Extension of modular functions and measures

Hans Weber¹

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Abstract We prove extension theorems for group-valued modular functions defined on orthomodular lattices or modular complemented lattices and for modular measures defined on (pseudo-)D-lattices generalizing results of Riečan (Mat Časopis Sloven Akad Vied 19:44–49, 1969; 20:239–244, 1970; Comment Math Univ Carolin 20:309–316, 1979), Avallone and De Simone (Ital J Pure Appl Math 9:109–122, 2001) and Avallone et al. (Bollettino UMI 9-B(8):423–444, 2006; Math Slovaca 66:421–438, 2016). As basic tool, we first prove an extension theorem for lattice uniformities. This also yields as immediate consequence a result on the extension of modular functions on arbitrary lattices similar to that of Fox and Morales (Fundam Math 78:99–106, 1973) and Kranz (Fundam Math 91:171–178, 1976).

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1 Introduction

Carathéodory's theorem on the extension of a σ -additive positive measure from an algebra of sets on the generated σ -algebra has been generalized in different directions. Sion [16] was the first who generalized this theorem for group-valued measures. Riečan [12–14] proved extension theorems for “measures”¹ defined on a more general domain, namely on orthomodular lattices or modular complemented lattices. Avallone and De Simone have given in [2] another proof (using a topological approach) of Riečan's extension theorem for positive

¹ The “measures” considered in these articles of Riečan are exactly σ -order continuous modular functions μ (as defined in Sect. 2) satisfying $\mu(0) = 0$. To see this in case [14] use [15, Lemma 1].

✉ Hans Weber
hans.weber@uniud.it

¹ Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università degli Studi di Udine, Via delle Scienze 206, 33100 Udine, Italy

measures on orthomodular lattices. As a consequence, they obtained an extension theorem for measures which can be “controlled” by a family of positive measures; in particular, this implies (employing a control measure theorem) an extension theorem for measures defined on an orthomodular lattice with values in a locally convex space. These results of [2] are generalized in [3] for modular measures on D-lattices² and recently for modular measures on pseudo-D-lattices [4].

The main aim of this article is to prove these extension theorems for measures (defined on orthomodular lattices, more general on D-lattices, or on modular complemented lattices) with values in topological groups (in particular in not necessarily locally convex topological linear spaces). For that, we first prove extension theorems for lattice uniformities; the measure extension one gets then easily as continuous extension with respect to a suitable lattice uniformity. With this procedure, the problem to extend measures (or modular functions) can completely be reduced to the problem of extension of lattice uniformities. In particular, this shows that here the structure of the range space plays an unimportant role. More precisely, let S be a sublattice of a σ -continuous σ -complete lattice L and $\sigma(S)$ be the σ -order closed sublattice generated by S . We first give a necessary and sufficient condition for extending an exhaustive σ -order continuous (σ -o.c. for short) lattice uniformity u on S to a σ -o.c. lattice uniformity $\bar{u}$ on $\sigma(S)$ (Corollary 4.6). This condition is satisfied, e.g. if L is an orthomodular lattice and S a sub-orthomodular lattice of L (Corollary 4.9), or if L is a modular complemented lattice and S a sublattice of L containing all complements of any $x \in S$ (Corollary 4.8). This is applied in Sect. 5 where u is the μ -uniformity of an exhaustive σ -o.c. group-valued modular function $\mu : S \rightarrow G$. Since S is dense in $(\sigma(S), \bar{u})$, μ can be extended by continuity to a σ -o.c. modular function on $\sigma(S)$. This yields always an extension theorem for modular functions when L is an orthomodular lattice (Theorem 5.3) or a modular complemented lattice (Theorem 5.2). When L is an arbitrary lattice, it yields a necessary and sufficient condition for extending an exhaustive σ -o.c. modular function $\mu : S \rightarrow G$ to a σ -o.c. modular function on $\sigma(S)$ (Theorem 5.1); (2) $\Leftrightarrow$ (3) of 5.1 is essentially equivalent to the result of Kranz [11] who generalized a result of Fox and Morales [10] replacing lattices of sets by abstract lattices.

The case that L is a D-lattice we treat separately in Sect. 6.1. We first show that any exhaustive σ -o.c. D-lattice uniformity on a D-sublattice S of a σ -continuous σ -complete D-lattice L can be extended to a σ -o.c. D-lattice uniformity on $\sigma(S)$ (Corollary 6.6). An immediate consequence (Corollary 6.7) is an extension theorem for group-valued modular measures from S to $\sigma(S)$ which generalizes results of Avallone et al. [3]. The system $\sigma(S)$ turns out to be a σ -complete D-sublattice of L (see Theorem 6.4 or [3, Theorem 3.4]). In Sect. 6.2, we briefly explain how to transfer the results of Sect. 6.1 in the setting of pseudo-D-lattices.

Throughout, let L be a lattice and $G = (G, +, \tau)$ a commutative Hausdorff topological group.

2 Preliminaries

Let (x_n) be a sequence in L and $x \in L$. We write $x_n \uparrow$ if (x_n) is increasing, and $x_n \uparrow x$ if (x_n) is increasing and $x = \sup_{n \in \mathbb{N}} x_n$. Analogously, $x_n \downarrow$ and $x_n \downarrow x$ are defined.

² D-lattices are a common generalization of orthomodular lattices and MV-algebras. For the significance of this structure, see [7].

For $A \subseteq L$ let

$$A_\sigma = \left\{ x \in L : \exists (x_n) \in A^{\mathbb{N}} \quad x_n \uparrow x \right\},$$

$$A_\delta = \left\{ x \in L : \exists (x_n) \in A^{\mathbb{N}} \quad x_n \downarrow x \right\}.$$

L is called σ -continuous if $x_n \uparrow x$ implies $x_n \wedge y \uparrow x \wedge y$ and $x_n \downarrow x$ implies $x_n \vee y \downarrow x \vee y$ for any sequence (x_n) in L and $x, y \in L$. If L is σ -continuous and A a sublattice of L , then A_σ and A_δ are also sublattices of L .

For $A \subseteq L$, we denote by $\sigma(A)$ the smallest subset M of L containing A with $M = M_\sigma = M_\delta$.

Proposition 2.1 *If L is σ -continuous and A a sublattice of L , then also $\sigma(A)$ is a sublattice of L .*

Proof Let $\mathfrak{A}$ be the set of all subsets T of $\sigma(S)$ containing A such that $T \vee T \subseteq \sigma(S)$ and $T \wedge T \subseteq \sigma(S)$. Then $S \in \mathfrak{A}$. Let M be a maximal element of $\mathfrak{A}$. Obviously, $M_\sigma \vee M_\sigma \subseteq (M \vee M)_\sigma \subseteq \sigma(S)_\sigma = \sigma(S)$. Moreover, since L is σ -continuous, $M_\sigma \wedge M_\sigma \subseteq (M \wedge M)_\sigma \subseteq \sigma(S)_\sigma = \sigma(S)$. Hence $M_\sigma \in \mathfrak{A}$, and thus $M_\sigma = M$ by the maximality of M . Dually, we have $M_\delta = M$. Therefore $\sigma(M) = M$ and finally $M = \sigma(S)$, since $S \subseteq M \subseteq \sigma(S)$. $\square$

Let u be a lattice uniformity on L , i.e. a uniformity on L making the lattice operations $\wedge$ and $\vee$ uniformly continuous. We call u *exhaustive* if every monotone sequence (equivalently, every monotone net) in L is u -Cauchy (cf. [17, Definition 1.2.5], [18, Proposition 6.1]). A subset $A \subseteq L$ is *lower dense* (*upper dense*) in (L, u) if for every $x \in L$ and $U \in u$ there exists an element $a \in A$ with $(a, x) \in U$ and $a \leq x$ (respectively, $a \geq x$).

