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Second-order Lagrangians admitting a first-order Hamiltonian formalism

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Abstract Second-order Lagrangian densities admitting a first-order Hamiltonian formalism are studied; namely, (1) for each second-order Lagrangian density on an arbitrary fibred manifold $p\colon E \to N$ the Poincaré–Cartan form of which is projectable onto J^1E, by using a new notion of regularity previously introduced, a first-order Hamiltonian formalism is developed for such a class of variational problems; (2) the existence of first-order equivalent Lagrangians is discussed from a local point of view as well as global; (3) this formalism is then applied to classical Hilbert–Einstein Lagrangian and a generalization of the BF theory. The results suggest that the class of problems studied is a natural variational setting for GR.

Keywords Hilbert–Einstein Lagrangian · Hamilton–Cartan formalism · Jacobi fields · Jet bundles · Poincaré–Cartan form · Presymplectic structure

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1 Preliminaries

1.1 Jet-bundle formalism

Below, a fibred manifold $p\colon E \to N$ is considered over a connected n-dimensional manifold N oriented by a volume form $v = dx^1 \wedge \cdots \wedge dx^n$. The bundle of k-jets of local sections of p is denoted by $p^k\colon J^k E \to N$, with natural projections $p^k_l\colon J^k E \to J^l E$, $k \geq l$.

Every fibred coordinate system (x^j, y^α), $1 \leq j \leq n$, $1 \leq \alpha \leq m = \dim E - n$, for the submersion p, induces a coordinate system (x^j, y^α_I) $(I = (i_1, \ldots, i_n)$ being a multi-index in $\mathbb{N}^n$ of order $|I| = i_1 + \ldots + i_n \leq r)$ on $J^r E$ defined by,

$$y^\alpha_I\left(j^r_x s\right) = \frac{\partial^{|I|}(y^\alpha \circ s)}{\partial (x^1)^{i_1} \ldots \partial (x^n)^{i_n}}(x),$$

where s is a local section of p. We also set $(j) = (0, \ldots, 0, \overset{(j}{1}, 0, \ldots, 0) \in \mathbb{N}^n$, $(jk) = (j) + (k)$, etc., and $y^\alpha_{(j)} = y^\alpha_j$.

Hence $j^r_x s$ codifies the partial derivatives up to the order r at the point $x \in N$ of the section s of p, determining the first r terms of the Taylor series of the coordinates $s^\alpha = y^\alpha \circ s$ of s at x.

From the earlier seventies (e.g., see [12,15]), jet bundles constitute the natural geometric setting to develop calculus of variations and Lagrangian and Hamiltonian formalisms, as well as to study the presymplectic structure attached to a variational problem. For more details on this topic, we refer the reader to more recent articles, such as [1, Chapter 6], [26, §1.3], [35, Chapters 2 & 3], [36, §§0.1, 0.2], [37].

As is known, classical fields can be viewed as the sections of fibred manifolds and the Lagrangian formalism are then formulated in terms of jet manifolds.

A Lagrangian density Lv of order r is the product of a volume form v on N and a smooth function on J^r; i.e., L is a function of the $n+m\binom{n+r}{r}$ coordinate functions x^j, y^α_I, $1 \leq j \leq n$, $1 \leq \alpha \leq m$, $|I| \leq r$, where m denotes, as above, the dimension of the fibres $p^{-1}(x)$, $\forall x \in N$, of the projection $p\colon E \to N$.

1.2 Legendre and Poincaré–Cartan forms

The *Legendre form* of a second-order Lagrangian density $\Lambda = Lv$, defined on $p\colon E \to N$, $L \in C^\infty(J^2 E)$, is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on $J^3 E$ locally given by (e.g., see [28,31,38]),

$$\omega_\Lambda = (-1)^{i-1} L^{i0}_\alpha v_i \otimes dy^\alpha + (-1)^{i-1} L^{ij}_\alpha v_i \otimes dy^\alpha_j,$$

where $v_i = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$, and

$$L^{ij}_\alpha = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y^\alpha_{(ij)}}, \tag{1}$$

$$L^{i0}_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \frac{1}{2 - \delta_{ij}} D_j \left(\frac{\partial L}{\partial y^\alpha_{(ij)}} \right), \tag{2}$$

and $D_j = \frac{\partial}{\partial x^j} + \sum_{|I|=0}^{\infty} \sum_{\alpha=1}^{m} y^\alpha_{I+(j)} \frac{\partial}{\partial y^\alpha_I}$ denotes the total derivative with respect to the coordinate x^j. The *Poincaré–Cartan form* (or P–C form for short) attached to Λ is the ordinary n-form on $J^3 E$ given by $\Theta_\Lambda = (p^3_2)^* \theta^2 \wedge \omega_\Lambda + \Lambda$ (e.g., see [28,38]), where θ^1, θ^2 are the first- and second-order structure forms on $J^1 E$, $J^2 E$, locally given by (cf. [27,37]), $\theta^1 = \theta^\alpha \otimes \frac{\partial}{\partial y^\alpha}$,

$\theta^2 = \theta^\alpha \otimes \frac{\partial}{\partial y^\alpha} + \theta_h^\alpha \otimes \frac{\partial}{\partial y_h^\alpha}$, respectively, and $\theta^\alpha = dy^\alpha - y_k^\alpha dx^k$, $\theta_i^\alpha = dy_i^\alpha - y_{(ik)}^\alpha dx^k$, is the standard basis of contact 1-forms, and the exterior product of $(p_2^3)^*\theta^2$ and the Legendre form, is taken with respect to the standard pairing $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$.

The theory of the P–C form for a second-order Lagrangian density is different from that of a first-order density, due to the appearance of operator D_j in the formula (2), which is essential for the formula (9) to hold. Because of this, such problem has motivated many works among which we should mention [7,9,13,17,20,23,24,28,38].

1.3 Projecting onto J^2E or J^1E

The most outstanding difference with a first-order Lagrangian density is that the Legendre and Poincaré–Cartan forms associated with a second-order Lagrangian density are generally defined on J^3E, thus increasing by one the order of the Lagrangian density Λ.

Although the Legendre form ω_Λ of a second-order Lagrangian density $\Lambda = Lv$ depends on the third derivatives of L, it may happen that for certain second-order Lagrangian densities the sum $\Theta_\Lambda = (p_2^3)^*\theta^2 \wedge \omega_\Lambda + \Lambda$ depends on the second derivatives only. In this case, the P–C form Θ_Λ of Λ is said to be projectable onto J^2E, e.g., see [13].

