2, \\ (1+t)^{-1}(\log(e+t))^{-2} & \text{for } \mu = 1/2, \\ (1+t)^{-1-\sqrt{1-4\mu^2}} & \text{for } \mu < 1/2, \end{cases} \tag{5}$$

where $f \lesssim g$ with positive functions f and g denotes that there exists a positive constant C such that the estimate $f \le Cg$ is valid. Moreover, $f \simeq g$ denotes that f and g satisfy $f \lesssim g$ and $g \lesssim f$. Thus we observe that the influence of the potential to the energy is quite different if $0 < \mu < 1/2$ or $\mu > 1/2$; the potential satisfying the former is called "non-effective." Generally, the time-dependent potential $M(t)u$ is "non-effective" if $\limsup_{t\to\infty}(1+t)\int_t^\infty M(s)\,ds < 1/4$. Here the notion of effective and non-effective coefficients originated from [13,14] for the classification of the time-dependent dissipations to dissipative wave equations

$$\tilde{u}_{tt} - \Delta\tilde{u} + 2b(t)\tilde{u}_t = 0, \tag{6}$$

which is identified with the Klein–Gordon equation of (1) by

$$\tilde{u} = \exp\left(-\int_0^t b(s)\,ds\right)u \quad \text{and } M = -b' - b^2. \tag{7}$$

It is known that the "oscillations" of variable coefficients have a crucial influence on energy estimates for hyperbolic equations. For instance, it is shown in [11,12] that the energy to the solution of the Cauchy problem for the wave equation with time-dependent propagation speed

$$u_{tt} - a(t)\Delta u = 0 \tag{8}$$

can be unbounded as $t \to \infty$ if $a(t)$ is oscillating very fast though $a(t)$ is bounded and strictly positive. Precisely, the energy is not bounded in general if the estimate $|a'(t)| \lesssim (1+t)^{-\beta}$ with $\beta < 1$ holds; on the other hand the energy is bounded uniformly with respect to t if $\beta > 1$. Moreover, it is proved in [11] that the energy is also uniformly bounded with respect to t if $|a'(t)| \lesssim (1+t)^{-1}$ and $|a''(t)| \lesssim (1+t)^{-2}$ with $a \in C^2([0,\infty))$. Here the oscillations in the coefficient $a(t)$ satisfying $|a'(t)| \lesssim (1+t)^{-\beta}$ with $\beta < 1$ and $\beta \ge 1$ are called "very fast" and "very slow," respectively. Thus we expect that if the oscillations in the coefficient are very slow, then the asymptotic behavior of the energy is the same as in the case without any oscillations. The notion of oscillations can be introduced to dissipative wave equation (6) with

$$\frac{a'(t)}{2a(t)^2} = b\left(\int_0^t a(s)\,\mathrm{d}s\right) \tag{9}$$

and to the Klein–Gordon equation of (1) with (7). Energy estimates with very slow oscillating coefficients in the dissipation $b(t)u_t$ and in the potential $M(t)u$, which were described by $|b(t)| \lesssim (1+t)^{-1}$ and $|M(t)| \lesssim (1+t)^{-2}$, were studied in [13,14], and [6], respectively.

Generally, very fast oscillating coefficients may destroy estimates, which are valid for slow oscillating coefficients (see [4]). However, some additional assumptions to the coefficient enable the estimates, even though the oscillations become very fast. Indeed, the energy to the solution of (8) can be bounded uniformly with respect to t although the oscillations of $a(t)$ are very fast if $a \in C^m([0,\infty))$ with $m \geq 2$ and there exist positive constants $a_\infty > 0$ and $\alpha \in [0, 1)$ such that

$$\int_0^t |a(s) - a_\infty|\,\mathrm{d}s = O(t^\alpha) \quad (t \to \infty), \tag{10}$$

here (10) is called the stabilization property for (8) (see [5,7,9]). Corresponding stabilization properties were studied in [8] for dissipative wave equations (6) with non-effective dissipation and in [3] for Klein–Gordon equations of (1) with effective potential. Briefly, the aim of the present paper is to prove energy estimate (4) to the solutions of (1) with non-effective and very fast oscillating coefficient in the potential to introduce a suitable stabilization property.

This paper is organized as follows. In Sect. 2 we give the main theorem and corresponding examples. In Sect. 3 we introduce the strategy of the proof and some estimates to be used in the proof. In Sect. 4 we prove some estimates of the micro-energy in restricted phase spaces. In Sect. 5 we give the proof of our main theorem. In Sect. 6 we give some concluding remarks, and Sect. 7 is an appendix.

2 Main result

In this paper we restrict ourselves to the following special structure of the coefficient in the potential:

$$M(t) = \frac{\mu^2}{(1+t)^2} + \delta(t) \tag{11}$$

as a perturbed model of scale-invariant potential (3). For the perturbation $\delta = \delta(t)$ we introduce the following hypothesis:

Hypothesis 1 (*Non-effective condition*) The principal part of $M(t)$ is non-effective, that is, μ satisfies

$$0 < \mu < \frac{1}{2}. \tag{12}$$

Hypothesis 2 (*Oscillation condition*) There exists a real number β satisfying $\beta < 1$ such that

$$|\delta(t)| \lesssim (1+t)^{-2\beta}. \tag{13}$$

Hypothesis 3 (*Stabilization condition*) There exists a real number γ satisfying $\gamma > 1$ such that

$$\left|\int_t^\infty \delta(s)\,\mathrm{d}s\right| \lesssim (1+t)^{-\gamma}. \tag{14}$$

Then our main theorem is given as follows:

Theorem 1 *Let $\delta \in C^0([0,\infty))$ and Hypotheses 1, 2 and 3 be fulfilled. If the following conditions hold*:

$$2\beta \begin{cases} \geq -\gamma+3 & \text{for } \gamma \neq 2, \\ > 1 & \text{for } \gamma = 2, \end{cases} \tag{15}$$

then energy estimate (4) with (5) is established.

Remark 1 By Hypotheses 1 and 3 we see that

$$\limsup_{t\to\infty}(1+t)\left|\int_t^\infty M(s)\,\mathrm{d}s\right| \leq \mu^2 + \limsup_{t\to\infty}(1+t)\left|\int_t^\infty \delta(s)\,\mathrm{d}s\right| < \frac{1}{4}.$$

It follows that $M(t)u$ is non-effective.

Remark 2 Hypothesis 2 and 3 do not require the asymptotic behavior $\delta(t) = O(t^{-2})$ $(t \to \infty)$. Hence, (11) is not always a small perturbation of scale-invariant model (3) in the sense of the L^∞ norm. In other words, $M(t)$ is allowed to possess very fast oscillations if one reduces the Klein–Gordon equation to (6) and (8) by (7) and (9). Indeed, we will introduce some examples of $\delta(t)$, to which one can apply Theorem 1. These examples satisfy

$$\liminf_{t\to\infty} t^\beta \delta(t) < 0. \tag{16}$$

It follows that for any $T > 0$ there exists T_1 satisfying $T_1 > T$ such that $M(T_1) < 0$. Here we note that models with negative time-dependent potential appear in some cosmology models (see [15]).

Example 1 We define $M(t)$ by $M(t) = \mu^2(1+t)^{-2} + \delta(t)$ with

$$\delta(t) = \frac{\sin((1+t)^\kappa)}{(1+t)^{2\beta}},$$

where β, μ and κ are real numbers satisfying $0 < \mu < 1/2$, $\kappa > 0$ and $1 - \kappa/4 \leq \beta < 1$. Thus Hypotheses 1 and 2 are satisfied. By Lemma 7 from "Appendix" we have

$$\left|\int_t^\infty \delta(s)\,\mathrm{d}s\right| = \frac{1}{\kappa}\left|\int_{(1+t)^\kappa}^\infty \frac{\sin\theta}{\theta^{1+\frac{2\beta-1}{\kappa}}}\,\mathrm{d}\theta\right| \lesssim (1+t)^{-(2\beta+\kappa-1)}.$$

Thus Hypothesis 3 is satisfied with $\gamma = 2\beta+\kappa-1 \geq 1+\kappa/2 > 1$. Therefore, Theorem 1 is applicable since (15) holds, that is, if

$$\beta \begin{cases} \geq 1 - \dfrac{\kappa}{4} & \text{for } 2\beta+\kappa \neq 3, \\ > \dfrac{1}{2} & \text{for } 2\beta+\kappa = 3. \end{cases} \tag{17}$$

Example 2 Let $\{t_j\}_{j=1}^{\infty} = \{2\pi j\}_{j=1}^{\infty}$. We define $M(t)$ by $M(t) := \mu^2(1+t)^{-2} + \delta(t)$ with

$$\delta(t) := \begin{cases} t_j^{-2\beta} \sin\left(t_j^{\kappa}(t-t_j)\right) & t \in \left[t_j, t_j + 2\pi t_j^{-\kappa}\right], \\ 0 & t \in [0,\infty) \setminus \bigcup_{j=1}^{\infty} \left[t_j, t_j + 2\pi t_j^{-\kappa}\right], \end{cases}$$

where β, μ and κ are real numbers satisfying $0 < \mu < 1/2, \kappa > 0$ and $1-(1+\kappa)/4 \le \beta < 1$. Thus Hypotheses 1 and 2 are satisfied. Here we note that $t_j + 2\pi t_j^{-\kappa} < t_{j+1}$. For an arbitrarily given positive real number t satisfying $t \ge t_1$ we take $N \in \mathbb{N}$ satisfying $t_N \le t < t_{N+1}$. Then we have

$$|\delta(t)| \le t_N^{-2\beta} = \left(\frac{N+1}{N}\right)^{2\beta} t_{N+1}^{-2\beta} \simeq (1+t)^{-2\beta}.$$

