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# Functional capacities on the Grushin space $\mathbb{G}^n_\alpha$

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**Abstract** Functional capacities on the Grushin space $\mathbb{G}^n_\alpha$ are introduced, developed, and subsequently applied to the theory of Sobolev embeddings.

**Keywords** Isoperimetric inequality · Carnot-Carathéodory space · Grushin spaces · Functional capacity

**Mathematics Subject Classification** 32U20 · 53C17

## Introduction

The isoperimetric inequality on the Euclidean space $\mathbb{R}^n$ has been investigated by many scholars (cf. [16,34–37], etc). This inequality has been appropriately extended to the Carnot-Carathéodory spaces in [3,10,12,21,32,42] without sharp constants and extremal sets in general. However, on the Grushin plane (existing as the simplest example of the Carnot-Carathéodory spaces and appearing in the hypoelliptic operator theory; cf. [18,19,25,26,44]), Monti and Morbidelli found the sharp constants and extremal sets for the isoperimetric inequality in [39]. Recently, Franceschi and Monti [17] studied the isoperimetric problem on the Grushin space $\mathbb{G}^n_\alpha$ (regarded as the high-dimensional case of the Grushin plane)—under

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a symmetry assumption that depends on the dimension, they proved the existence, additional symmetry, and regularity of an isoperimetric set. On the other hand, in [45,46] Xiao split the isoperimetric inequality twice via the directional capacity on the Euclidean space, thereby exploring its applications in handling the sharp Sobolev inequalities via the variational capacities and their affine counterparts; see also [47,48] for more information. This paper, as a continuation of Liu's paper [33] discussing the BV-capacity on the Gushin plane, shows that the isoperimetric inequality over any given Grushin space can be also split twice, thereby discovering several new results through considering the so-called functional capacities on $\mathbb{G}^n_\alpha$ in three sections:

▷ The first section presents several fundamental properties of the functional capacities;
▷ The second section gives some geometric estimates for the functional capacities;
▷ The third section provides certain applications of the functional capacities to the Sobolev-type imbeddings.

*Notation* Throughout this paper, unless otherwise indicated, we use $C$ to denote constants that depend on the homogeneous dimension of a given Grushin space and are not necessarily the same at each occurrence. And, $\mathsf{A} \sim \mathsf{B}$ means that there exist $C > 0$ and $c > 0$ such that $c \le \frac{\mathsf{A}}{\mathsf{B}} \le C$.

## 1 Basics of functional capacities

### 1.1 Grushin spaces and their metrics

From now on, let $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$, where $h, k \ge 1, n = h + k$ are integers. For a given real number $\alpha \ge 0$, define a family of vector fields on $\mathbb{R}^n$ by

$$\begin{cases} X_i = \partial_{x_i} \ \forall\, i \in \{1, 2, \dots, h\}; \\ X_{h+j} = |x|^\alpha \partial_{y_j} \ \forall\, j \in \{1, 2, \dots, k\}; \\ |x| = \left(\sum\limits_{i=1}^{h} x_i^2\right)^{\frac{1}{2}}. \end{cases}$$

It should be noted that the above vector fields satisfy Hörmander's condition when $\alpha = 2m$ are evens (cf. [28]). These vector fields induce the following distance between two points $g, g'$ in $\mathbb{R}^n$:

$$\begin{aligned} d_\alpha(g, g') = \inf \Big\{ & T \mid \exists \text{ Lipschitz continuous curve } \gamma : [0, T] \to \mathbb{R}^n \text{ such that} \\ & \gamma(0) = g, \gamma(T) = g' \text{and} \\ & \gamma'(t) = \sum_{i=1}^{n} a_i(t) X_i(\gamma(t)) \text{ with } \sum_{i=1}^{n} |a_i(t)|^2 \le 1, \text{ for a.e. } t \in [0, T] \Big\}. \end{aligned}$$

As a metric in $\mathbb{R}^n$, $d_\alpha(g, g')$ is well defined and coincides with the Carnot-Carathéodory distance—namely—one has:

$$d_\alpha(g, g') = \inf_{\gamma=(x,y)\in\Gamma_{g,g'}} \int_0^1 \sqrt{\sum_{i=1}^{h} |\dot{x_i}(t)|^2 + |x(t)|^{-2\alpha} \sum_{j=1}^{k} |\dot{y_j}(t)|^2 \mathrm{d}t},$$

where $\Gamma_{g,g'}$ is the set of all Lipschitz continuous curves

$$\gamma : [0,1] \to \mathbb{R}^n \ \text{ with } \ \gamma(0) = g \ \& \ \gamma(1) = g'.$$

The resulting space $\mathbb{G}^n_\alpha = (\mathbb{R}^n, d_\alpha)$ is the completion of the Riemannian metric space $\{(x,y) \in \mathbb{R}^n : x \neq 0\}$ equipped with the Riemannian metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + |x|^{-2\alpha}\mathrm{d}y^2.$$

Moreover, for $n = 2$, we can clearly see the geometric structure of $\mathbb{G}^2_\alpha$—in fact—[2, Lemma 2] yields that the Gaussian curvature and the Riemannian density are

$$\mathscr{K} = (-\alpha^2 - \alpha)|x|^{-2} \ \& \ dA = |x|^{-\alpha}\mathrm{d}x\mathrm{d}y \ \text{ on } \ \mathbb{G}^2_\alpha \setminus \{(x,y) \in \mathbb{R}^2 : x \neq 0\},$$

respectively, and they explode while approaching the $y$-axis.

By [18] or [44], we know that for any $g = (x,y)$, $g' = (x', y') \in \mathbb{G}^n_\alpha$ one has:

$$d_\alpha(g, g') \sim \sum_{i=1}^{h} |x_i - x'_i| + \sum_{j=1}^{k} \min\left\{|y_j - y'_j|^{\frac{1}{1+\alpha}}, |y_j - y'_j||x|^{-\alpha}\right\}. \tag{1}$$

The ball with radius $r$ and center $g$ under the metric $d_\alpha$ is given by

$$B_\alpha(g,r) = \{g' \in \mathbb{G}^n_\alpha : \ d_\alpha(g,g') \ < r\}.$$

According to [18] there are two positive constants $c_1 < c_2$ such that

$$Q_\alpha(g, c_1 r) \subseteq B_\alpha(g,r) \subseteq Q_\alpha(g, c_2 r) \quad \forall \quad g = (x,y) \in \mathbb{G}^n_\alpha, \tag{2}$$

where

$$Q_\alpha(g,r) = \prod_{i=1}^{h}[x_i - r, x_i + r] \times \prod_{j=1}^{k}[y_j - r(|x| + r)^\alpha, y_j + r(|x| + r)^\alpha].$$

Inclusion (2) implies that there exists a positive constant $C$ such that

$$|B_\alpha(g, 2r)| \le C|B_\alpha(g,r)| \quad \forall \quad (g,r) \in \mathbb{G}^n_\alpha \times (0, \infty).$$

The dilation on $\mathbb{G}^n_\alpha$ is given by

$$\delta_\lambda(x,y) = (\lambda x, \lambda^{\alpha+1} y), \ \lambda > 0.$$

The standard measure on $\mathbb{G}^n_\alpha$ is the usual Lebesgue measure $\mathrm{d}g = \mathrm{d}x\mathrm{d}y$, so one has (cf. [39]):

$$|\delta_\lambda(E)| = \lambda^Q |E| \quad \forall \quad \text{measurable set} \quad E \subseteq \mathbb{G}^n_\alpha,$$

where $Q = h + (\alpha+1)k$ is called the homogeneous dimension of $\mathbb{G}^n_\alpha$. Via the anisotropic dilation we introduce the following quasimetric $\rho$:

$$\rho(g,g') = \left(|x - x'|^{2\alpha+2} + (\alpha+1)^2|y - y'|^2\right)^{\frac{1}{2\alpha+2}} \ \forall \, g = (x,y), \ g' = (x', y') \in \mathbb{G}^n_\alpha.$$

The ball with radius $r$ and center $g$ under $\rho$ is given by

$$B(g,r) = \{g' \in \mathbb{G}^n_\alpha : \ \rho(g,g') \ < r\}.$$

According to (1), there are two positive constants $C_1$ and $C_2$ such that

$$\begin{cases} \frac{1}{C_1}d_\alpha(g,g') \le \rho(g,g') \le C_1 d_\alpha(g,g') \quad \text{as} \quad \max\{|x|,|x'|\} \le |y-y'|^{\frac{1}{\alpha+1}}; \\ d_\alpha(g,g') \le C_2\rho(g,g') \quad \text{as} \quad \max\{|x|,|x'|\} \ge |y-y'|^{\frac{1}{\alpha+1}}, \end{cases}$$

holds for any two points $g=(x,y), g'=(x',y') \in \mathbb{G}^n_\alpha$. Moreover, if $E$ is a bounded subset of $\mathbb{G}^n_\alpha$, then there exists a positive constant $C_3$ (depending on the set $E$) such that

$$\rho(g,g') \le C_3\big(d_\alpha(g,g')\big)^{\frac{1}{\alpha+1}} \quad \text{as} \quad \max\{|x|,|x'|\} \ge |y-y'|^{\frac{1}{\alpha+1}}$$

holds for any two points $g=(x,y), g'=(x',y')$ in $E$.

### 1.2 Definitions of BV and functional capacities

The divergence of a vector-valued function $\varphi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ is defined as

$$\mathrm{div}_\alpha\varphi = \sum_{i=1}^{h} \partial_{x_i}\varphi_i + |x|^\alpha \sum_{j=1}^{k} \partial_{y_j}\varphi_{h+j}.$$

The corresponding gradient operator is defined as

$$\nabla_\alpha = (\partial_{x_1},\ldots,\partial_{x_h}, |x|^\alpha\partial_{y_1},\ldots,|x|^\alpha\partial_{y_k}).$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The $X_\alpha$-variation of $f \in L^1(\Omega)$ is determined by

$$\| X_\alpha f \|(\Omega) = \sup_{\varphi\in\mathcal{F}} \int_\Omega f(g)\mathrm{div}_\alpha\varphi(g)\mathrm{d}g,$$

where $\mathcal{F}$ is the class of all functions $\varphi=(\varphi_1,\ldots,\varphi_n) \in C^1_0(\Omega;\mathbb{R}^n)$ such that

$$\| \varphi \|_\infty = \sup_{g\in\mathbb{G}^n_\alpha} (|\varphi_1(g)|^2 + \cdots + |\varphi_n(g)|^2)^{\frac{1}{2}} \le 1.$$

An $L^1$ function $f$ is said to have a bounded $X_\alpha$-variation on $\Omega$ provided $\| X_\alpha f \|(\Omega) < \infty$, and the collection of all such functions is denoted by $\mathcal{BV}(\Omega)$. The space $\mathcal{BV}_{\mathrm{loc}}(\Omega)$ is the set of functions belonging to $\mathcal{BV}(U)$ for each open set $U \subset\subset \Omega$. A measurable set $E \subseteq \mathbb{R}^n$ is of locally finite $\alpha$-perimeter in $\Omega$ (or an $X_\alpha$-Caccioppoli set) if the indicator $1_E$ of $E \subseteq \mathbb{G}^n_\alpha$ belongs to $\mathcal{BV}_{\mathrm{loc}}(\Omega)$—namely—if

$$\| \partial E \|_\alpha (U) := \| X_\alpha 1_E \|(U) < \infty$$

for every open set $U \subset\subset \Omega$. From [40] we see that $\| X_\alpha f \|$ is a Radon measure on $\Omega$ whenever $f \in \mathcal{BV}_{\mathrm{loc}}(\Omega)$.