More precisely, as it is known, the P–C form of a second-order Lagrangian projects onto J^2E if and only if the following system of PDEs holds (cf. [7,13]):

$$\frac{1}{2-\delta_{ib}}\frac{\partial^2 L}{\partial y_{ac}^\beta \partial y_{ib}^\alpha} + \frac{1}{2-\delta_{ia}}\frac{\partial^2 L}{\partial y_{bc}^\beta \partial y_{ia}^\alpha} + \frac{1}{2-\delta_{ic}}\frac{\partial^2 L}{\partial y_{ab}^\beta \partial y_{ic}^\alpha} = 0,$$

for all indices $1 \leq a \leq b \leq c \leq n$, $\alpha, \beta = 1, \dots, m$.

More surprisingly, there exist second-order Lagrangians for which the associated P–C form projects not only on J^2E but also on J^1E. Notably, this is the case of the Hilbert–Einstein Lagrangian in General Relativity [cf. formula (25)].

As is well known (e.g., see [15, (1.3)], [31, 2.1]), $p_{r-1}^r \colon J^rE \to J^{r-1}E$ admits an affine bundle structure modelled over the vector bundle

$$W^r = (p^{r-1})^* S^r T^* N \otimes (p_0^{r-1})^* V(p) \to J^{r-1}E. \tag{3}$$

Proposition 1.1 (cf. [24,33]) *The Poincaré–Cartan form attached to a Lagrangian $L \in C^\infty(J^2E)$ projects onto J^1E if and only if L is an affine function with respect to the affine structure of $p_1^2 \colon J^2E \to J^1E$, namely*

$$L = L_\alpha^{ij} y_{(ij)}^\alpha + L_0, \quad L_\alpha^{ji} = L_\alpha^{ij} \in C^\infty(J^1E), L_0 \in C^\infty(J^1E), \tag{4}$$

and the following equations hold:

$$\frac{\partial L_\beta^{ih}}{\partial y_a^\alpha} = \frac{\partial L_\alpha^{ia}}{\partial y_h^\beta}, \quad a, h, i = 1, \dots, n, \ \alpha, \beta = 1, \dots, m. \tag{5}$$

Equation (5) admit a variational meaning. The Euler–Lagrange (or E–L for short) operator of an arbitrary second-order Lagrangian can be written in terms of the coefficients of the P–C form [see the formulas (1), (2)] as follows:

$$\begin{aligned}\mathcal{E}_\alpha(L) &= \sum_{i \leq j} D_i D_j \left(\frac{\partial L}{\partial y_{(ij)}^\alpha}\right) - D_i\left(\frac{\partial L}{\partial y_i^\alpha}\right) + \frac{\partial L}{\partial y^\alpha} \\ &= \frac{\partial L}{\partial y^\alpha} - D_i\left(L_\alpha^{i0}\right), \quad 1 \leq \alpha \leq m.\end{aligned}$$

The E–L equations for an affine second-order Lagrangian L, given as in the formula (4), are of third order and they are of second order if and only if equation (5) hold (cf. [33, Proposition 2.2]).

As the projection $p^r_{r-1}\colon J^rE \to J^{r-1}E$ admits an affine bundle structure, a natural vector-bundle isomorphism is obtained,

$$I^r\colon (p^r_{r-1})^* W^r = (p^r)^* S^r T^* N \otimes (p^r_0)^* V(p) \xrightarrow{\cong} V(p^r_{r-1}), \tag{6}$$

where the vector bundle W^r is defined in (3). Given an arbitrary vector bundle $W \to N$, there exists an antiderivation

$$d_{E/N}\colon \Gamma(E, \wedge^r V^*(p) \otimes p^* W) \to \Gamma(E, \wedge^{r+1} V^*(p) \otimes p^* W)$$

of degree $+1$—called the fibre differential (e.g., see [15, (1.9)])—such that $d_{E/N}(fp^*\xi) = df|_{V(p)} \otimes \xi$, for all $f \in C^\infty(E)$ and all $\xi \in \Gamma(E, W)$. (In the previous paragraph, the relevant fact is that the vector bundle $W \to N$ is defined over the base manifold N, and not over the fibred manifold E.)

In what follows, we are mainly concerned with the fibre derivative d_{J^1E/J^0E}, which will simply be denoted by d^1_0 for the sake of simplicity.

A Lagrangian $L \in C^\infty(J^2E)$ is an affine function with respect to the affine structure of $p^2_1\colon J^2E \to J^1E$ if there exists a linear form $w_L\colon W^2 \to \mathbb{R}$, which is unique, such that $L(\tau + j^2_x s) = w_L(\tau) + L(j^2_x s)$, $\forall \tau \in S^2 T^*_x N \otimes V_{s(x)}(p)$ and $\forall j^2_x s \in J^2E$.

By using the isomorphism $(W^2)^* \cong (p^1)^* S^2 TN \otimes (p^1_0)^* V^*(p)$, the linear form w_L defines a section of the vector bundle $(p^1)^* S^2 TN \otimes (p^1_0)^* V^*(p) \to J^1E$. If L is locally given by the formula (4), then $w_L = L^{hi}_\alpha \frac{\partial}{\partial x^h} \odot \frac{\partial}{\partial x^i} \otimes dy^\alpha|_{V(p)}$, where the symbol $\odot$ denotes symmetric product.

If $\iota^2\colon (W^2)^* \to (p^1)^* \otimes^2 TN \otimes (p^1_0)^* V^*(p)$ is the natural embedding, then we consider the section

$$w'_L = \frac{1}{2}\left(\tilde{I}^1 \circ \iota^2 \circ w_L\right)\colon J^1E \to (p^1)^* TN \otimes V^*(p^1_0) \tag{7}$$

obtained by composing the following mappings:

$$\begin{aligned} J^1E \xrightarrow{w_L} (p^1)^* S^2 TN \otimes (p^1_0)^* V^*(p) = (W^2)^* \xrightarrow{\iota^2} (p^1)^* \otimes^2 TN \otimes (p^1_0)^* V^*(p) \\ = (p^1)^* TN \otimes \left[(p^1)^* TN \otimes (p^1_0)^* V^*(p)\right] \xrightarrow{\tilde{I}^1} (p^1)^* TN \otimes V^*(p^1_0), \end{aligned}$$

where $\tilde{I}^1 = 1_{(p^1)^*TN} \otimes ((I^1)^*)^{-1}$ is the isomorphism deduced from (6) for $r = 1$. As $I^1(dx^a \otimes \partial/\partial y^\alpha) = \partial/\partial y^\alpha_a$, dually we obtain $(I^1)^*(d^1_0 y^\alpha_a) = \partial/\partial x^a \otimes dy^\alpha|_{V(p)}$.