Moreover, if we note $\int_{t_j}^{t_j+2\pi t_j^{-\kappa}} \delta(s)\,\mathrm{d}s = \int_{t_j}^{t_{j+1}} \delta(s)\,\mathrm{d}s = 0$ for any j, we have

$$\begin{aligned} \left|\int_t^{\infty} \delta(s)\,\mathrm{d}s\right| &= \left|\int_t^{t_{N+1}} \delta(s)\,\mathrm{d}s\right| \le \int_{t_N}^{t_N+2\pi t_N^{-\kappa}} |\delta(s)|\,\mathrm{d}s \le 4t_N^{-(2\beta+\kappa)} \\ &= 4\left(\frac{N+1}{N}\right)^{2\beta+\kappa} t_{N+1}^{-(2\beta+\kappa)} \simeq (1+t)^{-(2\beta+\kappa)}. \end{aligned}$$

Thus Hypothesis 3 is satisfied with $\gamma = 2\beta + \kappa \ge (3+\kappa)/2 > 1$. Therefore, Theorem 1 is applicable since (15) holds, that is, if

$$\beta \ge \begin{cases} 1 - \dfrac{1+\kappa}{4} & \text{for } 2\beta+\kappa \ne 2, \\ \dfrac{1}{2} & \text{for } 2\beta+\kappa = 2. \end{cases} \tag{18}$$

3 Strategy of the proof and some estimates

3.1 Reduction to the dissipative wave equation

The basic strategy for the proof of our main theorem is the reduction to dissipative wave equation (6). The change of variables $u(t,x) = \eta(t)w(t,x)$ with $\eta = \eta(t) \in C^2([0,\infty))$ transforms the Klein–Gordon equation into the following dissipative wave equation:

$$w_{tt} - \Delta w + \frac{2\eta'(t)}{\eta(t)} w_t + \left(\frac{\eta''(t)}{\eta(t)} + M(t)\right) w = 0. \tag{19}$$

If we take η as the solution to the following Liouville-type equation:

$$\eta''(t) + M(t)\eta(t) = 0, \tag{20}$$

then (19) corresponds to (6) with $b = \eta'/\eta$. Hence, we may apply arguments for treating dissipative wave equations with very fast oscillations in [8]. The idea was introduced in [6] for very slow oscillating potentials. Thus the main task for the proof of our main theorem is to develop their method for very fast oscillating potentials.

Let T be a nonnegative number and $\{q_k(t)\}_{k=1}^{\infty}$ be defined by

$$q_k(t) := \begin{cases} M(t) & \text{for } k = 1, \\ \sum_{j=1}^{k-1} \left(\int_t^{\infty} q_j(s)\,\mathrm{d}s\right)\left(\int_t^{\infty} q_{k-j}(s)\,\mathrm{d}s\right) & \text{for } k \ge 2. \end{cases} \tag{21}$$

Then a solution to (20) is represented formally as follows:

$$\eta(t) = \exp\left(\sum_{j=1}^{\infty} \int_T^t \int_\tau^\infty q_j(s)\,\mathrm{d}s\,\mathrm{d}\tau\right) \tag{22}$$

(see Lemma 8 in Sect. 7). Therefore, we shall consider the following problems to realize our strategy:

(i) Uniform convergence of $\eta(t)$ on any compact interval of $[0, \infty)$.
(ii) Estimates of the oscillations and the stabilization to $b(t) = \eta'(t)/\eta(t)$.

3.2 Convergence of $\eta(t)$

Let μ be a real number satisfying (12), and γ_k be kth Catalan's number defined by

$$\gamma_k := \frac{(2k)!}{k!(k+1)!} \quad (k = 0, 1, \ldots).$$

If we define ν and $\mu_k(t)$ by

$$\nu := \frac{1 - \sqrt{1 - 4\mu^2}}{2} \tag{23}$$

and

$$\mu_k(t) := \gamma_{k-1}\mu^{2k}(1+t)^{-2},$$

then we have the following lemma:

Lemma 1 *The following equalities are established*:

$$\sum_{k=1}^{\infty} \mu_k(t) = \nu(1+t)^{-2} \tag{24}$$

and

$$\sum_{j=1}^{k-1} \left(\int_t^\infty \mu_j(s)\,ds\right)\left(\int_t^\infty \mu_{k-j}(s)\,ds\right) = \mu_k(t) \tag{25}$$

for any $k \ge 2$.

Proof By the generating function of the sequence of Catalan's numbers:

$$\sum_{j=0}^{\infty} \gamma_j r^j = \frac{1 - \sqrt{1 - 4r}}{2r} \tag{26}$$

(see Lemma 9 in Sect. 7), we have

$$\sum_{k=1}^{\infty} \mu_k(t) = \mu^2 \sum_{j=0}^{\infty} \gamma_j \mu^{2j}(1+t)^{-2} = \frac{1 - \sqrt{1 - 4\mu^2}}{2}(1+t)^{-2}.$$

By the equalities $\int_t^\infty \mu_k(s)\,ds = (1+t)\mu_k(t)$ and

$$\gamma_{k-1} = \sum_{j=0}^{k-1} \gamma_{j-1}\gamma_{k-j-1},$$

it follows that

$$\sum_{j=1}^{k-1} \mu_j(t)\mu_{k-j}(t) = \gamma_{k-1}\mu^{2k}(1+t)^{-4} = \mu_k(t)(1+t)^{-2} \tag{27}$$

for $k \geq 2$. Hence we have

$$\sum_{j=1}^{k-1} \left(\int_t^\infty \mu_j(s)\,\mathrm{d}s\right)\left(\int_t^\infty \mu_{k-j}(s)\,\mathrm{d}s\right) = (1+t)^2 \sum_{j=1}^{k-1} \mu_j(t)\mu_{k-j}(t) = \mu_k(t).$$

□

Remark 3 If $\delta(t) = 0$ and $0 < \mu < 1/2$, then $q_k(t) = \mu_k(t)$, and thus $\eta(t) = (1+t)^\nu$ and $b(t) = \nu(1+t)^{-1}$ with $0 < \nu < 1/2$.

Let $\sigma_k(t)$ with $k = 1, 2, \ldots$ be the error of $q_k(t)$ from $\mu_k(t)$, that is,

$$\sigma_k(t) := q_k(t) - \mu_k(t).$$

Then we have the following lemmas:

Lemma 2 *For any positive real number ρ there exist constants $T \geq 0$ and $\omega_0 > 0$ such that*

$$|\sigma_k(t)| \leq \omega_0((1+\rho)^k - 1)(1+t)^{-\gamma+1}\mu_k(t) \quad (k = 1, 2, \ldots) \tag{28}$$

and

$$\left|\int_t^\infty \sigma_k(s)\,ds\right| \leq \omega_0((1+\rho)^k - 1)(1+t)^{-\gamma+2}\mu_k(t) \quad (k = 1, 2, \ldots) \tag{29}$$

for any $t \geq T$.

Proof By (14) there exists a positive constant ρ_0 such that

$$\rho_0 = \max\left\{\rho\mu^2, \sup_{t\geq 0}\left\{(1+t)^\gamma \left|\int_t^\infty \delta(s)\,\mathrm{d}s\right|\right\}\right\}.$$

For a given positive real number ρ we define ω_0 and T by

$$\omega_0 := \frac{\rho_0}{\rho\mu^2} \quad \text{and } T := \omega_0^{\frac{1}{\gamma-1}} - 1. \tag{30}$$

Then for any $t \geq T$ we have

$$\omega_0(1+t)^{-\gamma+1} \leq \omega_0(1+T)^{-\gamma+1} = 1 \tag{31}$$

and

$$\left|\int_t^\infty \sigma_1(s)\,\mathrm{d}s\right| = \left|\int_t^\infty \delta(s)\,\mathrm{d}s\right| \leq \rho_0(1+t)^{-\gamma} = \omega_0\rho(1+t)^{-\gamma+2}\mu_1(t)$$

for any $t \geq T$. It follows that (29) is valid for $k = 1$. Let $j \geq 3$ and suppose that (29) is valid for $k = 2, \ldots, j-1$. Then we have

$$\begin{aligned}
&\left|\left(\int_t^\infty \mu_{j-k}(s)\,\mathrm{d}s\right)\left(\int_t^\infty \sigma_k(s)\,\mathrm{d}s\right) + \left(\int_t^\infty \mu_k(s)\,\mathrm{d}s\right)\left(\int_t^\infty \sigma_{j-k}(s)\,\mathrm{d}s\right)\right| \\
&= \left|(1+t)\mu_{j-k}(t)\left(\int_t^\infty \sigma_k(s)\,\mathrm{d}s\right) + (1+t)\mu_k(t)\left(\int_t^\infty \sigma_{j-k}(s)\,\mathrm{d}s\right)\right| \\
&\leq \omega_0((1+\rho)^k + (1+\rho)^{j-k} - 2)(1+t)^{-\gamma+3}\mu_k(t)\mu_{j-k}(t)
\end{aligned}$$

and

$$\begin{aligned}&\left|\left(\int_t^\infty \sigma_k(s)\,\mathrm{d}s\right)\left(\int_t^\infty \sigma_{j-k}(s)\,\mathrm{d}s\right)\right|\\&\le \omega_0^2((1+\rho)^k-1)((1+\rho)^{j-k}-1)(1+t)^{-2\gamma+4}\mu_k(t)\mu_{j-k}(t).\end{aligned}$$