As in [21] or [17], the $\alpha$-perimeter of a measurable set $E \subseteq \mathbb{G}^n_\alpha$ is given by

$$P_\alpha(E) = \sup\left\{\int_E \mathrm{div}_\alpha\varphi(g)\mathrm{d}g : \varphi\in\mathcal{F}\right\}.$$

Naturally, $P_\alpha(E)$ has the following lower semicontinuity, i.e., if $(E_h)_{h\in\mathbb{N}}$ is a sequence of measurable sets whose characteristic functions converge in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to the indicator $1_E$ of $E$, then

$$P_\alpha(E) \le \liminf_{h\to\infty} P_\alpha(E_h). \tag{3}$$

For a set $E \subseteq \mathbb{G}_\alpha^n$ let $\mathcal{A}(E, \mathcal{BV}(\mathbb{G}_\alpha^n))$ be the class of admissible functions on $\mathbb{G}_\alpha^n$, i.e., all functions $f \in \mathcal{BV}(\mathbb{G}_\alpha^n)$ satisfying $0 \le f \le 1$ and $f = 1$ in a neighborhood of $E$ (an open set containing $E$). Then the BV-capacity of $E$ is determined by

$$\operatorname{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) := \inf\{\| X_\alpha f \| (\mathbb{G}_\alpha^n) : f \in \mathcal{A}(E, \mathcal{BV}(\mathbb{G}_\alpha^n))\}.$$

For $1 \le p < Q$ and $p* = \frac{Qp}{Q-p}$, define

$$V_p = \left\{ f \in L^{p*}(\mathbb{G}_\alpha^n) : \int_{\mathbb{G}_\alpha^n} | \nabla_\alpha f(g) |^p \, \mathrm{d}g < \infty \right\}.$$

For $E \subseteq \mathbb{G}_\alpha^n$, set

$$\mathcal{K}_E^p = \{f \in V_p : f \ge 0, E \subseteq \operatorname{int}\{g : f(g) \ge 1\}\}.$$

Then the $p$-capacity of $E$ is defined as

$$\operatorname{cap}_{\alpha,p}(E) = \inf\{\| \nabla_\alpha f \|_p^p : f \in \mathcal{K}_E^p\}.$$

In particular, if $E \subseteq \mathbb{G}_\alpha^n$ be a compact set, the $p$-capacity of $E$ is given by

$$\operatorname{cap}_{\alpha,p}(E) = \inf\{\| \nabla_\alpha f \|_p^p : f \in \mathfrak{A}(E)\},$$

where

$$\mathfrak{A}(E) = \{f \in C_0^\infty(\mathbb{G}_\alpha^n) : f \ge 1_E\}$$

and $C_0^\infty(\mathbb{G}_\alpha^n)$ is the class of all $C^\infty$ functions with compact support in $\mathbb{G}_\alpha^n$.

### 1.3 Basic facts on functional capacities

Previously, several capacities on the general metric spaces were studied in [7,11,13,14,23,24,29,31]. While the last two references [7,11] tell us that the methods used to develop the theory of $p$-capacities are closely related to those used in certain variational minimization problems on Grushin spaces and Carnot-Carathéodory spaces, there are still some problems left open for the $p$-capacities on Carnot-Carathéodory spaces, even on Grushin spaces. Nevertheless, below is a list of the metric properties on $\operatorname{cap}_{\alpha,p}$.

**Theorem 1** *Let $A, B \subseteq \mathbb{G}_\alpha^n$.*

(i) *If $A \subseteq B$, then* $\operatorname{cap}_{\alpha,p}(A) \le \operatorname{cap}_{\alpha,p}(B)$.
(ii) $\operatorname{cap}_{\alpha,p}(\delta_\lambda A) = \lambda^{Q-p}\operatorname{cap}_{\alpha,p}(A)$.
(iii) $\operatorname{cap}_{\alpha,p}(L_{y'} A) = \operatorname{cap}_{\alpha,p}(A)\ \forall$ *vertical translations $L_{y'}$ with $y' \in \mathbb{R}^k$.*
(iv) $\operatorname{cap}_{\alpha,p}(B((0, y), r)) = r^{Q-p}\operatorname{cap}_{\alpha,p}(B((0, 0), 1))\ \forall\ y \in \mathbb{R}^k$.
(v) *If $A$ and $B$ are compact subsets of $\mathbb{G}_\alpha^n$, then*

$$\operatorname{cap}_{\alpha,p}(A \cup B) + \operatorname{cap}_{\alpha,p}(A \cap B) \le \operatorname{cap}_{\alpha,p}(A) + \operatorname{cap}_{\alpha,p}(B).$$

(vi) *If $\{A_i\}_{i=1}^\infty$ is a sequence of compact subsets of $\mathbb{G}_\alpha^n$ with $A_1 \supseteq A_2 \supseteq \cdots A_k \supseteq \cdots$, then*

$$\lim_{k\to\infty} \operatorname{cap}_{\alpha,p}(A_k) = \operatorname{cap}_{\alpha,p}(\cap_{k=1}^\infty A_k).$$

(vii) *If* $\{A_k\}_{k=1}^{\infty}$ *is a sequence of subsets of* $\mathbb{G}_\alpha^n$, *then*

$$\mathrm{cap}_{\alpha,p}\Big(\overset{\infty}{\underset{k=1}{\cup}} A_k\Big) \le \sum_{k=1}^{\infty} \mathrm{cap}_{\alpha,p}(A_k).$$

(viii) *If* $\{A_k\}_{k=1}^{\infty}$ *is a sequence of subsets of* $\mathbb{G}_\alpha^n$ *with* $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, *then*

$$\lim_{k\to\infty} \mathrm{cap}_{\alpha,p}(A_k) = \mathrm{cap}_{\alpha,p}\Big(\overset{\infty}{\underset{k=1}{\cup}} A_k\Big).$$

*Proof* (i) This is an obvious consequence of the definition of $p$-capacity.

(ii) For any $\varepsilon > 0$, there exists a function $f \in \mathcal{K}_A^p$, such that

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f|^p \mathrm{d}g < \mathrm{cap}_{\alpha,p}(A) + \varepsilon.$$

Let $\phi(g) = f(\delta_{\lambda^{-1}} g)$. Then $\phi \in \mathcal{K}^p_{\delta_\lambda A}$. Since

$$\begin{aligned}\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \phi|^p \mathrm{d}g &= \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f\left(\frac{x}{\lambda}, \frac{y}{\lambda^{\alpha+1}}\right)|^p \mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{G}_\alpha^n} \left(\sum_{i=1}^{h}\left(\partial_{x_i} f\left(\frac{x}{\lambda}, \frac{y}{\lambda^{\alpha+1}}\right)\right)^2 + |x|^{2\alpha}\sum_{j=1}^{k}\left(\partial_{y_j} f\left(\frac{x}{\lambda}, \frac{y}{\lambda^{\alpha+1}}\right)\right)^2\right)^{\frac{p}{2}} \mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{G}_\alpha^n} \left(\frac{1}{\lambda^2}\sum_{i=1}^{h}\left(\partial_{\xi_i} f(\xi,\eta)\right)^2 + \frac{1}{\lambda^2}|\xi|^{2\alpha}\sum_{j=1}^{k}\left(\partial_{\eta_j} f(\xi,\eta)\right)^2\right)^{\frac{p}{2}} \lambda^Q \mathrm{d}\xi\mathrm{d}\eta\\ &= \lambda^{Q-p}\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f|^p \mathrm{d}g,\end{aligned}$$

one has

$$\mathrm{cap}_{\alpha,p}(\delta_\lambda A) \le \lambda^{Q-p}(\mathrm{cap}_{\alpha,p}(A) + \varepsilon),$$

whence

$$\mathrm{cap}_{\alpha,p}(\delta_\lambda A) \le \lambda^{Q-p}\mathrm{cap}_{\alpha,p}(A).$$

The converse inequality of this last inequality can be similarly proved. So the desired equality is verified.

(iii) This follows from the integral formula

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f(x, y+y')|^p \mathrm{d}x\mathrm{d}y = \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f(x,y)|^p \mathrm{d}x\mathrm{d}y.$$

(iv) This follows from (ii) and (iii).

(v) For any $\varepsilon > 0$, there are two functions $\phi \in \mathcal{K}_A^p$, $\psi \in \mathcal{K}_B^p$ such that

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \phi| \mathrm{d}g < \mathrm{cap}_{\alpha,p}(A) + \frac{\varepsilon}{2} \ \ \& \ \ \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \psi| \mathrm{d}g < \mathrm{cap}_{\alpha,p}(B) + \frac{\varepsilon}{2}.$$

Let

$$\varphi_1 = \max\{\phi, \psi\} \ \ \& \ \ \varphi_2 = \min\{\phi, \psi\}.$$

Then

$$\varphi_1 \in \mathcal{K}^p_{A\cup B} \ \ \& \ \ \varphi_2 \in \mathcal{K}^p_{A\cap B}.$$

Using the proof of Lemma 2.4 in [30], we can obtain

$$|\nabla_\alpha \varphi_1|^p + |\nabla_\alpha \varphi_2|^p = |\nabla_\alpha \phi|^p + |\nabla_\alpha \psi|^p \text{ a.e } g \in \mathbb{G}_\alpha^n,$$

thereby getting

$$\begin{aligned}
&\mathrm{cap}_{\alpha,p}(A \cup B) + \mathrm{cap}_{\alpha,p}(A \cap B)\\
&\le \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \varphi_1|^p \mathrm{d}g + \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \varphi_2|^p \mathrm{d}g\\
&= \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \phi|^p \mathrm{d}g + \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \psi|^p \mathrm{d}g\\
&\le \mathrm{cap}_{\alpha,p}(A) + \mathrm{cap}_{\alpha,p}(B) + \varepsilon,
\end{aligned}$$

as desired.

(vi)–(vii)–viii) These follow readily from validating their counterparts for the Sobolev $p$-capacities on metric spaces presented in [29] (cf. [13]). □

### 1.4 Direct formulas for functional capacities

The next result indicates that $\mathrm{cap}_{\alpha,p}$ can be evaluated among some different function spaces.

**Theorem 2** *Let $K$ be a compact subset of $\mathbb{G}_\alpha^n$.*

(i)
$$\mathrm{cap}_{\alpha,p}(K) = \inf\{\| \nabla_\alpha f \|_p^p \colon f \in \mathfrak{B}(K)\},$$

*where $\mathfrak{B}(K)$ is the class of all functions $f \in C_0^\infty(\mathbb{G}_\alpha^n)$ with $f = 1$ in a neighborhood of $K$ and $0 \le f \le 1$ on $\mathbb{G}_\alpha^n$.*

(ii)
$$\mathrm{cap}_{\alpha,p}(K) = \inf\{\| \nabla_\alpha f \|_p^p \colon f \in \Lambda(K)\},$$

*where $\Lambda(K)$ is the class of all functions $f \in C_0^1(\mathbb{G}_\alpha^n)$ with $f = 1$ in a neighborhood of $K$ and $0 \le f \le 1$ on $\mathbb{G}_\alpha^n$.*

(iii)
$$\mathrm{cap}_{\alpha,p}(K) = \inf_{f \in \mathfrak{A}(K)} \left\{ \left( \int_0^1 \frac{\mathrm{d}\tau}{\left(\int_{\mathcal{E}_\tau} |\nabla_\alpha f|^{p-1} \mathrm{d}\mu_\tau\right)^{\frac{1}{p-1}}} \right)^{1-p} \right\},$$

*where $\mathcal{E}_\tau = \{g \in \mathbb{G}_\alpha^n : |f(g)| = \tau\}$ and $\mathrm{d}\mu_\tau = \| \partial \mathcal{E}_\tau \|_\alpha$ is the $\alpha$-perimeter measure of $\mathcal{E}_\tau$.*

(iv)
$$\mathrm{cap}_{\alpha,p}(K) = \inf_{f \in \mathfrak{B}(K)} \left\{ \left( \int_0^1 \frac{\mathrm{d}\tau}{\left(\int_{\mathcal{E}_\tau} |\nabla_\alpha f|^{p-1} \mathrm{d}\mu_\tau\right)^{\frac{1}{p-1}}} \right)^{1-p} \right\}.$$

(v)
$$\mathrm{cap}_{\alpha,p}(K) = \inf_{f \in \Lambda(K)} \left\{ \int_0^1 \left( \int_{\mathcal{E}_\tau} |\nabla_\alpha f|^{p-1} \mathrm{d}\mu_\tau \right)^{-\frac{Q}{p}} \mathrm{d}\tau \right\}^{-\frac{p}{Q}} \quad \textit{for } \frac{Q}{Q-1} \le p < Q.$$

*Proof* The proofs of (i)–(ii)–(iii)–(iv) are standard (cf. Sections 2.2.1& 2.2.2 in [35]), so they are omitted.