Hence $w'_L = L^{hi}_\alpha d^1_0\left(y^\alpha_h\right) \otimes \frac{\partial}{\partial x^i}$.

Remark 1.1 Equation (5) simply mean that for every index h the form $\eta^h = L^{hi}_\alpha dy^\alpha_i$ is d^1_0-closed, namely $d^1_0 \eta^h = 0$. Hence, there exist $L^i \in C^\infty(J^1E)$ such that locally,

$$\text{(i) } L^{ih}_\alpha = \frac{\partial L^i}{\partial y^\alpha_h}, \quad \text{(ii) } \frac{\partial L^h}{\partial y^\alpha_i} = \frac{\partial L^i}{\partial y^\alpha_h}, \qquad 1 \le \alpha \le m,\ h, i = 1, \ldots, n, \tag{8}$$

the equations (ii) above being a consequence of the symmetry $L^{hi}_\alpha = L^{ih}_\alpha$.

Letting $W = TN$ in the definition of the fibre differential above, recalling that the Poincaré lemma also holds for fibre differentiation (e.g., see [29]) and recalling that the fibres of $p^1_0\colon J^1E \to E$ are simply connected as they are diffeomorphic to $\mathbb{R}^{mn}$, the following global characterization of second-order variational problems with a P–C form projecting onto J^1E, is obtained:

Proposition 1.2 (see [33, Proposition 3.1]) *The Poincaré–Cartan form of a Lagrangian* $L \in C^\infty(J^2E)$ *projects onto* J^1E *if and only if* L *is an affine function with respect to the affine structure of* $p_1^2\colon J^2E \to J^1E$ *and the* TN*-valued* 1*-form* w'_L *defined in the formula* (7) *is* d_0^1*-closed. In this case, for every global (smooth) section* $\sigma\colon E \to J^1E$ *of* p_0^1*, there exists a unique globally defined section* $w_L^\sigma \in \Gamma(J^1E, (p^1)^*TN)$ *such that* $d_0^1\left(w_L^\sigma\right) = w'_L$*,* $w_L^\sigma(\sigma(e)) = 0, \forall e \in E$*.*

Remark 1.2 A general procedure to obtain global sections $\sigma\colon E \to J^1E$ of p_0^1 is to use Ehresmann (or nonlinear) connections, i.e., to use a differential 1-form γ on E taking values in the vertical subbundle $V(p)$ such that $\gamma(X) = X, \forall X \in V(p)$; hence, locally (cf. [32]), $\gamma = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha}$, $\gamma_j^\alpha \in C^\infty(E)$. The vertical differential of a section $s\colon U \to E$ (defined on a neigbourhood U of $x \in N$) at $e = s(x)$ is defined to be the linear mapping $(d^v s)_e\colon T_eE \to V_e(p)$, $(d^v s)_e X = X - s_* p_*(X), \forall X \in T_eE$. We claim that for every $e \in E$, there exists a unique $j_x^1 s \in J^1E$ such that i) $s(x) = e$, where $x = p(e)$, and ii) $(d^v s)_e = \gamma_e$. In fact, one has $(\partial(y^\alpha \circ s)/\partial x^j)(x) = -\gamma_j^\alpha(e)$, and the section σ^γ attached to γ is defined by, $\sigma^\gamma(e) = j_x^1 s$.

1.4 Summary of contents

Bearing the previous definitions and notations in mind, the paper is organized as follows: in Sect. 2 the Hamiltonian function, the momenta, and the Hamilton–Cartan equations attached to each of the aforementioned Lagrangians are introduced as a consequence of a normal form for their P–C form.

In Theorem 2.1, the local expression for the momenta, Hamiltonian and exterior differential of P–C form attached to a second-order Lagrangian density with P–C form projectable onto J^1E, are given and in the case that the momenta are functionally independent, the corresponding Hamiltonian equations are written explicitly. In Proposition 2.2, the holonomy of a solution to these equations is proved. In Corollary 2.3 this result is stated intrinsically in terms of the bilinear form b_Λ previously introduced, which is symmetric (see Proposition 2.4). As each Lagrangian L with P–C form projectable onto J^1E is affine with respect to the affine structure of $p_1^2\colon J^2E \to J^1E$ [see formula (4)], its Hessian matrix vanishes identically and hence L cannot be regular in the usual sense. Accordingly, a suitable notion of regularity is required for such class of Lagrangians; this new notion is precisely the aim of Proposition 2.2 and Corollary 2.3.

In [33], the study of the formal integrability—in the sense of Goldschmidt–Spencer—of the field equations for second-order Lagrangians with projectable P–C form to first order in their Hamiltonian form, is developed. In the real analytic case, this allows one to solve the Cauchy initial value problem for this class of Lagrangians.

The previous sections and the results of [33] are then applied to GR in Sect. 3, thus showing how the theory developed fits very well to the standard Lagrangians in this setting. Specifically, Sect. 3.1 studies Hilbert–Einstein Lagrangian from this point of view, proving its regularity and giving a new statement for the initial value problem. We have included explicit formulas in local coordinates of the P–C form for H–E Lagrangian in Section 3.1, as well as the values of the momenta $(L_{HE})_{rs}^{ij}$ and $(L_{HE})_{ab}^{i0}$.

Section 3.2 provides a strong generalization of the classical Lagrangians in BF theory, again showing that the results obtained above can naturally be applied to these new Lagrangians. The main result of this section is Theorem 3.2, characterizing the H–E

Lagrangian density among all the Lagrangian densities Λ_β defined by formula (27), and computing the Euler–Lagrange equations for anyone of such densities explicitly.

In Sect. 4, the existence of first-order Lagrangians variationally equivalent to a second-order Lagrangian admitting a first-order Hamiltonian formalism is studied, both from local and global point of view; see Theorem 4.1. This generalizes previous results obtained for the H–E Lagrangian in [4] (cf. Lemma 4.2 and Proposition 4.3).

Section 5 introduces the notions of symmetry and Noether invariant for the class of variational problems dealt with throughout the paper. Let us note that Theorem 5.1—the main result of the section—is completely new.

Section 6 discusses in particular the concepts introduced in the previous section for the H–E Lagrangian. Here we should highlight Theorem 6.1, which provides an interesting characterization of infinitesimal symmetries for Λ_{HE} when $n = \dim N \geq 3$.

Finally, in Section 7 the notion of a Jacobi field along an extremal s is introduced and the presymplectic structure $(\omega_2)_s$ defined on the space of Jacobi fields along s is defined. For the case of the H–E Lagrangian, in (44) the explicit formula of a Jacobi field along an extremal metric g is written in terms of the Levi-Civita connection of g and its curvature tensor. Two explicit cases are also developed in detail; see Examples 7.1 and 7.2. In Theorem 7.1, we make a contribution to the study of non-degeneracy of the presymplectic structure attached to a variational problem, by giving a sufficient condition for the radical of $(\omega_2)_s$ to vanish. In particular (see Corollary 7.2), this implies that the 2-form $(\omega_2)_g$ associated to Λ_{HE} along an Einstein metric g is non-degenerate.

