CrossMark

The Frölicher–Nijenhuis bracket and the geometry of G_2-and Spin(7)-manifolds

Kotaro Kawai[1,2] · Hông Vân Lê[3] ·
Lorenz Schwachhöfer[4]

Abstract We extend the characterization of the integrability of an almost complex structure J on differentiable manifolds via the vanishing of the Frölicher–Nijenhuis bracket $[J, J]^{FN}$ to an analogous characterization of torsion-free G_2-structures and torsion-free Spin(7)-structures. We also explain the Fernández–Gray classification of G_2-structures and the Fernández classification of Spin(7)-structures in terms of the Frölicher–Nijenhuis bracket.

Keywords Frölicher–Nijenhuis bracket · G_2-manifold · Spin(7)-manifold · Fernández–Gray classification · Fernández classification

Mathematics Subject Classification 53C25 · 53C29

✉ Lorenz Schwachhöfer
lschwach@math.tu-dortmund.de

Kotaro Kawai
kkawai@ms.u-tokyo.ac.jp; kkawai@math.gakushuin.ac.jp

Hông Vân Lê
hvle@math.cas.cz

1   Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro, Tokyo 153-8914, Japan

2   Present Address: Gakushuin University, 1-5-1, Mejiro, Toshima, Tokyo 171-8588, Japan

3   Institute of Mathematics CAS, Zitna 25, 11567 Prague 1, Czech Republic

4   Fakultät für Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, 44221 Dortmund, Germany

🗍 Springer

1 Introduction

A G_2-structure on a 7-dimensional manifold M^7 is a 3-form $\varphi \in \Omega^3(M^7)$ which at each point $p \in M^7$ is contained in a certain open subset of $\Lambda^3 T_p^* M^7$; similarly, a Spin(7)-structure on an 8-dimensional manifold M^8 is given by a 4-form $\Phi \in \Omega^4(M^8)$ which at each point is contained in a certain subset of $\Lambda^4 T_p^* M^8$. Such structures induce both an orientation and a Riemannian metric on the underlying manifold, denoted by g_φ and g_Φ, respectively, and Φ is self-dual w.r.t. this metric.

Manifolds with G_2-structures have first been investigated by Fernández and Gray [5], and Spin(7)-structures by Fernández [4] who showed that the covariant derivatives $\nabla\varphi \in \Omega^1(M^7, \Lambda^3 T^* M^7)$ and $\nabla\Phi \in \Omega^1(M^8, \Lambda^4 T^* M^8)$, respectively, decompose into four irreducible components in case of G_2-structures and into two irreducible components in the case of Spin(7)-structures. Thus, the conditions of the vanishing of some of these components yield $2^4 = 16$ classes of G_2-structure and $2^2 = 4$ classes of Spin(7)-structures, respectively, and the underlying geometries were discussed in [4,5]; see also Sect. 5.

A G_2-structure (Spin(7)-structure, respectively) is called *torsion-free*, if φ (Φ, respectively) is parallel. As it turns out, the parallelity of φ and Φ, respectively, is equivalent to φ and Φ being harmonic forms.

Alternatively, G_2- and Spin(7)-structures may be characterized via certain *(2-fold or 3-fold) cross products on the tangent bundle*. These are given as the sections

$$Cr_\varphi := \delta_{g_\varphi}\varphi \in \Omega^2(M^7, TM^7), \quad \chi_\varphi := -\delta_{g_\varphi} * \varphi \in \Omega^3(M^7, TM^7)$$

in case of G_2-structures, and as

$$P_\Phi := -\delta_{g_\Phi}\Phi \in \Omega^3(M^8, TM^8),$$

where $\delta_g : \Omega^{k+1}(M) \to \Omega^k(M, TM)$ is the contraction of a differential form with the Riemannian metric g, and they have natural interpretations via octonian multiplication. The triple cross product $\chi \in \Omega^3(M^7, TM^7)$ on a manifold with a G_2-structure has been introduced by Harvey–Lawson [8] and was used in many papers on deformation of associative submanifolds, see e.g., [11,13,14]. The 3-fold cross product P on $\mathbb{R}^8$ has been first explicitly constructed by Brown and Gray [1]. They also proved that (up to the G_2-action) there are exactly two non-equivalent 3-fold cross products on $\mathbb{R}^8 = \mathbb{O}$. In [8] Harvey and Lawson intensively used the 3-fold cross product on $\mathbb{R}^8$ which is related to the Cayley 4-form and hence is invariant under the action of Spin(7). Fernández showed the uniqueness of a Spin(7)-invariant 4-form on $\mathbb{R}^8$ (up to a multiplicative constant) and used the associated 3-fold cross product to classify Spin(7)-structures on 8-manifolds [4].

In this article, we view these cross products as elements of the *Frölicher–Nijenhuis Lie algebra* $\Omega^*(M, TM)$. Namely, it was shown by Frölicher–Nijenhuis in [6] that $\Omega^*(M, TM)$ can be given the structure of a graded Lie algebra using the *Frölicher–Nijenhuis bracket* $[\ ,\ ]^{FN}$ in a natural way. Thus, given a manifold with a G_2-structure (M^7, φ), we may consider the Frölicher–Nijenhuis brackets

$$[Cr_\varphi, \chi_\varphi]^{FN} \in \Omega^5(M^7, TM^7), \quad [\chi_\varphi, \chi_\varphi]^{FN} \in \Omega^6(M^7, TM^7),$$

(observe that $[Cr, Cr]^{FN} = 0$ due to graded skew symmetry), and analogously, for a manifold with a Spin(7)-structure (M^8, Φ) we may consider

$$[P_\Phi, P_\Phi]^{FN} \in \Omega^6(M^8, TM^8).$$

These brackets may be regarded as a natural generalization of the Nijenhuis tensor of an almost complex structure J. Indeed, regarding such a structure as an element $J \in \Omega^1(M, TM)$, it turns out that $[J, J]^{FN} \in \Omega^2(M, TM)$ coincides – up to a constant multiple – with the Nijenhuis tensor of J, whence J is integrable if and only if $[J, J]^{FN} = 0$ [7].

Our main result is that the Frölicher–Nijenhuis bracket also characterizes the torsion-freeness of G_2- and Spin(7)-structures, respectively. Namely, we show the following.

Theorem 1.1 *Let (M^7, φ) be a manifold with a G_2-structure and the associated Riemannian metric $g = g_\varphi$, and let ∇ be the Levi-Civita connection of g. Then, for every $p \in M^7$ the following are equivalent.*

(1) The G_2-structure is torsion-free at p, i.e., $(\nabla\varphi)_p = 0$.
(2) $[Cr_\varphi, \chi_\varphi]_p^{FN} = 0 \in \Lambda^5 T_p^* M^7 \otimes T_p M^7$.
(3) $[\chi_\varphi, \chi_\varphi]_p^{FN} = 0 \in \Lambda^6 T_p^* M^7 \otimes T_p M^7$.

In fact, we show in Theorem 3.7 that $(\nabla\varphi)_p$ is characterized by either $[\chi_\varphi, \chi_\varphi]_p^{FN}$, or by the projection of $[Cr_\varphi, \chi_\varphi]_p^{FN}$ onto a subspace isomorphic to $T_p M^7 \otimes T_p M^7$.

Theorem 1.2 *Let (M^8, Φ) be a manifold with a Spin(7)-structure, and let ∇ be the Levi-Civita connection of the associated Riemannian metric $g = g_\Phi$. Then, for every $p \in M^8$ the following are equivalent.*

(1) The Spin(7)-structure is torsion-free at p, i.e., $(\nabla\Phi)_p = 0$.
(2) $[P_\Phi, P_\Phi]_p^{FN} = 0 \in \Lambda^6 T_p^* M^8 \otimes T_p M^8$.

Namely, we show in Theorem 4.6 that $(\nabla\Phi)_p$ is characterized by the projection of $[P_\Phi, P_\Phi]_p^{FN}$ onto a subspace isomorphic to $W_p^7 \otimes T_p M^8$ for some rank-7 bundle $\Lambda^6 T^* M^8 \supset W^7 \to M^8$.

These explicit descriptions also allow us to give a complete characterization of the 16 cases of G_2-structures in terms of $[Cr_\varphi, \chi_\varphi]_p^{FN}$ and $[\chi_\varphi, \chi_\varphi]_p^{FN}$, and of the 4 classes of Spin(7)-structures in terms of $[P_\Phi, P_\Phi]_p^{FN}$; cf. Sect. 5.

Our paper is organized as follows. In Sect. 2, we recall the Frölicher–Nijenhuis bracket on $\Omega^*(M, TM)$. Then, we turn to the case of G_2-structures in Sect. 3, characterizing the torsion endomorphism and showing the results that lead us to Theorem 1.1. In Sect. 4, we repeat this discussion for the case of Spin(7)-structures which leads to Theorem 1.2. Finally, the characterization of the 16 classes of G_2-structures and the 4 classes of Spin(7)-structures in terms of the Frölicher–Nijenhuis bracket is given in Sect. 5. The Appendix then contains the proofs of some identities on representations of G_2 and Spin(7) which are used throughout the paper.

2 Preliminaries

2.1 The Frölicher–Nijenhuis bracket

Let M be a manifold and $(\Omega^*(M), \wedge) = (\bigoplus_{k \geq 0} \Omega^k(M), \wedge)$ be the graded algebra of differential forms. We shall use superscripts to indicate the degree of a form, i.e., α^k denotes an element of $\Omega^k(M)$.

Evidently, contraction $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ with a vector field $X \in \mathfrak{X}(M)$ is a derivation of degree -1. More generally, for $K \in \Omega^k(M, TM)$ we define $\iota_K \alpha^l$ as the

contraction of K with $\alpha^l \in \Omega^l(M)$ pointwise by

$$\iota_{\kappa^k \otimes X} \alpha^l := \kappa^k \wedge (\iota_X \alpha^l) \in \Omega^{k+l-1}(M),$$

where $\kappa^k \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$ is a vector field, and this is a derivation of $\Omega^*(M)$ of degree $k - 1$. Thus, the *Nijenhuis-Lie derivative along* $K \in \Omega^k(M, TM)$ defined as

$$\mathcal{L}_K(\alpha^l) := [\iota_K, d](\alpha^l) = \iota_K(d\alpha^l) + (-1)^k d(\iota_K \alpha^l) \in \Omega^{k+l}(M) \qquad (2.1)$$

is a derivation of $\Omega^*(M)$ of degree k.

Observe that for $k = 0$ in which case $K \in \Omega^0(M, TM)$ is a vector field, both ι_K and $\mathcal{L}_K$ coincide with the standard notion of contraction with and Lie derivative along a vector field.