Therefore, by (25), (27), (31) and the equality

$$(1+\rho)^k+(1+\rho)^{j-k}-2+((1+\rho)^k-1)((1+\rho)^{j-k}-1)=(1+\rho)^j-1$$

we have

$$\begin{aligned}|\sigma_j(t)| &= |q_j(t)-\mu_j(t)|\\&=\left|\sum_{k=1}^{j-1}\left(\int_t^\infty(\mu_k(s)+\sigma_k(s))\,\mathrm{d}s\right)\left(\int_t^\infty(\mu_{j-k}(s)+\sigma_{j-k}(s))\,\mathrm{d}s\right)-\mu_j(t)\right|\\&\le\omega_0\sum_{k=1}^{j-1}((1+\rho)^k+(1+\rho)^{j-k}-2)(1+t)^{-\gamma+3}\mu_k(t)\mu_{j-k}(t)\\&\quad+\omega_0^2\sum_{k=1}^{j-1}((1+\rho)^k-1)((1+\rho)^{j-k}-1)(1+t)^{-2\gamma+4}\mu_k(t)\mu_{j-k}(t)\\&\le\omega_0((1+\rho)^j-1)(1+t)^{-\gamma+1}\mu_j(t)\end{aligned}$$

for any $t\ge T$. Moreover, we have

$$\begin{aligned}\left|\int_t^\infty\sigma_j(s)\,\mathrm{d}s\right| &\le \omega_0((1+\rho)^j-1)\int_t^\infty(1+s)^{-\gamma+1}\mu_j(s)\,\mathrm{d}s\\&=\frac{\omega_0}{\gamma}((1+\rho)^j-1)(1+t)^{-\gamma+2}\mu_j(t)\\&\le\omega_0((1+\rho)^j-1)(1+t)^{-\gamma+2}\mu_j(t).\end{aligned}$$

Thus (28) and (29) are valid for $k=j$. Consequently, (28) and (29) are valid for any $k\in\mathbb{N}$. □

By Lemmas 1 and 2 we have the following two propositions, which ensure the convergence of $\eta(t)$.

Proposition 1 *The following estimates are established on* $[T,\infty)$*:*

$$\sum_{j=1}^\infty q_j(t)-M(t)\lesssim(1+t)^{-2} \tag{32}$$

and

$$\sum_{j=1}^\infty\int_t^\infty q_j(s)\,ds-\nu(1+t)^{-1}\lesssim(1+t)^{-\gamma}. \tag{33}$$

Proof Let $\rho > 0$ satisfy $(1+\rho)\mu^2 < 1/4$. By (21), (26), (31), Lemmas 1 and 2 we have

$$\begin{aligned}\left|\sum_{j=1}^{\infty} q_j(t) - M(t)\right| &= \left|\sum_{j=2}^{\infty} q_j(t)\right| = \left|\sum_{j=2}^{\infty} \mu_j(t) + \sum_{j=2}^{\infty} \sigma_j(t)\right| \\ &\le \nu(1+t)^{-2} + \omega_0 \sum_{j=2}^{\infty}((1+\rho)^j - 1)(1+t)^{-\gamma+1}\mu_j(t) \\ &\le \left(\nu + \omega_0 \sum_{j=2}^{\infty} \gamma_{j-1}((1+\rho)\mu^2)^j (1+t)^{-\gamma+1}\right)(1+t)^{-2} \\ &\le \left(\nu + \frac{1-\sqrt{1-4(1+\rho)\mu^2}}{2}\right)(1+t)^{-2} \\ &\lesssim (1+t)^{-2}.\end{aligned}$$

Moreover, by (14), Lemmas 1 and 2 we have

$$\begin{aligned}&\left|\sum_{j=1}^{\infty}\int_t^{\infty} q_j(s)\,\mathrm{d}s - \nu(1+t)^{-1}\right| = \left|\sum_{j=1}^{\infty}\int_t^{\infty} \sigma_j(s)\,\mathrm{d}s\right| \\ &\le \left|\int_t^{\infty}\delta(s)\,\mathrm{d}s\right| + \omega_0\sum_{j=2}^{\infty}((1+\rho)^j - 1)(1+t)^{-\gamma+2}\mu_j(t) \\ &\lesssim (1+t)^{-\gamma} + \omega_0 \frac{1-\sqrt{1-4(1+\rho)\mu^2}}{2}(1+t)^{-\gamma} \\ &\simeq (1+t)^{-\gamma}.\end{aligned}$$

Thus the proof is completed. □

We note that $\eta(t)$ is the solution to the following initial value problem:

$$\begin{cases}\eta''(t) + M(t)\eta(t) = 0, & t \in (T,\infty),\\ (\eta(T),\eta'(T)) = (1,\tilde{\eta}),\end{cases} \tag{34}$$

where T was defined by (30) and $\tilde{\eta} = \sum_{j=1}^{\infty}\int_T^{\infty} q_j(s)\,\mathrm{d}s$. Let us continue the solution $\eta = \eta(t)$ on the interval $[0, T)$ as the solution to the backward Cauchy problem

$$\begin{cases}\eta''(t) + M(t)\eta(t) = 0, & t \in [0,T),\\ (\eta(T),\eta'(T)) = (1,\tilde{\eta}).\end{cases} \tag{35}$$

We define $b(t)$ and $\sigma(t)$ on $[0,\infty)$ by

$$b(t) := \frac{\eta'(t)}{\eta(t)} \tag{36}$$

and

$$\sigma(t) := b(t) - \nu(1+t)^{-1}. \tag{37}$$

Then we have the following proposition.

Proposition 2 *If Hypotheses 1, 2 and 3 are fulfilled, then the following estimates are established*:

$$\left|b'(t)\right| \lesssim (1+t)^{-2\beta} \tag{38}$$

and

$$|\sigma(t)| \lesssim (1+t)^{-\gamma}. \tag{39}$$

It follows that

$$\eta(t) \simeq (1+t)^{\nu}, \tag{40}$$

and that for any ν_0 *and* ν_1 *satisfying* $0 < \nu_0 < \nu < \nu_1 \leq 1/2$ *there exists* $T_0 \geq 0$ *such that*

$$\nu_0(1+t)^{-1} \leq b(t) \leq \nu_1(1+t)^{-1} \tag{41}$$

for any $t \geq T_0$.

Proof Estimates (38) and (39) are trivial on the finite interval $[0, T]$. Suppose that $t \geq T$. Then by Proposition 1, (13), (22) and (77) we have

$$\begin{aligned}\left|b'(t)\right| &= \left|\frac{\eta''(t)}{\eta(t)} - \frac{\eta'(t)^2}{\eta(t)^2}\right| = \left|-M(t) - \left(\sum_{j=1}^{\infty}\int_t^{\infty} q_j(s)\mathrm{d}s\right)^2\right| = \left|\sum_{k=1}^{\infty} q_k(t)\right| \\ &\lesssim |M(t)| + (1+t)^{-2} \lesssim (1+t)^{-2\beta}\end{aligned}$$

and

$$|\sigma(t)| = \left|\sum_{j=1}^{\infty}\int_t^{\infty} q_j(s)\mathrm{d}s - \nu(1+t)^{-1}\right| \lesssim (1+t)^{-\gamma}.$$

Moreover, by (33), (36), (37) and (39) we have

$$\eta(t) = \eta(T)\exp\left(\nu\int_T^t (1+s)^{-1}\,\mathrm{d}s + \int_T^t \sigma(s)\,\mathrm{d}s\right) \simeq (1+t)^{\nu}$$

and

$$b(t)(1+t) = \nu + \sigma(t)(1+t)\begin{cases}\leq \nu_1 \\ \geq \nu_0\end{cases}$$

for any $t \geq T_0$ with $\min\{\nu_1 - \nu, \nu - \nu_0\} \geq \sup_{t \geq T_0}\{|\sigma(t)|(1+t)\}$. Thus the proof is complete. □

4 Construction of the fundamental solution

4.1 Micro-energy and zones

For the nonnegative constants T and T_0 in Propositions 1 and 2 estimate (4) is trivial on the finite interval $[0, T_1]$ with $T_1 = \max\{T; T_0\}$ by application of the usual energy method. Thus we can suppose that $T_1 = 0$ from now on without loss of generality.