2 Regularity and Hamiltonian formalism

In the usual (i.e., first-order) calculus of variations, a section s is an extremal of the Lagrangian density Λ on J^1E if and only if it satisfies the so-called Hamilton–Cartan equations (or H–C for short, e.g., see [15, (3.8)], [13, (1)]), namely if and only if the following equation holds: $(j^1 s)^*(i_X d\Theta_\Lambda) = 0$ for every p^1-vertical vector field X on J^1E.

If $\Lambda = Lv$ is an arbitrary second-order Lagrangian density on E, then the following formula holds (e.g., see [28]):

$$d\Theta_\Lambda = \mathcal{E}_\alpha(L)\theta^\alpha \wedge v + \eta_{n+1}(L), \tag{9}$$

where $\eta_{n+1}(L) = (-1)^i \eta_2^i(L) \wedge v_i$ and $\eta_2^i(L)$ is the 2-contact 2-form given by,

$$\begin{aligned}\eta_2^i(L) = &\frac{\partial L_\alpha^{i0}}{\partial y^\beta}\theta^\alpha \wedge \theta^\beta + \left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta} - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}\right)\theta^\alpha \wedge \theta_j^\beta \\ &+ \sum_{j\leq k}\frac{\partial L_\alpha^{i0}}{\partial y_{(jk)}^\beta}\theta^\alpha \wedge \theta_{(jk)}^\beta + \sum_{i\leq k\leq l}\frac{\partial L_\alpha^{i0}}{\partial y_{(jkl)}^\beta}\theta^\alpha \wedge \theta_{(jkl)}^\beta \\ &+ \frac{\partial L_\alpha^{ij}}{\partial y_k^\beta}\theta_j^\alpha \wedge \theta_k^\beta + \sum_{k\leq l}\frac{\partial L_\alpha^{ij}}{\partial y_{(kl)}^\beta}\theta_j^\alpha \wedge \theta_{(kl)}^\beta.\end{aligned}$$

From the formula (9), it follows that the H–C equations also characterize critical sections for a second-order density Λ; i.e., s is an extremal for Λ if and only if, $(j^3 s)^*(i_X d\Theta_\Lambda) = 0$ for every p^3-vertical vector field X on J^3E.

Remark 2.1 If the P–C form of a second-order density Λ projects onto J^1E, then its H–C equations have the same formal expression of a first-order density (see the formula (14)), although there is no first-order density having Θ_Λ as its P–C form. In fact, the P–C form of a first-order Lagrangian density $\tilde{\Lambda} = \tilde{L}v$, $\tilde{L} \in C^\infty(J^1E)$, is given by,

$$\Theta_{\tilde{\Lambda}} = (-1)^{i-1}\frac{\partial \tilde{L}}{\partial y_i^\alpha}dy^\alpha \wedge v_i + \tilde{H}v, \quad \tilde{H} = \tilde{L} - \frac{\partial \tilde{L}}{\partial y_i^\alpha}y_i^\alpha. \tag{10}$$

If $\Theta_\Lambda = \Theta_{\tilde{\Lambda}}$, then the following three equations are obtained:

$$(1)\ L_\alpha^{ih} = 0, \qquad (2)\ L_0 - y_i^\alpha L_\alpha^{i0} = \tilde{L} - \frac{\partial \tilde{L}}{\partial y_i^\alpha}y_i^\alpha, \qquad (3)\ L_\alpha^{i0} = \frac{\partial \tilde{L}}{\partial y_i^\alpha}.$$

From (4) and (1) it follows $L = L_0$; hence, L is of first order.

Moreover, taking (2) into account, the formulas (2) and (3) above are, respectively, rewritten as $L_0 - \tilde{L} = y_i^\alpha \frac{\partial(L_0 - \tilde{L})}{\partial y_i^\alpha}$, $\frac{\partial(L_0 - \tilde{L})}{\partial y_i^\alpha} = 0$. Hence $\tilde{L} = L$.

Theorem 2.1 (see [33, Theorem 4.1]) *If $\Lambda = Lv$ is a second-order Lagrangian density on E whose Poincaré–Cartan form projects onto J^1E, then letting*

$$p_\alpha^i = L_\alpha^{i0} - \frac{\partial L^i}{\partial y^\alpha}, \quad 1 \le \alpha \le m, \ 1 \le i \le n, \tag{11}$$

$$H = L_0 - y_i^\alpha L_\alpha^{i0} - \frac{\partial L^i}{\partial x^i}, \tag{12}$$

where the functions L^i are defined by the formulas (8)-(i)*, the following formula holds:*

$$d\Theta_\Lambda = (-1)^{i-1}dp_\alpha^i \wedge dy^\alpha \wedge v_i + dH \wedge v. \tag{13}$$

Furthermore, if the linear forms $d_0^1(p_\alpha^i)\colon V(p_0^1) \to \mathbb{R}$, $1 \le \alpha \le m$, $1 \le i \le n$, are linearly independent, then a section $s\colon N \to E$ is an extremal for Λ if and only if it satisfies the following equations:

$$\begin{cases} 0 = \dfrac{\partial(p_\alpha^i \circ j^1 s)}{\partial x^i} - \dfrac{\partial H}{\partial y^\alpha} \circ j^1 s, & 1 \le \alpha \le m, \\ 0 = \dfrac{\partial(y^\alpha \circ s)}{\partial x^i} + \dfrac{\partial H}{\partial p_\alpha^i} \circ j^1 s, & 1 \le \alpha \le m, \ 1 \le i \le n. \end{cases} \tag{14}$$

As is well known (e.g., see [15]), if the Hessian metric Hess(L) of a first-order density $\Lambda = Lv$ is non-singular, then every section $s^1\colon N \to J^1E$ of the projection $p^1\colon J^1E \to N$ that satisfies the P–C equation for Λ is holonomic; i.e., s^1 coincides with the 1-jet extension of the section $s = p_0^1 \circ s^1$ of the projection p. Namely, $(s^1)^*(i_X d\Theta_\Lambda) = 0$ for every p^1-vertical vector field X on J^1E, implies $s^1 = j^1 s$.