In [6,7], it was shown that $\Omega^*(M, TM)$ can be given a unique graded Lie algebra structure, called the *Frölicher–Nijenhuis bracket* and denoted by $[\cdot, \cdot]^{FN}$, such that $\mathcal{L}$ defines an action of $\Omega^*(M, TM)$ on $\Omega^*(M)$, that is,

$$\mathcal{L}_{[K_1,K_2]^{FN}} = [\mathcal{L}_{K_1}, \mathcal{L}_{K_2}] =: \mathcal{L}_{K_1} \circ \mathcal{L}_{K_2} - (-1)^{|K_1||K_2|} \mathcal{L}_{K_2} \circ \mathcal{L}_{K_1}. \qquad (2.2)$$

It is given by the following formula for $\alpha^k \in \Omega^k(M)$, $\beta^l \in \Omega^l(M)$, $X_1, X_2 \in \mathfrak{X}(M)$ [12, Theorem 8.7 (6), p. 70]:

$$\begin{aligned}
[\alpha^k \otimes X_1, \beta^l \otimes X_2]^{FN} &= \alpha^k \wedge \beta^l \otimes [X_1, X_2] \\
&+ \alpha^k \wedge \mathcal{L}_{X_1}\beta^l \otimes X_2 - \mathcal{L}_{X_2}\alpha^k \wedge \beta^l \otimes X_1 \\
&+ (-1)^k \left(d\alpha^k \wedge (\iota_{X_1}\beta^l) \otimes X_2 + (\iota_{X_2}\alpha^k) \wedge d\beta^l \otimes X_1 \right). \qquad (2.3)
\end{aligned}$$

In particular, for a vector field $X \in \mathfrak{X}(M)$ and $K \in \Omega^*(M, TM)$ we have [12, Theorem 8.16 (5), p. 75]

$$\mathcal{L}_X(K) = [X, K]^{FN},$$

that is, the Frölicher–Nijenhuis bracket with a vector field coincides with the Lie derivative of the tensor field $K \in \Omega^*(M, TM)$ which means that $\exp(tX) : \Omega^*(M, TM) \to \Omega^*(M, TM)$ is the action induced by (local) diffeomorphisms of M.

Example 2.1 Let $A \in \Omega^1(M, TM)$ be an endomorphism field on M. Then, [12, Remark 8.17, p. 75]

$$[A, A]^{FN} = 2[A, A]_N,$$

where $[A, A]_N$ is the Nijenhuis tensor of A. W.r.t. a local frame (e_i) with dual frame (e^i) we can write $A = e^i \otimes Ae_i$, whence

$$\begin{aligned}
\mathcal{L}_A \alpha^k &\overset{(2.1)}{=} e^i \wedge (\iota_{Ae_i} d\alpha^k) - d(e^i \wedge (\iota_{Ae_i} \alpha^k)) \\
&= A \cdot d\alpha^k - d(A \cdot \alpha^k), \qquad (2.4)
\end{aligned}$$

where we denote by $\cdot$ the pointwise action of $A_p \in End(T_pM)$ on $\Lambda^k T_p^* M$. Observe that by (2.2) we have $\mathcal{L}_{[A,A]^{FN}} = 2(\mathcal{L}_A)^2$, so that the derivation $\mathcal{L}_A : \Omega^k(M) \to \Omega^{k+1}(M)$ is a differential iff $[A, A]_N = 0$.

For instance, if $A = Id$ then $I \cdot \alpha^k = e^i \wedge (\iota_{e_i}\alpha^k) = k\alpha^k$, so that

$$\mathcal{L}_I \alpha^k \overset{(2.4)}{=} I \cdot d\alpha^k - d(I \cdot \alpha^k) = (k+1)d\alpha^k - d(k\alpha^k) = d\alpha^k.$$

To see another example, let $A = J$ be an almost complex structure. Then, $[J, J]^{FN} = 2[J, J]_N = 0$ iff J is integrable, and in this case one calculates from (2.4) that $\mathcal{L}_J = -d^c = $

$i(\partial - \bar{\partial})$ is the negative of the complex differential, where $d = \partial + \bar{\partial}$ is the decomposition into the holomorphic and anti-holomorphic part of d. In particular, $H_J^*(\Omega^*(M)) \cong H_{dR}^*(M)$ coincides with the deRham cohomology.

We end this section by providing a formula for the Frölicher–Nijenhuis bracket for those types of forms which we shall be concerned with. Recall from the introduction that on a Riemannian manifold (M, g) we define the map

$$\delta = \delta_g : \Lambda^k V^* \longrightarrow \Lambda^{k-1} V^* \otimes V, \qquad \delta_g(\alpha^k) := (\iota_{e_i} \alpha^k) \otimes (e^i)^\#, \qquad (2.5)$$

taking the sum over some basis (e_i) of $T_p M$ with dual basis (e^i) of $T_p^* M$. This implies that to each $\Psi \in \Omega^{k+1}(M)$ we may associate a section $\delta_g(\Psi) \in \Omega^k(M, TM)$.

Proposition 2.2 *Let (M, g) be an n-dimensional Riemannian manifold of dimension n and let $\Psi_l \in \Omega^{k_l+1}(M)$, $l = 1, 2$. Moreover, let*

$$K_l := \delta_g(\Psi_l) \in \Omega^{k_l}(M, TM)$$

with the map δ_g from (2.5).

Then, the Frölicher–Nijenhuis bracket at $p \in M$ is given as

$$[K_1, K_2]_p^{FN} = \Big((\iota_{e_i} \Psi_1) \wedge (\iota_{e_j} \nabla_{e_i} \Psi_2) - (-1)^{k_1} (\iota_{e_j} \iota_{e_i} \Psi_1) \wedge e^k \wedge \iota_{e_i} \nabla_{e_k} \Psi_2$$

$$-(\iota_{e_j} \nabla_{e_i} \Psi_1) \wedge (\iota_{e_i} \Psi_2) - (-1)^{k_1} e^k \wedge \iota_{e_i} \nabla_{e_k} \Psi_1 \wedge (\iota_{e_j} \iota_{e_i} \Psi_2) \Big) \otimes (e^j)^\#,$$

where (e_i) is an arbitrary basis of $T_p M$ with dual basis (e^i) of $T_p^* M$. In particular, if $K_1 = K_2 =: K$ and $k_1 = k_2$ is odd, then

$$[K, K]_p^{FN} = 2 \Big((\iota_{e_i} \Psi) \wedge (\iota_{e_j} \nabla_{e_i} \Psi) + (\iota_{e_j} \iota_{e_i} \Psi) \wedge e^k \wedge \iota_{e_i} \nabla_{e_k} \Psi \Big) \otimes (e^j)^\#.$$

Remark 2.3 If $K_1 = K_2 = K$ and $k_1 = k_2$ is even, then $[K, K]^{FN} = 0$ due to the graded skew symmetry of the bracket. Furthermore, observe that $(e^j)^\# = e_j$ in case (e_i) is an orthonormal basis.

Proof Evidently, if this formula holds for *some* basis (e_j) with dual basis (e^j), then it holds for *any* basis. Therefore, it suffices to show the assertion for an orthonormal basis (e_j) in which case $(e^j)^\# = e_j$.

Choose geodesic normal coordinates (x^i) around $p \in M$ in such a way that $(\partial_i)_p := (\partial/\partial x^i)_p$ is an orthonormal basis of $T_p M$. The dual basis of ∂_i is dx^i, whence $(dx^i)^\# = g^{ij} \partial_j$. Thus,

$$K_l = (\iota_{\partial_i} \Psi_l) \otimes (dx^i)^\# = g^{ij} (\iota_{\partial_i} \Psi_l) \otimes \partial_j.$$

Thus, by (2.3)

$$[K_1, K_2]^{FN} = [g^{ij} (\iota_{\partial_i} \Psi_1) \otimes \partial_j, \, g^{rs} (\iota_{\partial_r} \Psi_2) \otimes \partial_s]^{FN}$$

$$= (g^{ij} (\iota_{\partial_i} \Psi_1) \wedge \mathcal{L}_{\partial_j} (g^{rs} (\iota_{\partial_r} \Psi_2)) \otimes \partial_s$$

$$- \mathcal{L}_{\partial_s} (g^{ij} (\iota_{\partial_i} \Psi_1)) \wedge g^{rs} (\iota_{\partial_r} \Psi_2) \otimes \partial_j$$

$$+ (-1)^{k_1} d(g^{ij} (\iota_{\partial_i} \Psi_1)) \wedge \iota_{\partial_j} (g^{rs} (\iota_{\partial_r} \Psi_2)) \otimes \partial_s$$

$$+ (-1)^{k_1} (\iota_{\partial_s} (g^{ij} (\iota_{\partial_i} \Psi_1)) \wedge d(g^{rs} (\iota_{\partial_r} \Psi_2)) \otimes \partial_j.$$

Since at p, $g_{ij} = g^{ij} = \delta_{ij}$, $\partial_r g_{ij} = 0$, $\mathcal{L}_{\partial_j} \Psi = \nabla_{e_j} \Psi$, $\nabla_{\partial_i} \partial_j = 0$, and $\partial_j = (e^j)^\#$, the asserted formula follows. $\square$

3 Cross products and G_2-structures

3.1 G_2-structures and associated cross products

In this section, we collect some basic facts on G_2-structures, see e.g., [2,5,8,9] for references.

Let M be an oriented 7-manifold. A G_2-*structure on M* is a 3-form $\varphi \in \Omega^3(M)$ such that at each $p \in M$ there is a positively oriented basis (e_i) of $T_p M$ with dual basis (e^i) such that

$$\varphi_p = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \tag{3.1}$$

where $e^{i_1 \cdots i_k}$ is short for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. We call such a basis a G_2-*frame*. The stabilizer of φ_p is isomorphic to the exceptional group G_2, and there is a unique G_2-invariant Riemannian metric g_φ on M such that each G_2-frame is orthonormal. In particular, the Hodge-dual of φ w.r.t. g_φ is given by

$$*_{g_\varphi} \varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{3.2}$$

The set of G_2-frames yields a principal G_2-bundle

$$G_2(M) = G_2(M, \varphi) \longrightarrow M,$$

whence for each G_2-module V we denote by

$$V(M) := G_2(M) \times_{G_2} V \longrightarrow M \tag{3.3}$$

the associated vector bundle over M. For instance, if V_7 denotes the irreducible 7-dimensional G_2-module, then

$$V_7(M) \cong TM \cong T^*M.$$

Definition 3.1 [8] Let (M, φ) be an oriented manifold with a G_2-structure φ and the induced Riemannian metric $g = g_\varphi$. Then, the TM-valued forms $Cr_\varphi \in \Omega^2(M, TM)$ and $\chi_\varphi \in \Omega^3(M, TM)$ are defined by

$$Cr_\varphi := \delta_{g_\varphi}(\varphi) \quad \text{and} \quad \chi_\varphi := -\delta_g(*\varphi),$$

and are called the 2-*fold* and 3-*fold cross product on M*, respectively. That is, for $x, y, z, w \in TM$ we have

$$g_\varphi(Cr_\varphi(x, y), z) = \varphi(x, y, z), \quad g_\varphi(\chi_\varphi(x, y, z), w) = *\varphi(x, y, z, w).$$

We shall usually suppress the indices φ for g, Cr and χ if it is clear from the context which G_2-structure φ is used.