(v) As showed in [48, Theorem 1], for $f \in \Lambda(K)$, applying the co-area formula on $\mathbb{G}^n_\alpha$ (cf. [40, Theorem 4.2]) and the Hölder inequality, we have

$$\| \nabla_\alpha f \|_p^p = \int_0^1 \Big( \int_{\mathcal{E}_t} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_t \Big) \mathrm{d}t \geq \Big( \int_0^1 \big( \int_{\mathcal{E}_t} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_t \big)^{-\frac{Q}{p}} \mathrm{d}t \Big)^{-\frac{p}{Q}},$$

thereby getting

$$\mathrm{cap}_{\alpha,p}(K) \geq \inf_{f \in \Lambda(K)} \left( \int_0^1 \left( \int_{\mathcal{E}_t} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_t \right)^{-\frac{Q}{p}} \mathrm{d}t \right)^{-\frac{p}{Q}}.$$

In what follows we consider the reverse form of the above inequality. For any $f \in \Lambda(K)$ and $s \in (0, 1]$, let us choose

$$f_s(g) = \begin{cases} 1 - \gamma_s \int_0^{f(g)} \big( \int_{\mathcal{E}_r} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_r \big)^{-\frac{1+\frac{Q}{p}}{p}} \mathrm{d}r & \text{as} \quad f(g) \leq s; \\ 0 \quad \text{as} \quad f(g) \geq s, \end{cases}$$

where

$$\gamma_s = \left( \int_0^s \Big( \int_{\mathcal{E}_r} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_r \Big)^{-\frac{Q}{p}} \mathrm{d}r \right)^{-\frac{1+\frac{Q}{p}}{p}}.$$

By the co-area formula on $\mathbb{G}^n_\alpha$ again we have

$$\mathrm{cap}_{\alpha,p}(K) \leq \| \nabla_\alpha f \|_p^p \leq \left( \int_0^1 \Big( \int_{\mathcal{E}_t} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_t \Big)^{-\frac{Q}{p}} \mathrm{d}t \right)^{-\frac{p}{Q}}.$$

However, we are required to verify $f_s \in \Lambda(K)$. Of course, it is enough to check $0 \leq f_s \leq 1$—in fact—this follows from the Hölder inequality-implied estimate:

$$\int_0^s \Big( \int_{\mathcal{E}_r} | \nabla_\alpha f|^{p-1} \mathrm{d}\mu_r \Big)^{-\frac{1+\frac{Q}{p}}{p}} \mathrm{d}r \leq \gamma_s^{-1} \quad \text{under} \quad \frac{1}{Q} + \frac{1}{p} \leq 1.$$

□

**Corollary 3** *Let* $1 < p < Q$.

(i) *For almost all* $t \geq 0$ *and any* $f \in C_0^\infty(\mathbb{G}^n_\alpha)$ *with its level set* $\mathcal{L}_t = \{g \in \mathbb{G}^n_\alpha : |f(g)| > t\}$ *one has:*

$$\big(P_\alpha(\mathcal{L}_t)\big)^{\frac{p}{p-1}} \leq \left[ -\frac{\mathrm{d}}{\mathrm{d}t} |\mathcal{L}_t| \right] \left( \int_{\mathcal{E}_t} |\nabla_\alpha f|^{p-1} \mathrm{d}\mu_\tau \right)^{\frac{1}{p-1}}. \tag{4}$$

(ii) *The inequality*

$$\mathrm{cap}_{\alpha,p}(K) \geq \inf_{f \in \mathfrak{B}(K)} \left( - \int_0^1 \Big( \frac{\mathrm{d}|\mathcal{L}_\tau|}{\mathrm{d}\tau} \Big) \frac{\mathrm{d}\tau}{\big(P_\alpha(\mathcal{L}_\tau)\big)^{\frac{p}{p-1}}} \right)^{1-p}$$

*is valid for any compact subset* $K$ *of* $\mathbb{G}^n_\alpha$.

(iii) *Let $\mathscr{C}(r)$ denote the infimum $P_\alpha(E)$ for all bounded open sets $E$ in $\mathbb{G}_\alpha^n$ with $C^\infty$ boundary such that $|E| \geq r$. Then the inequality*

$$\mathrm{cap}_{\alpha,p}(K) \geq \left(\int_{|K|}^{\infty} \frac{\mathrm{d}r}{(\mathscr{C}(r))^{\frac{p}{p-1}}}\right)^{1-p}$$

*is valid for any compact set $K$ of $\mathbb{G}_\alpha^n$.*

*Proof* (i) It is enough to check (4). In fact, a combination of (4) and (iv) of Theorem 2 derives that (ii) is valid and (iii) can be deduced from (ii).

By Hölder's inequality, for almost all $t$ and $T$ with $t < T$,

$$\left(\int_{\mathcal{L}_t\setminus\mathcal{L}_T} |f|^{p-1}|\nabla_\alpha f|\mathrm{d}g\right)^{\frac{p}{p-1}} \leq \left(\int_{\mathcal{L}_t\setminus\mathcal{L}_T} |f|^p\mathrm{d}g\right)\left(\int_{\mathcal{L}_t\setminus\mathcal{L}_T} |\nabla_\alpha f|^p\mathrm{d}g\right)^{\frac{1}{p-1}}.$$

Using the co-area formula (cf. Theorem 5.2 in [21]) on the left side and another co-area formula (cf. Theorem 4.2 in [40]) on the right side, we have

$$\left(\int_t^T \tau^{p-1}P_\alpha(\mathcal{L}_\tau)\mathrm{d}\tau\right)^{\frac{p}{p-1}} \leq \left(\int_{\mathcal{L}_t\setminus\mathcal{L}_T} |f|^p\mathrm{d}g\right)\left(\int_t^T \mathrm{d}\tau\int_{\mathcal{E}_\tau} |\nabla_\alpha f|^{p-1}\mathrm{d}\mu_\tau\right)^{\frac{1}{p-1}}.$$

We divide both sides of the above inequality by $(T-t)^{\frac{p}{p-1}}$ and estimate the first factor on the right-hand side to obtain

$$\left(\int_t^T \tau^{p-1}P_\alpha(\mathcal{L}_\tau)\frac{\mathrm{d}\tau}{T-t}\right)^{\frac{p}{p-1}} \leq T^p\frac{|\mathcal{L}_t\setminus\mathcal{L}_T|}{T-t}\left(\frac{1}{T-t}\int_t^T \mathrm{d}\tau\int_{\mathcal{E}_\tau} |\nabla_\alpha f|^{p-1}\mathrm{d}\mu_\tau\right).$$

Passing to the lower limit as $T \to t$ and via lower semicontinuity property (3), we conclude that the left side of (4) is valid for almost $t > 0$. Using Theorem 5.2 in [21] again, we know that $\mathcal{E}_t$ is an $X_\alpha$-Caccioppoli set for a.e. $t \in \mathbb{R}$. The right side of (4) is the analogue of the Lebesgue density theorem and the weak convergence for Radon measures. □

### 1.5 Equilibrium potentials for functional capacities

A function $f$ on $\mathbb{G}_\alpha^n$ is called $p$-quasicontinuous provided that for each $\varepsilon > 0$ there exists an open set $U$ such that $f\mid_{\mathbb{G}_\alpha^n\setminus U}$ is continuous and $\mathrm{cap}_{\alpha,p}(U) < \varepsilon$. The following proposition reveals the continuous property of any $V_p$-function; see [24,30] for the cases of metric spaces.

**Proposition 4** *For any $f \in V_p$ there exists a function $h \in V_p$ such that $f(g) = h(g)$ for almost every $g \in \mathbb{G}_\alpha^n$ and $h$ is $p$-quasicontinuous. Denote by $f^*$ the representative of $f$, which is defined by*

$$f^*(g) = \begin{cases} \lim_{r\to 0}\frac{1}{|B(g,r)|}\int_{B(g,r)} f(g')\mathrm{d}g' & \text{if the limit exists;} \\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

*Then $f^*$ is also $p$-quasicontinuous if $f \in V_p$, and the limit in (5) exists* $\mathrm{cap}_{\alpha,p}$ *a.e. on $\mathbb{G}_\alpha^n$.*

**Lemma 5** *Assume that the sequence $\{f_k\}_{k=1}^\infty$ is precompact on $V_p$ and every function in the sequence is $p$-quasicontinuous. Then there exists a subsequence $\{f_{k_i}\}_{i=1}^\infty$ and a $p$-quasicontinuous function $f \in V_p$ such that for each $\delta > 0$ there exists an open set $U$ with the properties $f_{k_i} \to f$ uniformly on $G_\alpha^n\setminus U$ and* $\mathrm{cap}_{\alpha,p}(U) < \delta$.

By the above lemma we can easily extract a subsequence such that $f_{k_i} \to f$ $(i \to \infty)$ for $\mathrm{cap}_{\alpha,p}$ a.e. $g \in \mathbb{G}_\alpha^n$. We omit the proof of Lemma 5, while [8,24,29] and [43] have investigated the quasicontinuity on metric spaces.

**Lemma 6** (cf. [41]) *Let $\xi, \eta$ be any two vectors in $\mathbb{R}^n$. Then*

(i) *For $p \geq 2$,*

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq 2^{1-p}|\xi - \eta|^p.$$

(ii) *For $1 < p \leq 2$,*

$$(|\xi| + |\eta|)^{2-p}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq (p-1)|\xi - \eta|^2.$$

Denote by $V_p'$ the dual space of $V_p$. Then we define an operator $A$ from $V_p$ into $V_p'$ by

$$(A(u), v) = \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^{p-2}\nabla_\alpha u \cdot \nabla_\alpha v \mathrm{d}g \quad \forall \quad u, v \in V_p.$$

In fact, the operator $A$ is the $p$-Laplacian-type operator on the Grushin space which is investigated in [6,7]. Moreover, if

$$D_E = \{f \in V_p : f^*(g) \geq 1 \ for \ \mathrm{cap}_{\alpha,p} \text{ a.e. } g \in \mathbb{G}_\alpha^n\},$$

then we are led to find $u \in D_E$ such that

$$(A(u), v - u) = \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^{p-2}\nabla_\alpha u \cdot \nabla_\alpha (v - u)\mathrm{d}g \geq 0 \quad \forall \quad v \in D_E. \tag{6}$$

Equation (6) is closely related to the $p$-capacity, so it is a question which deserves a serious consideration on the Grushin space. Moreover, the coming-up-next lemma can be verified by Lemmas 5–6; see [49, Lemma 3.4] for the Euclidean case.

**Lemma 7** *Assume that $p > 1$. Then there exists a unique solution to* (6).

**Lemma 8** *Let $p > 1$. Then $u \in D_E$ is a solution of* (6) *if and only if*

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g = \inf\left\{\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha v|^p \mathrm{d}g : v \in D_E\right\}. \tag{7}$$

*Proof* If $u \in D_E$ is a solution of (6), then we have, for any $v \in D_E$,

$$\begin{aligned}\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \,\mathrm{d}g &\leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^{p-2}\nabla_\alpha u \cdot \nabla_\alpha v \,\mathrm{d}g \\ &\leq \frac{1}{2}\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g + \frac{1}{2}\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^{p-2}|\nabla_\alpha v|^2 \mathrm{d}g.\end{aligned}$$

Furthermore, by the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g &\leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^{p-2}|\nabla_\alpha v|^2 \mathrm{d}g \\ &\leq \left(\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g\right)^{1-\frac{2}{p}} \left(\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha v|^p \mathrm{d}g\right)^{\frac{2}{p}},\end{aligned}$$

whence (7) is valid.

Conversely, assume that (7) holds. Fix $v \in D_E$ and set $\phi = v - u$. Let $0 < \varepsilon \leq 1$. Then it is easy to see that $u + \varepsilon\phi \in D_E$. Therefore,

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g \leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha (u + \varepsilon\phi)|^p \mathrm{d}g$$

and moreover,

$$\int_{\mathbb{G}_\alpha^n} \frac{|\nabla_\alpha (u + \varepsilon\phi)|^p - |\nabla_\alpha u|^p}{\varepsilon} \mathrm{d}g \geq 0.$$

Since

$$\lim_{\varepsilon \to 0} \frac{|\nabla_\alpha (u + \varepsilon\phi)|^p - |\nabla_\alpha u|^p}{\varepsilon} = p|\nabla_\alpha u|^{p-2} \nabla_\alpha u \cdot \nabla_\alpha \phi$$

holds for a.e. $g \in \mathbb{G}_\alpha^n$, the Lebesgue dominated convergence theorem is used to derive that $u \in D_E$ is a solution of (6). In fact, if we choose the desired majorant

$$\varphi = |\nabla_\alpha \phi||\nabla_\alpha u|^{p-1} + |\nabla_\alpha \phi|^p,$$

then we can utilize $u, \phi \in V_p$ to check

$$\frac{|\nabla_\alpha (u + \varepsilon\phi)|^p - |\nabla_\alpha u|^p}{\varepsilon} \leq C\varphi \ \ \& \ \ \varphi \in L^1(\mathbb{G}_\alpha^n).$$

□

**Theorem 9** *Let $p > 1$. If $u$ is a solution of* (6)*, then*

$$\mathrm{cap}_{\alpha,p}(E) = \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g.$$

*Such $u$ is called an equilibrium potential for* $\mathrm{cap}_{\alpha,p}(E)$.