In the case of a second-order density with a P–C form projecting onto J^1E, the following result holds:

Proposition 2.2 [33] *If $\Lambda = Lv$ is a second-order Lagrangian on E such that* (i) *its Poincaré–Cartan form Θ_Λ projects onto J^1E,* (ii) *the linear forms $d_0^1(p_\alpha^i)\colon V(p_0^1) \to \mathbb{R}$, $1 \le \alpha \le m$, $1 \le i \le n$, where the functions p_α^i are introduced in* (11)*, are linearly independent, then every solution to its H–C equations, is holonomic.*

As $p_0^1\colon J^1E \to E$ is an affine bundle modelled over $W^1 = p^*(T^*N) \otimes V(p)$ [cf. (3)], there is a canonical isomorphism $I\colon (p_0^1)^*W^1 \overset{\cong}{\to} V(p_0^1)$ locally given by, $I(j_x^1 s, (dx^i)_x \otimes (\partial/\partial y^\alpha)_{s(x)}) = (\partial/\partial y_i^\alpha)_{j_x^1 s}$.

According to the previous lemma, we can define a bilinear form

$$\begin{cases} b_\Lambda \colon (p_0^1)^* W^1 \times_{J^1E} (p_0^1)^* W^1 \to \mathbb{R}, \\ b_\Lambda \left(j_x^1 s; w_0 \otimes Y_0, w_1 \otimes Y_1 \right) = \left\langle w_0, (\phi_v^1)^{-1} \left(i_{Y_0} i_Y (d\Theta_\Lambda) \right) \right\rangle, \\ w_a \in T_x^* N, Y_a \in V_{s(x)}(p), a = 0, 1;\ Y = I(j_x^1 s, w_1 \otimes Y_1), \end{cases} \tag{15}$$

where ϕ_v^k is the isomorphism defined by

$$\phi_v^k \colon \wedge^k T_x N \to \wedge^{n-k} T_x^* N \tag{16}$$

for every $1 \leq k \leq n-1$, obtained by contracting with v, namely

$$\phi_v^k (X_1 \wedge \cdots \wedge X_k) = i_{X_1} \dots i_{X_k} v, \quad \forall X_1, \dots, X_k \in T_x N.$$

If $w_0 = (dx^i)_x$ and $Y_0 = (\partial/\partial y^\alpha)_{s(x)}$, then one readily obtains,

$$i_{Y_0} i_Y (d\Theta_\Lambda) = (-1)^{i-1} \left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s) \right) (v_i)_x,$$

$$\left\langle w_0, (\phi_v^1)^{-1} \left(i_{Y_0} i_Y (d\Theta_\Lambda) \right) \right\rangle = \frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s).$$

In other words,

$$b_\Lambda \left(j_x^1 s; \left(dx^i \right)_x \otimes \left(\frac{\partial}{\partial y^\alpha} \right)_{s(x)}, \left(dx^j \right)_x \otimes \left(\frac{\partial}{\partial y^\beta} \right)_{s(x)} \right) = \frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s).$$

Hence, the next result follows:

Corollary 2.3 *Let Λ be a second-order density on E whose P–C form projects onto $J^1 E$. If the bilinear form defined in (15) is non-singular, then every solution to the H–C equations for Λ is holonomic.*

Proposition 2.4 (see [33, Proposition 5.4]) *The bilinear form b_Λ defined in (15) is symmetric.*

In fact, if $\bar{L}$ is the Lagrangian defined by

$$\bar{L} = L_0 - \frac{\partial L^i}{\partial x^i} - y_i^\alpha \frac{\partial L^i}{\partial y^\alpha}, \tag{17}$$

then, as a calculation shows,

$$p_\alpha^i = \frac{\partial \bar{L}}{\partial y_i^\alpha}. \tag{18}$$

3 Applications to GR

3.1 Hilbert–Einstein Lagrangian

Below, we follow [33]. Let $p_M \colon M = M(N) \to N$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n$. Every coordinate system $(x^i)_{i=1}^n$ on an open domain $U \subseteq N$ induces a coordinate system (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where the functions $y_{jk} = y_{kj}$ are defined by,

$$g_x = y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \quad \forall g_x \in (p_M)^{-1}(U).$$

Following the notations in [19], the Ricci tensor field attached to a symmetric connection Γ is given by $S^{\gamma}(X,Y)=\mathrm{trace}(Z\mapsto R^{\gamma}(Z,X)Y)$, where R^{γ} denotes the curvature tensor field of the covariant derivative ∇^{γ} associated to Γ on the tangent bundle; hence, $S^{\gamma}=(R^{\gamma})_{jl}dx^{l}\otimes dx^{j}$, where $(R^{\gamma})_{jl}=(R^{\gamma})^{k}_{jkl}$, and $(R^{\gamma})^{i}_{jkl}=\partial\Gamma^{i}_{jl}/\partial x^{k}-\partial\Gamma^{i}_{jk}/\partial x^{l}+\Gamma^{m}_{jl}\Gamma^{i}_{km}-\Gamma^{m}_{jk}\Gamma^{i}_{lm}$.

The H–E Lagrangian density is given by

$$(\Lambda_{HE})_{j^2_x g}=g^{ij}(x)(R^{g})^{h}_{ihj}(x)v_{g}(x)=L_{HE}(j^2_x g)v_x,$$

where v is the standard volume form, R^{g} is the curvature tensor of the Levi-Civita connection Γ^{g} of the metric g, and v_g denotes the Riemannian volume form attached to g; i.e., in coordinates, $v_g=\sqrt{|\det((g_{ab})^{n}_{a,b=1})|}v$. Hence,

$$L_{HE}\circ j^2 g=(\rho\circ g)(y^{ij}\circ g)(R^{g})^{h}_{ihj},\quad \rho=\sqrt{|\det((y_{ab})^{n}_{a,b=1})|}. \tag{19}$$

The local expression for L_{HE} is readily seen to be

$$\begin{aligned}
L_{HE}&=\rho\sum\nolimits_{a,b}\sum\nolimits_{c,d}\left(y^{ac}y^{bd}-y^{ab}y^{cd}\right)y_{ab,cd}+(L_{HE})_0,\\
(L_{HE})_0&=\frac{\rho}{2}\sum_{r\le s}\sum_{k\le l}\frac{1}{(1+\delta_{kl})(1+\delta_{rs})}\Big(2y^{rs}\left(y^{ki}y^{jl}+y^{li}y^{jk}\right)-2y^{kl}y^{sr}y^{ji}\\
&\quad+2y^{kl}\left(y^{jr}y^{si}+y^{js}y^{ri}\right)+3y^{ij}\left(y^{kr}y^{ls}+y^{ks}y^{lr}\right)\\
&\quad-y^{ir}\left(y^{ks}y^{jl}+y^{ls}y^{jk}\right)-y^{is}\left(y^{kr}y^{jl}+y^{lr}y^{jk}\right)\\
&\quad-2y^{ki}\left(y^{sl}y^{jr}+y^{rl}y^{js}\right)-2y^{li}\left(y^{sk}y^{jr}+y^{rk}y^{js}\right)\Big)y_{kl,i}y_{rs,j}.
\end{aligned} \tag{20}$$