Remark 3.2 If we use the G_2-structure to identify each $T_p M \cong Im\mathbb{O}$ with the imaginary octonians, then Cr and χ can be interpreted w.r.t. the octonian product $\cdot : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ as

$$Cr(x, y) := (x \cdot y)_{Im\mathbb{O}} \quad \text{and} \quad \chi(x, y, z) := ((x \cdot y) \cdot z - x \cdot (y \cdot z))_{Im\mathbb{O}}.$$

We summarize important known facts about the decomposition of tensor products of G_2-modules into irreducible summands which are well known, see e.g., [10, Section 2]. We denote by V_k the k-dimensional irreducible G_2-module if there is a unique such module. For instance, V_7 is the irreducible 7-dimensional G_2-module from above, and $V_7^* \cong V_7$. For its exterior powers, we obtain the decompositions

$$\begin{aligned}
\Lambda^0 V_7 &\cong \Lambda^7 V_7 \cong V_1, \quad \Lambda^2 V_7 \cong \Lambda^5 V_7 \cong V_7 \oplus V_{14}, \\
\Lambda^1 V_7 &\cong \Lambda^6 V_7 \cong V_7, \quad \Lambda^3 V_7 \cong \Lambda^4 V_7 \cong V_1 \oplus V_7 \oplus V_{27},
\end{aligned} \tag{3.4}$$

where $\Lambda^k V_7 \cong \Lambda^{7-k} V_7$ due to the Hodge isomorphism. We denote by $\Lambda_l^k V_7 \subset \Lambda^k V_7$ the subspace isomorphic to V_l in the above notation. Evidently, $\Lambda_1^3 V_7$ and $\Lambda_1^4 V_7$ are spanned by φ and $*\varphi$, respectively. For the decompositions of $\Lambda^2 V_7$ and $\Lambda^5 V_7$ the following descriptions are well known.

$$\Lambda_7^2 V_7 = \{\iota_v \varphi \mid v \in V_7\},$$
$$\Lambda_{14}^2 V_7 = \{\alpha^2 \in \Lambda^2 V_7 \mid *\varphi \wedge \alpha^2 = 0\},$$
$$\Lambda_1^3 V_7 = \mathbb{R}\varphi,$$
$$\Lambda_7^3 V_7 = \{\iota_v * \varphi \mid v \in V_7\},$$
$$\Lambda_1^4 V_7 = \mathbb{R} * \varphi,$$
$$\Lambda_7^4 V_7 = \varphi \wedge V_7 = \{\varphi \wedge v \mid v \in V_7\},$$
$$\Lambda_7^5 V_7 = *\varphi \wedge V_7 = \{*\varphi \wedge v \mid v \in V_7\},$$
$$\Lambda_{14}^5 V_7 = \{\alpha^5 \in \Lambda^5 V_7 \mid \alpha^5 \wedge (\iota_v \varphi) = 0 \text{ for all } v \in V_7\}. \tag{3.5}$$

We also point out that all representations of G_2 are of real type, meaning that for any real irreducible representation V of G_2 the complexified space $V^{\mathbb{C}} := V \otimes \mathbb{C}$ is (complex) irreducible; equivalently, a real irreducible representation of G_2 does not admit a G_2-invariant complex structure.

These decompositions are used in appendix to obtain many formulas which will be used in the sequel.

3.2 The torsion of manifolds with a G_2-structure

Let (M, φ) be a manifold with a G_2-structure with the corresponding Riemannian metric $g = g_\varphi$, and let ∇ be the Levi-Civita connection of g. In general, φ and $*\varphi$ will not be parallel w.r.t. ∇, and the failure of their parallelity can be described in the following way which is essentially a reformulation of the intrinsic torsion of a G_2-structure discussed in [3, 10].

Proposition 3.3 *Let (M, φ) be a manifold with a G_2-structure with associated Riemannian metric $g = g_\varphi$ and Levi-Civita connection ∇. Then, there is a section $T \in \Omega^1(M, TM) = \Gamma(\mathrm{End}(TM))$ such that for all $v \in TM$ we have*

$$\nabla_v \varphi = \iota_{T(v)} * \varphi \quad \text{and} \quad \nabla_v * \varphi = -(T(v))^\flat \wedge \varphi. \tag{3.6}$$

Thus, the section $T \in \Omega^1(M, TM)$ measures how φ fails to be parallel, and this has been described in Fernández and Gray [5] by slightly different means. In fact, it contains the same information as the intrinsic torsion of the G_2-structure in the sense of [3], whence we use the following terminology.

Definition 3.4 Let (M, φ) be a manifold with a G_2-structure. The section $T \in \Omega^1(M, TM)$ for which (3.6) holds is called the *torsion endomorphism* of the G_2-structure.

For an orthonormal frame (e_i) of $T_p M$ we define the coefficients of T by

$$t_{ij} := \langle T(e_i), e_j \rangle, \quad \text{so that} \quad T(e_i) = t_{ij} e_j. \tag{3.7}$$

Furthermore, we define the form

$$\tau := t_{ij} e^{ij} = \frac{1}{2}(t_{ij} - t_{ji}) e^{ij} = e^i \wedge T(e_i)^\flat \in \Lambda^2 V_7^*. \tag{3.8}$$

For the exterior derivatives of φ and $*\varphi$, we have

$$d\varphi_p = T_p^\top (e_i)^\flat \wedge (\iota_{e_i} * \varphi_p) \quad \text{and} \quad d * \varphi_p = -\tau_p \wedge \varphi_p, \tag{3.9}$$

where we sum over an orthonormal basis (e_i) of T_pM in the first equation and where $T_p^\top$ denotes the transpose matrix of T_p. In particular, it is now a straightforward calculation to show that (M, φ) is torsion-free at $p \in M$ (i.e., $T_p = 0$) iff $d\varphi_p = 0$ and $d * \varphi_p = 0$ (cf. [5]).

3.3 The Frölicher–Nijenhuis brackets on a manifold with a G_2-structure

In this section, we shall compute part of the Frölicher–Nijenhuis brackets of the sections $Cr = \delta_g \varphi \in \Omega^2(M, TM)$ and $\chi = -\delta_g * \varphi \in \Omega^3(M, TM)$ from Definition 3.1.

The Frölicher–Nijenhuis bracket $[Cr, Cr]^{FN}$ vanishes identically due to the graded skew symmetry of the bracket. On the other hand, the Frölicher–Nijenhuis brackets $[Cr, \chi]^{FN}$ and $[\chi, \chi]^{FN}$ are elements of $\Omega^5(M, TM)$ and $\Omega^6(M, TM)$, respectively.

Due to the decomposition $\Lambda^5 V_7 = \Lambda_7^5 V_7 \oplus \Lambda_{14}^5 V_7$ as a G_2-module, we may decompose

$$\Omega^5(M, TM) = \Gamma(M, \Lambda_7^5 T^*M \otimes TM) \oplus \Gamma(M, \Lambda_{14}^5 T^*M \otimes TM),$$

and we denote the projections onto the two summands by π_7 and π_{14}, respectively. We now wish to show Theorem 1.1 from the introduction. In order to work toward the proof, we first calculate $\pi_7([Cr, \chi]^{FN})$.

Proposition 3.5 *Let (M, g, φ) be a manifold with a G_2-structure and let $T \in \Omega^1(M, TM)$ be its torsion endomorphism. Then, for each $p \in M$,*

$$\pi_7\left([Cr, \chi]_p^{FN}\right) = 2 * \varphi \wedge \left(\left(T_p^\top - 2T_p - tr(T_p)\right)e_i\right)^\flat \otimes e_i, \tag{3.10}$$

summing over an orthonormal basis (e_i) of T_pM, where $T^\top$ denotes the transpose of T. In particular, $\pi_7([Cr, \chi]_p^{FN}) = 0$ if and only if $T_p = 0$ if and only if $[Cr, \chi]_p^{FN} = 0$.

Proof We fix $p \in M$ and use normal coordinates around p. Then, in order to calculate $[Cr, \chi]_p^{FN}$ we apply Proposition 2.2 to $\Psi_1 = \varphi$ and $\Psi_2 = *\varphi$ and obtain

$$- [Cr, \chi]_p^{FN} = [(\iota_{e_i}\varphi) \otimes e_i, (\iota_{e_j} * \varphi) \otimes e_j]_p^{FN} =: \beta_j \otimes e_j,$$

where

$$\beta_j = (\iota_{e_k}\varphi) \wedge (\iota_{e_j} \nabla_{e_k} * \varphi) - (\iota_{e_j} \iota_{e_k}\varphi) \wedge e^l \wedge (\iota_{e_k} \nabla_{e_l} * \varphi)$$
$$- (\iota_{e_j} \nabla_{e_k}\varphi) \wedge (\iota_{e_k} * \varphi) - e^l \wedge (\iota_{e_k} \nabla_{e_l}\varphi) \wedge (\iota_{e_j} \iota_{e_k} * \varphi). \tag{3.11}$$

Decomposing $\Lambda^5 T_p^*M$ according to (3.5), we write $\beta_j = *\varphi \wedge v_j^\flat + \beta_j^{14}$ with $v_j \in T_pM$ and $\beta_j^{14} \in \Lambda_{14}^5 T_p^*M$, so that $\pi_7(-[Cr, \chi]^{FN}) = *\varphi \wedge v_j^\flat \otimes e_j$. Let

$$b_{ij} = *((\iota_{e_i}\varphi) \wedge \beta_j). \tag{3.12}$$

Then, as $(\iota_{e_i}\varphi) \wedge \beta_j^{14} = 0$ by (3.5) and $(\iota_{e_i}\varphi) \wedge *\varphi \wedge v_j^\flat = 3\langle e_i, v_j \rangle \text{vol}$ by (6.1), it follows that

$$\pi_7\left(-[Cr, \chi]_p^{FN}\right) = \frac{1}{3} b_{ij}\left(*\varphi \wedge e^i\right) \otimes e_j. \tag{3.13}$$

In order to determine the coefficients b_{ij}, we decompose β_j into the four summands from (3.11). Then, from the first summand, we get

$$(\iota_{e_i}\varphi) \wedge (\iota_{e_k}\varphi) \wedge (\iota_{e_j}\nabla_{e_k} * \varphi) \overset{(3.6)}{=} -(\iota_{e_i}\varphi) \wedge (\iota_{e_k}\varphi) \wedge (\iota_{e_j}(T(e_k)^\flat \wedge \varphi))$$
$$= -(\iota_{e_i}\varphi) \wedge (\iota_{e_k}\varphi) \wedge (t_{kj}\varphi - T(e_k)^\flat \wedge (\iota_{e_j}\varphi))$$
$$\overset{(6.3),(6.4)}{=} -6t_{kj}\delta_{ik}\text{vol} + 2(\delta_{ik}t_{kj} + t_{ki}\delta_{kj} + t_{kk}\delta_{ij})\text{vol}$$
$$= 2(t_{ji} - 2t_{ij} + tr(T)\delta_{ij})\text{vol}. \tag{3.14}$$

From the second summand, we obtain

$$-(\iota_{e_i}\varphi) \wedge (\iota_{e_j}\iota_{e_k}\varphi) \wedge e^l \wedge (\iota_{e_k}\nabla_{e_l} * \varphi)$$
$$\overset{(3.6)}{=} (\iota_{e_i}\varphi) \wedge (\iota_{e_j}\iota_{e_k}\varphi) \wedge e^l \wedge (\iota_{e_k}(T(e_l)^\flat \wedge \varphi))$$
$$= (\iota_{e_i}\varphi) \wedge (\iota_{e_j}\iota_{e_k}\varphi) \wedge e^l \wedge (t_{lk}\varphi - T(e_l)^\flat \wedge (\iota_{e_k}\varphi))$$
$$\overset{(6.6)}{=} t_{lk}\Big(2(\delta_{kl}\delta_{ji} - \delta_{jl}\delta_{ki})\text{vol} - 2e^{jkli} \wedge \varphi\Big)$$
$$\quad - (\iota_{e_i}\varphi) \wedge (\iota_{e_j}\iota_{e_k}\varphi) \wedge e^l \wedge T(e_l)^\flat \wedge (\iota_{e_k}\varphi)$$
$$= 2(tr(T)\delta_{ij} - t_{ji})\text{vol} - 2e^{ij} \wedge \tau \wedge \varphi$$
$$\quad - \frac{1}{2}(\iota_{e_i}\varphi) \wedge \iota_{e_j}((\iota_{e_k}\varphi) \wedge (\iota_{e_k}\varphi)) \wedge \tau$$
$$\overset{(6.12)}{=} 2(tr(T)\delta_{ij} - t_{ji})\text{vol} - 2e^{ij} \wedge \tau \wedge \varphi$$
$$\quad - 3(\iota_{e_i}\varphi) \wedge (\iota_{e_j} * \varphi) \wedge \tau$$
$$\overset{(6.10)}{=} 2(tr(T)\delta_{ij} - t_{ji})\text{vol} - 2e^{ij} \wedge \tau \wedge \varphi$$
$$\quad - 3(-2 * e^{ji} + e^{ji} \wedge \varphi) \wedge \tau$$
$$= 2(tr(T)\delta_{ij} - t_{ji})\text{vol} + e^{ij} \wedge \tau \wedge \varphi$$
$$\quad + 6t_{kl}(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik})\text{vol}$$
$$= 2(tr(T)\delta_{ij} + 2t_{ji} - 3t_{ij})\text{vol} + e^{ij} \wedge \tau \wedge \varphi. \tag{3.15}$$