By partial Fourier transformation with respect to x and denoting $v(t,\xi)=\widehat{u}(t,\xi)$, (1) is reduced to the following Cauchy problem:

$$\begin{cases} v_{tt}+|\xi|^2 v+M(t)v=0, & (t,\xi)\in(0,\infty)\times\mathbb{R}^n,\\ v(0,\xi)=\widehat{u}_0(\xi), \quad v_t(0,\xi)=\widehat{u}_1(\xi), & \xi\in\mathbb{R}^n. \end{cases} \tag{42}$$

For a positive large number N, which will be chosen later, we divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into three zones: the *pseudo-differential zone* $Z_\Psi=Z_\Psi(N)$, the *hyperbolic zone* $Z_H=Z_H(N)$ and the *intermediate zone* $Z_I=Z_I(N)$ as follows:

$$\begin{aligned} Z_\Psi(N)&:=\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ (1+t)|\xi|\le N\},\\ Z_H(N)&:=\left\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ (1+t)^{2-\gamma}|\xi|\ge N\right\},\\ Z_I(N)&:=\left\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ (1+t)^{2-\gamma}|\xi|\le N\le(1+t)|\xi|\right\}. \end{aligned}$$

Here we note that $2-\gamma\le 2\beta-1<1$ by Hypothesis 2, 3 and (15). It follows that $Z_I\not\subseteq Z_H$. We define $\theta_1=\theta_1(r)$ and $\theta_2=\theta_2(r)$ on $[0,\infty)$ by

$$\theta_1(r):=\begin{cases}\infty & \text{for } r=0,\\ \max\left\{Nr^{-1}-1,0\right\} & \text{for } r>0,\end{cases}$$

and

$$\theta_2(r):=\begin{cases}\infty & \text{for } r=0,\\ \max\left\{(Nr^{-1})^{\frac{1}{2-\gamma}}-1,0\right\} & \text{for } r>0 \text{ and } \gamma\ne 2,\\ \max\left\{(Nr^{-1})^{\frac{1}{2\beta-1}}-1,0\right\} & \text{for } r>0 \text{ and } \gamma=2.\end{cases}$$

Then $(\theta_1(|\xi|),\xi)$ and $(\theta_2(|\xi|),\xi)$ denote the *separating hypersurfaces* between Z_Ψ and Z_I, and Z_I and Z_H, respectively. Indeed, the zones can be represented as follows:

$$\begin{aligned} Z_\Psi(N)&=\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ 0\le t\le\theta_1(|\xi|)\},\\ Z_H(N)&=\begin{cases}\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ t\ge\theta_2(|\xi|)\} & \text{for } \gamma\le 2,\\ \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ 0\le t\le\theta_2(|\xi|)\} & \text{for } \gamma>2,\end{cases}\\ Z_I(N)&=\begin{cases}\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ \theta_1(|\xi|)\le t\le\theta_2(|\xi|)\} & \text{for } \gamma\le 2,\\ \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n\ ;\ t\ge\theta_1(|\xi|) \text{ and } t\ge\theta_2(|\xi|)\} & \text{for } \gamma>2\,.\end{cases} \end{aligned}$$

Let $\chi\in C^\infty(\mathbb{R}_+)$ such that $\chi'(r)\le 0$, $\chi(r)=1$ for $r\le 1$, $\chi(3/2)=1/2$ and $\chi(r)=0$ for $r\ge 2$. We define the micro-energy $U(t,\xi)$ by

$$U(t,\xi):=(h(t,\xi)v(t,\xi),\, v_t(t,\xi)-b(t)v(t,\xi))^T, \tag{43}$$

where

$$h(t,\xi):=(1+t)^{-1}\chi(N^{-1}(1+t)|\xi|)+i|\xi|(1-\chi(N^{-1}(1+t)|\xi|)). \tag{44}$$

Then we have the following lemma.

Lemma 3 *We have that* $h(t,\xi)=(1+t)^{-1}$ *in* $Z_\Psi(N)$ *and* $|h(t,\xi)|\simeq|\xi|$ *in* $Z_I(N)\cup Z_H(N)$. *Moreover, there exists a positive constant* N_0 *such that for any* $N\ge N_0$ *the following estimate holds*:

$$|U(t,\xi)|^2\simeq|\xi|^2\,|v(t,\xi)|^2+|v_t(t,\xi)|^2+(1+t)^{-2}\,|v(t,\xi)|^2 \tag{45}$$

uniformly with respect to (t,ξ).

Proof We have $h(t,\xi)=(1+t)^{-1}$ in $Z_\Psi(N)$ by (44). Let $(t,\xi)\in Z_I(N)\cup Z_H(N)$, that is, $N^{-1}(1+t)|\xi|\ge 1$. If $1\le N^{-1}(1+t)|\xi|\le 3/2$, then we have

$$|h(t,\xi)|\begin{cases}\ge \frac{1}{2}(1+t)^{-1}\ge \frac{1}{3N}|\xi|,\\ \le \sqrt{(1+t)^{-2}+\frac{1}{4}|\xi|^2}\le \sqrt{\frac{1}{N^2}+\frac{1}{4}}|\xi|.\end{cases}$$

If $N^{-1}(1+t)|\xi|\ge 3/2$, then we have

$$|h(t,\xi)|\begin{cases}\ge \frac{1}{2}|\xi|,\\ \le \sqrt{\frac{1}{4}(1+t)^{-2}+|\xi|^2}\le \sqrt{\frac{1}{9N^2}+1}|\xi|.\end{cases}$$

Therefore, by (41), Cauchy–Schwarz inequality and recalling Proposition 2, we have in $Z_\Psi(N)$ that

$$\begin{aligned}|U(t,\xi)|^2&=(1+t)^{-2}|v|^2+|v_t|^2+b(t)^2|v|^2-2b(t)\Re(v_t\overline{v})\\&\ge((1+t)^{-2}-b(t)^2)|v|^2+\frac{1}{2}|v_t|^2\\&\ge(1-\nu_1^2)(1+t)^{-2}|v|^2+\frac{1}{2}|v_t|^2\\&\ge\frac{1}{2}(1-\nu_1^2)(1+t)^{-2}|v|^2+\frac{1}{2N^2}(1-\nu_1^2)|\xi|^2|v|^2+\frac{1}{2}|v_t|^2\end{aligned}$$

and

$$\begin{aligned}|U(t,\xi)|^2&\le(1+t)^{-2}|v|^2+2|v_t|^2+2b(t)^2|v|^2\\&\le(1+2\nu_1^2)(1+t)^{-2}|v|^2+2|v_t|^2\\&\le|\xi|^2|v|^2+(1+2\nu_1^2)(1+t)^{-2}|v|^2+2|v_t|^2.\end{aligned}$$

On the other hand, in $Z_I(N)\cup Z_H(N)$ with $N\ge 2/3$ we have

$$\begin{aligned}|U(t,\xi)|^2&\ge\frac{1}{9N^2}|\xi|^2|v|^2+|v_t|^2+b(t)^2|v|-\frac{36\nu_1^2}{36\nu_1^2+1}|v_t|^2-\frac{36\nu_1^2+1}{36\nu_1^2}b(t)^2|v|^2\\&=\frac{1}{18N^2}|\xi|^2|v|^2+\frac{1}{36\nu_1^2+1}|v_t|^2+\left(\frac{1}{18N^2}|\xi|^2-\frac{1}{36\nu_1^2}b(t)^2\right)|v|^2\\&\ge\frac{1}{18N^2}|\xi|^2|v|^2+\frac{1}{36\nu_1^2+1}|v_t|^2+\frac{1}{36}(1+t)^{-2}|v|^2\end{aligned}$$

and

$$\begin{aligned}|U(t,\xi)|^2&\le 2|\xi|^2|v|^2+2|v_t|^2+2b(t)^2|v|^2\\&\le 2|\xi|^2|v|^2+2|v_t|^2+2\nu_1^2(1+t)^{-2}|v|^2.\end{aligned}$$

Thus (45) is proved. □

For the micro-energy $U=U(t,\xi)$ defined by (43) we define $V=V(t,\xi)$ by

$$V(t,\xi):=\eta(t)^{-1}U(t,\xi). \tag{46}$$

Then we have the following lemma.

Lemma 4 *The vector V is a solution to the following first-order system*:

$$\partial_t V = AV, \quad A = A(t,\xi) = \begin{pmatrix} \frac{h_t(t,\xi)}{h(t,\xi)} & h(t,\xi) \\ -\frac{|\xi|^2}{h(t,\xi)} & -2b(t) \end{pmatrix}. \tag{47}$$

Proof The proof is straight-forward from the definitions of U, η and (36). □

We shall consider the fundamental solution $E = E(t,s,\xi)$ to (47), that is, the solution to

$$\partial_t E = A(t,\xi)E, \quad E(s,s,\xi) = I. \tag{48}$$

4.2 Considerations in the pseudo-differential zone $Z_\Psi(N)$

We shall prove the following statement.

Proposition 3 *Assume Hypothesis 1. The fundamental solution to (48) satisfies the following estimate*:

$$\|E(t,0,\xi)\| \lesssim (1+t)^{-2\nu}$$

uniformly for $(t,\xi) \in Z_\Psi$.

Proof Let $(t,\xi) \in Z_\Psi$, that is, $0 \le t \le \theta_1(|\xi|)$. We consider (48) with

$$A = \begin{pmatrix} -(1+t)^{-1} & (1+t)^{-1} \\ -(1+t)|\xi|^2 & -2b(t) \end{pmatrix}.$$

If we put $E(t,0,\xi) = (e_{ij}(t,\xi))_{i,j=1,2}$, then we can write for $j = 1, 2$ the following system of coupled integral equations of Volterra type:

$$e_{1j}(t,\xi) = (1+t)^{-1}\left(\delta_{1j} + \int_0^t e_{2j}(\tau,\xi)\,\mathrm{d}\tau\right), \tag{49}$$

$$e_{2j}(t,\xi) = \eta(t)^{-2}\left(\delta_{2j} - |\xi|^2 \int_0^t (1+\tau)\eta(\tau)^2 e_{1j}(\tau,\xi)\,\mathrm{d}\tau\right). \tag{50}$$

By substituting (50) into (49) and integrating by parts we get

$$\begin{aligned} e_{1j}(t,\xi) &= (1+t)^{-1}\left(\delta_{1j} + \delta_{2j}\int_0^t \eta(\tau)^{-2}\,\mathrm{d}\tau\right) \\ &\quad - |\xi|^2(1+t)^{-1}\int_0^t \eta(\tau)^{-2}\int_0^\tau (1+s)\eta(s)^2 e_{1j}(s,\xi)\,\mathrm{d}s\,\mathrm{d}\tau \\ &= (1+t)^{-1}\left(\delta_{1j} + \delta_{2j}\int_0^t \eta(\tau)^{-2}\,\mathrm{d}\tau\right) \\ &\quad - |\xi|^2(1+t)^{-1}\int_0^t \left(\int_\tau^t \eta(s)^{-2}\,\mathrm{d}s\right)(1+\tau)\eta(\tau)^2 e_{1j}(\tau,\xi)\,\mathrm{d}\tau. \end{aligned}$$