Hence L_{HE} is an affine function and according to Proposition 1.1 its P–C form projects onto J^1M if and only if the following equations hold:

$$0=2\frac{\partial(L_{HE})^{hi}_{rs}}{\partial y_{ht,a}}-\frac{\partial(L_{HE})^{ai}_{ht}}{\partial y_{rs,h}}-\frac{\partial(L_{HE})^{ah}_{ht}}{\partial y_{rs,i}},$$

where

$$\begin{aligned}
(L_{HE})^{ij}_{rs}&=\frac{1}{2-\delta_{ij}}\frac{\partial L_{HE}}{\partial y_{rs,ij}}\\
&=\frac{1}{1+\delta_{rs}}\rho\left(y^{ir}y^{js}+y^{jr}y^{is}-2y^{rs}y^{ij}\right),
\end{aligned} \tag{21}$$

and the result follows immediately as $(L_{HE})^{ij}_{rs}$ does not depend on the variables $y_{ij,k}$.

Furthermore, in the present case, one has

$$\begin{aligned}
\theta^{2}&=\sum_{a\le b}\theta^{ab}\otimes\frac{\partial}{\partial y_{ab}}+\sum_{a\le b}\theta^{ab}_{i}\otimes\frac{\partial}{\partial y_{ab,i}},\\
\theta^{ab}&=dy_{ab}-y_{ab,k}dx^{k},\qquad \theta^{ab}_{i}=dy_{ab,i}-y_{ab,il}dx^{l},\\
p^{i}_{kl}&=\sum_{r\le s}\left(\frac{\partial^{2}(L_{HE})_0}{\partial y_{rs,j}\partial y_{kl,i}}-\frac{\partial(L_{HE})^{ij}_{kl}}{\partial y_{rs}}-\frac{\partial(L_{HE})^{ij}_{rs}}{\partial y_{kl}}\right)y_{rs,j}\\
&=\sum_{r\le s}Y^{i;rs,j}_{kl}y_{rs,j},
\end{aligned} \tag{22}$$

$$
\begin{aligned}
Y_{kl}^{i;rs,j} &= \frac{\rho}{(1+\delta_{kl})(1+\delta_{rs})}\Big[2y^{rs}y^{kl}y^{ij} - \left(y^{rk}y^{sl}+y^{rl}y^{sk}\right)y^{ij} \\
&\quad + \left(y^{sk}y^{lj}+y^{sl}y^{kj}\right)y^{ri} + \left(y^{rk}y^{lj}+y^{rl}y^{kj}\right)y^{si} \\
&\quad - \left(y^{ki}y^{lj}+y^{li}y^{kj}\right)y^{rs} - \left(y^{ri}y^{sj}+y^{rj}y^{si}\right)y^{kl}\Big], \qquad (23)\\
H &= \rho\sum_{k\le l}\sum_{r\le s}\frac{1}{(1+\delta_{rs})(1+\delta_{kl})}\Big(-y^{ij}y^{kl}y^{rs} \\
&\quad + y^{kl}\left(y^{ir}y^{js}+y^{is}y^{jr}\right) + \frac{1}{2}y^{ij}\left(y^{ks}y^{lr}+y^{kr}y^{ls}\right) \\
&\quad - \frac{1}{2}y^{ir}\left(y^{jl}y^{ks}+y^{jk}y^{ls}\right) - \frac{1}{2}y^{is}\left(y^{jl}y^{kr}+y^{jk}y^{lr}\right)\Big)\, y_{rs,j}y_{kl,i}. \qquad (24)
\end{aligned}
$$

Hence the P–C form of the H–E Lagrangian is given by

$$
\begin{aligned}
\Theta_{\Lambda_{HE}} &= \sum\nolimits_{a\le b}(-1)^{h-1}(L_{HE})_{ab}^{h0}dy_{ab}\wedge v_h \\
&\quad + \sum\nolimits_{a\le b}(-1)^{h-1}(L_{HE})_{ab}^{hj}dy_{ab,j}\wedge v_h + Hv, \qquad (25)
\end{aligned}
$$

where $H = -(L_{HE})_{ab}^{h0}y_{ab,h} - \sum_{h\le j}(L_{HE})_{ab}^{hj}y_{ab,jh} + L_{HE}$ is the Hamiltonian defined in the formula (24) and $(L_{HE})_{ab}^{i0}$ is given by the formula

$$
\begin{aligned}
(L_{HE})_{ab}^{i0} &= \rho\sum\nolimits_{r\le s}\frac{1}{(1+\delta_{ab})(1+\delta_{rs})}\Big[y^{rs}\left(y^{ai}y^{bj}+y^{bi}y^{aj}\right) \\
&\quad + y^{ji}\left(y^{ar}y^{bs}+y^{as}y^{br}\right) \\
&\quad - y^{ai}\left(y^{bs}y^{jr}+y^{br}y^{js}\right) - y^{bi}\left(y^{as}y^{jr}+y^{ar}y^{js}\right)\Big]\, y_{rs,j}. \qquad (26)
\end{aligned}
$$

Remark 3.1 As a calculation shows, from the expression in (24) for the Hamiltonian of the H–E Lagrangian, for every $j_x^1 g \in J^1M$ the following formula holds true: $H(j_x^1 g) = \rho(x)g^{ij}(x)((\Gamma^g)_{ij}^r(x)(\Gamma^g)_{hr}^h(x) - (\Gamma^g)_{hi}^r(x)(\Gamma^g)_{jr}^h(x))$. Hence the function H—considered as a first-order Lagrangian—not only provides the H–C equations for Λ_{HE} but also its own E–L equations, e.g., see [5, 3.3.1].