From the third term in (3.11), we get

$$-(\iota_{e_i}\varphi) \wedge (\iota_{e_j}\nabla_{e_k}\varphi) \wedge (\iota_{e_k} * \varphi)$$
$$\overset{(3.6)}{=} -(\iota_{e_i}\varphi) \wedge (\iota_{e_j}\iota_{T(e_k)} * \varphi) \wedge (\iota_{e_k} * \varphi)$$
$$\overset{(6.7)}{=} -2(t_{kk}\delta_{ji} - \delta_{jk}t_{ki})\text{vol} - e^j \wedge T(e_k)^\flat \wedge e^{ki} \wedge \varphi$$
$$= -2(tr(T)\delta_{ij} - t_{ji})\text{vol} - e^{ij} \wedge \tau \wedge \varphi. \tag{3.16}$$

Finally, from the last term in (3.11), we get

$$-(\iota_{e_i}\varphi) \wedge e^l \wedge (\iota_{e_k}\nabla_{e_l}\varphi) \wedge (\iota_{e_j}\iota_{e_k} * \varphi)$$
$$\overset{(3.6)}{=} -(\iota_{e_i}\varphi) \wedge e^l \wedge (\iota_{e_k}\iota_{T(e_l)} * \varphi) \wedge (\iota_{e_j}\iota_{e_k} * \varphi)$$

$$\overset{(6.13)}{=} 2(\iota_{e_i}\varphi) \wedge e^l \wedge (\iota_{T(e_l)}\varphi) \wedge (\iota_{e_j}\varphi)$$

$$\overset{(6.4)}{=} 4(t_{li}\delta_{lj} + \delta_{li}t_{lj} + t_{ll}\delta_{ij})\text{vol}$$

$$= 4(t_{ji} + t_{ij} + tr(T)\delta_{ij})\text{vol}. \tag{3.17}$$

Thus, adding (3.14) through (3.17), we get from (3.11) that

$$
\begin{aligned}
b_{ij}\text{vol} = (\iota_{e_i}\varphi) \wedge \beta_j &= 2(t_{ji} - 2t_{ij} + tr(T)\delta_{ij})\text{vol} \\
&\quad + 2(tr(T)\delta_{ij} + 2t_{ji} - 3t_{ij})\text{vol} + e^{ij} \wedge \tau \wedge \varphi \\
&\quad - 2(tr(T)\delta_{ij} - t_{ji})\text{vol} - e^{ij} \wedge \tau \wedge \varphi \\
&\quad + 4(t_{ji} + t_{ij} + tr(T)\delta_{ij})\text{vol} \\
&= 6(2t_{ji} - t_{ij} + tr(T)\delta_{ij})\text{vol},
\end{aligned}
$$

and hence, (3.13) implies (3.10).

Thus, $\pi_7([Cr, \chi]_p^{FN}) = 0$ iff $T_p^\top - 2T_p - tr(T_p)Id = 0$. Taking the trace, this implies that $tr(T_p) - 2tr(T_p) - 7tr(T_p) = 0$ and hence, $tr(T_p) = 0$, and $T_p^\top - 2T_p = 0$ evidently implies that $T_p = 0$. That is, $\pi_7([Cr, \chi]_p^{FN}) = 0$ iff $T_p = 0$, showing the last statement. $\quad\square$

Next, let us consider the bracket $[\chi, \chi]^{FN}$.

Proposition 3.6 *Let (M, g, φ) be a manifold with a G_2-structure and let $T \in \Omega^1(M, TM)$ be its torsion endomorphism with the associated form $\tau \in \Omega^2(M)$ from (3.8). Then for each $p \in M$,*

$$[\chi, \chi]_p^{FN} = -4 * (T_p + T_p^\top)(e_i) \otimes e_i + 6e^i \wedge \tau_p \wedge \varphi \otimes e_i, \tag{3.18}$$

summing over an orthonormal basis (e_i) of T_pM. In particular, $[\chi, \chi]_p^{FN} = 0$ if and only if $T_p = 0$.

Proof According to Proposition 2.2, we have

$$[\chi, \chi]_p^{FN} = \gamma_j \otimes e_j,$$

where

$$\gamma_j = 2((\iota_{e_k} * \varphi) \wedge (\iota_{e_j} \nabla_{e_k} * \varphi) + (\iota_{e_j}\iota_{e_k} * \varphi) \wedge e^l \wedge (\iota_{e_k} \nabla_{e_l} * \varphi)). \tag{3.19}$$

Now let $c_{ij} := *(e^i \wedge \gamma_j) = \langle e_i, (*\gamma_j)^\flat \rangle$. Then,

$$[\chi, \chi]_p^{FN} = c_{ij} * e^i \otimes e_j, \tag{3.20}$$

In order to evaluate the coefficients c_{ij}, we consider the two summands in (3.19) separately, and obtain from the first one

$$
\begin{aligned}
e^i \wedge (\iota_{e_k} * \varphi) \wedge (\iota_{e_j} \nabla_{e_k} * \varphi) &\overset{(3.6)}{=} -e^i \wedge (\iota_{e_k} * \varphi) \wedge (\iota_{e_j}(T(e_k)^\flat \wedge \varphi)) \\
&= -t_{kj}e^i \wedge (\iota_{e_k} * \varphi) \wedge \varphi + e^i \wedge (\iota_{e_k} * \varphi) \wedge T(e_k)^\flat \wedge (\iota_{e_j}\varphi) \\
&\overset{(6.2),(6.8)}{=} -4t_{kj}\delta_{ik}\text{vol} + 2(\delta_{ik}t_{kj} - t_{kk}\delta_{ij})\text{vol} + T(e_k)^\flat \wedge e^{ikj} \wedge \varphi \\
&= -2(t_{ij} + tr(T)\delta_{ij})\text{vol} + e^{ij} \wedge \tau \wedge \varphi. \tag{3.21}
\end{aligned}
$$

From the second summand in (3.19), we calculate

$$e^i \wedge (\iota_{e_j} \iota_{e_k} * \varphi) \wedge e^l \wedge (\iota_{e_k} \nabla_{e_l} * \varphi)$$

$$\overset{(3.6)}{=} -e^{il} \wedge (\iota_{e_j} \iota_{e_k} * \varphi) \wedge (\iota_{e_k}(T(e_l)^\flat \wedge \varphi))$$

$$= -t_{lk} e^{il} \wedge (\iota_{e_j} \iota_{e_k} * \varphi) \wedge \varphi + e^i \wedge (\iota_{e_j} \iota_{e_k} * \varphi) \wedge \tau \wedge (\iota_{e_k} \varphi)$$

$$\overset{(6.9),(6.14)}{=} -t_{lk} \big(2(\delta_{lj}\delta_{ik} - \delta_{ij}\delta_{lk})\mathrm{vol} - e^{iljk} \wedge \varphi \big) + 3e^{ij} \wedge \tau \wedge \varphi$$

$$= -2(t_{ji} - tr(T)\delta_{ij})\mathrm{vol} + 2e^{ij} \wedge \tau \wedge \varphi. \tag{3.22}$$

Thus, adding (3.21) and (3.22), Eq. (3.19) yields

$$c_{ij}\mathrm{vol} = e^i \wedge \gamma_j = 2\Big(-2(t_{ij} + tr(T)\delta_{ij})\mathrm{vol} + e^{ij} \wedge \tau \wedge \varphi$$

$$-2(t_{ji} - tr(T)\delta_{ij})\mathrm{vol} + 2e^{ij} \wedge \tau \wedge \varphi \Big)$$

$$= 2(-2(t_{ij} + t_{ji}) + 3\langle *e^i, e^j \wedge \tau \wedge \varphi \rangle)\mathrm{vol},$$

and from this and (3.20), the formula (3.18) follows.

In order to show the last statement, observe that $[\chi, \chi]_p^{FN} = 0$ iff $c_{ij} = 0$ for all i, j. Since then $c_{ij} + c_{ji} = -8(t_{ij} + t_{ji})$, it follows that $t_{ij} + t_{ji} = 0$ and hence $0 = c_{ij} = 6*(e^{ij} \wedge \tau \wedge \varphi)$ for all i, j which implies that $\tau = t_{kl}e^{kl} = 0$ and hence, $t_{kl} = t_{lk}$. All of this together implies that $t_{ij} = 0$ for all i, j, and hence, $T_p = 0$ as asserted. □

We are now ready to show the following result which immediately implies Theorem 1.1 from the introduction.

Theorem 3.7 *Let (M^7, φ) be a manifold with a G_2-structure with associated metric $g = g_\varphi$, let ∇ be the Levi-Civita connection of g, and let $T \in \Omega^1(M^7, TM^7)$ be its torsion endomorphism defined in Definition 3.4. Then, for every $p \in M^7$ the following are equivalent.*

(1) $T_p = 0 \in T_p^* M^7 \otimes T_p M^7$.
(2) The G_2-structure is torsion-free at p, i.e., $(\nabla\varphi)_p = 0$.
(3) $\pi_7([Cr, \chi]_p^{FN}) = 0 \in \Lambda_7^5 T_p^* M^7 \otimes T_p M^7$.
(4) $[Cr, \chi]_p^{FN} = 0 \in \Lambda^5 T_p^* M^7 \otimes T_p M^7$.
(5) $[\chi, \chi]_p^{FN} = 0 \in \Lambda^6 T_p^* M^7 \otimes T_p M^7$.

Proof The equivalence of the first two statements is well known, see e.g., [5]. Proposition 3.5 shows the equivalence of the first and the third, whereas Proposition 3.6 shows the equivalence of the first and the last statement. That $(\nabla\varphi)_p = 0$ implies $[Cr, \chi]_p^{FN} = 0$ is immediate from the formula of the bracket in Proposition 2.2, and obviously, $[Cr, \chi]_p^{FN} = 0$ implies $\pi_7([Cr, \chi]_p^{FN}) = 0$. □

4 Cross products and Spin(7)-structures

4.1 Spin(7)-structures and associated cross products

The exposition in this section mainly follows the references [2,4,8].

Let M be an oriented 8-manifold. A Spin(7)-*structure on M is a 4-form $\Phi \in \Omega^4(M)$ such that at each $p \in M$ there is a positively oriented basis $(e_\mu)_{\mu=0}^7$ of $T_p M$ with dual basis

$(e^\mu)_{\mu=0}^7$ such that $\Phi_p \in \Lambda^4 T_p M$ is of the form

$$\Phi_p := e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \qquad (4.1)$$
$$+ e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

Throughout this section, we shall use Greek indices $\mu, \nu, \ldots$ to run over $0, \ldots, 7$, whereas Latin indices $i, j, \ldots$ range over $1, \ldots, 7$.