We define $f_j(t,\xi)$ by

$$f_j(t,\xi) := \eta(t)^2|e_{1j}(t,\xi)|.$$

By (40), there exists a constant $C \ge 1$ such that

$$C^{-1}(1+t)^\nu \le \eta(t) \le C(1+t)^\nu.$$

Then we have

$$\begin{aligned}
f_j(t,s,\xi) &\le (1+t)^{-1}\eta(t)^2\left(1+\int_0^t \eta(\tau)^{-2}\,\mathrm{d}\tau\right)\\
&\quad + |\xi|^2(1+t)^{-1}\eta(t)^2\int_0^t\left(\int_\tau^t \eta(s)^{-2}\,\mathrm{d}s\right)(1+\tau)f_j(\tau,\xi)\,\mathrm{d}\tau\\
&\le C^2(1+t)^{-1+2\nu}\left(1+C^2\int_0^t (1+\tau)^{-2\nu}\,\mathrm{d}\tau\right)\\
&\quad + C^4|\xi|^2(1+t)^{-1+2\nu}\int_0^t\left(\int_\tau^t (1+s)^{-2\nu}\,\mathrm{d}s\right)(1+\tau)f_j(\tau,\xi)\,\mathrm{d}\tau\\
&\le C^2(1+t)^{-1+2\nu}\left(1+\frac{C^2}{1-2\nu}(1+t)^{1-2\nu}\right)+\frac{C^4}{1-2\nu}|\xi|^2\int_0^t (1+\tau)f_j(\tau,\xi)\,\mathrm{d}\tau\\
&\le C_1 + C_2|\xi|^2\int_0^t (1+\tau)f_j(\tau,\xi)\,\mathrm{d}\tau,
\end{aligned}$$

where $C_1 = C^2 + C^4/(1-2\nu)$ and $C_2 = C^4/(1-2\nu)$. By Gronwall's inequality we conclude

$$f_j(t,\xi) \le C_1\exp\left(C_2|\xi|^2\int_0^t (1+\tau)\,\mathrm{d}\tau\right) \le C_1\exp\left(\frac{C_2N^2}{2}\right).$$

Thus we get $|e_{1j}(t,\xi)| \lesssim \eta(t)^{-2}$. Moreover, by (50) we have

$$|e_{2j}(t,\xi)| \lesssim \eta(t)^{-2}\left(1+|\xi|^2\int_s^t (1+\tau)\mathrm{d}\tau\right) \lesssim \eta(t)^{-2}.$$

Summarizing the above estimates and (40) we conclude the proof. □

4.3 Considerations in the hyperbolic zone $Z_H(N)$

We shall prove the following statement.

Proposition 4 *Assume Hypotheses 1, 2 and 3. The fundamental solution $E(t,s,\xi)$ to (48) satisfies the following estimate*:

$$\|E(t,s,\xi)\| \lesssim \frac{(1+s)^\nu}{(1+t)^\nu} \tag{51}$$

uniformly for $(t,\xi), (s,\xi) \in Z_H(N)$ with $s \le t$ and $N \ge \nu_1/\sqrt{2}$.

Proof Estimate (51) is trivial for $s \le t \le 2^{1/(\gamma-1)} - 1$. Hence, we suppose that $t \ge 2^{1/(\gamma-1)} - 1$ from now on. Then we see that

$$N^{-1}(1+t)|\xi| \ge 2N^{-1}(1+t)^{2-\gamma}|\xi| \ge 2.$$

It follows that

$$A = \begin{pmatrix} 0 & i|\xi| \\ i|\xi| & -2b(t) \end{pmatrix}.$$

Let M_0 be the diagonalizer of the principal part with respect to powers of $|\xi|$ of A given by

$$M_0 := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{52}$$

We put $E_1 = E_1(t, s, \xi) := M_0^{-1} E(t, s, \xi)$. Then (48) is reduced to

$$\partial_t E_1 = A_1 E_1, \quad E_1(s, s, \xi) = M_0^{-1}, \tag{53}$$

whereas

$$A_1 := M_0^{-1} A M_0 = \begin{pmatrix} i|\xi| - b(t) & -b(t) \\ -b(t) & -i|\xi| - b(t) \end{pmatrix}.$$

Let $M_1 = M_1(t, \xi)$ be the diagonalizer of the principal part of A_1 given by

$$M_1 := \begin{pmatrix} 1 & -\frac{(A_1)_{12}}{(A_1)_{11}-(A_1)_{22}} \\ -\frac{(A_1)_{21}}{(A_1)_{22}-(A_1)_{11}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{ib(t)}{2|\xi|} \\ \frac{ib(t)}{2|\xi|} & 1 \end{pmatrix}.$$

By (41) we have

$$\det M_1 = 1 - \frac{b(t)^2}{4|\xi|^2} \geq 1 - \frac{\nu_1^2}{4N^2} \geq \frac{1}{2}$$

for $(1+t)|\xi| \geq N$, that is, for $(t, \xi) \in Z_H(N) \cup Z_I(N)$ with $N \geq \nu_1/\sqrt{2}$. It follows that M_1 is invertible. Moreover, we have

$$\|M_1(t, \xi)\| \leq \max_{i,j=1,2}\{|(M_1)_{i,j}(t, \xi)|\} \leq \max\left\{1, \frac{\nu_1}{2N}\right\} = 1. \tag{54}$$

We put

$$E_2 = E_2(t, s, \xi) := M_1^{-1}(t, \xi) E_1(t, s, \xi).$$

Then (53) is reduced to

$$\partial_t E_2 = A_2(t, \xi) E_2, \quad E_2(s, s, \xi) = M_1^{-1}(s, \xi) M_0^{-1}, \tag{55}$$

where

$$A_2 = A_2(t, \xi) := M_1^{-1} A_1 M_1 - M_1^{-1}(\partial_t M_1).$$

Here we note the following representations:

$$(A_2)_{11} = \overline{(A_2)_{22}} = i|\xi| - b(t) - \frac{i}{\det M_1}\left(\frac{b(t)^2}{2|\xi|} + \frac{b(t)b'(t)}{4|\xi|^2}\right)$$

and

$$(A_2)_{21} = \overline{(A_2)_{12}} = \frac{1}{\det M_1}\left(-\frac{b(t)^3}{4|\xi|^2} + \frac{ib'(t)}{2|\xi|}\right).$$

By (38), (41) and noting that $\beta < 1 < \gamma$ we have

$$\begin{aligned} |(A_2)_{21}| &\lesssim |\xi|^{-1}(|\xi|^{-1}(1+t)^{-3} + (1+t)^{-2\beta}) \\ &\lesssim |\xi|^{-1}((1+t)^{-\gamma-1} + (1+t)^{-2\beta}) \simeq |\xi|^{-1}(1+t)^{-2\beta} \end{aligned} \tag{56}$$

in Z_H. Moreover, for $(t, \xi), (s, \xi) \in Z_H$ and $s \leq t$, by (36) we have

$$\left|\exp\left(\int_s^t (A_2)_{jj}(\tau, \xi)\, d\tau\right)\right| = \exp\left(-\int_s^t b(\tau)\, d\tau\right) = \frac{\eta(s)}{\eta(t)} \tag{57}$$

for $j=1,2$. We define $\Phi_2(t,s,\xi)$ by

$$\Phi_2(t,s,\xi):=\begin{pmatrix}\exp\left(\int_s^t (A_2)_{11}(\tau,\xi)\,\mathrm{d}\tau\right) & 0\\ 0 & \exp\left(\int_s^t (A_2)_{22}(\tau,\xi)\,\mathrm{d}\tau\right)\end{pmatrix},$$

and we put $E_3(t,s,\xi):=\Phi_2^{-1}(t,s,\xi)E_2(t,s,\xi)$. Then (55) is reduced to

$$\partial_t E_3=R_3(t,s,\xi)E_3,\quad E_3(s,s,\xi)=M_1^{-1}(s,\xi)M_0^{-1}, \tag{58}$$

where

$$\begin{aligned}R_3(t,s,\xi)&:=\Phi_2^{-1}(t,s,\xi)A_2(t,\xi)\Phi_2(t,s,\xi)-\Phi_2^{-1}(t,s,\xi)(\partial_t\Phi_2)(t,s,\xi)\\&=\begin{pmatrix}0 & \overline{r_3(t,s,\xi)}\\ r_3(t,s,\xi) & 0\end{pmatrix}\end{aligned}$$

and

$$r_3(t,s,\xi)=(A_2)_{21}(t,\xi)\exp\left(i\int_s^t \Im\{(A_2)_{11}(\tau,\xi)\}\,\mathrm{d}\tau\right).$$

Here $E_3(t,s,\xi)$ can be represented as a Peano–Baker series in the form

$$\begin{aligned}E_3(t,s,\xi)&=E_3(s,s,\xi)+\int_s^t R_3(t_1,s,\xi)\mathrm{d}t_1\\&\quad+\sum_{k=2}^{\infty}\int_s^t\int_s^{t_1}\cdots\int_s^{t_{k-1}} R_3(t_1,s,\xi)\cdots R_3(t_k,s,\xi)\,\mathrm{d}t_k\cdots\mathrm{d}t_1.\end{aligned}$$