Let $D(L)$ be the system of all semimetrics $d : L^2 \rightarrow [0, +\infty[$ satisfying $d(x \vee z, y \vee z) \leq 2d(x, y)$ and $d(x \wedge z, y \wedge z) \leq 2d(x, y)$ for all $x, y, z \in L$. Obviously, the uniformity induced by a nonempty subset of $D(L)$ is a lattice uniformity. Vice versa, every lattice uniformity on L is generated by a nonempty subset of $D(L)$, see [19, Corollary 1.4].

A semimetric $d \in D(L)$ is called *exhaustive* if so is the d -uniformity.

Lemma 2.2 *Let $d \in D(L)$.*

- (a) $d(\sup_{i \leq n} x_i, \sup_{i \leq n} y_i) \leq \sum_{i=1}^n 2^i d(x_i, y_i)$ and $d(\inf_{i \leq n} x_i, \inf_{i \leq n} y_i) \leq \sum_{i=1}^n 2^i d(x_i, y_i)$ whenever $x_i, y_i \in L$ for $i = 1, \dots, n$.
- (b) If $x, y, z \in L$ with $x \leq y \leq z$, then $d(x, y) \leq 2d(x, z)$ and $d(y, z) \leq 2d(x, z)$.

Proof (a) For the first inequality, one verifies by induction

$$d\left(\sup_{i \leq m} x_i, \sup_{i \leq m} y_i\right) \leq \sum_{i=1}^{m-1} 2^i d(x_i, y_i) + 2^{m-1} d(x_m, y_m) \quad \text{for } 1 \leq m \leq n.$$

The second inequality then follows by duality.

- (b) $d(x, y) = d(x \wedge y, z \wedge y) \leq 2d(x, z)$ and $d(y, z) = d(x \vee y, z \vee y) \leq 2d(x, z)$.

$\square$

Lemma 2.3 *Let $d \in D(L)$ and K be a lower-dense sublattice of (L, d) . Let (x_n) be a decreasing sequence in L and $\varepsilon > 0$. Then there is a decreasing sequence (z_n) in K such that $z_n \leq x_n$ and $d(x_n, z_n) \leq \varepsilon$ for all $n \in \mathbb{N}$.*

Proof Choose $y_n \in K$ such that $y_n \leq x_n$ and $d(y_n, x_n) \leq 4^{-n}\varepsilon$ for $n \in \mathbb{N}$. Then $z_n := \inf_{i \leq n} y_i$ defines a decreasing sequence in K and

$$d(x_n, z_n) = d\left(\inf_{i \leq n} x_i, \inf_{i \leq n} y_i\right) \leq \sum_{i=1}^n 2^i d(x_i, y_i) \leq \sum_{i=1}^n 2^i 4^{-i} \varepsilon \leq \varepsilon.$$

□

Let $\mu : L \rightarrow G$ be a modular function, i.e. $\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y)$ for all $x, y \in L$. Then the sets

$$\{(a, b) \in L^2 : \mu(x) - \mu(y) \in U \text{ whenever } x, y \in [a \wedge b, a \vee b]\} \text{ where } U$$

is a 0-neighbourhood in G

form a base of the weakest lattice uniformity making μ uniformly continuous (see [9]); this uniformity is called the μ -uniformity. The proof of the next proposition is contained in [9].

Proposition 2.4 *Let $\mu : L \rightarrow G$ be a modular function and $(|\cdot|_\gamma)_{\gamma \in \Gamma}$ be a system of bounded group seminorms generating the topology τ of G . Then*

$$d_\gamma(a, b) := \sup \{|\mu(x) - \mu(y)|_\gamma : x, y \in [a \wedge b, a \vee b], x \leq y\}, \quad \gamma \in \Gamma,$$

defines a system of semimetrics on L which generates the μ -uniformity. Moreover,

$d_\gamma(x \vee z, y \vee z) \leq d_\gamma(x, y)$ and $d_\gamma(x \wedge z, y \wedge z) \leq d_\gamma(x, y)$ for all $x, y, z \in L$ and $\gamma \in \Gamma$, in particular $d_\gamma \in D(L)$ for $\gamma \in \Gamma$.

A modular function $\mu : L \rightarrow G$ is *exhaustive* if $(\mu(x_n))_{n \in \mathbb{N}}$ is Cauchy for every monotone sequence (x_n) in L . By [21, 3.6] μ is exhaustive iff the μ -uniformity is exhaustive.

Let u be a lattice uniformity on a sublattice M of L . We call u *lower σ -order continuous* (*upper σ -order continuous*) if $x_n \uparrow x$ (respectively $x_n \downarrow x$)³ implies $x_n \rightarrow x$ (u) for any sequence (x_n) in M and $x \in M$. u is *σ -order continuous* if u is both, lower σ -o.c. and upper σ -o.c. A semimetric $d \in D(M)$ is, respectively, σ -o.c., lower σ -o.c., upper σ -o.c. if the d -uniformity is so.

A modular function $\mu : M \rightarrow G$ is *lower σ -o.c.* (*upper σ -o.c.*) if $x_n \uparrow x$ (respectively $x_n \downarrow x$) implies $\mu(x_n) \rightarrow \mu(x)$ (τ) for any sequence (x_n) in M and $x \in M$. If μ is both, lower σ -o.c. and upper σ -o.c., then μ is called σ -o.c..

More general, one can define for an infinite cardinal number κ (lower, upper) κ -order continuity replacing in the definition sequences by nets $(x_\gamma)_{\gamma \in \Gamma}$ in M with $|\Gamma| \leq \kappa$. The proof of [21, Proposition 3.5] shows that for a modular function $\mu : M \rightarrow G$ the μ -uniformity is lower κ -o.c. (upper κ -o.c.) iff $\mu(x_\gamma) \rightarrow \mu(x)$ whenever $x \in M$ and $(x_\gamma)_{\gamma \in \Gamma}$ is a net in M such that $|\Gamma| \leq \kappa$ and $x_\gamma \uparrow x$ (respectively $x_\gamma \downarrow x$). Here we are interested only in the case that κ is countable:

Proposition 2.5 *Let $\mu : M \rightarrow G$ be a modular function defined on a sublattice of L . Then μ is lower σ -o.c. (upper σ -o.c.) iff the μ -uniformity is lower σ -o.c. (respectively, upper σ -o.c.).*

Lemma 2.6 *Let L be σ -complete, M and K sublattices of L and $d \in D(M)$.*

- (a) *Suppose that $K_\delta \subseteq M$ and K is lower dense in (M, d) . If $d(y_n, y) \rightarrow 0$ ($n \rightarrow +\infty$) whenever (y_n) is a decreasing sequence in K with $y_n \downarrow y \in L$, then d is upper σ -o.c.*

³ In this article the supremum or infimum of a subset of a sublattice M of L is always taken in L (not in M).

- (b) Suppose that $K_\sigma \subseteq M$ and K is upper-dense in (M, d) . If $d(y_n, y) \rightarrow 0$ ($n \rightarrow +\infty$) whenever (y_n) is a increasing sequence in K with $y_n \uparrow y \in L$, then d is lower σ -o.c.