A basis (e_μ) of $T_p M$ whose dual basis (e^μ) satisfies (4.1) is called a Spin(7)-*frame*. Observe that if we define for a Spin(7)-frame (e_μ) the forms φ_p and $*_7 \varphi_p$ on $V_p :=$ span$(e_i)_{i=1}^7 \subset T_p M$ as in (3.1) and (3.2), then

$$\Phi_p = e^0 \wedge \varphi_p + *_7 \varphi_p.$$

The stabilizer of Φ_p is the group Spin(7) acting on $T_p M$ via the spinor representation, and there is a unique Spin(7)-invariant Riemannian metric g_Φ on M such that each Spin(7)-frame is orthonormal. In particular, Φ is self-dual w.r.t. g_Φ. The set of all Spin(7)-frames forms a principal Spin(7)-bundle

$$\text{Spin}(7)_M = \text{Spin}(7)_{(M,\Phi)} \longrightarrow M,$$

and again, for each Spin(7)-module W we obtain the associated vector bundle

$$W(M) := \text{Spin}(7)_M \times_{\text{Spin}(7)} W \longrightarrow M. \qquad (4.2)$$

For instance, if we denote the k-dimensional irreducible Spin(7)-module by W_k (in case the dimension uniquely specifies this module), then

$$W_8(M) \cong TM \cong T^*M.$$

It is well known that the action of Spin(7) on W_8 is transitive on the unit sphere $S^7 \subset W_8$, and the stabilizer of an element is isomorphic to $G_2 \subset \text{Spin}(7)$. In analogy of the products Cr and χ on manifolds with a G_2-structure in Definition 3.1, we define on a Spin(7)-manifold M a triple product as follows.

Definition 4.1 Let (M, Φ) be manifold with a Spin(7)-structure, and let $g = g_\Phi$ be the induced Riemannian metric. Then, the TM-valued form $P = P_\Phi \in \Omega^3(M, TM)$ is defined by

$$P_\Phi := -\delta_{g_\Phi}(\Phi),$$

and is called the 3-*fold cross product on* M. That is, for $x, y, z, w \in TM$ we have

$$g(P(x, y, z), w) = \Phi(x, y, z, w). \qquad (4.3)$$

We shall usually suppress the indices Φ for g and P if it is clear from the context which Spin(7)-structure Φ is used.

4.2 Spin(7)-representations

In this section, we shall discuss the decomposition of symmetric and anti-symmetric powers of W_8 as Spin(7)-modules. For its exterior powers, we obtain the decompositions

$$\Lambda^0 W_8 \cong \Lambda^8 W_8 \cong W_1, \qquad\qquad \Lambda^2 W_8 \cong \Lambda^6 W_8 \cong W_7 \oplus W_{21},$$
$$\Lambda^1 W_8 \cong \Lambda^7 W_8 \cong W_8, \qquad\qquad \Lambda^3 W_8 \cong \Lambda^5 W_8 \cong W_8 \oplus W_{48},$$

$$\Lambda^4 W_8 \cong W_1 \oplus W_7 \oplus W_{27} \oplus W_{35} \tag{4.4}$$

where $\Lambda^k W_8 \cong \Lambda^{8-k} W_8$ via the Hodge-$*$. Again, we denote by $\Lambda_l^k W_8 \subset \Lambda^k W_8$ the subspace isomorphic to W_l in the above notation.

Moreover, there are also irreducible decompositions of the symmetric powers of W_7 and W_8 as

$$\odot^2 W_7 \cong W_1 \oplus W_{27}, \qquad \odot^2 W_8 \cong W_1 \oplus W_{35} \tag{4.5}$$

into the induced Spin(7)-invariant metric and the trace-free symmetric tensors; see [9].

Lemma 4.2 *Let $e^0 \in W_8$ be a unit vector, let $V_7 := e_0^\perp$ on which* Spin(7) *acts as the double cover of $SO(7)$, so that $V_7 \cong W_7$ as a* Spin(7)-*module. Then, the following maps are* Spin(7)-*equivariant embeddings.*

$$\lambda^k : W_7 \longrightarrow \Lambda^k W_8, \qquad \begin{aligned} \lambda^2(v) &:= e^0 \wedge v^\flat + (\iota_v \varphi) \\ \lambda^4(v) &:= e^0 \wedge (\iota_v *_7 \varphi) - v^\flat \wedge \varphi \\ \lambda^6(v) &:= \Phi \wedge \lambda^2(v) = 3 * \lambda^2(v) \end{aligned} \tag{4.6}$$

Here, $*$ and $*_7$ denote the Hodge-$*$ in W_8 and V_7, respectively.

Proof The decompositions in (4.5) imply that there are Spin(7)-equivariant maps $\lambda^k : W_7 \to \Lambda^k W_8$, and these are unique up to rescaling.

The equivariance of λ^2 follows from [15, p.68], and thus, $\Phi \wedge \lambda^2(v) \in \Lambda_7^6 W_8$, whence $\Phi \wedge \lambda^2(v) = 3 * \lambda^2(v)$ follows from [15, Theorem 9.8]. This shows the statement on λ^6.

By [15, Theorem 9.8], $\Lambda_7^4 W_8$ is the infinitesimal orbit of Φ under the action of $\mathfrak{so}(W_8) \cong \Lambda^2 W_8$. That is,

$$\Lambda_7^4 W_8 = \{(u^\flat \wedge v^\flat) \cdot \Phi \mid u, v \in W_8\} = \{u^\flat \wedge (\iota_v \Phi) - v^\flat \wedge (\iota_u \Phi) \mid u, v \in W_8\}.$$

Setting $u := e_0$ and picking $v \in e_0^\perp \cong W_7$ for a Spin(7)-frame (e_μ), it follows that the image of λ^4 equals $\Lambda_7^4 W_8$, and since λ^4 is evidently G_2-equivariant, it must coincide with the Spin(7)-equivariant map $W_7 \to \Lambda_7^4 W_8$. □

From this lemma, we obtain the following descriptions of the decompositions, which essentially recapitulates [15, Theorem 9.8].

$$\begin{aligned}
\Lambda_7^k W_8 &= \{\lambda^k(v) \mid v \in V_7\} \quad \text{for } k = 2, 4, 6, \\
\Lambda_{21}^2 W_8 &= \{\alpha^2 \in \Lambda^2 W_8 \mid \alpha^2 \wedge \Phi \wedge \lambda^2(v) = 0 \text{ for all } v \in V_7\}, \\
\Lambda_{21}^6 W_8 &= \{\alpha^6 \in \Lambda^6 W_8 \mid \alpha^6 \wedge \lambda^2(v) = 0 \text{ for all } v \in V_7\}, \\
\Lambda_8^3 W_8 &= \{\iota_a \Phi \mid a \in W_8\}, \\
\Lambda_8^5 W_8 &= \{a^\flat \wedge \Phi \mid a \in W_8\}, \\
\Lambda_{48}^3 W_8 &= \{\alpha^3 \in \Lambda^3 W_8 \mid \Phi \wedge \alpha^3 = 0\}, \\
\Lambda_{48}^5 W_8 &= \{\alpha^5 \in \Lambda^3 W_8 \mid \Phi \wedge *\alpha^5 = 0\}, \\
\Lambda_1^4 W_8 &= \mathbb{R}\Phi, \\
\Lambda_{27}^4 W_8 &= \text{span}\{\lambda^2(v) \wedge \lambda^2(w) \mid v, w \in V_7, \langle v, w \rangle = 0\} \\
\Lambda_{35}^4 W_8 &= \{\alpha^4 \mid *\alpha^4 = -\alpha^4\}.
\end{aligned} \tag{4.7}$$

We also recall the decomposition of the tensor product

$$Lin(W_8, W_7) := W_8^* \otimes W_7 = W_8 \oplus W_{48}. \tag{4.8}$$

Here, the summand isomorphic to W_8 is given as

$$\{(\iota_a \lambda^2(e_i)) \otimes e_i \mid a \in W_8\}, \tag{4.9}$$

where the sum is taken over an orthonormal basis (e_i) of $V_7 \cong W_7$. Finally, we define the Spin(7)-invariant tensor $\sigma \in (W_8 \otimes W_7 \otimes W_8 \otimes W_7)^*$ by

$$\sigma(a, u, b, v) := \frac{1}{2} * (a^\flat \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v)). \tag{4.10}$$

Contraction with the inner products on W_7 and W_8 induces a Spin(7)-equivariant map

$$\phi_\sigma : Lin(W_8, W_7) \longrightarrow Lin(W_8, W_7) \tag{4.11}$$

$$\phi_\sigma(A)(a) := \sigma(a, A(e_\mu), e_\mu, e_i) \, e_i.$$

We calculate

$$\begin{aligned}
\lambda^4(u) \wedge \lambda^2(v) &= (e^0 \wedge (\iota_u *_7 \varphi) - u^\flat \wedge \varphi) \wedge (e^0 \wedge v^\flat + \iota_v \varphi) \\
&= e^0 \wedge (\iota_u *_7 \varphi) \wedge (\iota_v \varphi) + e^0 \wedge u^\flat \wedge v^\flat \wedge \varphi - u^\flat \wedge \varphi \wedge (\iota_v \varphi) \\
&\overset{(6.10),(6.11)}{=} 2u^\flat \wedge v^\flat \wedge (e^0 \wedge \varphi - *_7 \varphi) - 2e^0 \wedge (*_7(u^\flat \wedge v^\flat)). \tag{4.12}
\end{aligned}$$

Lemma 4.3 *The map ϕ_σ has eigenvalues -1 and 6 with multiplicity 48 and 8, respectively.*

Proof Observe that the Spin(7)-invariant inner products on W_7 and W_8 induce an inner product on $Lin(W_8, W_7) = W_8^* \otimes W_7$ for which $(e^\mu \otimes e_i)$ is an orthonormal basis whenever (e_μ) is an orthonormal basis of W_8 so that $V_7 = e_0^\perp$ is spanned by (e_i). This induced inner product satisfies

$$\langle \phi_\sigma(e^\mu \otimes e_i), e^\nu \otimes e_j \rangle_{Lin(W_8, W_7)} = \langle \phi_\sigma(e^\mu \otimes e_i)(e_\nu), e_j \rangle_{W_7} = \sigma(e_\nu, e_i, e_\mu, e_j),$$

and since $\sigma(e_\mu, e_i, e_\mu, e_j) = 0$, it follows that the matrix representation of ϕ_σ w.r.t. the basis $(e^\mu \otimes e_i)$ has 0's on the diagonal, whence $tr(\phi_\sigma) = 0$. Furthermore, ϕ_σ is self-adjoint since $\sigma(a, u, b, v) = \sigma(b, v, a, u)$ by (4.10) and (4.12), whence has real eigenvalues.