*Proof* If $f \in \mathcal{K}_E^p$, then it follows from (5) that $f^* \geq 1_E$ $\mathrm{cap}_{\alpha,p}$-almost everywhere on $G_\alpha^n$. Thus, $\mathcal{K}_E^p \subseteq D_E$. By Lemma 8, we have

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g \leq \mathrm{cap}_{\alpha,p}(E).$$

We next consider the reverse inequality. Assume that $u$ is a solution of (6). Define a function $\theta(t)$ as follows:

$$\theta(t) = \begin{cases} 2 & \text{if } t \geq 1; \\ 1 + t & \text{if } 0 < t < 1; \\ 1 & \text{if } t \leq 0. \end{cases}$$

Clearly,

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha \theta(u)|^p \mathrm{d}g \leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g. \tag{8}$$

It is easy to check that $(\theta(u))^* \geq 1$ $\mathrm{cap}_{\alpha,p}$-almost everywhere on $G_\alpha^n$. Moreover,

$$\theta(u) \in D_E \cap \{u : u \geq 0\}.$$

If we can show that $\mathcal{K}_E^p$ is dense in $D_E \cap \{u : u \geq 0\}$, then it follows from (8) that

$$\mathrm{cap}_{\alpha,p}(E) \leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u|^p \mathrm{d}g.$$

Therefore, fix $u \in D_E$ with $u \geq 0$. By Proposition 4, $u^*$ is $p$-quasicontinuous. So for each $\varepsilon > 0$ there is a open set $U_\varepsilon$ such that

$$\mathrm{cap}_{\alpha,p}(U_\varepsilon) < \varepsilon$$

and $u^* \mid_{\mathbb{G}_\alpha^n \setminus U_\varepsilon}$ is continuous. Denote by

$$B_\varepsilon = \{g \in \mathbb{G}_\alpha^n \setminus U_\varepsilon : u^* > 1 - \varepsilon\}.$$

By the above facts, $B_\varepsilon$ is a relatively open subset of $\mathbb{G}_\alpha^n \setminus U_\varepsilon$; thus, there exists an open set $M_\varepsilon$ such that

$$B_\varepsilon = M_\varepsilon \cap \{\mathbb{G}_\alpha^n \setminus U_\varepsilon\}.$$

Without losing generality, we may assume that $u^*(g) \geq 1_E$ for every $g \in \mathbb{G}_\alpha^n$. The definition of $p$-capacity implies that there exists a function $v_\varepsilon \in V_p$ such that

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha v_\varepsilon|^p \, \mathrm{d}g < \varepsilon \ \text{ with } \ v_\varepsilon \geq 0 \ \& \ U_\varepsilon \subseteq Int\{g : v_\varepsilon(g) \geq 1\}.$$

If

$$u_\varepsilon = (1 - \varepsilon)^{-1} u^* + v_\varepsilon,$$

then it is the function which we are looking for. It is easy to check that $u_\varepsilon \in \mathcal{K}_E^p$ and $u_\varepsilon \to u$ in $V_p$ as $\varepsilon \to 0$. This completes the proof. □

## 2 Geometric estimates for functional capacities

### 2.1 Isoperimetric and isocapacitary inequalities

These inequalities are presented in the following assertion.

**Theorem 10** *Let*

$$\begin{cases} \alpha > 0 \ \& \ Q = h + (\alpha+1)k; \\ c(\alpha) = \begin{cases} \frac{\alpha+1}{\alpha+2}\left(2\int_0^\pi \sin^\alpha \theta \, \mathrm{d}\theta\right)^{-\frac{1}{\alpha+1}} & as\ h = 1 = k \\ \frac{|E_\alpha|}{(P_\alpha(E_\alpha))^{\frac{Q}{Q-1}}} = \frac{c_{hk}\int_0^1 f(r)^k \mathrm{d}r}{k\left(c_{hk}\int_0^1 \sqrt{f'(r)^2 + r^{2\alpha}}\, f(r)^{k-1}\mathrm{d}r\right)^{\frac{Q}{Q-1}}} & as\ h = 1 < k; \end{cases} \\ c_{hk} = \frac{hk\pi^{\frac{h}{2}}\pi^{\frac{k}{2}}}{\Gamma(1+\frac{h}{2})\Gamma(1+\frac{k}{2})} \ \& \ f(r) = \int_{\arcsin r}^{\pi/2} \sin^{\alpha+1}(t)\mathrm{d}t. \end{cases}$$

(i) *There exists a constant $c(\alpha) > 0$ such that for any measurable set $E \subseteq \mathbb{G}_\alpha^n$ with finite measure*

$$| E | \leq c(\alpha)(P_\alpha(E))^{\frac{Q}{Q-1}}. \tag{9}$$

*When $h = 1$, the equality holds in (9) for the isoperimetric set*

$$E_\alpha = \left\{(x, y) \in \mathbb{G}_\alpha^n : |y| < \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\mathrm{d}t, |x| < 1\right\}.$$

*Moreover, the isoperimetric sets are unique up to dilations $\delta_\lambda$ and vertical translations $L_{y'}(x, y) = (x, y + y') \ \ \forall \ \ y' \in \mathbb{R}^k$.*

(ii) *If* $1 \le p < q < Q$, *then*

$$\big(\mathrm{cap}_{\alpha,p}(\cdot)\big)^{\frac{1}{Q-p}} \le c(p,q,\alpha)\big(\mathrm{cap}_{\alpha,q}(\cdot)\big)^{\frac{1}{Q-q}},$$

*where*

$$c(p,q,\alpha) = \left(\frac{q-p}{Q-q}\cdot\frac{Q}{p}+1\right)^{\frac{p}{Q-p}}\left(\big(c(\alpha)\big)^{\frac{Q-1}{Q}}\frac{q(Q-1)}{Q-q}\right)^{\frac{Q(q-p)}{(Q-q)(Q-p)}}.$$

(iii) *For any compact set* $K \subseteq \mathbb{G}^n_\alpha$,

$$|K|^{\frac{Q-p}{Q}} \le \big(c(\alpha)\big)^{\frac{(Q-1)p}{Q}}\left(\frac{Q-p}{Q(p-1)}\right)^{-(p-1)}\mathrm{cap}_{\alpha,p}(K)\ \ \forall\ \ p\in(1,Q).$$

(iv) *For any compact set* $K \subseteq \mathbb{G}^n_\alpha$,

$$\mathrm{cap}_{\alpha,1}(K) = \inf_{\mathfrak{g}\supseteq K} P_\alpha(\mathfrak{g}),$$

*where the infimum is taken over all bounded open sets* $\mathfrak{g}$ *with* $C^\infty$ *boundary in* $\mathbb{G}^n_\alpha$ *containing* $K$.

*Proof* (i) This is exactly the isoperimetric inequality on $\mathbb{G}^n_\alpha$ presented in [17,39].

(ii) For any compact set $K \subseteq \mathbb{G}^n_\alpha$, let $g = f^\delta$ with the positive constant $\delta$ to be fixed later, where

$$f \in \mathfrak{A}(K) = \{f \in C^\infty_0(\mathbb{G}^n_\alpha) : f \ge 1_K\}.$$

Then $g \in \mathfrak{A}(K)$. Moreover, an application of $\nabla_\alpha g = \delta f^{\delta-1}\nabla_\alpha f$ deduces

$$\begin{aligned}\int_{\mathbb{G}^n_\alpha} |\nabla_\alpha g|^p\, \mathrm{d}g &= \delta^p\int_{\mathbb{G}^n_\alpha} |f|^{(\delta-1)p}|\nabla_\alpha f|^p\, \mathrm{d}g\\ &\le \delta^p\Big(\int_{\mathbb{G}^n_\alpha}|\nabla_\alpha f|^q\,\mathrm{d}g\Big)^{\frac{p}{q}}\Big(\int_{\mathbb{G}^n_\alpha}|f|^{p(\delta-1)(\frac{q}{p})'}\,\mathrm{d}g\Big)^{1-\frac{p}{q}}.\end{aligned}$$

Note that

$$p(\delta-1)\left(\frac{q}{p}\right)' = p(\delta-1)\frac{\frac{q}{p}}{\frac{q}{p}-1} = \frac{qQ}{Q-q}.$$

At this time,

$$\delta = \frac{q-p}{Q-q}\cdot\frac{Q}{p}+1.$$

Therefore, by (15),

$$\begin{aligned}\Big(\int_{\mathbb{G}^n_\alpha}|f|^{p(\delta-1)(\frac{q}{p})'}\,\mathrm{d}g\Big)^{1-\frac{p}{q}} &= \left[\Big(\int_{\mathbb{G}^n_\alpha}|f|^{\frac{Qq}{Q-q}}\,\mathrm{d}g\Big)^{\frac{Q-q}{Qq}}\right]^{\frac{Q(q-p)}{Q-q}}\\ &\le \left(\big(c(\alpha)\big)^{\frac{Q-1}{Q}}\frac{q(Q-1)}{Q-q}\,\|\nabla_\alpha f\|_q\right)^{\frac{Q(q-p)}{Q-q}}.\end{aligned}$$

The above inequality implies

$$\int_{\mathbb{G}_\alpha^n} \mid \nabla_\alpha g \mid^p \mathrm{d}g$$
$$\leq \left(\frac{q-p}{Q-q}\cdot\frac{Q}{p}+1\right)^p \left(\left(c(\alpha)\right)^{\frac{Q-1}{Q}} \frac{q(Q-1)}{Q-q}\right)^{\frac{Q(q-p)}{Q-q}} \parallel \nabla_\alpha f \parallel_q^p \left(\parallel \nabla_\alpha f \parallel_q\right)^{\frac{Q(q-p)}{Q-q}}$$
$$= \left(\frac{q-p}{Q-q}\cdot\frac{Q}{p}+1\right)^p \left(\left(c(\alpha)\right)^{\frac{Q-1}{Q}} \frac{q(Q-1)}{Q-q}\right)^{\frac{Q(q-p)}{Q-q}} \parallel \nabla_\alpha f \parallel_q^{\frac{q(Q-p)}{Q-q}}.$$

Hence

$$\mathrm{cap}_{\alpha,p}(K) \leq \left(\frac{q-p}{Q-q}\cdot\frac{Q}{p}+1\right)^p \left(\left(c(\alpha)\right)^{\frac{Q-1}{Q}} \frac{q(Q-1)}{Q-q}\right)^{\frac{Q(q-p)}{Q-q}} \left(\mathrm{cap}_{\alpha,q}(K)\right)^{\frac{Q-p}{Q-q}}.$$

This implies the desired inequality

$$\left(\mathrm{cap}_{\alpha,p}(K)\right)^{\frac{1}{Q-p}} \leq \frac{\left(\frac{q-p}{Q-q}\cdot\frac{Q}{p}+1\right)^{\frac{p}{Q-p}}}{\left(\left(c(\alpha)\right)^{\frac{Q-1}{Q}} \frac{q(Q-1)}{Q-q}\right)^{\frac{Q(p-q)}{(Q-q)(Q-p)}}} \left(\mathrm{cap}_{\alpha,q}(K)\right)^{\frac{1}{Q-q}}.$$

(iii) Via isoperimetric inequality (9), we have

$$P_\alpha(K) \geq \left(c(\alpha)\right)^{-\frac{Q-1}{Q}} |K|^{\frac{Q-1}{Q}}.$$

Using (iii) of Corollary 3 and noticing

$$\mathscr{C}(r) = \left(c(\alpha)\right)^{-\frac{Q-1}{Q}} r^{\frac{Q-1}{Q}},$$

we obtain

$$\mathrm{cap}_{\alpha,p}(K) \geq \left(c(\alpha)\right)^{-\frac{(Q-1)p}{Q}} \left(\frac{Q-p}{Q(p-1)}\right)^{p-1} |K|^{\frac{Q-p}{Q}}.$$

(iv) Let $f \in \mathfrak{B}(K)$. Using the coarea formula in Theorem 5.2 of [21], we have

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f| \mathrm{d}g = \int_0^1 P_\alpha\{g \in \mathbb{G}_\alpha^n : |f(g)| \geq t\}\mathrm{d}t \geq \inf_{\mathfrak{g}\supseteq K} P_\alpha(\mathfrak{g}).$$

Conversely, suppose that $\mathfrak{g}$ is a bounded open set with $C^\infty$ boundary and containing $K$. Let

$$d(g) = \mathrm{dist}_{\mathbb{R}^n}(g, \mathbb{G}_\alpha^n \backslash \mathfrak{g}) \ \ \& \ \ \mathfrak{g}_t = \{g \in \mathbb{G}_\alpha^n : d(g) > t\}.$$

Let $\varphi$ denote a nondecreasing function, infinitely differentiable on $[0, \infty)$, equal to unity for $d \geq 2\varepsilon$ and equal to zero for $d \leq \varepsilon$, where $\varepsilon$ is a sufficiently small positive number. Denote by $u_\varepsilon(g) = \varphi(d(g))$. Since $u_\varepsilon \in \mathfrak{B}(K)$, we can use the coarea formula in the Euclidian context to obtain

$$\mathrm{cap}_{\alpha,1}(K) \leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u_\varepsilon(g)| \,\mathrm{d}g = \int_0^{2\varepsilon} \varphi'(t) \int_{\partial\mathfrak{g}_t} \frac{|\nabla_\alpha d(g)|}{|\nabla_{\mathbb{R}^n} d(g)|} \,\mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t.$$

Upon letting $\varepsilon \to 0$, we conclude that the right side of the above inequality tends to $P_\alpha(\mathfrak{g})$ by Proposition 2.1 in [17] and the proof of Theorem 2.3 in [21], and so $\mathrm{cap}_{\alpha,1}(K) \leq P_\alpha(\mathfrak{g})$.