Proof We only proof (a); (b) then follows by duality. Let $x_n, x \in M$ with $x_n \downarrow x$ and $\varepsilon > 0$. Choose $z \in K$ such that $z \leq x$ and $d(z, x) \leq \varepsilon$, and a decreasing sequence (z_n) in K such that $z_n \leq x_n$ and $d(z_n, x_n) \leq \varepsilon$ for $n \in \mathbb{N}$ (see Lemma 2.3). Let $y_n := z_n \vee z$ and $y := \inf_{n \in \mathbb{N}} y_n \in K_\delta \subseteq M$. Then $d(x_n, y_n) = d(x_n \vee z, z_n \vee z) \leq 2d(x_n, z_n) \leq 2\varepsilon$ and $d(y, x) \leq 2d(z, x) \leq 2\varepsilon$ by Lemma 2.2(b). It follows $d(x_n, x) \leq d(x_n, y_n) + d(y_n, y) + d(y, x) \leq 4\varepsilon + d(y_n, y)$. Since $d(y_n, y) \rightarrow 0$ ($n \rightarrow +\infty$), we have $d(x_n, x) \leq 5\varepsilon$ eventually. $\square$

3 A uniqueness theorem

We are interested in extending a lattice uniformity (respectively, a modular function) on S to a σ -o.c. lattice uniformity (a σ -o.c. modular function) on $\sigma(S)$. Hereby, the uniqueness part can be proved very easily and without any additional assumption.

Theorem 3.1 *Let S be a sublattice of L .*

- (a) *If w a σ -o.c. lattice uniformity on $\sigma(S)$, then S is dense in $(\sigma(S), w)$.*
- (b) *If u and v are σ -o.c. lattice uniformities on $\sigma(S)$ which induce on S the same uniformity (in symbols $u|_S = v|_S$), then $u = v$.*
- (c) *If d_1 and d_2 are σ -o.c. semimetrics belonging to $D(\sigma(S))$ which coincide on S , then $d_1 = d_2$.*
- (d) *If $\mu, \nu : \sigma(S) \rightarrow G$ are σ -o.c. modular functions such that $\mu|_S = \nu|_S$, then $\mu = \nu$.*

Proof (a) Let $C := \bar{S}$ be the closure of S in $(\sigma(S), w)$. Since w is σ -o.c., $C \ni x_n \uparrow x \in L$ or $C \ni x_n \downarrow x \in L$ implies $x \in \sigma(S)$ and $x_n \rightarrow x$ (w), hence $x \in \bar{C} = C$, i.e. $C_\sigma = C = C_\delta$. Thus $C = \sigma(C) \supseteq \sigma(S)$.

- (b) Since $w := u \vee v$ is σ -o.c., S is dense in $(\sigma(S), w)$ by (a). The assertion now follows from [21, 3.8]. We repeat the proof: since $u \subseteq w$, the closures $\bar{U}^{w \times w}$ of $U \in u|_S$ taken in $(\sigma(S), w) \times (\sigma(S), w)$ form a base of u . Since $u|_S = v|_S$, these sets also form a base of v . Hence $u = v$.
- (c) Let $A := \{x \in \sigma(S) : d_1(x) = d_2(x)\}$. Then $S \subseteq A = A_\sigma = A_\delta$. Hence $A = \sigma(S)$.
- (d) In the proof of (c), replace d_i by μ_i .

$\square$

4 Extension of lattice uniformities

In this section, let L be a σ -complete σ -continuous lattice and S a sublattice of L .

We are interested in extending a lattice uniformity from S to a σ -o.c. uniformity on $\sigma(S)$ (see Theorem 4.6 and its corollaries). This is then applied in the next section to obtain extension theorems for modular functions.

Beginning with semimetrics $d \in D(S)$, we extend d , in the first step, from S to S_σ .

Proposition 4.1 *Let $d \in D(S)$ be exhaustive and lower σ -o.c.*

- (a) *Then there is a unique lower σ -order continuous $d_\sigma \in D(S_\sigma)$ extending d . Moreover, S is lower dense in (S_σ, d_σ) , and d_σ is exhaustive.*

- (b) If K is a lower-dense subset of (S, d) , then K is lower dense in (S_σ, d_σ) , too.
 (c) If K is a upper-dense subset of (S, d) , then K_σ is upper dense in (S_σ, d_σ) .

Proof (a) Uniqueness: let $\bar{d} \in D(S_\sigma)$ be a lower σ -o.c. extension of d and $x, y \in S_\sigma$. Let $x_n, y_n \in S$ with $x_n \uparrow x$ and $y_n \uparrow y$. Then $x_n \rightarrow x$ ($\bar{d}$) and $y_n \rightarrow y$ ($\bar{d}$), hence $\bar{d}(x, y) = \lim \bar{d}(x_n, y_n) = \lim d(x_n, y_n)$.

Existence: let $x, y \in S_\sigma$ and $x_n, y_n \in S$ with $x_n \uparrow x$ and $y_n \uparrow y$. We first show that we can define d_σ by $d_\sigma(x, y) := \lim d(x_n, y_n)$. Since (x_n) and (y_n) are Cauchy sequences, the $\lim d(x_n, y_n)$ exists. Next we have to show that this limit does not depend on the choice of (x_n) and (y_n) . For that, it is enough to show that $d(x_n, x_n^*) \rightarrow 0$ if $S \ni x_n^* \uparrow x$: let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $d(x_i, x_j) \leq \varepsilon$ and $d(x_i^*, x_j^*) \leq \varepsilon$ for $i, j \geq n_0$. We show that $d(x_n, x_n^*) \leq 6\varepsilon$ for any $n \geq n_0$. Fix $n \geq n_0$. Since L is σ -continuous, $x_j \wedge x_n^* \uparrow x \wedge x_n^* = x_n^*$ (for $j \rightarrow +\infty$). Therefore $x_j \wedge x_n^* \rightarrow x_n^*$ (d). Let $m \geq n$ with $d(x_m \wedge x_n^*, x_n^*) \leq \varepsilon$. Then

$$d(x_n \wedge x_n^*, x_n^*) \leq d(x_n \wedge x_n^*, x_m \wedge x_n^*) + d(x_m \wedge x_n^*, x_n^*) \leq 2d(x_n, x_m) + \varepsilon \leq 2\varepsilon + \varepsilon = 3\varepsilon.$$

Thus we also have $d(x_n \wedge x_n^*, x_n) \leq 3\varepsilon$ and finally $d(x_n, x_n^*) \leq d(x_n, x_n \wedge x_n^*) + d(x_n \wedge x_n^*, x_n^*) \leq 6\varepsilon$.

We have proved that d_σ is well defined. Obviously, d_σ is an $[0, +\infty[$ -valued extension of d and d_σ is symmetric. To verify the triangle inequality, let $x, y, z \in S_\sigma$. Let $x_n, y_n \in S$ as above and $z_n \in S$ with $z_n \uparrow z$. Then

$$d_\sigma(x, y) = \lim d(x_n, y_n) \leq \lim (d(x_n, z_n) + d(z_n, y_n)) = d_\sigma(x, z) + d_\sigma(z, y).$$

Moreover, by the σ -continuity of L we have $x_n \wedge z_n \uparrow x \wedge z$ and $y_n \wedge z_n \uparrow y \wedge z$, hence

$$d_\sigma(x \wedge z, y \wedge z) = \lim d(x_n \wedge z_n, y_n \wedge z_n) \leq 2 \lim d(x_n, y_n) = 2d_\sigma(x, y).$$

Similarly, $d_\sigma(x \vee z, y \vee z) \leq 2d_\sigma(x, y)$. Thus $d_\sigma \in D(S_\sigma)$.