Therefore, by (56) and the equalities $\|R_3(\tau,s,\xi)\|=|r_3(t,s,\xi)|=|(A_2)_{21}(t,\xi)|$ there exists a positive constant C such that

$$\begin{aligned}\|E_3(t,s,\xi)\|&\le\|E_3(s,s,\xi)\|+\exp\left(\int_s^t\|R_3(\tau,s,\xi)\|\,\mathrm{d}\tau\right)\\&\lesssim 1+\exp\left(C|\xi|^{-1}\int_s^t(1+\tau)^{-2\beta}\,\mathrm{d}\tau\right).\end{aligned}$$

We define $\phi(t,s):=\int_s^t(1+\tau)^{-2\beta}\,\mathrm{d}\tau$. Then we shall estimate $|\xi|^{-1}\phi(t,s)$ in $Z_H(N)$. If $\gamma<2$, then by (15) we have $2\beta-1\ge-\gamma+2>0$. It follows

$$|\xi|^{-1}\phi(t,s)\le|\xi|^{-1}\phi(\infty,\theta_2)=\frac{(1+\theta_2)^{-2\beta-\gamma+3}}{N(2\beta-1)}\le\frac{1}{N(2\beta-1)}.$$

If $\gamma>2$ and $\beta<1/2$, then we have

$$\begin{aligned}|\xi|^{-1}\phi(t,s)\le|\xi|^{-1}\phi(\theta_2,0)&\le\frac{|\xi|^{-1}(1+\theta_2)^{-2\beta+1}}{1-2\beta}\\&=\frac{(1+\theta_2)^{-2\beta-\gamma+3}}{N(1-2\beta)}\le\frac{1}{N(1-2\beta)}.\end{aligned}$$

If $\gamma>2$ and $\beta\ge1/2$, then by (15) we have

$$\begin{aligned}|\xi|^{-1}\phi(t,s)&\le|\xi|^{-1}\int_0^{\theta_2}(1+\tau)^{\gamma-3}\,\mathrm{d}\tau\\&\le\frac{|\xi|^{-1}}{\gamma-2}(1+\theta_2)^{\gamma-2}=\frac{1}{N(\gamma-2)}.\end{aligned}$$

If $\gamma = 2$ and $|\xi| \le N$, then by (15) we have

$$|\xi|^{-1}\phi(t,s) \le |\xi|^{-1}\phi(\infty,\theta_2) = \frac{1}{2\beta-1}|\xi|^{-1}(1+\theta_2)^{-2\beta+1} = \frac{1}{N(2\beta-1)}.$$

If $\gamma = 2$ and $|\xi| \ge N$, then by (15) we have

$$|\xi|^{-1}\phi(t,s) \le N^{-1}\phi(\infty,0) = \frac{1}{N(2\beta-1)}.$$

Thus we have $\|E_3(t,s,\xi)\| \lesssim 1$ uniformly in Z_H. Consequently, by (54) and (57) we obtain

$$\begin{aligned}\|E(t,s,\xi)\| &= \|M_0 M_1(t,\xi)\Phi_2(t,s,\xi)E_3(t,s,\xi)\|\\ &\simeq \|\Phi_2(t,s,\xi)E_3(t,s,\xi)\| = \frac{\eta(s)}{\eta(t)}\|E_3(t,s,\xi)\|\\ &\lesssim \frac{\eta(s)}{\eta(t)}.\end{aligned}$$

Thus the proof of Proposition 4 is concluded by (40). □

4.4 Considerations in the intermediate zone $Z_I(N)$

We shall prove the following statement.

Proposition 5 *Assume Hypotheses 1 and 3. The fundamental solution $E = E(t,s,\xi)$ to (48) satisfies the following estimate:*

$$\|E(t,s,\xi)\| \lesssim \frac{(1+s)^\nu}{(1+t)^\nu} \tag{59}$$

uniformly for $(t,\xi), (s,\xi) \in Z_I$ with $s \le t$.

We shall introduce some lemmas in order to prove Proposition 5.

Lemma 5 *Estimate (59) holds uniformly for*

$$\theta_1(|\xi|) \le s \le t \le \tilde{\theta}_1(|\xi|) := 2\theta_1(|\xi|) + 1.$$

Proof Let $\theta_1 \le t \le \tilde{\theta}_1$, that is, $1 \le N^{-1}(1+t)|\xi| \le 2$. By Lemma 3 and the estimates

$$\left|\frac{h_t(t,\xi)}{h(t,\xi)}\right| \lesssim |\xi|^{-1}(|\xi|(1+t)^{-1} + |\xi|^2) \simeq (1+t)^{-1},$$

there exists a positive constant C such that

$$\|A(t,\xi)\| \lesssim C(1+t)^{-1}.$$

It follows that

$$\int_s^t \|A(\tau,\xi)\|\,\mathrm{d}\tau \le C(1+\theta_1)^{-1}\int_{\theta_1}^{\tilde{\theta}_1}\mathrm{d}\tau = C$$

for $\theta_1 \le s \le t \le \tilde{\theta}_1$. Therefore, by the same way to estimate $E_3 = E_3(t,s,\xi)$ in Z_H we have $\|E(t,s,\xi)\| \lesssim 1$ uniformly for $(t,\xi), (s,\xi) \in Z_I$. Moreover, by (40) and the estimates

$$1 \le \frac{(1+t)^\nu}{(1+s)^\nu} \le \frac{\left(1+\widetilde{\theta}_1\right)^\nu}{(1+\theta_1)^\nu} = 2^\nu$$

we have (59) what we wanted to prove. □

We define $B = B(t,\xi)$ by

$$B(t,\xi) := \begin{pmatrix} 0 & i|\xi| \\ i|\xi| & -2\nu(1+t)^{-1} \end{pmatrix}.$$

Let us consider the fundamental solution $\mathcal{E}(t,s,\xi)$ to

$$\partial_t \mathcal{E}(t,s,\xi) = B(t,\xi)\mathcal{E}(t,s,\xi), \quad \mathcal{E}(s,s,\xi) = I. \tag{60}$$

Then we have the following lemma.

Lemma 6 *Assume Hypothesis 1. The fundamental solution to (60) satisfies the following estimates*:

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \frac{(1+s)^\nu}{(1+t)^\nu} \quad \textit{and} \; \|\mathcal{E}^{-1}(t,s,\xi)\| \lesssim \frac{(1+t)^\nu}{(1+s)^\nu} \tag{61}$$

uniformly for $(t,\xi), (s,\xi) \in Z_I$.

Proof We put $\mathcal{E}_1 = \mathcal{E}_1(t,s,\xi) := M_0^{-1}\mathcal{E}(t,s,\xi)$, where M_0 was defined in (52). Then (60) is reduced to

$$\partial_t \mathcal{E}_1 = B_1(t,\xi)\mathcal{E}_1, \quad \mathcal{E}_1(s,s,\xi) = M_0^{-1}, \tag{62}$$

where

$$B_1 := M_0^{-1} B M_0 = \begin{pmatrix} i|\xi| - \nu(1+t)^{-1} & -\nu(1+t)^{-1} \\ -\nu(1+t)^{-1} & -i|\xi| - \nu(1+t)^{-1} \end{pmatrix}.$$

Let $\widetilde{M}_1$ be the diagonalizer of the principal part of B_1 given by

$$\widetilde{M}_1 := \begin{pmatrix} 1 & -\frac{(B_1)_{12}}{(B_1)_{11}-(B_1)_{22}} \\ -\frac{(B_1)_{21}}{(B_1)_{22}-(B_1)_{11}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{i\nu(1+t)^{-1}}{2|\xi|} \\ \frac{i\nu(1+t)^{-1}}{2|\xi|} & 1 \end{pmatrix}.$$

Here we see that $\widetilde{M}_1$ is invertible and $\det \widetilde{M}_1 \ge 1/2$ in $Z_H(N) \cup Z_I(N)$ with $N \ge \nu/\sqrt{2}$. We put $\mathcal{E}_2 = \mathcal{E}_2(t,s,\xi) := \widetilde{M}_1^{-1}(t,\xi)\mathcal{E}_1(t,s,\xi)$. Then (62) is reduced to

$$\partial_t \mathcal{E}_2 = B_2(t,\xi)\mathcal{E}_2, \quad \mathcal{E}_2(s,s,\xi) = \widetilde{M}_1^{-1}(s,\xi)M_0^{-1}, \tag{63}$$

where

$$B_2 = B_2(t,\xi) := \widetilde{M}_1^{-1} B_1 \widetilde{M}_1 - \widetilde{M}_1^{-1}(\partial_t \widetilde{M}_1),$$

$$(B_2)_{11} = \overline{(B_2)_{22}} = i|\xi| - \nu(1+t)^{-1} - \frac{i}{\det \widetilde{M}_1}\left(\frac{\nu^2(1+t)^{-2}}{2|\xi|} - \frac{\nu^2(1+t)^{-3}}{4|\xi|^2}\right)$$

and

$$(B_2)_{21} = \overline{(B_2)_{12}} = \frac{1}{\det \widetilde{M}_1}\left(-\frac{\nu^3(1+t)^{-3}}{4|\xi|^2} - \frac{i\nu(1+t)^{-2}}{2|\xi|}\right).$$

Here we note that for $\theta_1 \le s < t$ we have

$$\left|\exp\left(\int_s^t (B_2)_{jj}(\tau,\xi)\,\mathrm{d}\tau\right)\right| = \frac{(1+s)^\nu}{(1+t)^\nu} \tag{64}$$

for $j = 1, 2$ and

$$|(B_2)_{21}| \lesssim |\xi|^{-2}(1+t)^{-3} + |\xi|^{-1}(1+t)^{-2} \lesssim |\xi|^{-1}(1+t)^{-2}.$$