Decomposing $Lin(W_8, W_7) \overset{(4.8)}{\cong} W_8 \oplus W_{48}$, (4.9) implies that the elements in the summand congruent to W_8 are given by the maps

$$A_a : W_8 \longrightarrow W_7, \qquad A_a(b) := \langle (\iota_a \lambda^2(e_i))^\#, b \rangle e_i$$

for a fixed $a \in W_8$. In order to calculate $\phi_\sigma(A_a)$, observe that Spin(7) acts transitively on the unit sphere, whence we may assume w.l.o.g. that $a = e_0$, so that

$$A_{e_0}(b) = \langle e_i, b \rangle e_i = pr_{e_0^\perp}(b),$$

where $pr_{e_0^\perp} : W_8 \to e_0^\perp = V_7$ is the orthogonal projection. Thus,

$$\begin{aligned}
\phi_\sigma(A_{e_0})(b) &= \sigma(b, A_{e_0}(e_\mu), e_\mu, e_i) \, e_i = \sigma(b, e_j, e_j, e_i) \, e_i \\
&\overset{(4.12)}{=} * \left(b^\flat \wedge e^j \wedge \left(e^j \wedge e^i \wedge (e^0 \wedge \varphi - *_7 \varphi) - e^0 \wedge (*_7 e^{ji}) \right) \right) e_i \\
&= * \left(e^0 \wedge b^\flat \wedge e^j \wedge (*_7 e^{ij}) \right) e_i \\
&= (1 - \delta_{ij}) * (e^0 \wedge b^\flat \wedge *_7 e^i) e_i \\
&= 6 \langle b, e_i \rangle e_i = 6 A_{e_0}(b),
\end{aligned}$$

so that $\phi_\sigma(A_a) = 6A_a$ for all A_a. By Schur's lemma and since ϕ_σ is self-adjoint, $\phi_\sigma|_{W_{48}} = cId_{W_{48}}$ for some $c \in \mathbb{R}$, whence

$$0 = tr(\phi_\sigma) = 6 \dim W_8 + c \dim W_{48},$$

and from this, $c = -1$ and lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

For a manifold with a Spin(7)-structure (M, Φ) and induced metric $g = g_\Phi$, the covariant derivative $g_\Phi \nabla_v \Phi$ w.r.t. the Levi–Civita connection is contained in the infinitesimal orbit of $\mathfrak{so}(T_pM, g_p)$ [2] and hence in $\Lambda^4_7 T^*M$. That is, there is a section $T \in \Omega^1(M, W_7(M)) = \Gamma(Lin(TM, W_7(M)))$ such that

$$\nabla_v \Phi = \lambda^4(T(v)) = e^0 \wedge (\iota_{T(v)} * \varphi) - (T(v))^\flat \wedge \varphi \qquad (4.13)$$

with the map $\lambda^4 : W_7 \rightarrow \Lambda^4_7 T_pM$ from (4.6). In analogy to Definition 3.4, we use the following terminology.

Definition 4.4 Let (M, Φ) be a manifold with a Spin(7)-structure. The section $T \in \Omega^1(M, W_7(M))$ for which (4.13) holds is called the *torsion endomorphism* of the Spin(7)-structure.

4.3 The Frölicher–Nijenhuis brackets on a manifold with a Spin(7)-structure

Recall the section $P = -\delta_g \Phi \in \Omega^3(M, TM)$ on a manifold with a Spin(7)-structure (M, Φ) from Definition 4.1. We wish to relate its Frölicher–Nijenhuis bracket to its torsion. In order to do this, recall that $[P, P]^{FN} \in \Omega^6(M, TM)$.

Due to the decomposition $\Lambda^6 W_8 = \Lambda^6_7 W_8 \oplus \Lambda^6_{21} W_8$ as a Spin(7)-module, we may decompose

$$\Omega^6(M, TM) = \Gamma(M, \Lambda^6_7 T^*M \otimes TM) \oplus \Gamma(M, \Lambda^6_{21} T^*M \otimes TM),$$

and we denote the projections onto the two summands by π_7 and π_{21}, respectively.

Proposition 4.5 *Let (M, Φ) be a manifold with a Spin(7)-structure with the torsion endomorphism $T \in \Omega^1(M, W_7(M))$ from (4.13), and let $P = -\delta_g \Phi \in \Omega^3(M, TM)$ be as before. Then, for $p \in M$,*

$$\pi_7([P, P]^{FN}_p) = -\frac{2}{3} \Phi \wedge \lambda^2 \Big((4T_p + \phi_\sigma(T_p))(e_\mu) \Big) \otimes e_\mu, \qquad (4.14)$$

summing over an orthonormal basis (e_μ) of T_pM.

Proof By Proposition 2.2, $[P, P]^{FN}_p = \gamma_\mu \otimes e_\mu$, where $\gamma_\mu \in \Lambda^6 T^*_pM$ is given by

$$\gamma_\mu = 2((\iota_{e_v} \Phi) \wedge \iota_{e_\mu}(\nabla_{e_v} \Phi) + (\iota_{e_\mu} \iota_{e_v} \Phi) \wedge e^\rho \wedge (\iota_{e_v} \nabla_{e_\rho} \Phi)). \qquad (4.15)$$

If we decompose $\gamma_\mu = \Phi \wedge \lambda^2(v_\mu) + \gamma^{21}_\mu$ with $\gamma^{21}_\mu \in \Lambda^6_{21} T^*_pM$, then for any $v \in V_7 = e^\perp_0$ we have $\gamma^{21}_\mu \wedge \lambda^2(v) = 0$ by (4.7) and hence,

$$\begin{aligned}
\gamma_\mu \wedge \lambda^2(v) &= \Phi \wedge \lambda^2(v_\mu) \wedge \lambda^2(v) \\
&= (e^0 \wedge \varphi + *_7\varphi) \wedge (e^0 \wedge v^\flat_\mu + \iota_{v_\mu}\varphi) \wedge (e^0 \wedge v^\flat + \iota_v\varphi) \\
&= e^0 \wedge \varphi \wedge (\iota_{v_\mu}\varphi) \wedge (\iota_v\varphi) + *_7\varphi \wedge e^0 \wedge v^\flat_\mu \wedge (\iota_v\varphi) \\
&\quad + *_7\varphi \wedge (\iota_{v_\mu}\varphi) \wedge e^0 \wedge v^\flat
\end{aligned}$$

$$\overset{(6.3),(6.1)}{=} 6\langle v_\mu, v\rangle\text{vol} + 3\langle v_\mu, v\rangle\text{vol} + 3\langle v_\mu, v\rangle\text{vol} = 12\langle v_\mu, v\rangle\text{vol}.$$

Thus,

$$\pi_7([P, P]_p^{FN}) = \frac{1}{12} * (\gamma_\mu \wedge \lambda^2(e_i))\, \Phi \wedge \lambda^2(e_i) \otimes e_\mu. \tag{4.16}$$

For arbitrary $v \in V_7 = e_0^\perp$ we compute

$$\begin{aligned}
\gamma_\mu \wedge \lambda^2(v) &= 2((\iota_{e_v}\Phi) \wedge \iota_{e_\mu}(\nabla_{e_v}\Phi) \wedge \lambda^2(v) \\
&\quad + (\iota_{e_\mu}\iota_{e_v}\Phi) \wedge e^\rho \wedge (\iota_{e_v}\nabla_{e_\rho}\Phi) \wedge \lambda^2(v)) \\
&= 2((\iota_{e_v}\Phi) \wedge \iota_{e_\mu}\lambda^4(T_p(e_v)) \wedge \lambda^2(v) \\
&\quad + (\iota_{e_\mu}\iota_{e_v}\Phi) \wedge e^\rho \wedge (\iota_{e_v}\lambda^4(T_p(e_\rho))) \wedge \lambda^2(v)) \\
&\overset{(6.17),(6.18)}{=} 2((-4\delta_{\mu v}\langle T(e_v), v\rangle\text{vol} + e^{v\mu} \wedge \lambda^4(T_p(e_v)) \wedge \lambda^2(v)) \\
&\quad + (-12\delta_{\mu\rho}\langle T_p(e_\rho), v\rangle\text{vol} + e^{\rho\mu} \wedge \lambda^4(T_p(e_\rho)) \wedge \lambda^2(v)) \\
&= 2(-16\langle T_p(e_\mu), v\rangle\text{vol} - 2e^{\mu v} \wedge \lambda^4(T_p(e_v)) \wedge \lambda^2(v)) \\
&= 2\Big(-16\langle T_p(e_\mu), v\rangle - 4\sigma(e_\mu, T_p(e_v), e_v, v)\Big)\text{vol} \\
&\overset{(4.11)}{=} -8\langle(4T_p + \phi_\sigma(T_p))(e_\mu), v\rangle\text{vol},
\end{aligned}$$

and this together with (4.16) implies (4.14) and completes the proof. □

With this, we are now ready to prove the following which immediately implies Theorem 1.2 from the introduction.

Theorem 4.6 *Let (M^8, Φ) be a manifold with a* Spin(7)*-structure (M^8, Φ), let ∇ be the Levi-Civita connection of $g = g_\Phi$, and let $T \in \Omega^1(M^8, W_7(M^8))$ be its torsion endomorphism defined in Definition* 4.4. *Then, for every $p \in M^8$ the following are equivalent.*

(1) $T_p = 0 \in T_p^* M^8 \otimes W^7(M)_p$.
(2) The Spin(7)*-structure is torsion-free at p, i.e., $(\nabla\Phi)_p = 0$.*
(3) $\pi_7([P, P]_p^{FN}) = 0 \in \Lambda_7^6 T_p^* M^8 \otimes T_p M^8$.
(4) $[P, P]_p^{FN} = 0 \in \Lambda^6 T_p^* M^8 \otimes T_p M^8$.

Proof The equivalence of the first two statements was shown in [4]. Also, $T_p = 0$ implies $(\nabla\Phi)_p = 0$, whence by Proposition 2.2, $[P, P]_p^{FN} = 0$, and this trivially implies $\pi_7([P, P]_p^{FN}) = 0$.

By (4.14), $\pi_7([P, P]_p^{FN}) = 0$ iff $4T_p + \phi_\sigma(T_p) = 0$, and since ϕ_σ does not have -4 as an eigenvalue by Lemma 4.3, this implies that $T_p = 0$. □

5 The 16 classes of G_2- and 4 classes of Spin(7)-structures

In this section, we shall interpret the classification of G_2-structures and of Spin(7)-structures [4,5] in terms of the Frölicher–Nijenhuis bracket.

For the G_2-case, this classification is given by determining which components of the torsion endomorphism T vanish, where T is regarded as a section of the endomorphism bundle $\text{End}(TM^7)$ which is G_2-equivariantly isomorphic to

$$V_7(M^7) \otimes V_7(M^7) \cong V_1(M^7) \oplus V_7(M^7) \oplus V_{14}(M^7) \oplus V_{27}(M^7). \qquad (5.1)$$

Since this decomposition has 4 summands, the classification consists of $2^4 = 16$ cases.

Observe that both $\Lambda_7^5 T^* M^7 \otimes T M^7$ and $\Lambda^6 T^* M^7 \otimes T M^7$ are G_2-equivariantly isomorphic to $V_7(M^7) \otimes V_7(M^7)$, where explicit isomorphisms are given by

$$K : \Lambda_7^5 T^* M^7 \otimes T M^7 \ni (*\varphi \wedge v^\flat) \otimes w \longmapsto v^\flat \otimes w \in T^* M^7 \otimes T M^7$$
$$L : \Lambda^6 T_p^* M^7 \otimes T M^7 \ni (*v^\flat) \otimes w \longmapsto v^\flat \otimes w \in T^* M^7 \otimes T M^7.$$

If (M^7, φ) is a manifold with a G_2-structure and the cross products Cr and χ, then we define the sections

$$K_{\pi_7([Cr,\chi]^{FN})}, \ L_{[\chi,\chi]^{FN}} \in \Gamma(\mathrm{End}(TM)).$$

Therefore, by Propositions 3.5, 3.6 there are G_2-equivariant vector bundle isomorphisms

$$\tau_1, \tau_2 : \mathrm{End}(TM^7) \longrightarrow \mathrm{End}(TM^7)$$

such that for the torsion endomorphism $T \in \Gamma(\mathrm{End}(TM))$ we have

$$\tau_1(T) = K_{\pi_7([Cr,\chi]^{FN})} \quad \text{and} \quad \tau_2(T) = L_{[\chi,\chi]^{FN}}, \qquad (5.2)$$

where by a slight abuse of notation we denote the map $\tau_i : \Gamma(\mathrm{End}(TM)) \to \Gamma(\mathrm{End}(TM))$ applying τ_i pointwise by the same symbol.