By the definition of d_σ and the exhaustivity of d , we have for $S \ni x_n \uparrow x$

$$d_\sigma(x_n, x) = \lim_{j \rightarrow +\infty} d(x_n, x_j) \rightarrow 0 \quad (n \rightarrow +\infty),$$

in particular, S is lower dense in S_σ .

d_σ is lower σ -o.c.: let now $S_\sigma \ni x_n \uparrow x$ and $(x_{n,i})_i$ for each $n \in \mathbb{N}$ be a sequence in S with $x_{n,i} \uparrow x_n$ (for $i \rightarrow +\infty$). Then $S \ni y_n := \sup_{i,j \leq n} x_{i,j} \uparrow x$. Therefore $d_\sigma(y_n, x) \rightarrow 0$ ($n \rightarrow +\infty$) as we have seen before, hence $d_\sigma(x_n, x) \rightarrow 0$ ($n \rightarrow +\infty$) as $y_n \leq x_n \leq x$.

Since S is dense in (S_σ, d_σ) and d is exhaustive, by [18, Proposition 6.14] d_σ is exhaustive, too.

(b) Let $x \in S_\sigma$ and $\varepsilon > 0$. By (a) there is an $y \in S$ with $y \leq x$ and $d_\sigma(x, y) \leq \varepsilon$. By the assumption in (b), there is a $z \in K$ with $z \leq y$ and $d_\sigma(y, z) \leq \varepsilon$. Therefore $x \geq z \in K$ and $d_\sigma(x, z) \leq 2\varepsilon$.

(c) Let $x \in S_\sigma$ and $\varepsilon > 0$. Let $x_n \in S$ with $x_n \uparrow x$ and (z_n) an increasing sequence in K with $d(x_n, z_n) \leq \varepsilon$ and $z_n \geq x_n$ for all $n \in \mathbb{N}$. (Such a sequence exists by the dual version of Lemma 2.3.) Then $x \leq z := \sup z_n \in K_\sigma$ and $d_\sigma(x, z) = \lim d(x_n, z_n) \leq \varepsilon$. $\square$

By duality, we obtain from Proposition 4.1 the following result.

Proposition 4.2 *Let $d \in D(S)$ be exhaustive and upper σ -o.c..*

- (a) *Then there is a unique upper σ -o.c. $d_\delta \in D(S_\delta)$ extending d . Moreover, S is upper dense in (S_δ, d_δ) , and d_δ is exhaustive.*
 (b) *If K is a upper-dense subset of (S, d) , then K is upper dense in (S_δ, d_δ) , too.*

(c) If K is a lower-dense subset of (S, d) , then K_δ is lower dense in (S_δ, d_δ) .

Remark 4.3 If in Proposition 4.1 d is σ -o.c., then $x_n \rightarrow x$ (d_σ) whenever $S_\sigma \ni x_n \downarrow x \in S$. On the other hand, in general d_σ is not σ -o.c. as shown by Example 4.4.

Proof $S_\sigma \ni x_n \downarrow x \in S$, $\varepsilon > 0$ and (z_n) be a decreasing sequence in S such that $z_n \leq x_n$ and $d(z_n, x_n) \leq \varepsilon$ for all $n \in \mathbb{N}$, see Lemma 2.3. Then $S \ni y_n := x \vee z_n \downarrow x$, and thus $\lim d(y_n, x) = 0$. With Lemma 2.2(b), we get

$$d(x_n, x) \leq d(x_n, y_n) + d(y_n, x) \leq 2d(x_n, z_n) + d(y_n, x) \leq 2\varepsilon + d(y_n, x),$$

and hence $d(x_n, x) \leq 3\varepsilon$ eventually. $\square$

Example 4.4 Let $S := [0, 1] \cup [2, 3] \subseteq L := [0, 1] \cup [2, 3] (\subseteq \mathbb{R})$ and $\bar{d}(x, y) := |x - y|$ for $x, y \in L$. Then $S_\sigma = L$, $\bar{d}$ is the unique lower σ -o.c. extension of $d := \bar{d}|_{S \times S}$ belonging to $D(L)$, d is σ -o.c., but $\bar{d}$ is not.

$\bar{\mu}(x) := x$ defines a modular function on L . Let $\mu := \bar{\mu}|_S$. Then the $\bar{\mu}$ -uniformity (μ -uniformity) coincides with the $\bar{d}$ -uniformity (respectively, with the d -uniformity). The function μ is σ -o.c., $\bar{\mu}$ is the unique lower σ -o.c. $\mathbb{R}$ -valued modular extension of μ , but $\bar{\mu}$ is not σ -o.c.: $2 + \frac{1}{n} \downarrow 1$ in L , but $\mu(2 + \frac{1}{n}) \rightarrow \bar{\mu}(1)$.

Theorem 4.5 Let $d \in D(S)$ be exhaustive. Then the following conditions are equivalent:

- (1) $\lim d(x_n \vee y_n, x_n) = 0$ whenever $(x_n), (y_n)$ are sequences in S and $x, y \in L$ such that $x_n \uparrow x$, $y_n \downarrow y$ and $y \leq x$.
- (2) d admits a σ -o.c. extension $d_\sigma \in D(S_\sigma)$ onto S_σ .
- (3) d admits a σ -o.c. extension $\bar{d} \in D(\sigma(S))$ onto $\sigma(S)$.

Proof (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): First observe that by Proposition 4.1 the extension d_σ is uniquely determined, d_σ is exhaustive and S is lower dense in (S_σ, d_σ) . By Proposition 4.2 applied to S_σ, d_σ instead of S, d , there is a (unique) exhaustive upper σ -o.c. $\bar{d} \in D(S_{\sigma\delta})$ extending d_σ such that S_σ is upper dense in $(S_{\sigma\delta}, \bar{d})$. Moreover, 4.2(c) applied with $K := S$ yields that S_δ is lower dense in $(S_{\sigma\delta}, \bar{d})$.

Let $\mathfrak{A}$ be the set of all pairs (T, d_T) where T is a sublattice of $\sigma(S)$ containing $S_{\sigma\delta}$, $d_T \in D(T)$ is an extension of $\bar{d}$, S_σ is upper dense in (T, d_T) and S_δ is lower dense in (T, d_T) . As we have seen before $(S_{\sigma\delta}, \bar{d}) \in \mathfrak{A}$. By Zorn's lemma, $\mathfrak{A}$ contains a maximal element $(M, \bar{d})$. Applying Lemma 2.6(a) for $S_\delta, M, \bar{d}$ instead of K, M, d , we get that $\bar{d}$ is upper σ -o.c., and applying Lemma 2.6(b) for $S_\sigma, M, \bar{d}$ instead of K, M, d , we get that $\bar{d}$ is lower σ -o.c.; thus $\bar{d}$ is σ -o.c. Since S_σ is dense in $(M, \bar{d})$ and $d_\sigma = \bar{d}|_{S_\sigma}$ is exhaustive, $\bar{d}$ is by [18, Proposition 6.14] exhaustive, too.

It remains to show that $M = \sigma(S)$. We first show that $M_\sigma = M$: by Proposition 4.1(a) there is a unique lower σ -o.c. $\hat{d} \in D(M_\sigma)$ extending $\bar{d}$. Moreover, applying 4.1(b) with S_δ, M instead of K, S and 4.1(c) with S_σ, M instead of K, S , we obtain that S_δ is lower dense and S_σ is upper dense in $(M_\sigma, \hat{d})$. Thus $(M_\sigma, \hat{d}) \in \mathfrak{A}$ and therefore $M_\sigma = M$ by the maximality of M . Similarly, one sees that $M_\delta = M$. Therefore $M = \sigma(S)$.