We define $\widetilde{\Phi}_2(t,s,\xi)$ by

$$\widetilde{\Phi}_2(t,s,\xi) := \begin{pmatrix} \exp\left(\int_s^t (B_2)_{11}(\tau,\xi)\,\mathrm{d}\tau\right) & 0 \\ 0 & \exp\left(\int_s^t (B_2)_{22}(\tau,\xi)\,\mathrm{d}\tau\right) \end{pmatrix},$$

and we put $\mathcal{E}_3(t,s,\xi) := \widetilde{\Phi}_2^{-1}(t,s,\xi)\mathcal{E}_2(t,s,\xi)$. Then (63) is reduced to

$$\partial_t \mathcal{E}_3 = \widetilde{R}_3(t,s,\xi)\mathcal{E}_3, \quad \mathcal{E}_3(s,s,\xi) = \widetilde{M}_1^{-1}(s,\xi)M_0^{-1}, \tag{65}$$

where

$$\begin{aligned} \widetilde{R}_3(t,s,\xi) &:= \widetilde{\Phi}_2^{-1}(t,s,\xi)B_2(t,\xi)\widetilde{\Phi}_2(t,s,\xi) - \widetilde{\Phi}_2^{-1}(t,s,\xi)(\partial_t\widetilde{\Phi}_2)(t,s,\xi) \\ &= \begin{pmatrix} 0 & \overline{\widetilde{r}_3(t,s,\xi)} \\ \widetilde{r}_3(t,s,\xi) & 0 \end{pmatrix}, \end{aligned}$$

and

$$\widetilde{r}_3(t,s,\xi) = (B_2)_{21}(t,\xi)\exp\left(i\int_s^t \Im\{(B_2)_{11}(\tau,\xi)\}\,\mathrm{d}\tau\right).$$

Therefore, there exists a positive constant C such that

$$\begin{aligned} \|\mathcal{E}_3(t,s,\xi)\| &\le \|\mathcal{E}_3(s,s,\xi)\| + \exp\left(\int_s^t \|\widetilde{R}_3(\tau,s,\xi)\|\,\mathrm{d}\tau\right) \\ &\lesssim 1 + \exp\left(C|\xi|^{-1}\int_{\theta_1}^\infty (1+\tau)^{-2}\,\mathrm{d}\tau\right) \\ &= 1 + \exp\left(C|\xi|^{-1}(1+\theta_1)^{-1}\right) \lesssim 1. \end{aligned}$$

Consequently, by (64) we have

$$\|\mathcal{E}(t,s,\xi)\| \simeq \left\|\widetilde{\Phi}_2(t,s,\xi)\mathcal{E}_3(t,s,\xi)\right\| \lesssim \frac{(1+s)^\nu}{(1+t)^\nu}.$$

Analogously, by the equality

$$\left|\exp\left(-\int_s^t (B_2)_{jj}(\tau,\xi)\,\mathrm{d}\tau\right)\right| = \frac{(1+t)^\nu}{(1+s)^\nu} \tag{66}$$

we have $\|\mathcal{E}^{-1}(t,s,\xi)\| \lesssim (1+t)^\nu/(1+s)^\nu$. Thus the proof is complete. □

Proof of Proposition 5. For $\tilde{\theta}_1 \le s \le t \le \theta_2$ we consider the fundamental solution $E = E(t,s,\xi)$ to (48) with

$$A = \begin{pmatrix} 0 & i|\xi| \\ i|\xi| & -2b(t) \end{pmatrix}.$$

We shall prove the equivalence between $\|E(t,s,\xi)\|$ and $\|\mathcal{E}(t,s,\xi)\|$ by using the stabilization condition in Hypothesis 3. We define $\Lambda_0 = \Lambda_0(t,\xi)$ by

$$\Lambda_0(t) := \begin{pmatrix} 1 & 0 \\ 0 & \psi(t) \end{pmatrix}, \quad \psi(t) = \exp\left(2\int_t^\infty \sigma(\tau)\,\mathrm{d}\tau\right).$$

We put $\widetilde{E}_1 = \widetilde{E}_1(t,s,\xi) := \Lambda_0^{-1}(t)E(t,s,\xi)$. Then (48) is reduced to

$$\partial_t \widetilde{E}_1 = \widetilde{A}_1(t,\xi)\widetilde{E}_1, \quad \widetilde{E}_1(s,s,\xi) = \Lambda_0^{-1}(s), \tag{67}$$

where

$$\widetilde{A}_1 := \begin{pmatrix} 0 & i|\xi|\psi(t) \\ i|\xi|\psi(t)^{-1} & -2\nu(1+t)^{-1} \end{pmatrix}. \tag{68}$$

We put $\widetilde{E}_2 = \widetilde{E}_2(t,s,\xi) := \mathcal{E}^{-1}(t,s,\xi)\widetilde{E}_1(t,s,\xi)$. Then (67) is reduced to

$$\partial_t \widetilde{E}_2 = \widetilde{A}_2(t,s,\xi)\widetilde{E}_2, \quad \widetilde{E}_2(s,s,\xi) = \widetilde{\Lambda}_0^{-1}(s), \tag{69}$$

where

$$\begin{aligned}\widetilde{A}_2(t,s,\xi) &:= \mathcal{E}^{-1}(t,s,\xi)(\widetilde{A}_1(t,\xi) - B(t,\xi))\mathcal{E}(t,s,\xi) \\ &= \mathcal{E}^{-1}(t,s,\xi)\begin{pmatrix} 0 & i|\xi|(\psi(t)-1) \\ i|\xi|(\psi(t)^{-1}-1) & 0 \end{pmatrix}\mathcal{E}(t,s,\xi).\end{aligned}$$

By (39) we have $\lim_{t\to\infty}\psi(t) = 1$. It follows that

$$|\psi(t) - 1| \lesssim \left|2\int_t^\infty \sigma(\tau)\,\mathrm{d}\tau\right| \lesssim (1+t)^{-\gamma+1}. \tag{70}$$

Analogously, we have

$$\left|\psi(t)^{-1} - 1\right| \lesssim (1+t)^{-\gamma+1}. \tag{71}$$

Therefore, by Lemma 6, (70) and (71) we have

$$\int_s^t \left\|\widetilde{A}_2(\tau,s,\xi)\right\|\mathrm{d}\tau \lesssim |\xi|\widetilde{\phi}(t,s), \quad \widetilde{\phi}(t,s) := \int_s^t (1+\tau)^{-\gamma+1}\,\mathrm{d}\tau.$$

If $\gamma < 2$, then we have

$$|\xi|\widetilde{\phi}(t,s) \le |\xi|\widetilde{\phi}(\theta_2,0) \le \frac{|\xi|(1+\theta_2)^{2-\gamma}}{2-\gamma} = \frac{N}{2-\gamma}.$$

If $\gamma > 2$ and $|\xi| \le N$, then we have

$$|\xi|\widetilde{\phi}(t,s) \le N\widetilde{\phi}(\infty,0) = \frac{N}{\gamma-2}.$$

If $\gamma > 2$ and $|\xi| \ge N$, then we have

$$|\xi|\widetilde{\phi}(t,s) \le |\xi|\widetilde{\phi}(\infty,\theta_2) = \frac{N}{\gamma-2}.$$

If $\gamma = 2$, then we have

$$\begin{aligned}|\xi|\int_s^t (1+\tau)^{-1}\,\mathrm{d}\tau &\le |\xi|\int_s^t (1+\tau)^{2\beta-2}\,\mathrm{d}\tau \le |\xi|\int_{\theta_1}^{\theta_2}(1+\tau)^{2\beta-2}\,\mathrm{d}\tau \\ &\le \frac{|\xi|(1+\theta_2)^{2\beta-1}}{2\beta-1} = \frac{N}{2\beta-1}.\end{aligned}$$

Therefore, by the same way to estimate $E_3 = E_3(t,s,\xi)$ in Z_H we have

$$\left\|\widetilde{E}_2(t,s,\xi)\right\| \lesssim \left\|\widetilde{E}_2(s,s,\xi)\right\| \lesssim 1.$$

Consequently, by Lemma 6 we have

$$\|E(t,s,\xi)\| = \left\|\widetilde{\Lambda}_0(t)\mathcal{E}(t,s,\xi)\widetilde{E}_2(t,s,\xi)\right\| \lesssim \frac{(1+s)^\nu}{(1+t)^\nu}.$$

Thus the proof of Proposition 5 is concluded. □

5 Proof of Theorem 1

5.1 In the case $\gamma \le 2$

Let $\gamma \le 2$. We note that if $\gamma \le 2$, then $Z_\Psi(N) \cup Z_I(N) \subset \{(t,\xi)\,;\ |\xi| \le N\}$. If $(t,\xi) \in Z_\Psi(N)$, then by Lemma 3, Proposition 3 and (40) we have

$$\begin{aligned}
&|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 + p(t)|v(t,\xi)|^2\\
&\le (1+t)^{2\nu}(|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 + (1+t)^{-2}|v(t,\xi)|^2)\\
&\simeq (1+t)^{2\nu}|U(t,\xi)|^2 \simeq (1+t)^{4\nu}\,|V(t,\xi)|\\
&= (1+t)^{4\nu}\,|E(t,0,\xi)V(0,\xi)|^2\\
&\lesssim |V(0,\xi)|^2\,.
\end{aligned}$$