□

## 2.2 Functional capacity of a ball

This is helpful and useful for a better understanding of the geometry of $\mathrm{cap}_{\alpha,p}(\cdot)$.

**Theorem 11** *Let $\alpha > 0$ and $1 \le p < \infty$. Suppose $g_0 = (0, y_0) \in \mathbb{G}_\alpha^n$.*

(i) $\mathrm{cap}_{\alpha,p}(B(g_0,r)) = \begin{cases} Q\left(\frac{Q-p}{p-1}\right)^{p-1} \sigma_p r^{Q-p} & as\ 1 < p < Q; \\ 0 & as\ p \ge Q, \end{cases}$

*where* $\sigma_p = \int_{B(g_0,1)} |\nabla_\alpha \rho(g_0, g)|^p \,\mathrm{d}g$.

(ii) $P_\alpha(B(g_0,r)) = \frac{c_{hk}}{2(1+\alpha)^k} \frac{\Gamma(\frac{\alpha+h}{2\alpha+2})\Gamma(\frac{k}{2})}{\Gamma(\frac{Q+\alpha}{2\alpha+2})} r^{Q-1}$, *where $c_{hk}$ is the constant appearing in Theorem 10.*

(iii) *If $h = 1$, then* $\mathrm{cap}_{\alpha,1}(\bar{B}(g_0,r)) = \mathrm{cap}_{\alpha,1}(B(g_0,r)) = \frac{c_{1k}}{2(1+\alpha)^k} \frac{\sqrt{\pi}\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})} r^{Q-1}$.

*Proof* (i) This follows from [7].

(ii) Firstly, we show that

$$P_\alpha(B(o,r)) = \frac{2\pi}{1+\alpha} r^{Q-1} \ \&\ P_\alpha(\bar{B}(o,r)) = \frac{2\pi}{1+\alpha} r^{Q-1}$$

hold, where $o = (0,0) \in \mathbb{G}_\alpha^n$.
Note that

$$\begin{cases} B(o,r) = \{(x,y) \in \mathbb{G}_\alpha^n : |y| < \phi(|x|)\}; \\ \phi(|x|) = (1+\alpha)^{-1}\sqrt{r^{2\alpha+2} - |x|^{2\alpha+2}}. \end{cases}$$

So, via [17] we have

$$\begin{aligned} P_\alpha(B(o,r)) &= c_{hk} \int_0^r (\phi'(s)^2 + s^{2\alpha})^{\frac{1}{2}} s^{h-1} \phi(s)^{k-1} \mathrm{d}s \\ &= \frac{c_{hk} r^{Q-1}}{(1+\alpha)^{k-1}} \int_0^1 s^{\alpha+h-1}(1 - s^{2\alpha+2})^{\frac{k}{2}-1} \mathrm{d}s \\ &= \frac{c_{hk}}{2(1+\alpha)^k} \frac{\Gamma(\frac{\alpha+h}{2\alpha+2})\Gamma(\frac{k}{2})}{\Gamma(\frac{Q+\alpha}{2\alpha+2})} r^{Q-1}. \end{aligned}$$

Since $B(o,r)$ and $\bar{B}(o,r)$ are equivalent,

$$P_\alpha(\bar{B}(o,r)) = P_\alpha(B(o,r)) = \frac{c_{hk}}{2(1+\alpha)^k} \frac{\Gamma(\frac{\alpha+h}{2\alpha+2})\Gamma(\frac{k}{2})}{\Gamma(\frac{Q+\alpha}{2\alpha+2})} r^{Q-1}.$$

By [17] again, we know that $P_\alpha(\cdot)$ is invariant under a vertical translation $L_y$ with $y \in \mathbb{R}^k$. Therefore,

$$\begin{cases} P_\alpha(B((0,y),r)) = \frac{c_{hk}}{2(1+\alpha)^k} \frac{\Gamma(\frac{\alpha+h}{2\alpha+2})\Gamma(\frac{k}{2})}{\Gamma(\frac{Q+\alpha}{2\alpha+2})} r^{Q-1} \\ P_\alpha(\bar{B}((0,y),r)) = \frac{c_{hk}}{2(1+\alpha)^k} \frac{\Gamma(\frac{\alpha+h}{2\alpha+2})\Gamma(\frac{k}{2})}{\Gamma(\frac{Q+\alpha}{2\alpha+2})} r^{Q-1} \end{cases} \quad \forall \quad y \in \mathbb{R}^k.$$

(iii) Theorem 1(iii) implies that it suffices to show that

$$\mathrm{cap}_{\alpha,1}(B(o,r)) = \frac{c_{1k}}{2(1+\alpha)^k} \frac{\sqrt{\pi}\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})} r^{Q-1}.$$

On the one hand, by Theorem 10(iv) and Theorem 11(ii) we conclude that if $h = 1$ then

$$\mathrm{cap}_{\alpha,1}(B(o,r)) \le P_\alpha(B(o,r)) = \frac{c_{1k}}{2(1+\alpha)^k} \frac{\sqrt{\pi}\,\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})} r^{Q-1}.$$

On the other hand, for any set $A \subseteq \mathbb{G}^n_\alpha$ with its interior $int A \supseteq B(o,r)$ it is sufficient to prove that

$$P_\alpha(B(o,r)) \le P_\alpha(A).$$

As in [17], we consider the following functions $\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}^n$:

$$\Phi(\xi,\eta) = \big(\mathrm{sgn}(\xi)|(\alpha+1)\xi|^{\frac{1}{\alpha+1}}, \eta\big) \ \ \& \ \ \Psi(x,y) = \left(\mathrm{sgn}(x)\frac{|x|^{\alpha+1}}{\alpha+1}, y\right).$$

Clearly, the function $\Psi$ is a homeomorphism and $\Phi$ is its inverse. Let $\tilde{B} = \Psi(B(o,r))$. Then it is easy to see that $\tilde{B}$ is an Euclidean ball in $\mathbb{R}^n$ with the radius $\frac{r^{\alpha+1}}{\alpha+1}$. Via [17, Proposition 2.5], we know

$$P(\tilde{B}) = P_\alpha(B(o,r)),$$

where $P(\cdot)$ is the Euclidean perimeter of a set in $\mathbb{R}^n$. Also,

$$P(\Psi(A)) = P_\alpha(A).$$

Clearly,

$$\tilde{B} \subseteq \Psi(A) \quad \& \quad P(\tilde{B}) \le P(\Psi(A)),$$

so

$$P_\alpha(B(o,r)) \le P_\alpha(A).$$

Consequently,

$$\mathrm{cap}_{\alpha,1}(B(o,r)) \le P_\alpha(B(o,r)) = \frac{c_{1k}}{2(1+\alpha)^k} \frac{\sqrt{\pi}\,\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})} r^{Q-1}.$$

□

The following rough estimates for balls (related to the different metrics) can be obtained and may be usually sufficient for applications.

**Theorem 12** *Let* $1 \le p < Q$.

(i) *There are two positive constants* $C_1$ *and* $C_2$ *depending only on* $Q$ *and* $p$ *such that*

$$C_1|B_\alpha(g,r)|^{\frac{Q-p}{Q}} \le \mathrm{cap}_{\alpha,p}(B_\alpha(g,r)) \le C_2 r^{-p}|B_\alpha(g,r)| \quad \forall \quad g \in \mathbb{G}^n_\alpha.$$

(ii) *There are two positive constants* $C_3$ *and* $C_4$ *depending only on* $Q$ *and* $p$, *such that*

$$C_3|B(g,r)|^{\frac{Q-p}{Q}} \le \mathrm{cap}_{\alpha,p}(B(g,r)) \le C_4 r^{-p}|B_\alpha(g,r)| \quad \forall \quad g = (x,y) \in \mathbb{G}^n_\alpha.$$

*Proof* (i) From Theorem 3.3 in [15] (cf. [20,22]), we can choose the cutoff function $u \in C_0^\infty(B_\alpha(g, 2r))$ satisfying

$$u = 1 \ \text{ on } \ B_\alpha(g, r) \ \text{ and } \ |\nabla_\alpha u(h)| \leq Cr^{-1} \ \text{ a.e. } \ h \in \mathbb{G}_\alpha^n.$$

Then the definition of $p$-capacity and the doubling property derive

$$\operatorname{cap}_{\alpha,p}(B_\alpha(g, r)) \leq \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha u(h)|^p \mathrm{d}h \leq Cr^{-p}|B_\alpha(g, 2r)| \leq C_2 r^{-p}|B_\alpha(g, r)|.$$

On the other hand, for any $u \in C_0^\infty(G_\alpha^n)$ with $u = 1$ in a neighborhood of $B_\alpha(g, r)$ and $0 \leq u \leq 1$ on $G_\alpha^n$, using the Sobolev inequality in Proposition 17 we have

$$|B_\alpha(g, r)|^{\frac{Q-p}{Qp}} \leq \Big(\int_{B_\alpha(g,r)} |u|^{\frac{Qp}{Q-p}} \mathrm{d}h\Big)^{\frac{Q-p}{Qp}} \leq \frac{p(Q-1)}{Q-p}\big(c(\alpha)\big)^{\frac{Q-1}{Q}}\Big(\int_{\mathbb{G}_\alpha^n} | \nabla_\alpha u |^p \ \mathrm{d}g\Big)^{\frac{1}{p}}.$$

A further application of the definition of $p$-capacity derives

$$C_1|B_\alpha(g, r)|^{\frac{Q-p}{Q}} \leq \operatorname{cap}_{\alpha,p}(B_\alpha(g, r)),$$

where

$$C_1 = \left(\frac{p(Q-1)}{Q-p}\,(c(\alpha))^{\frac{Q-1}{Q}}\right)^{-p}.$$

(ii) By the relation between $d_\alpha$ and $\rho$ we know that for any point $g \in \mathbb{G}_\alpha^n$

$$B(g, r) \subseteq B_\alpha(g, Cr),$$

where the constant $C > 1$ is independent of $g$. Then we apply the monotonicity of $p$-capacity to obtain

$$\operatorname{cap}_{\alpha,p}(B(g, r)) \leq \operatorname{cap}_{\alpha,p}(B_\alpha(g, Cr)) \leq C_4 r^{-p}|B_\alpha(g, r)|.$$

Utilizing the Sobolev inequality in Proposition 17 again, we have, for any $u \in \mathfrak{A}(B(g, r))$,

$$|B(g, r)|^{\frac{Q-p}{Qp}} \leq \left(\int_{B(g,r)} |u|^{\frac{Qp}{Q-p}} \mathrm{d}h\right)^{\frac{Q-p}{Qp}} \leq \frac{p(Q-1)}{Q-p}\big(c(\alpha)\big)^{\frac{Q-1}{Q}} \left(\int_{\mathbb{G}_\alpha^n} | \nabla_\alpha u |^p \ \mathrm{d}g\right)^{\frac{1}{p}}.$$

Using the definition of $p$-capacity, we get

$$C_3|B(g, r)|^{\frac{Q-p}{Q}} \leq \operatorname{cap}_{\alpha,p}(B(g, r)),$$

where

$$C_3 = \left(\frac{p(Q-1)}{Q-p}\big(c(\alpha)\big)^{\frac{Q-1}{Q}}\right)^{-p}.$$

□

If $r > \beta|x|$ with $\beta > 1$, then $|x| + r \sim r$, and hence it is easy to deduce

**Corollary 13** *Let* $1 \leq p < Q$.

(i) *There are two positive constants* $C_1'$ *and* $C_2'$ *depending only on* $Q$ *and* $p$ *such that if* $r > \beta|x|$ *with* $\beta > 1$ *then*

$$C_1' r^{-p}|B_\alpha(g, r)| \leq \operatorname{cap}_{\alpha,p}(B_\alpha(g, r)) \leq C_2' r^{-p}|B_\alpha(g, r)| \ \ \forall \ \ g = (x, y) \in \mathbb{G}_\alpha^n.$$

(ii) *There are two positive constants $C_3'$ and $C_4'$ depending only on $Q$ and $p$, such that if $r > \beta|x|$ with $\beta > 1$ then*

$$C_3'|B(g,r)|^{\frac{Q-p}{Q}} \le \mathrm{cap}_{\alpha,p}(B(g,r)) \le C_4'|B(g,r)|^{\frac{Q-p}{Q}} \quad \forall \quad g = (x, y) \in \mathbb{G}_\alpha^n.$$

### 2.3 1-Capacity versus BV-capacity

The 1-capacity has a close relationship with the BV-capacity. In what follows, we prove their equivalence by two-sided estimates. Similar arguments on the metric spaces have appeared in [27].