(2) $\Rightarrow$ (1): With x_n, y_n, x, y as in (1) we have

$$\begin{aligned} d(x_n \vee y_n, x_n) &\leq d_\sigma(x_n \vee y_n, x \vee y_n) + d_\sigma(x \vee y_n, x) + d_\sigma(x, x_n) \\ &\leq 3d_\sigma(x, x_n) + d_\sigma(x \vee y_n, x). \end{aligned}$$

Now observe that $\lim d_\sigma(x, x_n) = \lim d_\sigma(x \vee y_n, x) = 0$ since $x_n \uparrow x$, $x \vee y_n \downarrow x$ and d_σ is σ -o.c..

(1) $\Rightarrow$ (2): Applying (1) with $y_n = y = x \in S$, one sees that d is lower σ -o.c.. Let $d_\sigma \in D(S_\sigma)$ be the extension of d according to Proposition 4.1. It remains to show that d_σ is upper σ -o.c.: let $z_n, z \in S_\sigma$ with $z_n \downarrow z$ and $\varepsilon > 0$. Choose a decreasing sequence (y_n) in S such that $y_n \leq z_n$ and $d_\sigma(y_n, z_n) \leq \varepsilon$ for all $n \in \mathbb{N}$ and an increasing sequence (x_n) in S with $x_n \uparrow z$. Then $\inf y_n \leq \sup x_n$. Therefore $\lim d(x_n \vee y_n, x_n) = 0$ by (1). From

$$d_\sigma(z_n, z) \leq d_\sigma(z_n, x_n \vee y_n) + d_\sigma(x_n \vee y_n, x_n) + d_\sigma(x_n, z),$$

$d_\sigma(z_n, x_n \vee y_n) = d_\sigma(x_n \vee z_n, x_n \vee y_n) \leq 2d_\sigma(z_n, y_n) \leq 2\varepsilon$ and $\lim d(x_n \vee y_n, x_n) = \lim d_\sigma(x_n, z) = 0$ it follows that $d_\sigma(z_n, z) \leq 3\varepsilon$ eventually. $\square$

The next theorem is essentially a reformulation of Theorem 4.5.

Theorem 4.6 *Let u be an exhaustive lattice uniformity on S . Then the following conditions are equivalent:*

- (1) *For any $U \in \mathcal{U}$, we have $(x_n \vee y_n, x_n) \in U$ eventually whenever $(x_n), (y_n)$ are sequences in S and $x, y \in L$ such that $x_n \uparrow x$, $y_n \downarrow y$ and $y \leq x$.*
- (2) *There exists a σ -o.c. lattice uniformity u_σ on S_σ with $u_{\sigma|S} = u$.*
- (3) *There exists a σ -o.c. lattice uniformity $\bar{u}$ on $\sigma(S)$ with $\bar{u}|_S = u$.*

Proof (1) $\Rightarrow$ (3): Let D be a subset of $D(S)$ generating u . Each $d \in D$ has by Theorem 4.5 (1) $\Rightarrow$ (3) a σ -o.c. extension $\bar{d} \in D(\sigma(S))$. Then the uniformity $\bar{u}$ generated by $\{\bar{d} : d \in D\}$ satisfies (3).

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): Let $U \in \mathcal{u}$ and $\bar{d} \in D(S_\sigma)$ be continuous with respect to u_σ such that $\{(x, y) \in S : \bar{d}(x, y) \leq 1\} \subseteq U$. Now apply (2) $\Rightarrow$ (1) of Theorem 4.5 to $d := \bar{d}|_{S \times S}$. $\square$

We now consider the case that L is a complemented modular lattice or an orthomodular lattice. In this case, set $0 = \min L$ and $1 = \max L$.

Lemma 4.7 *Let L be a complemented modular lattice.*

- (a) *If $a, b \in L$ with $a \leq b$ and $a' \in L$ is a complement of a , then there is a complement b' of b with $b' \leq a'$.*
- (b) *If $z_n \uparrow z$ and $z'_n \downarrow z'$ in L such that z'_n is a complement of z_n for each $n \in \mathbb{N}$, then z' is a complement of z .*

Proof (a) By [6, Theorem I.14] L is relatively complemented. One easily sees that a relative complement b' of $b \wedge a'$ in $[0, a']$ is a complement of b in L with $b' \leq a'$.

(b) Since L is σ -continuous, $1 = z_n \vee z'_n \leq z \vee z'_n \downarrow z \vee z'$ and hence $z \vee z' = 1$, and dually $0 = z_n \wedge z'_n \geq z_n \wedge z' \uparrow z \wedge z'$ and hence $z \wedge z' = 0$. $\square$

Corollary 4.8 *Let L be a complemented modular lattice and S a sublattice of L which contains all complements for any $x \in S$. Let u be an exhaustive σ -o.c. lattice uniformity on S . Then there exists a unique σ -o.c. lattice uniformity $\bar{u}$ on $\sigma(S)$ with $\bar{u}|_S = u$.*

Proof The uniqueness statement is contained in Theorem 3.1.

Existence: since u is generated by a subset of $D(S)$, it is enough to consider the case that u is generated by a single $d \in D(S)$. By Proposition 4.1, d has a lower σ -o.c. extension $d_\sigma \in D(S_\sigma)$. By Theorem 4.5, it remains to show that d_σ is upper σ -o.c.: let $z_n, z \in S_\sigma$ with $z_n \downarrow z$, and (x_n) an increasing sequence in S with $x_n \uparrow z$. By Lemma 4.7(a), there is a decreasing sequence (x'_n) in S such that x'_n is a complement of x_n for each $n \in \mathbb{N}$. By Lemma

4.7 (b), $z' := \inf x'_n$ is a complement of z . Consequently, $S_\sigma \ni z_n \wedge x'_n \downarrow z \wedge z' = 0$ and therefore $d_\sigma(z_n \wedge x'_n) \rightarrow 0$ by Remark 4.3. It follows, using $x_n \leq z \leq z_n$ and the modular law, that

$$d_\sigma(z_n, z) \leq 2d_\sigma(z_n, x_n) = 2d_\sigma((z_n \wedge x'_n) \vee x_n, 0 \vee x_n) \leq 4d_\sigma(z_n \wedge x'_n, 0) \rightarrow 0.$$

□

Corollary 4.9 *Let $L = (L, \wedge, \vee, \perp)$ be an orthomodular lattice, S a sub-orthomodular lattice of L and u an exhaustive σ -o.c. lattice uniformity on S . Then there exists a unique σ -o.c. lattice uniformity $\bar{u}$ on $\sigma(S)$ with $\bar{u}|_S = u$.*

Proof The proof is almost identical with that of the proceeding corollary. One has to replace the complements x'_n by the orthocomplements $x_n^\perp$. By the rules in orthomodular lattices, we have $x_n^\perp \downarrow z^\perp$. Further observe that $(z_n \wedge x_n^\perp) \vee x_n = z_n$ by the orthomodular law. □

As well known, the monotone system generated by an algebra of sets is a σ -algebra. B. Riečan generalized this fact in [13, Lemma on p. 240] and [14, Proposition 5]:

Remark 4.10 (B. Riečan)

- (a) Under the assumption of Corollary 4.8, $\sigma(S)$ contains all complements for any $x \in \sigma(S)$.
- (b) Under the assumptions of Corollary 4.9, $\sigma(S)$ is a sub-orthomodular lattice.