Theorem 3.1 (cf. [4,10,33]) *We have*

- (i) *With the natural identification $V(p_M) \cong p_M^* S^2T^*N$, the bilinear form $b_{\Lambda_{HE}}$ is defined on $p_M^*(T^*N \otimes S^2T^*N)$.*
- (ii) *The Lagrangian function $\bar{L}_{HE}$ defined in (17) coincides with the opposite to the Hamiltonian function.*
- (iii) *The H–E Lagrangian satisfies the regularity condition of Corollary 2.3.*

Proof (i) From the formula

$$
\frac{\partial p_\alpha^i}{\partial y_h^\beta} = \frac{\partial L_\alpha^{i0}}{\partial y_h^\beta} - \frac{\partial L_\beta^{ih}}{\partial y^\alpha},
$$

and (22), (23) it follows that the matrix of $b_{\Lambda_{HE}}$ in the basis $(dx^i)_x \otimes (\partial/\partial y_{jk})_{g_x}$, $g_x \in p^{-1}(x)$, $1 \le i \le n$, $1 \le j \le k \le n$, at a point $j_x^1 g$ is

$$
\left((\partial p_{mr}^j/\partial y_{cd,h})(j_x^1 g)\right)_{c\le d,h}^{m\le r,j} = \left(Y_{mr}^{j;cd,h}(g_x)\right)_{c\le d,h}^{m\le r,j},
$$

and one can conclude.

(ii) It follows from the formulas (18), (22), (24) by means of a simple calculation.
(iii) The proof is similar to that of Proposition 5.1 in [4], as

$$\frac{\partial p^j_{mr}}{\partial y_{cd,h}} = \frac{\partial^2 \bar{L}_{HE}}{\partial y_{mr,j}\partial y_{cd,h}} = \frac{\partial^2 L^\nabla}{\partial y_{mr,j}\partial y_{cd,h}},$$

where L^∇ is the first-order Lagrangian variationally equivalent to L_{HE} introduced in [4].

In the present case, equation (14) become

$$\begin{cases} 0 = \dfrac{\partial(p^i_{kl}\circ j^1 s)}{\partial x^i} - \dfrac{\partial H}{\partial y_{kl}}\circ j^1 s, & 1 \le k \le l \le n, \\ 0 = \dfrac{\partial(y_{kl}\circ s)}{\partial x^i} + \dfrac{\partial H}{\partial p^i_{kl}}\circ j^1 s, & 1 \le i \le n,\ 1 \le k \le l \le n. \end{cases}$$

□

Remark 3.2 By using the previous theorem, in [33, Theorem 6.2] the following result has been obtained: "Given symmetric scalars $\gamma^i_{jk} = \gamma^i_{kj}$, $i, j, k = 1, \dots, n$, there exists a Ricci-flat (pseudo-)Riemannian metric g of signature (n^-, n^+) defined on a neighbourhood of $x_0 \in N$ such that $g_{ij}(x_0) = \delta_{ij}$, $(\Gamma^g)^i_{jk}(x_0) = \gamma^i_{jk}$, for all i, j, k."

3.2 BF field theory

In this section, we consider a new approach to BF Lagrangians (cf. [3,6,11,21,22]) generalizing the H–E functional.

Let $p_M\colon M \to N$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n$. A classical result ([41, Appendix II], also see [25,40]) states that the only Diff$N$-invariant Lagrangian on J^2M depending linearly on the second-order coordinates $y_{ab,ij}$ is of the form $\lambda\mathcal{L}_{EH} + \mu$, for scalars λ, μ.

Since the seventies, nonlinear models have been appearing of the H–E Lagrangian; see [8,14,18], and the references cited in these papers. The Lagrangians considered in such works are either of the form $f(L_{HE})$, $f'' \neq 0$, or are linear combinations of quadratic expressions of the curvature tensor. In both cases, L is not an affine function over $J^2M \to J^1M$ (cf. Proposition 1.1). Hence, these Lagrangians are outside the framework of the problems of second order whose P–C form projects onto the first-order jet bundle, thus possessing a true Hamiltonian formalism of first order.

The interest of the generalization of the BF theory that appears below lies on the fact that, while such Lagrangians are a remarkable generalization of the H–E Lagrangian, all of them satisfy the conditions of Proposition 1.1.

Let $\pi\colon F(N) \to N$ be the principal $Gl(n, \mathbb{R})$-bundle of linear frames on N. Given a metric g on N, let $\pi_g\colon F_g(N) \subset F(N) \to N$ be the subbundle of orthonormal linear frames with respect to g, i.e., $u = (X_1, \dots, X_n)$ belongs to $F_g(N)$ if and only if, $g(X_i, X_j) = \varepsilon_i\delta_{ij}$, with $\varepsilon_i = +1$ for $1 \le i \le n^+$ and $\varepsilon_i = -1$ for $1 + n^+ \le i \le n$. This is a principal bundle with structure group the orthogonal group $O(n^+, n^-)$, $n^+ + n^- = n$, associated to the quadratic form $q(x) = \sum_{a=1}^{n^+}(x^a)^2 - \sum_{b=n^++1}^{n^++n^-}(x^b)^2$.

By virtue of the symmetries of the curvature tensor R^g of the Levi-Civita connection of a metric g, for every $X, Y \in T_xN$ the endomorphism $R^g(X, Y)$ takes values in the vector subspace of skew-symmetric linear operators (with respect to g_x) in $\mathrm{End}(T_xN) = T^*_xN \otimes T_xN$. More generally, let $p_M\colon M \to N$ be the bundle of pseudo-Riemannian metrics of signature (n^+, n^-), and let

$$\mathcal{A}(TN) \subset (p_M)^* \operatorname{End}(TN) = M \times_N \operatorname{End}(TN)$$

be the vector subbundle of the pairs (g_x, A), $g_x \in (p_M)^{-1}(x)$ and $A \in \operatorname{End}(T_x N)$, such that $g_x(AX, Y) + g_x(X, AY) = 0$, $\forall X, Y \in T_x N$; i.e., A is skew-symmetric with respect to g_x. Pulling $\mathcal{A}(TN)$ back along a metric g, understood as a smooth section of $p_M : M \to N$, one obtains the adjoint bundle of the bundle of orthonormal frames with respect to g, i.e., the bundle associated to $F_g(N)$ under the adjoint representation of $O(n^+, n^-)$ on its Lie algebra $\mathfrak{o}(n^+, n^-)$, i.e., $g^*\mathcal{A}(TN) = \operatorname{ad} F_g(N) = (F_g(N) \times \mathfrak{o}(n^+, n^-))/O(n^+, n^-)$.