**Theorem 14** *For any compact set $E \subseteq \mathbb{G}_\alpha^n$, there exists a constant $C$ such that*

$$\mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) \le \mathrm{cap}_{\alpha,1}(E) \le C\mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)).$$

*Proof* From the definition of 1-capacity and BV-capacity it follows that

$$\mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) \le \mathrm{cap}_{\alpha,1}(E).$$

We next show that

$$\mathrm{cap}_{\alpha,1}(E) \le C\mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n))$$

holds. Assume that $\mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) < \infty$. Let $\varepsilon > 0$ and choose a function $u \in \mathcal{A}(E, \mathcal{BV}(\mathbb{G}_\alpha^n))$ such that

$$\| X_\alpha u \| (\mathbb{G}_\alpha^n) < \mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) + \varepsilon.$$

By the coarea formula again (cf. Theorem 5.2 in [21]) and the Cavalieri principle,

$$\| X_\alpha u \| (\mathbb{G}_\alpha^n) = \int_0^1 P_\alpha(\{g \in \mathbb{G}_\alpha^n : u(g) > t\})\mathrm{d}t$$

and there exists a $t_0 \in (0, 1)$ such that

$$P_\alpha(\{g \in \mathbb{G}_\alpha^n : u(g) > t_0\}) < \mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) + \varepsilon.$$

Applying isoperimetric inequality (9) derives

$$|\{g \in \mathbb{G}_\alpha^n : u(g) > t_0\}| < \infty.$$

From Theorem 3.1 in [31], we obtain a collection of disjoint balls $B_\alpha(g_i, r_i)$, $i = 1, 2, \ldots,$ such that

$$\{g \in \mathbb{G}_\alpha^n : u(g) > t_0\} \subseteq \cup_{i=1}^\infty B_\alpha(g_i, 5r_i)$$

and

$$\sum_{i=1}^\infty \frac{|B_\alpha(g_i, 5r_i)|}{5r_i} < CP_\alpha(\{g \in \mathbb{G}_\alpha^n : u(g) > t_0\}).$$

Using Theorem 1 and Theorem 12, we have

$$\begin{aligned}\mathrm{cap}_{\alpha,1}(E) &\le \sum_{i=1}^\infty \mathrm{cap}_{\alpha,1}(B_\alpha(g_i, 5r_i)) \\ &\le C\sum_{i=1}^\infty \frac{B_\alpha(g_i, 5r_i)}{5r_i} \\ &\le C(\mathrm{cap}(E, \mathcal{BV}(\mathbb{G}_\alpha^n)) + \varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the desired result. □

## 2.4 Relationship with Hausdorff capacity

The Hausdorff capacity and the Hausdorff measure on $\mathbb{R}^n$ and even on some metric spaces have been investigated in, e.g., [1,4,14,27,50]. Similarly, we can define the Hausdorff measure and capacity on $\mathbb{G}_\alpha^n$.

For $1 \le p < Q$ the Hausdorff capacity $H_\alpha^p(E)$ of $E \subseteq \mathbb{G}_\alpha^n$ with respect to the metric $d_\alpha$ is defined by

$$\begin{cases} \inf\left\{ \sum_{i=1}^{\infty} \frac{|B_\alpha(g_i,r_i)|}{r_i^p} : E \subseteq \cup_{i=1}^{\infty} B_\alpha(g_i,r_i) \right\} & \text{if } 1 \le p < h+k; \\ \inf\left\{ \sum_{i=1}^{\infty} |B_\alpha(g_i,r_i)|^{\frac{Q-p}{Q}} : E \subseteq \cup_{i=1}^{\infty} B_\alpha(g_i,r_i) \right\} & \text{if } h+k \le p < Q. \end{cases}$$

Moreover, the Hausdorff measure $\mathcal{H}_\alpha^p(E)$ of $E \subseteq \mathbb{G}_\alpha^n$ is defined by

$$\begin{cases} \sup_{\delta>0} \inf\left\{ \sum_{i=1}^{\infty} \frac{|B_\alpha(g_i,r_i)|}{r_i^p} : E \subseteq \cup_{i=1}^{\infty} B_\alpha(g_i,r_i), r_i \le \delta \right\} & \text{if } 1 \le p < h+k; \\ \sup_{\delta>0} \inf\left\{ \sum_{i=1}^{\infty} |B_\alpha(g_i,r_i)|^{\frac{Q-p}{Q}} : E \subseteq \cup_{i=1}^{\infty} B_\alpha(g_i,r_i), r_i \le \delta \right\} & \text{if } h+k \le p < Q. \end{cases}$$

It is obvious that

$$H_\alpha^p(E) \le \mathcal{H}_\alpha^p(E) \quad \forall \quad E \subseteq \mathbb{G}_\alpha^n.$$

It should be noted that $H_\alpha^p(E)$ and $\mathcal{H}_\alpha^p(E)$ are exactly the $Q-1$-dimensional Hausdorff capacity and measure in [33] when $p = 1$. Moreover, unlike the case of Euclidean spaces, the Hausdorff capacity and measure on $\mathbb{G}_\alpha^n$ are defined by different forms and depend on the range of the index $p$.

**Theorem 15** *Let $1 \le p < Q$.*

(i) *If $p \in [1, h+k)$, then for any $g \in \mathbb{G}_\alpha^n$ and $r > 0$ there exists a positive constant $C$ such that*
$$Cr^{-p}|B_\alpha(g,r)| \le H_\alpha^p(B_\alpha(g,r)) \le r^{-p}|B_\alpha(g,r)|. \tag{10}$$

(ii) *If $p \in [h+k, Q)$, then for any $g \in \mathbb{G}_\alpha^n$ and $r > 0$ there exists a positive constant $C$ such that*
$$C|B_\alpha(g,r)|^{\frac{Q-p}{Q}} \le H_\alpha^p(B_\alpha(g,r)) \le |B_\alpha(g,r)|^{\frac{Q-p}{Q}}. \tag{11}$$

(iii) *If $p \in [1, Q)$, there exists a positive constant $C$ such that*
$$\mathrm{cap}_{\alpha,p}(E) \le C H_\alpha^p(E) \tag{12}$$
*holds for any compact set $E \subseteq \mathbb{G}_\alpha^n$.*

(iv) *There exist two positive constants $C_5$ and $C_6$ such that*
$$C_5 H_\alpha^1(E) \le \mathrm{cap}_{\alpha,1}(E) \le C_6 H_\alpha^1(E) \tag{13}$$
*holds for any compact set $E \subseteq \mathbb{G}_\alpha^n$.*

*Proof* (i) It is obvious that

$$H_\alpha^p(B_\alpha(g,r)) \le r^{-p}|B_\alpha(g,r)|$$

by the definition of Hausdorff capacity. We only consider the lower bound. For any collection of balls $\{B_\alpha(g_j, r_j)\}$ covering $B_\alpha(g, r)$ with $r_j \le r$, $j = 1, 2, \ldots$, then

$$\sum_{j=1}^{\infty} \frac{|B_\alpha(g_j, r_j)|}{r_j^p} \ge \sum_{j=1}^{\infty} \frac{|B_\alpha(g_j, r_j)|}{r^p} \ge \frac{|B_\alpha(g, r)|}{r^p}.$$

In what follows, suppose first that there exists $j_0$ such that $r_{j_0} > r$. Without loss of generality, we may assume that

$$B_\alpha(g_j, r_j) \cap B_\alpha(g, r) \ne \varnothing \quad \forall \quad j.$$

Then

$$B_\alpha(g, r_{j_0}) \subseteq B_\alpha(g_{j_0}, 3r_{j_0}).$$

If $1 \le p < h + k$ and $g = (x, y)$, using (2) we have

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{|B_\alpha(g_j, r_j)|}{r_j^p} &\ge \frac{|B_\alpha(g_{j_0}, r_{j_0})|}{r_{j_0}^p}\\ &\ge C\frac{|B_\alpha(g_{j_0}, 3r_{j_0})|}{r_{j_0}^p}\\ &\ge C\frac{|B_\alpha(g, r_{j_0})|}{r_{j_0}^p}\\ &\ge C' r_{j_0}^{h+k-p}(|x| + r_{j_0})^{\alpha k}\\ &\ge C' r^{h+k-p}(|x| + r)^{\alpha k}\\ &\ge C''\frac{|B_\alpha(g, r)|}{r^p}.\end{aligned}$$

In short,

$$H_\alpha^p(B_\alpha(g, r)) \ge Cr^{-p}|B_\alpha(g, r)|,$$

where $C = \min\{C'', 1\}$. This completes the proof of (10).

(ii) Clearly,

$$H_\alpha^p(B_\alpha(g, r)) \le |B_\alpha(g_i, r_i)|^{\frac{Q-p}{Q}}.$$

An application of Theorem 12(i) and Theorem 15(iii) validates (11).

(iii) Take any covering balls $\{B_\alpha(g_i, r_i)\}$ such that $E \subseteq \cup_{i=1}^{\infty} B_\alpha(g_i, r_i)$. By Theorem 1(i) and (vii) and Theorem 12 we have

$$\mathrm{cap}_{\alpha,p}(E) \le \sum_{i=1}^{\infty} \mathrm{cap}_{\alpha,p}(B_\alpha(g_i, r_i)) \le C \sum_{i=1}^{\infty} r_i^{-p}|B_\alpha(g_i, r_i)|.$$

Therefore, (12) holds true by the definition of Hausdorff capacity.

(iv) We combine (iii) in this theorem with Theorem 14 and [31, Theorem 3.6] to derive (13). □

# 3 Applications to Sobolev-type imbeddings

## 3.1 Sobolev-type inequalities

The first is the endpoint case.

**Proposition 16** *For any $f \in C_0^\infty(\mathbb{G}_\alpha^n)$ one has*

$$\Big(\int_{\mathbb{G}_\alpha^n} |f|^{\frac{Q}{Q-1}} \mathrm{d}g\Big)^{\frac{Q-1}{Q}} \le \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \int_{\mathbb{G}_\alpha^n} |\ \nabla_\alpha f\ | \,\mathrm{d}g. \tag{14}$$

*The constant $\big(c(\alpha)\big)^{\frac{Q-1}{Q}}$ in the above inequality is sharp when $h = 1$.*

*Proof* Assume that $0 \le f \in C_0^\infty(\mathbb{G}_\alpha^n)$. By the coarea formula (cf. Theorem 5.2 in [21]) and Theorem 10, we have

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f| \mathrm{d}g = \int_0^\infty P_\alpha(E_t)\,\mathrm{d}t \ge \big(c(\alpha)\big)^{\frac{1-Q}{Q}} \int_0^\infty |E_t|^{1-\frac{1}{Q}} \mathrm{d}t,$$

where

$$E_t = \big(\{g \in \mathbb{G}_\alpha^n :\ f(g) > t\}\big).$$

Let

$$f_t = \min\{t, f\}\ \ \&\ \ \chi(t) = \Big(\int_{\mathbb{G}_\alpha^n} f_t^{\frac{Q}{Q-1}} \mathrm{d}g\Big)^{1-\frac{1}{Q}} \quad \forall\, t \in \mathbb{R}.$$

It is easy to see that

$$\lim_{t\to\infty} \chi(t) = \Big(\int_{\mathbb{G}_\alpha^n} |f|^{\frac{Q}{Q-1}} \mathrm{d}g\Big)^{1-\frac{1}{Q}}.$$

We can check that $\chi(t)$ is locally Lipschitz and $\chi'(t) \le |E_t|^{1-\frac{1}{Q}}$, a.e. $t$. Hence,

$$\Big(\int_{\mathbb{G}_\alpha^n} |f|^{\frac{Q}{Q-1}} \mathrm{d}g\Big)^{1-\frac{1}{Q}} = \int_0^\infty \chi'(t)\mathrm{d}t \le \int_0^\infty |E_t|^{1-\frac{1}{Q}} \mathrm{d}t \le \big(c(\alpha)\big)^{\frac{Q}{Q-1}} \int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f| \mathrm{d}g.$$