5 Extension of modular functions

In this section, let L be a σ -complete σ -continuous lattice, S a sublattice of L and G be complete.

Theorem 5.1 *Let $\mu : S \rightarrow G$ be an exhaustive modular function. Then the following conditions are equivalent:*

- (1) $\lim \mu(x_n \vee y_n) - \mu(x_n) = 0$ whenever $(x_n), (y_n)$ are sequences in S and $x, y \in L$ such that $x_n \uparrow x, y_n \downarrow y$ and $y \leq x$.
- (2) There exists a σ -o.c. modular function $\mu_\sigma : S_\sigma \rightarrow G$ extending μ .
- (3) There exists a σ -o.c. modular function $\bar{\mu} : \sigma(S) \rightarrow G$ extending μ .

Proof (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3): By Proposition 2.5, the μ_σ -uniformity u_σ on S_σ is σ -o.c.. Therefore by Theorem 4.6 (applied to S_σ, u_σ instead of S, u), there exists a σ -o.c. lattice uniformity $\bar{u}$ on $\sigma(S)$ with $\bar{u}|_{S_\sigma} = u_\sigma$. By Theorem 3.1, S is dense in $(\sigma(S), \bar{u})$. Therefore μ_σ admits a continuous extension $\bar{\mu}$ on $(\sigma(S), \bar{u})$ satisfying the condition (3).

(3) $\Rightarrow$ (1): Let x_n, y_n, x, y be given as in (1). Since the $\bar{\mu}$ -uniformity ($=: \bar{u}$) is σ -o.c., we have $x_n \rightarrow x$ ($\bar{u}$) and $y_n \rightarrow y$ ($\bar{u}$), hence $x_n \vee y_n \rightarrow x \vee y = x$ ($\bar{u}$). Consequently $\mu(x_n \vee y_n) - \mu(x_n) \rightarrow \bar{\mu}(x) - \bar{\mu}(x) = 0$ (τ).

(1) $\Rightarrow$ (2): Applying (1) with $y_n = y = x \in S$, one sees that μ is lower σ -o.c. Therefore the μ -uniformity u is lower σ -o.c. by Proposition 2.5. Let $(|\cdot|_\gamma)_{\gamma \in \Gamma}$ be a system of bounded group seminorms generating τ and $(d_\gamma)_{\gamma \in \Gamma}$ the corresponding system of semimetrics on S generating u according to Proposition 2.4. By Proposition 4.1, each d_γ has a lower σ -o.c. extension $\bar{d}_\gamma \in D(S_\sigma)$. Obviously, S is lower dense in S_σ with respect to the uniformity $\bar{u}$ generated by $(\bar{d}_\gamma)_{\gamma \in \Gamma}$. Then the continuous extension μ_σ of μ on (S_σ, u_σ) is a lower σ -o.c. modular function. We proceed as in the proof of 4.5 (1) $\Rightarrow$ (2) to show that μ_σ is also upper

σ -o.c.: let $z_n, z \in S_\sigma$ with $z_n \downarrow z$. Fix $\gamma \in \Gamma$ and $\varepsilon > 0$. Choose a decreasing sequence (y_n) in S such that $y_n \leq z_n$ and $\bar{d}_\gamma(y_n, z_n) \leq \varepsilon$ for all $n \in \mathbb{N}$ and an increasing sequence (x_n) in S with $x_n \uparrow z$. Then $\inf y_n \leq \sup x_n$. Therefore $\lim \mu(x_n \vee y_n) - \mu(x_n) = 0$ by (1). From

$$|\mu_\sigma(z_n) - \mu_\sigma(z)|_\gamma \leq |\mu_\sigma(z_n) - \mu_\sigma(x_n \vee y_n)|_\gamma + |\mu_\sigma(x_n \vee y_n) - \mu_\sigma(x_n)|_\gamma + |\mu_\sigma(x_n) - \mu_\sigma(z)|_\gamma,$$

$|\mu_\sigma(z_n) - \mu_\sigma(x_n \vee y_n)|_\gamma \leq \bar{d}_\gamma(x_n \vee z_n, x_n \vee y_n) \leq \bar{d}_\gamma(z_n, y_n) \leq \varepsilon$ and $\lim \mu(x_n \vee y_n) - \mu(x_n) = \lim \mu_\sigma(x_n) - \mu_\sigma(z) = 0$ follows that $|\mu_\sigma(z_n) - \mu_\sigma(z)|_\gamma \leq 2\varepsilon$ eventually.

We have seen that $\lim |\mu_\sigma(z_n) - \mu_\sigma(z)|_\gamma = 0$ for any $\gamma \in \Gamma$, i.e. $\mu_\sigma(z_n) - \mu_\sigma(z) \rightarrow 0$ (τ). $\square$

Theorem 5.2 *Let L be a complemented modular lattice and S a sublattice of L which contains all complements for any $x \in S$.*

Then for any σ -o.c. exhaustive modular function $\mu : S \rightarrow G$ there exists a unique σ -o.c. modular function $\bar{\mu} : \sigma(S) \rightarrow G$ extending μ .

Proof The uniqueness statement is contained in Theorem 3.1.

Let u be the μ -uniformity and $\bar{u}$ its extension on $\sigma(S)$ according to Corollary 4.8. Then S is dense in $(\sigma(S), \bar{u})$ by Theorem 3.1 and the continuous extension $\bar{\mu}$ of μ on $(\sigma(S), \bar{u})$ is σ -o.c. since $\bar{u}$ is σ -o.c. $\square$

Theorem 5.3 *Let $L = (L, \wedge, \vee, \perp)$ be an orthomodular lattice and S a sub-orthomodular lattice of L .*

Then for any σ -o.c. exhaustive modular function $\mu : S \rightarrow G$ there exists a unique σ -o.c. modular function $\bar{\mu} : \sigma(S) \rightarrow G$ extending μ .

Proof This can be proved as the proceeding theorem replacing Corollary 4.8 by Corollary 4.9. $\square$

Remark 5.4 Let M be an orthomodular lattice or a modular complemented lattice. Then $\mu : M \rightarrow G$ is additive (i.e. $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x, y \in M$ and $x \wedge y = 0$) iff μ is a modular function with $\mu(0) = 0$.

Proof The implication $\Leftarrow$ is obvious. $\Rightarrow$: In case of M being an orthomodular lattice see [20, Proposition 3.1]. If M is modular and complemented, one proceeds similarly: let $x, y \in M$. Since M is relatively complemented, $x \wedge y$ has a relative complement z in $[0, x]$. One easily sees that then z is a relative complement of y in $[0, x \vee y]$. Therefore

$$\mu(x \vee y) + \mu(x \wedge y) = (\mu(y) + \mu(z)) + \mu(x \wedge y) = \mu(y) + (\mu(z) + \mu(x \wedge y)) = \mu(y) + \mu(x).$$

$\square$

6 Extension of D-uniformities and modular measures on D-lattices and pseudo-D-lattices

6.1 D-lattices

In this section, let L be a σ -complete σ -continuous D-lattice, S a D-sublattice of L and G be complete.

Recall that a D-lattice $L = (L, \wedge, \vee, \ominus)$ is a lattice with a smallest element 0, a greatest element 1 and a partial operation $\ominus$ on L such that $b \ominus a$ is defined iff $a \leq b$ and for all $a, b, c \in L$ the following two rules are satisfied:

If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$,
if $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

A *D-sublattice* of L is a sublattice closed with respect to $\ominus$ and containing 1. Examples for D-lattices are MV-algebras and orthomodular lattices. An orthomodular lattice becomes a D-lattice with $b \ominus a := b \wedge a^\perp$ if $a \leq b$.