For an element $A = a_{ij} e^i \otimes e_j \in \mathrm{End}(V_7)$ let us denote its skew-symmetrization by

$$\sigma_A := a_{ij} e^{ij} \in \Lambda^2 V_7^*.$$

With this notation, it follows from (3.10) and (3.18) that τ_1 and τ_2 take the form

$$\tau_1(T) = 2\Big(T - 2T^\top - tr(T)id\Big),$$
$$\tau_2(T) = -4(T + T^\top) + 6 * (e^i \wedge \sigma_T \wedge \varphi) \otimes e_i,$$

summing over some basis (e_i) with dual basis (e^i).

The G_2-equivariance of τ_1 and τ_2 and (5.2) now implies that the $V_k(M)$-component of T vanishes if and only if the $V_k(M)$-component of $K_{\pi_7([Cr,\chi]^{FN})}$ vanishes if and only if the $V_k(M)$-component of $L_{[\chi,\chi]^{FN}}$ vanishes. Since the cases in the Fernández–Gray classification are determined by the vanishing of the components of T, we obtain the interpretation of these cases given in Table 1.

The interpretation of manifolds (M^8, Φ) with a Spin(7)-structure is analogous. Again, the torsion T and the projection $\pi_7([P, P]^{FN})$ are sections of the Spin(7)-equivariantly isomorphic bundles $T^* M^8 \otimes W_7(M^8)$ and $\Lambda_7^6 T^* M^8 \otimes T M^8$, respectively, with an explicit identification given by

$$H : \Lambda_7^6 T^* M^8 \otimes W_7(M^8) \ni \Phi \wedge (\lambda^2(a)) \otimes v \longmapsto v^\flat \otimes a \in T^* M^8 \otimes W_7(M^8),$$

and if (M^8, Φ) is a manifold with a Spin(7)-structure and the 3-fold product P, then by (4.14)

$$H_{\pi_7([P,P]^{FN})} = \tau_3(T), \quad \text{where} \quad \tau_3(T) = -\frac{2}{3}(4T + \phi_\sigma(T)).$$

As before by abuse of notation we regard ϕ_σ as the pointwise application of the map from (4.11) to sections of $W_7(M^8) \otimes T^* M^8 \cong Lin(W_8(M^8), W_7(M^8))$.

Table 1 Fernández-Gray classification of G_2-structures

Classes	Relation on $A \in \{T, K_{\pi_7([Cr,\chi]^{FN})}, L_{[\chi,\chi]^{FN}}\}$
$V_1(M) \oplus V_7(M) \oplus V_{14}(M) \oplus V_{27}(M)$	No relation
$V_7(M) \oplus V_{14}(M) \oplus V_{27}(M)$	$tr(A) = 0$
$V_1(M) \oplus V_{14}(M) \oplus V_{27}(M)$	$\sigma_A \in \Omega^2_{14}(M)$
$V_1(M) \oplus V_7(M) \oplus V_{27}(M)$	$\sigma_A \in \Omega^2_7(M)$
$V_1(M) \oplus V_7(M) \oplus V_{14}(M)$	$A + A^\top - \frac{2}{7}tr(A)\mathrm{id}_{TM} = 0$
$V_{14}(M) \oplus V_{27}(M)$	$\sigma_A \in \Omega^2_{14}(M), \quad tr(A) = 0$
$V_7(M) \oplus V_{27}(M)$	$\sigma_A \in \Omega^2_7(M), \quad tr(A) = 0$
$V_7(M) \oplus V_{14}(M)$	$A + A^\top = 0$
$V_1(M) \oplus V_{27}(M)$	$A - A^\top = 0$
$V_1(M) \oplus V_{14}(M)$	$A + A^\top - \frac{2}{7}tr(A)\mathrm{id}_{TM} = 0, \quad \sigma_A \in \Omega^2_{14}(M)$
$V_1(M) \oplus V_7(M)$	$A + A^\top - \frac{2}{7}tr(A)\mathrm{id}_{TM} = 0, \quad \sigma_A \in \Omega^2_7(M)$
$V_{27}(M)$	$A - A^\top = 0, \quad tr(A) = 0$
$V_{14}(M)$	$A + A^\top = 0, \quad \sigma_A \in \Omega^2_{14}(M)$
$V_7(M)$	$A + A^\top = 0, \quad \sigma_A \in \Omega^2_7(M)$
$V_1(M)$	$A = \frac{1}{7}tr(A)\mathrm{id}_{TM}$
$\{0\}$	$A = 0$

Table 2 Fernández classification of Spin(7)-structures

Classes	Relation on $A \in \{T, H_{\pi_7([P,P]^{FN})}\}$
$W_8 \oplus W_{48}$	No relation
W_{48}	$pr_8(A) = 0$
W_8	$pr_{48}(A) = 0$
$\{0\}$	$A = 0$

By (4.8), $W_7(M^8) \otimes T^*M^8$ can be decomposed as $W_8(M^8) \oplus W_{48}(M^8)$ whence by the Spin(7)-equivariance of τ_3, the $W_k(M^8)$-component of T vanishes if and only if the $W_k(M^8)$-component of $H_{\pi_7([P,P]^{FN})}$ does. Since the classification of Fernández [4] into $2^2 = 4$ different cases is given by the vanishing of the components of the torsion T, it follows that these cases can be also interpreted by the vanishing of the components of $H_{\pi_7([P,P]^{FN})}$, which leads to the interpretation of the classes of Spin(7)-manifolds given in Table 2, where $pr_k : W_7(M^8) \otimes T^*M^8 \to W_k(M^8)$ is the canonical projection.

6 Appendix

In this appendix, we shall collect some of the formulas which we needed in the calculations in this paper. Most of them are known and can be found in a similar form, e.g., in [15, Lemma 4.37], but we shall collect them here for the reader's convenience.

Lemma 6.1 *For all* $u, v, w, r \in V_7$ *and any orthonormal basis* (e_i) *of* V_7 *the following identities hold.*

$$u^\flat \wedge (\iota_v \varphi) \wedge *\varphi = 3 \langle u, v \rangle \mathrm{vol} \tag{6.1}$$

$$u^\flat \wedge (\iota_v * \varphi) \wedge \varphi = 4 \langle u, v \rangle \mathrm{vol} \tag{6.2}$$

$$(\iota_u \varphi) \wedge (\iota_v \varphi) \wedge \varphi = 6 \langle u, v \rangle \mathrm{vol} \tag{6.3}$$

$$u^\flat \wedge (\iota_v \varphi) \wedge (\iota_w \varphi) \wedge (\iota_r \varphi) = 2 \Big(\langle v, w \rangle \langle u, r \rangle$$
$$+ \langle u, v \rangle \langle w, r \rangle + \langle u, w \rangle \langle v, r \rangle \Big) \mathrm{vol} \tag{6.4}$$

$$(\iota_u \iota_v * \varphi) \wedge w^\flat \wedge *\varphi = -2 u^\flat \wedge v^\flat \wedge w^\flat \wedge *\varphi \tag{6.5}$$

$$(\iota_u \iota_v \varphi) \wedge w^\flat \wedge (\iota_r \varphi) \wedge \varphi = 2(\langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle) \mathrm{vol}$$
$$- 2 u^\flat \wedge v^\flat \wedge w^\flat \wedge r^\flat \wedge \varphi \tag{6.6}$$

$$(\iota_u \iota_v * \varphi) \wedge (\iota_w * \varphi) \wedge (\iota_r \varphi) = 2(\langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle) \mathrm{vol}$$
$$+ u^\flat \wedge v^\flat \wedge w^\flat \wedge r^\flat \wedge \varphi \tag{6.7}$$

$$u^\flat \wedge v^\flat \wedge (\iota_w * \varphi) \wedge (\iota_r \varphi) = 2(\langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle) \mathrm{vol}$$
$$+ u^\flat \wedge v^\flat \wedge w^\flat \wedge r^\flat \wedge \varphi \tag{6.8}$$

$$u^\flat \wedge v^\flat \wedge (\iota_w \iota_r * \varphi) \wedge \varphi = 2(\langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle) \mathrm{vol}$$
$$- u^\flat \wedge v^\flat \wedge w^\flat \wedge r^\flat \wedge \varphi \tag{6.9}$$

$$(\iota_u * \varphi) \wedge (\iota_v \varphi) = -2 * (u^\flat \wedge v^\flat) + u^\flat \wedge v^\flat \wedge \varphi \tag{6.10}$$

$$\varphi \wedge (\iota_u \varphi) = 2 u^\flat \wedge *\varphi \tag{6.11}$$

$$(\iota_{e_i} \varphi)^2 = 6 * \varphi \tag{6.12}$$

$$(\iota_u \iota_{e_i} * \varphi) \wedge (\iota_v \iota_{e_i} * \varphi) = 2(\iota_u \varphi) \wedge (\iota_v \varphi) \tag{6.13}$$

$$(\iota_u \iota_{e_i} * \varphi) \wedge (\iota_{e_i} \varphi) = 3 u^\flat \wedge \varphi \tag{6.14}$$

Proof For the proof of these identities, observe that the left-hand side of each equation is a G_2-invariant element of some tensor power of V_7, and therefore it has to be a linear combination of the summands on the right-hand side; the coefficients of this linear combination then can be determined by using the explicit formulas for φ and $*\varphi$ in (3.1) and (3.2).

To pick one explicit example which is not among the identities shown in [15], observe that the left-hand side of (6.4) is an element of $(V_7 \otimes \odot^3 V_7)^{G_2}$. Since $\odot^3 V_7 \cong V_7 \oplus V_{77}$, we have $\dim(V_7 \otimes \odot^3 V_7)^{G_2} = 1$, and there is one G_2-invariant element of $V_7 \otimes \odot^3 V_7$ given by deriving the square of the scalar product which lies in $\odot^4 V_7$. Thus,

$$u^\flat \wedge (\iota_v \varphi) \wedge (\iota_w \varphi) \wedge (\iota_r \varphi) = c \Big(\langle v, w \rangle \langle u, r \rangle$$
$$+ \langle u, v \rangle \langle w, r \rangle + \langle u, w \rangle \langle v, r \rangle \Big) \mathrm{vol} \tag{6.15}$$

for some constant $c \in \mathbb{R}$. Now setting $u = v = w = r =: e_1$ and using (3.1) implies that $c = 2$, showing (6.4).

The remaining identities are shown in a similar fashion.   □

The following two decompositions of G_2- and Spin(7)-representations is also well known, cf. [15, Theorem 8.5, 9.8], [10, (4.7), (4.8)].