Moreover, by Propositions 3, 4 and 5 we have

$$\|E(t,0,\xi)\| = \|E(t,\theta_1,\xi)E(\theta_1,0,\xi)\| \lesssim (1+t)^{-\nu}(1+\theta_1)^{-\nu}$$

for $(t,\xi) \in Z_I(N)$,

$$\|E(t,0,\xi)\| = \|E(t,\theta_2,\xi)E(\theta_2,\theta_1,\xi)E(\theta_1,0,\xi)\| \lesssim (1+t)^{-\nu}(1+\theta_1)^{-\nu}$$

for $(t,\xi) \in Z_H(N) \cap \{(t,\xi)\,;\ |\xi| \le N\}$ and

$$\|E(t,0,\xi)\| \lesssim (1+t)^{-\nu}$$

for $(t,\xi) \in Z_H(N) \cap \{(t,\xi)\,;\ |\xi| \ge N\}$. Therefore, if $(t,\xi) \in Z_I \cup (Z_H(N) \cap \{(t,\xi)\,;\ |\xi| \le N\})$, then we have

$$\begin{aligned}
|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 &\lesssim |U(t,\xi)|^2 \simeq (1+t)^{2\nu}\,|V(t,\xi)|\\
&= (1+t)^{2\nu}\,|E(t,0,\xi)V(0,\xi)|^2\\
&\lesssim (1+\theta_1)^{-2\nu}\,|V(0,\xi)|^2\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
p(t)|v(t,\xi)|^2 \le p(\theta_1)|v(t,\xi)|^2 &= N^{-2}(1+\theta_1)^{2\nu}|\xi|^2|v(t,\xi)|^2\\
&\lesssim |V(0,\xi)|^2.
\end{aligned}$$

On the other hand, if $(t,\xi) \in Z_H(N) \cap \{(t,\xi)\,;\ |\xi| \ge N\}$, then we have

$$|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \lesssim |V(0,\xi)|^2\,,$$

and thus

$$p(t)|v(t,\xi)|^2 \le |v(t,\xi)|^2 \le N^{-2}|\xi|^2|v(t,\xi)|^2 \lesssim |V(0,\xi)|^2.$$

Consequently, the following estimate is established uniformly with respect to (t,ξ):

$$|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 + p(t)|v(t,\xi)|^2 \lesssim |V(0,\xi)|^2. \tag{72}$$

Parseval's identity and (3) conclude energy estimate (4).

5.2 In the case $\gamma > 2$

Let $\gamma > 2$. We note that estimate (72) in $Z_\Psi(N) \cup (Z_I(N) \cap \{(t,\xi)\ ;\ |\xi| \le N\})$ is proved exactly by the same way as in the proof for $\gamma \le 2$. By Propositions 4 and 5 we have

$$\|E(t,0,\xi)\| = \|E(t,\theta_2,\xi)E(\theta_2,0,\xi)\| \lesssim (1+t)^{-\nu}$$

for $(t,\xi) \in Z_I(N) \cap \{(t,\xi)\ ;\ |\xi| \ge N\}$, and

$$\|E(t,0,\xi)\| \lesssim (1+t)^{-\nu}$$

for $(t,\xi) \in Z_H(N)$. Therefore, analogously to the corresponding estimate in the case $\gamma \le 2$ we have

$$|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \lesssim |V(0,\xi)|^2$$

for $(t,\xi) \in (Z_I(N) \cap \{(t,\xi)\ ;\ |\xi| \ge N\}) \cup Z_H(N)$, and thus

$$p(t)|v(t,\xi)|^2 \lesssim |V(0,\xi)|^2.$$

Consequently, estimate (72) is established uniformly with respect to (t,ξ), and thus energy estimate (4) is concluded.

6 Concluding remarks

Klein–Gordon equation (1) can be identified with dissipative wave equation (6) by means of (7). Hence, we may expect that the previous results for (6) and (8) will be directly reduced to Klein–Gordon equations. However, such a procedure is not straight-forward. Indeed, it is not easy to see whether the corresponding oscillation and stabilization conditions to b, which were introduced in previous papers, are satisfied or not by the solution of nonlinear equation (7).

The optimality of assumption (15) is an open problem. If we succeed to reduce our problem to previous results for dissipative wave equation (6) in [8] or the wave equation with variable propagation speed (8) in [9], we may expect that if $\delta \in C^m([0,\infty))$ with $m \ge 0$, then assumption (15) is weakened to

$$\beta \ge \frac{1}{2}\left(-\gamma + 3 - \frac{m(\gamma-1)}{m+2}\right) \tag{73}$$

under some suitable assumptions for the derivatives $\delta^{(k)}(t)$ $(k = 1, \cdots, m)$. Moreover, we may also expect that estimate (4) does not hold in general if $\beta < -\gamma + 2$.

In [6] the authors considered the following general model of the coefficient in the potential:

$$M(t) = \frac{\mu^2}{g(t)^2(1+t)^2} \tag{74}$$

with a non-effective potential having very slow oscillations. We may expect to extend (74) to very fast oscillations. However, the argument of proof for Theorem 1 can be applied only in the case $g(t) \equiv 1$.

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7 Appendix

7.1 An estimate for Example 1

Lemma 7 *The following estimate is valid for any* $\sigma > 0$:

$$\left| \int_\tau^\infty \frac{\sin\theta}{\theta^\sigma} d\theta \right| = O(\tau^{-\sigma}) \ \ (\tau \to \infty). \tag{75}$$

Proof Let τ be a real number satisfying $\tau \geq 2\pi$ and N a positive integer satisfying $2\pi(N-1) \leq \tau < 2\pi N$. Then we have

$$\begin{aligned}
\left| \int_\tau^\infty \frac{\sin\theta}{\theta^\sigma} \mathrm{d}\theta \right| &\leq \int_\tau^{2\pi N} \frac{|\sin\theta|}{\theta^\sigma} \mathrm{d}\theta + \sum_{k=N}^\infty \left| \int_{2\pi k}^{2\pi(k+1)} \frac{\sin\theta}{\theta^\sigma} \mathrm{d}\theta \right| \\
&\leq \frac{4}{\tau^\sigma} + \sum_{k=N}^\infty \left(\int_{2\pi k}^{2\pi k+\pi} \frac{\sin\theta}{\theta^\sigma} \mathrm{d}\theta + \int_{2\pi k+\pi}^{2\pi(k+1)} \frac{\sin\theta}{\theta^\sigma} \mathrm{d}\theta \right) \\
&\leq \frac{4}{\tau^\sigma} + \sum_{k=N}^\infty \left(\frac{2}{(2\pi k)^\sigma} - \frac{2}{(2\pi(k+1))^\sigma} \right) \\
&= \frac{4}{\tau^\sigma} + \frac{2}{(2\pi N)^\sigma} \leq \frac{6}{\tau^\sigma},
\end{aligned}$$

as we wanted to show. □

7.2 Representation of solutions for Liouville-type equations

Lemma 8 *The function* $\eta = \eta(t)$ *in* (22) *is a solution to* (20).

Proof By the Cauchy product formula

$$\left(\sum_{j=1}^\infty a_j \right) \left(\sum_{j=1}^\infty a_j \right) = \sum_{k=2}^\infty \sum_{j=1}^{k-1} a_j a_{k-j} \tag{76}$$

we have

$$\begin{aligned}
\left(\sum_{j=1}^\infty \int_t^\infty q_j(s) \mathrm{d}s \right)^2 &= \sum_{k=2}^\infty \sum_{j=1}^{k-1} \left(\int_t^\infty q_j(s) \mathrm{d}s \right) \left(\int_t^\infty q_{k-j}(s) \mathrm{d}s \right) \\
&= \sum_{k=2}^\infty q_k(t).
\end{aligned}$$

Hence, we obtain

$$\frac{\eta''(t)}{\eta(t)} + M(t) = \left(\sum_{j=1}^\infty \int_t^\infty q_j(s) \mathrm{d}s \right)^2 - \sum_{k=1}^\infty q_k(t) + M(t) = 0. \tag{77}$$

Thus $\eta = \eta(t)$ is a solution to (20). □

7.3 Catalan's numbers

Lemma 9 *Let γ_k be the kth Catalan's number which is defined by*

$$\gamma_k := \frac{(2k)!}{k!(k+1)!} = \begin{cases} 1 & \text{for } k = 0, \\ \sum_{j=0}^{k-1} \gamma_j \gamma_{k-j-1} & \text{for } k = 1, 2, \dots. \end{cases} \tag{78}$$

Moreover, for any $0 < r < 1/4$ *the following equality is established*:

$$\nu(r) := \sum_{j=1}^{\infty} r^j \gamma_{j-1} = \frac{1 - \sqrt{1-4r}}{2}. \tag{79}$$

Proof By (78) we have

$$\lim_{k\to\infty} \frac{r^{k+1}\gamma_k}{r^k \gamma_{k-1}} = \lim_{k\to\infty} \frac{2(2k-1)r}{k+1} = 4r \tag{80}$$

for any $r > 0$. It follows that the series $\sum_{j=1}^{\infty} r^j \gamma_{j-1}$ converges for $0 \le r < 1/4$. By (76) we have the following inequalities:

$$\nu(r)^2 = \left(\sum_{j=1}^{\infty} r^j \gamma_{j-1} \right)^2 = \sum_{k=2}^{\infty} r^k \sum_{j=1}^{k-1} \gamma_{j-1}\gamma_{k-j-1} = \sum_{k=2}^{\infty} r^k \gamma_{k-1} = \nu(r) - r. \tag{81}$$

Thus $\nu(r)$ is given by (79) as a solution to the quadratic equation $\nu^2 - \nu + r = 0$. This completes the proof. □

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