A *D-uniformity* on L is a lattice uniformity on L such that $\ominus$ is uniformly continuous, or equivalently, that $\triangle : L \times L \rightarrow L$ defined by $a \triangle b := (a \vee b) \ominus (a \wedge b)$ is uniformly continuous. Let $D_0(L)$ be the set of all $d \in D(L)$ such that $d(x \triangle z, y \triangle z) \leq 2d(x, y)$. It follows from [19, Proposition 1.2] that the D-uniformities on L are exactly the uniformities generated by a nonempty subset of $D_0(L)$.

We need the following estimation:

Lemma 6.1 $d(a, b) = d(b \triangle (b \ominus a), b \triangle 0) \leq 2d(b \ominus a, 0)$ if $d \in D_0(L)$ and $a \leq b$.

A modular measure $\mu : L \rightarrow G$ is a modular function such that $\mu(b \ominus a) = \mu(b) - \mu(a)$ whenever $a, b \in L$ and $a \leq b$. If μ is a modular measure, then the μ -uniformity is always a D-uniformity, see [1, 4.2]. D-uniformities and modular measures were intensively studied by A. Avallone and P. Vitolo. In a series of articles, they (partly with coauthors) transferred various theorems for measures on Boolean algebras to the setting of modular measures on D-lattices. Hereby, the D-uniformities play a similar role as FN-topologies in the study of measures on Boolean algebras.

If an orthomodular lattice is considered as D-lattice, then any lattice uniformity is a D-uniformity. This follows from the fact that a lattice uniformity on an orthomodular lattice also makes the orthocomplementation $x \mapsto x^\perp$ uniformly continuous, see [20, Theorem 1.1]. Moreover, a modular function μ on an orthomodular lattice is a modular measure iff $\mu(0) = 0$.

The following lemma is essentially contained in [7, Section 1.8] and in this formulation in [3, Lemma 3.2].

Lemma 6.2 Let $x_n, y_n, x, y \in L$ for $n \in \mathbb{N}$. Then

$y \geq x_n \uparrow x$ implies $y \ominus x_n \downarrow y \ominus x$, and $y \geq x_n \downarrow x$ implies $y \ominus x_n \uparrow y \ominus x$;
 $x \leq y_n \uparrow y$ implies $y_n \ominus x \uparrow y \ominus x$, and $x \leq y_n \downarrow y$ implies $y_n \ominus x \downarrow y \ominus x$

Lemma 6.3 (a) If $S \ni x \leq y \in S_\sigma$, then $y \ominus x \in S_\sigma$.

(b) If M is a sublattice of L , then $M_\sigma \triangle M_\sigma \subseteq (M \triangle M)_{\sigma\delta}$ and $M_\delta \triangle M_\delta \subseteq (M \triangle M)_{\sigma\delta}$.

Proof (a) If $S \ni y_n \uparrow y$, then $S \ni x \vee y_n \uparrow y$ and hence $S \ni (x \vee y_n) \ominus x \uparrow y \ominus x$ by Lemma 6.2.

(b) Set $A = M \triangle M$. Let $x, y \in M_\sigma$ and $x_n, y_n \in M$ with $x_n \uparrow x$ and $y_n \uparrow y$. For fixed $k \in \mathbb{N}$, we have by Lemma 6.2

$$A \ni (x_{k+n} \vee y_{k+n}) \ominus (x_k \wedge y_k) \uparrow (x \vee y) \ominus (x_k \wedge y_k) \in A_\sigma,$$

and therefore $A_\sigma \ni (x \vee y) \ominus (x_n \wedge y_n) \downarrow (x \vee y) \ominus (x \wedge y) \in A_{\sigma\delta}$. This proves $M_\sigma \triangle M_\sigma \subseteq A_{\sigma\delta}$.

Similarly, if $M \ni x_n \downarrow x$ and $M \ni y_n \downarrow y$, then

$$A \ni (x_k \vee y_k) \ominus (x_{k+n} \wedge y_{k+n}) \uparrow (x_k \vee y_k) \ominus (x \wedge y) \in A_\sigma,$$

therefore $A_\sigma \ni (x_n \vee y_n) \ominus (x \wedge y) \downarrow (x \vee y) \ominus (x \wedge y) \in A_{\sigma\delta}$. This proves $M_\delta \triangle M_\delta \subseteq A_{\sigma\delta}$. $\square$

The following result was first proved in [3, Theorem 3.4]. Here we give an essentially easier proof, which could be of interest also in the Boolean case.

Theorem 6.4 $\sigma(S)$ is a D -sublattice of L .

Proof Let $\mathfrak{A}$ be the set of all sublattices T of $\sigma(S)$ containing S such that $T \triangle T \subseteq \sigma(S)$. Then $S \in \mathfrak{A}$. Let M be a maximal element of $\mathfrak{A}$. By Lemma 6.3, we have $M_\sigma \triangle M_\sigma \subseteq (M \triangle M)_{\sigma\delta} \subseteq \sigma(S)_{\sigma\delta} = \sigma(S)$, hence $M_\sigma \in \mathfrak{A}$, and thus $M_\sigma = M$ by the maximality of M . Similarly, one sees $M_\delta = M$. Therefore $\sigma(M) = M$ and finally $M = \sigma(S)$, since $S \subseteq M \subseteq \sigma(S)$. $\square$

Theorem 6.5 Let $d \in D_0(S)$ be exhaustive and σ -o.c.. Then d has a unique σ -o.c. extension $\bar{d} \in D_0(\sigma(S))$.

Proof The uniqueness statement is contained in Theorem 3.1.

Existence: Let $d_\sigma \in D(S_\sigma)$ be according to Proposition 4.1. We first show that d_σ is upper σ -o.c. As in the proof of Corollary 4.8, let $z_n, z \in S_\sigma$ with $z_n \downarrow z$ and (x_n) an increasing sequence in S with $x_n \uparrow z$. Then $z_n \ominus x_n \in S_\sigma$ by Lemma 6.3(a) and $z_n \ominus x_n \downarrow 0$ by Lemma 6.2. Hence $d_\sigma(z_n \ominus x_n, 0) \rightarrow 0$ by Remark 4.3. Therefore, using $x_n \leq z \leq z_n$ and Lemma 6.1, we have $d_\sigma(z_n, z) \leq 2d_\sigma(z_n, x_n) \leq 4d_\sigma(z_n \ominus x_n, 0) \rightarrow 0$.