Lemma 6.2 *Decompose* $\Lambda^2 V_7^* = \Lambda_7^2 V_7^* \oplus \Lambda_{14}^2 V_7^*$ *according to* (3.5). *Then,*

$$\Lambda_7^2 V_7^* = \{\alpha^2 \in \Lambda^2 V_7^* \mid *(\alpha^2 \wedge \varphi) = 2\alpha^2\}, \quad and$$
$$\Lambda_{14}^2 V_7^* = \{\alpha^2 \in \Lambda^2 V_7^* \mid *(\alpha^2 \wedge \varphi) = -\alpha^2\}.$$

In particular,

$$\begin{aligned} \Lambda_7^2 V_7^* &= \{\alpha^2 + *(\alpha^2 \wedge \varphi) \mid \alpha^2 \in \Lambda^2 V_7^*\}, \quad and \\ \Lambda_{14}^2 V_7^* &= \{2\alpha^2 - *(\alpha^2 \wedge \varphi) \mid \alpha^2 \in \Lambda^2 V_7^*\}. \end{aligned} \tag{6.16}$$

Lemma 6.3 *Decompose* $\Lambda^2 W_8^* = \Lambda_7^2 W_8^* \oplus \Lambda_{21}^2 W_8^*$ *according to* (4.4). *Then,*

$$\Lambda_7^2 W_8^* = \{\alpha^2 \in \Lambda^2 W_8^* \mid \Phi \wedge \alpha^2 = 3 * \alpha^2\}, \quad and$$
$$\Lambda_{21}^2 W_8^* = \{\alpha^2 \in \Lambda^2 W_8^* \mid \Phi \wedge \alpha^2 = - * \alpha^2\}.$$

We shall also need the following result.

Lemma 6.4 *For all* $u, v \in W_7$ *and* $a, b \in W_8$ *the following formulas hold:*

$$(\iota_a \Phi) \wedge (\iota_b \lambda^4(u)) \wedge \lambda^2(v)$$
$$= -4 \langle a, b \rangle_{W_8} \langle u, v \rangle_{W_7} \text{vol} + a^\flat \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v), \tag{6.17}$$
$$a^\flat \wedge (\iota_b \iota_{e_\mu} \Phi) \wedge (\iota_{e_\mu} \lambda^4(u)) \wedge \lambda^2(v)$$
$$= -12 \langle a, b \rangle_{W_8} \langle u, v \rangle_{W_7} \text{vol} + a^\flat \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v) \tag{6.18}$$

where in (6.18) *the sum is taken over an orthonormal basis* (e_μ) *of* W_8.

Proof By (4.4) and (4.5), the decomposition of $W_8 \otimes W_8$ and $W_7 \otimes W_7$ into Spin(7)-irreducible summands yields

$$W_8 \otimes W_8 = \odot^2 W_8 \oplus \Lambda^2 W_8 \cong (W_1 \oplus W_{35}) \oplus (W_7 \oplus W_{21}),$$
$$W_7 \otimes W_7 = \odot^2 W_7 \oplus \Lambda^2 W_7 \cong (W_1 \oplus W_{27}) \oplus W_{21},$$

so that there are two inequivalent summands in common and hence, the space of Spin(7)-invariant tensors in $W_8 \otimes W_8 \otimes W_7 \otimes W_7$ is 2-dimensional. Since the left-hand sides of (6.17) and (6.18) describe such tensors, it follows that there must be constants $c_1, \ldots, c_4 \in \mathbb{R}$ such that

$$(\iota_a \Phi) \wedge (\iota_b \lambda^4(u)) \wedge \lambda^2(v)$$
$$= c_1 \langle a, b \rangle_{W_8} \langle u, v \rangle_{W_7} \text{vol} + c_2 a^\flat \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v), \tag{6.19}$$
$$a^\flat \wedge (\iota_b \iota_{e_\mu} \Phi) \wedge (\iota_{e_\mu} \lambda^4(u)) \wedge \lambda^2(v)$$
$$= c_3 \langle a, b \rangle_{W_8} \langle u, v \rangle_{W_7} \text{vol} + c_4 a^\flat \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v). \tag{6.20}$$

In order to determine these constants, we first let $a = b := e_0$, so that

$$(\iota_{e_0} \Phi) \wedge (\iota_{e_0} \lambda^4(u)) \wedge \lambda^2(v) = \varphi \wedge (\iota_u *_7 \varphi) \wedge (e^0 \wedge v^\flat + (\iota_v \varphi))$$

$$= - e^0 \wedge v^\flat \wedge (\iota_u *_7 \varphi) \wedge \varphi$$

$$\stackrel{(6.2)}{=} - 4\langle u, v\rangle \text{vol},$$

and from this, $c_1 = -4$ follows. Whence if we let $a := e_0$ and $b, u, v \in W_7 = e_0^\perp$ then

$$(\iota_{e_0}\Phi) \wedge (\iota_b \lambda^4(u)) \wedge \lambda^2(v) \stackrel{(4.6)}{=} \varphi \wedge (-e^0 \wedge (\iota_b \iota_u *_7 \varphi) - \iota_b(u^\flat \wedge \varphi))$$

$$\wedge (e^0 \wedge v^\flat + (\iota_v \varphi))$$

$$= - \varphi \wedge e^0 \wedge (\iota_b \iota_u *_7 \varphi) \wedge (\iota_v \varphi)$$

$$+ \varphi \wedge u^\flat \wedge (\iota_b \varphi) \wedge e^0 \wedge v^\flat$$

$$\stackrel{(6.11)}{=} 2e^0 \wedge (\iota_b \iota_u *_7 \varphi) \wedge v^\flat \wedge *_7 \varphi$$

$$+ 2b^\flat \wedge *_7 \varphi \wedge u^\flat \wedge e^0 \wedge v^\flat$$

$$\stackrel{(6.5)}{=} - 2e^0 \wedge b^\flat \wedge u^\flat \wedge v^\flat \wedge *_7 \varphi$$

$$\stackrel{(4.12)}{=} e^0 \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v),$$

so that $c_2 = 1$ follows. Now substituting $a = b := e_0$ and $u = v := e_1$ into (6.20) and using the index i to run from 1 to 7 yields

$$e^0 \wedge (-\iota_{e_i}\varphi) \wedge (\iota_{e_i}(-e^1 \wedge \varphi)) \wedge (\iota_{e_1}\varphi) = e^0 \wedge (\iota_{e_i}\varphi) \wedge (\delta_{1i}\varphi - e^1 \wedge (\iota_{e_i}\varphi)) \wedge (\iota_{e_1}\varphi)$$

$$= e^0 \wedge (\iota_{e_1}\varphi) \wedge (\iota_{e_1}\varphi) \wedge \varphi$$

$$- e^0 \wedge e^1 \wedge (\iota_{e_i}\varphi) \wedge (\iota_{e_i}\varphi) \wedge \iota_{e_1}\varphi$$

$$\stackrel{(6.3),(6.12)}{=} 6\text{vol} - 6e^0 \wedge e^1 \wedge \iota_{e_1}\varphi \wedge *\varphi$$

$$\stackrel{(6.1)}{=} -12\text{vol},$$

so that $c_3 = -12$ follows. Finally, for $a := e_0$, $u := e_1$ and $b, v \in W_7 = e_0^\perp$ (6.20) reads

$$e^0 \wedge (\iota_b \iota_{e_\mu}\Phi) \wedge (\iota_{e_\mu}\lambda^4(e_1)) \wedge \lambda^2(v)$$

$$= e^0 \wedge (\iota_b \varphi) \wedge (\iota_{e_1} *_7 \varphi) \wedge (\iota_v \varphi)$$

$$+ e^0 \wedge (\iota_b \iota_{e_i} *_7 \varphi) \wedge (-\iota_{e_i}(e^1 \wedge \varphi)) \wedge (\iota_v \varphi)$$

$$\stackrel{(*)}{=} 0 + e^0 \wedge (\iota_b \iota_{e_i} *_7 \varphi) \wedge (-\delta_{1i}\varphi + e^1 \wedge (\iota_{e_i}\varphi)) \wedge (\iota_v \varphi)$$

$$= - e^0 \wedge (\iota_b \iota_{e_1} *_7 \varphi) \wedge \varphi \wedge (\iota_v \varphi)$$

$$+ e^{01} \wedge (\iota_b \iota_{e_i} *_7 \varphi) \wedge (\iota_{e_i}\varphi) \wedge (\iota_v \varphi)$$

$$\stackrel{(6.11),(6.14)}{=} - 2e^0 \wedge (\iota_b \iota_{e_1} *_7 \varphi) \wedge v^\flat \wedge *_7 \varphi + 3e^{01} \wedge b^\flat \wedge \varphi \wedge (\iota_v \varphi)$$

$$\stackrel{(6.11),(6.5)}{=} 4e^0 \wedge b^\flat \wedge e^1 \wedge v^\flat \wedge *_7 \varphi + 6e^{01} \wedge b^\flat \wedge v^\flat \wedge *_7 \varphi$$

$$= - 2e^0 \wedge b^\flat \wedge e^1 \wedge v^\flat \wedge *_7 \varphi$$

$$\stackrel{(4.12)}{=} e^0 \wedge b^\flat \wedge \lambda^4(e_1) \wedge \lambda^2(v),$$

so that $c_4 = 1$ follows. At $(*)$ we have used that the map

$$(u, v, w) \longmapsto *\Big((\iota_u \varphi) \wedge (\iota_v *_7 \varphi) \wedge (\iota_w \varphi)\Big)$$

is a G_2-invariant element of $W_7 \otimes \odot^2 W_7$, and since by (4.5) $\odot^2 W_7 \cong W_1 \oplus W_{27}$ has no irreducible component isomorphic to W_7, there is no such element other than 0.   □

References

1. Brown, R.B., Gray, A.: Vector cross product. Comment. Math. Helv. **42**, 222–236 (1967)
2. Bryant, R.: Metrics with exceptional holonomy. Ann. Math. **126**, 525–576 (1987)
3. Bryant, R.: Some remarks on G_2-structures. In: Proceedings of Gökova Geometry-Topology Conference 2005, Gökova Geometry/Topology Conference (GGT), Gökova, pp. 75–109 (2006)
4. Fernández, M.: A classification of Riemannian manifolds with structure group Spin(7). Ann. Mat. Pura Appl. **143**, 101–122 (1986)
5. Fernández, M., Gray, A.: Riemannian manifolds with structure group G_2. Ann. Mat. Pura Appl. **32**, 19–45 (1982)
6. Frölicher, A., Nijenhuis, A.: Theory of vector-valued differential forms. I. Derivations of the graded ring of differential forms. Indag. Math. **18**, 338–359 (1956)
7. Frölicher, A., Nijenhuis, A.: Some new cohomology invariants for complex manifolds. I, II. Indag. Math. **18**(540–552), 553–564 (1956)
8. Harvey, R., Lawson, H.B.: Calibrated geometry. Acta Math. **148**, 47–157 (1982)
9. Humphreys, J.: Introduction to Lie Algebras and Representation Theory, Second Printing, Graduate Texts in Mathematics, vol. 9. Springer, New York (1978)
10. Karigiannis, S.: Deformations of G_2 and Spin(7)-structures. Can. J. Math. **57**, 1012–1055 (2005)
11. Kawai, K.: Deformations of homogeneous associative submanifolds in nearly parallel G_2-manifolds Asian J. Math. **21**, 429–462 (2017). arXiv: 1407.8046
12. Kolar, I., Michor, P.W., Slovak, J.: Natural Operators in Differential Geometry. Springer, Berlin (1993)
13. Lê , H. V., Vanžura, J.: McLean's second variation formula revisited. J. Geom. Phys. 113, 188–196. arXiv:1605.01267 (2017)
14. McLean, R.: Deformations of calibrated submanifolds. Commun. Anal. Geom. **6**, 705–747 (1998)
15. Salmon, D., Walpuski, T.: Notes on the octonions. arXiv: 1005.2820