Following [39] we prove the sharpness of the constant $c(\alpha)$ when $h = 1$, where we assume $\alpha \ge 1$ for technical reasons. We take a bounded open set $E \subseteq \mathbb{G}_\alpha^n$ with boundary of class $C^2$. For any $\varepsilon > 0$ let $\rho(p) = d_\alpha(p, E)$ and

$$f_\varepsilon(p) = \begin{cases} 1 & \text{if} \quad p \in \bar{E}; \\ 1 - \varepsilon^{-1}\rho(p) & \text{if} \quad 0 < \rho(p) < \varepsilon; \\ 0 & \text{if} \quad \rho(p) \ge \varepsilon. \end{cases}$$

Denote by

$$\mathcal{E}_\varepsilon = \{g \in \mathbb{G}_\alpha^n : \rho(p) < \varepsilon\}.$$

By Theorem 5.1 in [40], we obtain the identity

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} |\mathcal{E}_\varepsilon \backslash E| = P_\alpha(E).$$

Applying the Sobolev inequality to $f_\varepsilon$ and letting $\varepsilon \to 0$, we have

$$\begin{aligned}
|E|^{\frac{Q-1}{Q}} &= \lim_{\varepsilon\to 0} \| f_\varepsilon \|_{\frac{Q}{Q-1}} \\
&\le \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \left( \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{\mathcal{E}_\varepsilon \setminus E} |\nabla_\alpha \rho| \mathrm{d}p \right) \\
&= \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \left( \lim_{\varepsilon\to 0} \varepsilon^{-1} |\mathcal{E}_\varepsilon \setminus E| \right) \\
&= P_\alpha(E),
\end{aligned}$$

where we have used the Eikonal equation and the coarea formula (cf. Theorem 5.2 in [21]). Thus we get isoperimetric inequality (9) which implies the sharpness of (14). □

The second is the $1 < p < Q$ Sobolev inequality on Grushin spaces.

**Proposition 17** *Let $1 < p < Q$. For any $f \in C_0^\infty(\mathbb{G}_\alpha^n)$ one has*

$$\left( \int_{\mathbb{G}_\alpha^n} |f|^{\frac{Qp}{Q-p}} \mathrm{d}g \right)^{\frac{Q-p}{Qp}} \le \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \frac{p(Q-1)}{Q-p} \left( \int_{\mathbb{G}_\alpha^n} | \nabla_\alpha f |^p \, \mathrm{d}g \right)^{\frac{1}{p}}, \tag{15}$$

*where the constant $c(\alpha)$ appears in Theorem 10.*

*Proof* For some $\gamma > 1$ to be fixed later, we obtain, via (14) and the Hölder inequality,

$$\begin{aligned}
\left( \int_{\mathbb{G}_\alpha^n} |f|^{\frac{\gamma Q}{Q-1}} \mathrm{d}g \right)^{\frac{Q-1}{Q}} &\le \gamma \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \int_{\mathbb{G}_\alpha^n} | f |^{\gamma-1} | \nabla_\alpha f | \, \mathrm{d}g \\
&\le \gamma \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \left( \int_{\mathbb{G}_\alpha^n} |f|^{\frac{p(\gamma-1)}{p-1}} \right)^{1-\frac{1}{p}} \left( \int_{\mathbb{G}_\alpha^n} | \nabla_\alpha f |^p \, \mathrm{d}g \right)^{\frac{1}{p}}.
\end{aligned}$$

Choosing

$$\gamma = \frac{p(Q-1)}{Q-p}$$

and noting

$$\gamma - 1 = \frac{Q(p-1)}{Q-p},$$

we conclude that (15) holds. □

### 3.2 Splitting Sobolev-type inequalities

In order to split (14)–(15) via functional capacities, we need the following assertion.

**Theorem 18** (i) *The analytic inequality*

$$\| f \|_{\frac{Q}{Q-1}} \le \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \left( \int_0^\infty \big(\mathrm{cap}_{\alpha,1}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \ge t\})\big)^{\frac{Q}{Q-1}} \mathrm{d}t^{\frac{Q}{Q-1}} \right)^{\frac{Q-1}{Q}} \tag{16}$$

*for any Lebesgue measurable function $f$ with compact support in $\mathbb{G}_\alpha^n$, is equivalent to, the geometric inequality*

$$|M|^{\frac{Q-1}{Q}} \le \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \mathrm{cap}_{\alpha,1}(M) \tag{17}$$

*for any compact domain $M \subseteq \mathbb{G}_\alpha^n$.*
(ii) *Inequalities* (16) *and* (17) *are true. Moreover, they are sharp only when* $h = 1$.

*Proof* In what follows, we always adopt the short notation:

$$\Omega_t(f) = \{g \in \mathbb{G}_\alpha^n : |f(g)| \geq t\}$$

for a function $f$ defined on $\mathbb{G}_\alpha^n$ and a number $t > 0$.

(i) Given a compact domain $M \subseteq \mathbb{G}_\alpha^n$, let $f = 1_M$. Then

$$\| f \|_{\frac{Q}{Q-1}} = |M|^{\frac{Q-1}{Q}}$$

and

$$\Omega_t(f) = \begin{cases} M, & \text{if} \quad t \in (0, 1], \\ \emptyset, & \text{if} \quad t \in (1, \infty). \end{cases}$$

Hence

$$\int_0^\infty \frac{\mathrm{d}t^{\frac{Q}{Q-1}}}{(\mathrm{cap}_{\alpha,1}(\Omega_t(f)))^{\frac{Q}{1-Q}}} = \left(\int_0^1 + \int_1^\infty\right) \frac{\mathrm{d}t^{\frac{Q}{Q-1}}}{(\mathrm{cap}_{\alpha,1}(\Omega_t(f)))^{\frac{Q}{1-Q}}} = (\mathrm{cap}_{\alpha,1}(M))^{\frac{Q}{Q-1}}.$$

Therefore, (16) implies (17).

Conversely, we prove that (17) implies (16). Suppose that (17) holds for any compact subdomain of $\mathbb{G}_\alpha^n$. For $t > 0$ and $f$, a Lebesgue measurable function with compact support in $\mathbb{G}_\alpha^n$, we use the definition of the Lebesgue $\frac{Q}{Q-1}$-integral on a given metric space and (17) to get

$$\| f \|_{\frac{Q}{Q-1}}^{\frac{Q}{Q-1}} = \int_0^\infty |\Omega_t(f)| \mathrm{d}t^{\frac{Q}{Q-1}} \leq \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \int_0^\infty \big(\mathrm{cap}_{\alpha,1}(\Omega_t(f))\big)^{\frac{Q}{Q-1}} \mathrm{d}t^{\frac{Q}{Q-1}}.$$

(ii) Thanks to the equivalence between (16) and (17), it suffices to prove that (17) is valid. In fact, by application of the definition of $\mathrm{cap}_{\alpha,1}(\cdot)$ to (14) we have

$$|M|^{\frac{Q-1}{Q}} \leq \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \inf\{\| \nabla_\alpha f \|_1 \colon f \in C_0^\infty(\mathbb{G}_\alpha^n),\ f \geq 1_M\},$$

namely, (17) holds true.

In the sequel we consider the sharpness of (17) when $h = 1$. Via (17) and (19) we have

$$|M|^{\frac{Q-1}{Q}} \leq \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \mathrm{cap}_{\alpha,1}(M) \leq \big(c(\alpha)\big)^{\frac{Q-1}{Q}} P_\alpha(M).$$

Choose $M = E_\alpha$ as given in Theorem 10. Since $E_\alpha$ is the isoperimetric set, it follows from Theorem 10 that

$$|E_\alpha|^{\frac{Q-1}{Q}} = \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \mathrm{cap}_{\alpha,1}(E_\alpha) = \big(c(\alpha)\big)^{\frac{Q-1}{Q}} P_\alpha(E_\alpha).$$

This implies the sharpness of (17). □

Now, (14) can be separated according to the following formulation.

**Theorem 19** (i) *The analytic inequality*

$$\left(\int_0^\infty \big(\mathrm{cap}_{\alpha,1}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \geq t\})\big)^{\frac{Q}{Q-1}} \mathrm{d}t^{\frac{Q}{Q-1}}\right)^{\frac{Q-1}{Q}} \leq \| \nabla_\alpha f \|_1 \ \forall f \in C_0^1(\mathbb{G}_\alpha^n) \tag{18}$$

*is equivalent to the geometric inequality*

$$\mathrm{cap}_{\alpha,1}(M) \leq P_\alpha(M) \tag{19}$$

*for any compact domain* $M \subseteq \mathbb{G}_\alpha^n$ *with* $C^1$ *boundary.*

(ii) *Inequalities* (18) *and* (19) *are true. Moreover, they are sharp only when* $h = 1$.

*Proof* (i) For $\delta > 0$ and $M \subseteq \mathbb{G}_\alpha^n$ (a compact domain with $C^1$ boundary), let $R > 0$ be such that $M \subseteq B_\alpha(o, R)$, where $o = (0, 0) \in \mathbb{G}_\alpha^n$. Choose $\delta > 0$ such that

$$2\delta < \mathrm{dist}_{\mathbb{R}^n}(M, \partial B_\alpha(o, R)),$$

where $\mathrm{dist}_{\mathbb{R}^n}(\cdot, \partial B_\alpha(o, R))$ represents the Euclidean distance from $M$ to $B_\alpha(o, R)$.

Define the Lipschitz function

$$f_\delta(g) = \begin{cases} 1 - \delta^{-1}\mathrm{dist}_{\mathbb{R}^n}(g, M) & \text{if } \mathrm{dist}_{\mathbb{R}^n}(g, M) < \delta, \\ 0 & \text{if } \mathrm{dist}_{\mathbb{R}^n}(g, M) \geq \delta. \end{cases}$$

Let $A_\delta$ be the intersection of $B_\alpha(o, R)$ with a tubular neighborhood of $M$ of radius $\delta$. If (18) holds, then

$$M \subseteq \Omega_t(f_\delta) \quad \forall \quad t \in [0, 1]$$

is applied to derive

$$\mathrm{cap}_{\alpha,1}(M) \leq \left( \int_0^1 \left(\mathrm{cap}_{\alpha,1}(\Omega_t(f_\delta))\right)^{\frac{Q}{Q-1}} \mathrm{d}t^{\frac{Q}{Q-1}} \right)^{\frac{Q-1}{Q}} \leq \| \nabla_\alpha f_\delta \|_1 .$$

Meanwhile, using the coarea formula given in Theorem 5.2 in [21] yields

$$\begin{aligned} \| \nabla_\alpha f_\delta \|_1 &\leq \frac{1}{\delta} \int_{A_\delta} |\nabla_\alpha \mathrm{dist}_{\mathbb{R}^n}(g, M)| \mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{\delta} \int_0^\delta \int_{\{g \in B_\alpha(o,R):\ \mathrm{dist}_{\mathbb{R}^n}(g,M)=s\}} \frac{|\nabla_\alpha \mathrm{dist}_{\mathbb{R}^n}(\cdot, M)|}{|\nabla_{\mathbb{R}^n} \mathrm{dist}_{\mathbb{R}^n}(\cdot, M)|} \mathrm{d}\mathcal{H}^1 \mathrm{d}s, \end{aligned}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. Letting $\delta \to 0$ we conclude that the right side of the above inequality will tend to $P_\alpha(M)$ and we have used Proposition 2.1 in [17]. Then (19) is valid.