By Theorem 4.5 d has a σ -o.c. extension $\bar{d} \in D(\sigma(S))$. It remains to show that $\bar{d}(x \triangle y, y \triangle z) \leq 2\bar{d}(x, y)$ for all $x, y, z \in \sigma(S)$. Let $\mathfrak{A}$ be the set of all sublattices T of $\sigma(S)$ containing S such that $\bar{d}(x \triangle z, y \triangle z) \leq 2\bar{d}(x, y)$ for all $x, y, z \in T$. By Zorn's lemma, $\mathfrak{A}$ contains a maximal element M . Let $x, y, z \in M_\sigma$ and $x_n, y_n, z_n \in M$ such that $x_n \uparrow x$, $y_n \uparrow y$ and $z_n \uparrow z$. We show that $x_n \triangle z_n \rightarrow x \triangle z$ ($\bar{d}$). Since $x_i, z_i \in M \in \mathfrak{A}$, we have

$$\bar{d}((x_k \vee z_k) \triangle (x_n \wedge z_n), (x_n \vee z_n) \triangle (x_n \wedge z_n)) \leq 2\bar{d}(x_k \vee z_k, x_n \vee z_n).$$

Passing to the limit for a fixed $n \in \mathbb{N}$ and $n < k \rightarrow +\infty$ (see the previous formula), we get

$$\bar{d}((x \vee z) \triangle (x_n \wedge z_n), x_n \triangle z_n) \leq 2\bar{d}(x \vee z, x_n \vee z_n).$$

Therefore

$$\begin{aligned} \bar{d}(x \triangle z, x_n \triangle z_n) &\leq \bar{d}(x \triangle z, (x \vee z) \triangle (x_n \wedge z_n)) + \bar{d}((x \vee z) \triangle (x_n \wedge z_n), x_n \triangle z_n) \\ &\leq \bar{d}(x \triangle z, (x \vee z) \triangle (x_n \wedge z_n)) + 2\bar{d}(x \vee z, x_n \vee z_n). \end{aligned}$$

Since $(x \vee z) \triangle (x_n \wedge z_n) \downarrow x \triangle z$, $x_n \vee z_n \uparrow x \vee z$ and $\bar{d}$ is σ -o.c., we have

$$\lim \bar{d}(x \triangle z, (x \vee z) \triangle (x_n \wedge z_n)) = \lim \bar{d}(x \vee z, x_n \vee z_n) = 0$$

and hence $\lim \bar{d}(x \triangle z, x_n \triangle z_n) = 0$.

As we have seen, $x_n \triangle z_n \rightarrow x \triangle z$ ($\bar{d}$) and analogously $y_n \triangle z_n \rightarrow y \triangle z$ ($\bar{d}$); moreover, $x_n \rightarrow x$ ($\bar{d}$) and $y_n \rightarrow y$ ($\bar{d}$). Therefore from $\bar{d}(x_n \triangle z_n, y_n \triangle z_n) \leq 2\bar{d}(x_n, y_n)$ we obtain passing to the limit $\bar{d}(x \triangle z, y \triangle z) \leq 2\bar{d}(x, y)$.

This proves that $M_\sigma \in \mathfrak{A}$, and by the maximality of M that $M_\sigma = M$.

Similarly, one can prove that $M_\delta = M$, starting with the inequality

$$\bar{d}((x_n \vee z_n) \triangle (x_k \wedge z_k), (x_n \vee z_n) \triangle (x_n \wedge z_n)) \leq 2\bar{d}(x_k \wedge z_k, x_n \wedge z_n)$$

where $x_k, z_k \in M$ with $x_k \downarrow x$ and $z_k \downarrow z$.

Finally, $M_\sigma = M_\delta = M$ implies $M = \sigma(S)$, i.e. $\bar{d} \in D_0(\sigma(S))$. $\square$

Corollary 6.6 Let u be an exhaustive σ -o.c. D -uniformity on S . Then there exists a unique σ -o.c. D -uniformity $\bar{u}$ on $\sigma(S)$ with $\bar{u}|_S = u$.

This immediately follows from Proposition 3.1, Theorem 6.5 and the fact that any D -uniformity on S or $\sigma(S)$ is generated by semimetrics of $D_0(S)$ or $D_0(\sigma(S))$, respectively.

Corollary 6.7 *Let $\mu : S \rightarrow G$ be a σ -o.c. exhaustive modular measure. Then there exists a unique σ -o.c. modular measure $\bar{\mu} : \sigma(S) \rightarrow G$ extending μ .*

Proof We proceed as in the proofs of 5.2 and 5.3. Let u be the μ -uniformity and $\bar{u}$ its extension on $\sigma(S)$ according to Corollary 6.6. Then S is dense in $(\sigma(S), \bar{u})$ by Theorem 3.1. Define $\bar{\mu}$ as the continuous extension of μ on $(\sigma(S), \bar{u})$. The uniqueness statement is contained in Theorem 3.1. $\square$

Corollary 6.6 generalizes Corollary 4.9. Corollary 6.7 generalizes the extension theorem for measures on orthomodular lattices (Theorem 5.3) as well as for measures on MV-algebras [5, Theorem 3.2.2].

6.2 Pseudo-D-lattices

In this section, let L be a σ -complete σ -continuous pseudo-D-lattice, S a D-sublattice of L and G be complete.

Pseudo-D-lattices (= lattice ordered pseudoeffect algebras) were introduced by Dvurečenskij and Vetterlein [8] as “noncommutative version” of D-lattices. For the definition and basic properties, we refer to [8] and [4]. On L are defined besides the lattice operations $\vee, \wedge$ two partial binary operations $\setminus$ and $/$ which can be extended on $L \times L$ by

$$a \ast \Delta b = (a \vee b) \setminus (a \wedge b) \quad \text{and} \quad a \Delta^* b = (a \wedge b) / (a \vee b).$$

If L is a D-lattice, then $\ast \Delta$ and Δ^* coincide with Δ .

Let us recall the definition of a D-uniformity. This is a lattice uniformity on L making $\setminus$ and $/$ uniformly continuous, or equivalently, a lattice uniformity making $\ast \Delta$ and Δ^* uniformly continuous. Let $D_0(L)$ be the set of all $d \in D(L)$ such that $d(x \ast \Delta z, y \ast \Delta z) \leq 2d(x, y)$ and $d(x \Delta^* z, y \Delta^* z) \leq 2d(x, y)$. By [19, Proposition 1.2], the D-uniformities on L are exactly the uniformities generated by a nonempty subset of $D_0(L)$.

As 6.1 and 6.3 one can verify the following facts using the usual rules of calculation in pseudo-D-lattices, in particular the identity $(b \setminus a) / b = a$ for $a \leq b$ and rules for the two “differences” $\setminus$ and $/$ analogous to 6.2.

Lemma 6.8 (a) $d(a, b) \leq 2d(b \setminus a, 0)$ if $d \in D_0(L)$ and $a \leq b$.

(b) If $S \ni x \leq y \in S_\sigma$, then $y \setminus x \in S_\sigma$.

(c) If M is a sublattice of L , then $(M_\sigma \ast \Delta M_\sigma) \cup (M_\delta \ast \Delta M_\delta) \subseteq (M \ast \Delta M)_{\sigma\delta}$ and $(M_\sigma \Delta^* M_\sigma) \cup (M_\delta \Delta^* M_\delta) \subseteq (M \Delta^* M)_{\sigma\delta}$.

The next theorem, which coincides with [4, Theorem 4.2], can be proved in the same way as Theorem 6.4 using Lemma 6.8 (c) instead of 6.3(b). This leads to a proof simpler than that given in [4].

Theorem 6.9 $\sigma(S)$ is a D-sublattice of L .

It is near at hand how to modify proofs of Sect. 6.1 to obtain the following results.

Theorem 6.10 (a) Any σ -o.c. exhaustive $d \in D_0(S)$ has a unique σ -o.c. extension $\bar{d} \in D_0(\sigma(S))$.

(b) Any σ -o.c. exhaustive D-uniformity u on S has a unique σ -o.c. D-uniformity $\bar{u}$ on $\sigma(S)$ with $\bar{u}|_S = u$.

Corollary 6.11 *Let $\mu : S \rightarrow G$ be a σ -o.c. exhaustive modular measure. Then there exists a unique σ -o.c. modular measure $\bar{\mu} : \sigma(S) \rightarrow G$ extending μ .*

Such an extension theorem was proved in [4, Corollaries 5.5 and 5.6] under the additional assumption that G is a locally convex linear space, or that G is a normed group and μ has finite variation.

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