Suppose that (19) is true for any compact subdomain of $\mathbb{G}_\alpha^n$ with $C^1$ boundary. The monotonicity of $\mathrm{cap}_{\alpha,1}(\cdot)$ ensures that $t \to \mathrm{cap}_{\alpha,1}(\Omega_t(f))$ is a decreasing function on $[0, \infty)$ and so that

$$\begin{aligned} t^{\frac{1}{Q-1}} (\mathrm{cap}_{\alpha,1}(\Omega_t(f)))^{\frac{Q}{Q-1}} &= (t\mathrm{cap}_{\alpha,1}(\Omega_t(f)))^{\frac{1}{Q-1}} \mathrm{cap}_{\alpha,1}(\Omega_t(f)) \\ &\leq \left( \int_0^t \mathrm{cap}_{\alpha,1}(\Omega_r(f)) \mathrm{d}r \right)^{\frac{1}{Q-1}} \mathrm{cap}_{\alpha,1}(\Omega_t(f)) \\ &= \frac{Q-1}{Q} \frac{\mathrm{d}}{\mathrm{d}t} \left( \int_0^t \mathrm{cap}_{\alpha,1}(\Omega_r(f)) \mathrm{d}r \right)^{\frac{Q}{Q-1}} . \end{aligned}$$

Via (19) and the above estimate we obtain

$$\begin{aligned}\int_0^\infty (\mathrm{cap}_{\alpha,1}(\Omega_t(f)))^{\frac{Q}{Q-1}}\mathrm{d}t^{\frac{Q}{Q-1}} &= \frac{Q}{Q-1}\int_0^\infty (\mathrm{cap}_{\alpha,1}(\Omega_t(f)))^{\frac{Q}{Q-1}} t^{\frac{1}{Q-1}}\mathrm{d}t\\ &\le \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \mathrm{cap}_{\alpha,1}(\Omega_r(f))\mathrm{d}r\right)^{\frac{Q}{Q-1}}\right)\mathrm{d}t\\ &= \left(\int_0^\infty \mathrm{cap}_{\alpha,1}(\Omega_t(f))\mathrm{d}t\right)^{\frac{Q}{Q-1}}\\ &\le \left(\int_0^\infty P_\alpha(\Omega_t(f))\mathrm{d}t\right)^{\frac{Q}{Q-1}}\\ &= \| \nabla_\alpha f \|_1^{\frac{Q}{Q-1}},\end{aligned}$$

where we have used Theorem 5.2 in [21] again in the last step.

(ii) Due to the equivalence between (18) and (19), it is enough to check that (19) is valid for any compact subdomain of $\mathbb{G}_\alpha^n$. It is easy to discover that Theorem 10(iv) implies (19).

By Theorem 11, it is easy to see that for any $y \in \mathbb{R}^k$,

$$\mathrm{cap}_{\alpha,1}(\bar{B}((0,y),r)) = P_\alpha(\bar{B}((0,y),r)) \ \ \& \ \ \mathrm{cap}_{\alpha,1}(E_\alpha) = P_\alpha(E_\alpha),$$

which imply the sharpness of (19) when $h = 1$. As in [33], (19) has two kinds of minimizers and it is different from the setting for $\mathbb{R}^n$; see [45]. □

*Remark 1* Until now, it is uncertain that inequalities (16), (17), (18) and (19) are sharp under $h > 1$. Resolving this issue depends on the optimal constant of the isoperimetric problem on a given Grushin space for $h > 1$.

To separate (15), let $f \in C_0^\infty(\mathbb{G}_\alpha^n)$ and

$$T = \sup\{t > 0 : \mathrm{cap}_{\alpha,p}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \ge t\}) > 0\} > 0. \tag{20}$$

By some computations we obtain

$$\psi(t) = \int_0^t \frac{\mathrm{d}\tau}{[\phi(\tau)]^{\frac{1}{p-1}}} < \infty \quad \forall \quad t \in (0,T),$$

where

$$\phi(t) = \int_{\mathcal{E}_t} |\nabla_\alpha f|^{p-1}\mathrm{d}\mu_t$$

and $\mathrm{d}\mu_\tau = \| \partial\mathcal{E}_\tau \|_\alpha$ is the perimeter measure of the level set $\mathcal{E}_\tau$ of $f$. In a similar way to verify Lemma 2.3.1 in [35], we get the following result on Grushin spaces.

**Lemma 20** *Let $f \in C_0^\infty(\mathbb{G}_\alpha^n)$ and satisfy (20). Then the function $t(\psi)$ is absolutely continuous on any segment $[0, \psi(T-\delta)]$ for $\delta \in (0,T)$, and*

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f|^p \mathrm{d}g \ge \int_0^{\psi(t)} [t'(\psi)]^p \mathrm{d}\psi, \tag{21}$$

*where the function $t(\psi)$ is the inverse of $\psi(t)$ on the interval $[0, \psi(T)]$. If $T = \max|f|$, then the equality sign in (21) is valid.*

This last lemma is utilized to give a separation of (15).

**Theorem 21** *Let* $1 < p < Q$.

(i) *The analytic inequality*

$$\| f \|_{\frac{Qp}{Q-p}} \le c(p,\alpha)\left(\int_0^\infty \left(\mathrm{cap}_{\alpha,p}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \ge t\})\right)^{\frac{Q}{Q-p}} \mathrm{d}t^{\frac{Qp}{Q-p}}\right)^{\frac{Q-p}{Qp}} \tag{22}$$

*for any Lebesgue measurable function* $f$ *with compact support in* $\mathbb{G}_\alpha^n$, *is equivalent to, the geometric inequality*

$$|M|^{\frac{Q-p}{Qp}} \le c(p,\alpha)(\mathrm{cap}_{\alpha,p}(M))^{\frac{1}{p}} \tag{23}$$

*for any compact domain* $M \subseteq \mathbb{G}_\alpha^n$, *where*

$$c(p,\alpha) = \left(c(\alpha)\right)^{\frac{Q-1}{Q}} \left(\frac{Q-p}{Q(p-1)}\right)^{\frac{1}{p}-1}.$$

(ii) *Inequalities* (22) *and* (23) *are true.*

(iii) *The analytic inequality*

$$\left(\int_0^\infty \left(\mathrm{cap}_{\alpha,p}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \ge t\})\right)^{\frac{Q}{Q-p}} \mathrm{d}t^{\frac{Qp}{Q-p}}\right)^{\frac{Q-p}{Qp}} \le \frac{\| \nabla_\alpha f \|_p}{\psi_{p,Q}^{-\frac{1}{Q}}} \quad \forall f \in C_0^1(\mathbb{G}_\alpha^n) \tag{24}$$

*holds with*

$$\psi_{p,Q} = \frac{\Gamma(Q)}{\Gamma(\frac{Q}{p})\Gamma(1+Q-\frac{Q}{p})}.$$

*Proof* (i) Via the integral formula and (23), we conclude that (22) holds. Conversely, if (22) is true, then (23) follows from taking $f = 1_M$.

(ii) Theorem 10(iii) implies (23).

(iii) By [38, (40)] we get

$$\left(\int_0^\infty \left(\mathrm{cap}_{\alpha,p}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \ge t\})\right)^{\frac{Q}{Q-p}} \mathrm{d}t^{\frac{Qp}{Q-p}}\right)^{\frac{Q-p}{Qp}} \le C \| \nabla_\alpha f \|_p \quad \forall f \in C_0^1(\mathbb{G}_\alpha^n)$$

Furthermore, we prove that the constant $C$ has the form $\psi_{p,Q}$. Let

$$\psi(t) = \int_0^t \left(\int_{\{g:|f(g)|=\tau\}} |\nabla_\alpha f(g)|^{p-1} \mathrm{d}\mu_\tau\right)^{\frac{1}{1-p}} \mathrm{d}\tau$$

and denote by $t(\psi)$ the inverse function of $\psi(t)$. Then Lemma 20 deduces

$$\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f(g)|^p \mathrm{d}g = \int_0^\infty |t'(\psi)|^p \mathrm{d}\psi. \tag{25}$$

Via the Bliss inequality in [9] we have

$$\left(\int_0^\infty t(\psi)^{\frac{Qp}{Q-p}} \frac{\mathrm{d}\psi}{\psi^{1+\frac{Q(p-1)}{Q-p}}}\right)^{\frac{Q-p}{Qp}} \tag{26}$$
$$\leq \left(\frac{Q-p}{p-1}\right)^{\frac{Q-p}{Qp}} \left(\frac{\Gamma(Q)}{\Gamma(\frac{Q}{p})\Gamma(1+Q-\frac{Q}{p})}\right)^{\frac{1}{Q}} \left(\int_0^\infty |t'(\psi)|^p \mathrm{d}\psi\right)^{\frac{1}{p}}.$$

Using (25) and integrating by parts we deduce that (26) amounts to

$$\left(\int_0^\infty \frac{\mathrm{d}\big(t(\psi)^{\frac{Qp}{Q-p}}\big)}{\psi^{\frac{Q(p-1)}{Q-p}}}\right)^{\frac{Q-p}{Qp}} \leq \left(\frac{\Gamma(Q)}{\Gamma(\frac{Q}{p})\Gamma(1+Q-\frac{Q}{p})}\right)^{\frac{1}{Q}} \left(\int_{\mathbb{G}_\alpha^n} |\nabla_\alpha f(g)|^p \mathrm{d}g\right)^{\frac{1}{p}}. \tag{27}$$

From Theorem 2(iii) we have

$$\mathrm{cap}_{\alpha,p}(\{g \in \mathbb{G}_\alpha^n : |f(g)| \geq t\}) \leq [\psi(t)]^{1-p},$$

thereby getting (24) via (27). □

*Remark 2* It follows from Proposition 17 that

$$|M|^{\frac{Q-p}{Qp}} \leq \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \frac{p(Q-1)}{Q-p} (\mathrm{cap}_{\alpha,p}(M))^{\frac{1}{p}}$$

for any compact domain $M \subseteq \mathbb{G}_\alpha^n$. But a straightforward computation gives

$$\big(c(\alpha)\big)^{\frac{Q-1}{Q}} \frac{p(Q-1)}{Q-p} > \big(c(\alpha)\big)^{\frac{(Q-1)}{Q}} \left(\frac{Q-p}{Q(p-1)}\right)^{\frac{1}{p}-1}.$$

Moreover,

$$\lim_{\alpha\to 0} c(p,\alpha) = \left(\frac{p-1}{2-p}\right)^{1-\frac{1}{p}} (2^{-\frac{1}{p}}\pi^{-\frac{1}{2}}),$$

which appears in [45, (3.3)] for $h = 1 = k$.

**Corollary 22** *The analytic inequality*

$$\| f \|_{\frac{Qp}{Q-p}} \leq c(p,\alpha)\psi_{p,Q}^{\frac{1}{Q}} \| \nabla_\alpha f \|_p, \quad f \in C_0^\infty(\mathbb{G}_\alpha^n), \tag{28}$$

*holds, where $c(p,\alpha)$ and $\psi_{p,Q}$ are given in Theorem 21.*

*Proof* We only need to prove that the constant in (28) is strictly less than the constant in (15), that is,

$$c(p,\alpha)\psi_{p,Q}^{\frac{1}{Q}} < \big(c(\alpha)\big)^{\frac{Q-1}{Q}} \frac{p(Q-1)}{Q-p}.$$

To this end, we need the following inequality for the Gamma function:

$$\frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \leq \frac{(x+y)^{x+y}}{x^x y^y}\ \forall\, x,y \in \mathbb{R}^+.$$

If

$$x=\frac{Q}{p}-1\ \ \&\ \ y=Q-\frac{Q}{p},$$

then

$$\begin{aligned}
&\frac{\left(c(\alpha)\right)^{\frac{Q-1}{Q}\frac{p(Q-1)}{Q-p}}}{c(\alpha)^{\frac{(Q-1)}{Q}}\left(\frac{Q-p}{Q(p-1)}\right)^{-(1-\frac{1}{p})}\psi_{p,Q}^{\frac{1}{Q}}}\\
&\geq\frac{p(Q-1)}{(Q-p)^{\frac{1}{p}}(Q(p-1))^{1-\frac{1}{p}}}\left(\frac{\Gamma(Q)}{\Gamma\left(\frac{Q}{p}\right)\Gamma\left(1+Q-\frac{Q}{p}\right)}\right)^{-\frac{1}{Q}}\\
&\geq\frac{p(Q-1)}{(Q-p)^{\frac{1}{p}}(Q(p-1))^{1-\frac{1}{p}}}\frac{\left(\frac{Q}{p}-1\right)^{\frac{1}{p}-\frac{1}{Q}}\left(Q-\frac{Q}{p}\right)^{1-\frac{1}{p}}}{(Q-1)^{1-\frac{1}{Q}}}\\
&=\left(\frac{Q-1}{Q-p}\right)^{\frac{1}{Q}}p^{\frac{1}{Q}}>1,
\end{aligned}$$

due to $1<p<Q$ and $Q>2$. This completes the proof of this corollary. □

*Remark 3* The constant $c(p,\alpha)\psi_{p,Q}^{\frac{1}{Q}}$ in (28) is not sharp. As a matter of fact, if $\alpha=1$, $p=2$, $h=1$ and $k=1$, then $Q=3$ and

$$c(2,1)\psi_{2,3}^{\frac{1}{3}}=\left(\frac{16}{3\sqrt{3}}\right)^{\frac{1}{3}}\pi^{-\frac{1}{3}},$$

which is bigger than the constant $\pi^{-\frac{1}{3}}$ in [5, Theorem 1].

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