If β is an $\mathcal{A}(TN)$-valued p_M-horizontal $(n-2)$-form on M, then a second-order Lagrangian density Λ_β is defined on $J^2 M$ by setting,

$$\left(\Lambda_\beta\right)_{j^2_x g} = L_\beta(j^2_x g) v(x) = \operatorname{trace}\left(\beta(g_x) \wedge R^g(x)\right), \tag{27}$$

where R^g is considered as a ad $F_g(N)$-valued 2-form on N. Locally,

$$R^g = \sum_{k<l} (R^g)^i_{jkl} dx^k \wedge dx^l \otimes dx^j \otimes \frac{\partial}{\partial x^i},$$

$$\beta = \sum_{k<l} \beta^i_{kl,j} v_{kl} \otimes dx^j \otimes \frac{\partial}{\partial x^i}, \quad \beta^i_{kl,j} \in C^\infty(M), \tag{28}$$

where $v_{kl} = dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge \widehat{dx^l} \wedge \cdots \wedge dx^n$. Here and below, we identify the vector space $\operatorname{End}(T_x N)$ to $T^*_x N \otimes T_x N$ by agreeing that $w \otimes X$ is identified to the endomorphism given by, $(w \otimes X)(Y) = w(Y)X$, $\forall X, Y \in T_x N$, $w \in T^*_x N$. Hence

$$L_\beta(j^2_x g) = \sum_{k<l} (-1)^{k+l+1} \beta^i_{kl,j}(g_x)(R^g)^j_{ikl}(x). \tag{29}$$

If we set $\beta^j_{kl,i} = -\beta^j_{lk,i}$ for $k \geq l$, then, as a calculation shows, the following local expression holds:

$$L_\beta = (-1)^{k+l+1} \beta^j_{kl,i} y^{ih} y_{hl,jk} + L^0_\beta, \tag{30}$$

with

$$\begin{aligned} L^0_\beta = \sum_{k \leq l} \sum_{r \leq s} \frac{-1}{4(1+\delta_{kl})(1+\delta_{rs})} \Big\{ & \Big[(-1)^s \beta^{kl}_{st} y^{tr} + (-1)^r \beta^{kl}_{rt} y^{ts}\Big] y^{ij} \\ & + \Big[(-1)^j \beta^{li}_{jt} y^{tr} + (-1)^r \beta^{li}_{rt} y^{tj}\Big] y^{ks} + \Big[(-1)^j \beta^{ki}_{jt} y^{tr} + (-1)^r \beta^{ki}_{rt} y^{tj}\Big] y^{ls} \\ & + \Big[(-1)^j \beta^{li}_{jt} y^{ts} + (-1)^s \beta^{li}_{st} y^{tj}\Big] y^{kr} + \Big[(-1)^j \beta^{ki}_{jt} y^{ts} + (-1)^s \beta^{ki}_{st} y^{tj}\Big] y^{lr} \\ & - \Big[(-1)^s \beta^{li}_{st} y^{tr} + (-1)^r \beta^{li}_{rt} y^{ts}\Big] y^{kj} - \Big[(-1)^s \beta^{ki}_{st} y^{tr} + (-1)^r \beta^{ki}_{rt} y^{ts}\Big] y^{lj} \\ & - \Big[(-1)^k \beta^{rj}_{kt} y^{tl} + (-1)^l \beta^{rj}_{lt} y^{tk}\Big] y^{is} - \Big[(-1)^k \beta^{sj}_{kt} y^{tl} + (-1)^l \beta^{sj}_{lt} y^{tk}\Big] y^{ir} \Big\} \\ & \cdot y_{kl,i} y_{rs,j}, \end{aligned}$$

where $\beta^{jk}_{lt} = (-1)^k \beta^j_{kl,t} + (-1)^j \beta^k_{jl,t}$, and the equations $\beta^d_{ac,i} y^{ib} + \beta^b_{ac,i} y^{id} = 0$ have been used, which hold because β takes values in $\mathcal{A}(TN)$.

Remark 3.3 As the functions $\beta^j_{kl,i}$ and y^{hi} do not depend on the first partial derivatives $y_{ab,c}$, the formula (30) proves that the Lagrangian L_β satisfies the conditions of Proposition 1.1 and hence the theory developed here can be applied to all these Lagrangians.

Remark 3.4 Attached to each $\mathcal{A}(TN)$-valued p_M-horizontal $(n-2)$-form β on M, there exists a section $\tilde{\beta}$ of the vector bundle $(p_M)^*(\wedge^2 TN)\otimes\mathcal{A}(TN)$, given by

$$\tilde{\beta}(g_x)=\beta(g_x)\circ\left(\phi_v^2\otimes\mathrm{id}_{\mathcal{A}(TN)}\right)^{-1},\quad\forall g_x\in M,$$

where ϕ_v^2 is the isomorphism defined in (16). If β is locally given as in (28), then

$$\tilde{\beta}(g_x)=\textstyle\sum_{k<l}(-1)^{k+l}\beta^i_{kl,j}(g_x)\left(\frac{\partial}{\partial x^k}\right)_x\wedge\left(\frac{\partial}{\partial x^l}\right)_x\otimes(dx^j)_x\otimes\left(\frac{\partial}{\partial x^i}\right)_x,$$
$$\forall g_x\in M.$$

If $\mathrm{sym}_{14}\colon\otimes^4 T_xN\to\otimes^4 T_xN$ is the symmetrization operator of the arguments 1 and 4, i.e., $\mathrm{sym}_{14}(X_1\otimes X_2\otimes X_3\otimes X_4)=X_1\otimes X_2\otimes X_3\otimes X_4+X_4\otimes X_2\otimes X_3\otimes X_1$, for all $X_i\in T_xN$, $1\le i\le 4$, and for every $p\ge 0$, $q\ge 1$, the symbol $\sharp$ denotes the isomorphism $\otimes^{p+1}T_x^*N\otimes^{q-1}T_xN\to\otimes^p T_x^*N\otimes^q T_xN$ induced by the metric g_x, then

$$\mathrm{sym}_{14}\left(\tilde{\beta}^\sharp(g_x)\right)=(-1)^l\beta^{ik}_{lj}(g_x)g^{jt}(x)\left(\frac{\partial}{\partial x^k}\right)_x\otimes\left(\frac{\partial}{\partial x^l}\right)_x\otimes\left(\frac{\partial}{\partial x^t}\right)_x\otimes\left(\frac{\partial}{\partial x^i}\right)_x,$$

and the formula (30) can be rewritten as, $L_\beta=(-1)^{c+1}\beta^{ab}_{ci}y^{id}y_{ab,cd}+L^0_\beta$.

Theorem 3.2 *Let Λ_β be the Lagrangian density attached to a $\mathcal{A}(TN)$-valued p_M-horizontal $(n-2)$-form β as defined in (27). Then*

(i) *The Lagrangian function (29) coincides with the H–E Lagrangian (i.e., $L_\beta=L_{HE}$) if and only if the form β is given by,*

$$(\beta_{HE})^j_{kl,i}=(-1)^{k+l+1}\rho\left(\delta^{ik}y^{jl}-\delta^{il}y^{jk}\right),\tag{31}$$

where the function ρ is defined in (19).

(ii) *With the natural identification $V(p_M)\cong p_M^*S^2T^*N$, the bilinear form b_{Λ_β} is defined on $p_M^*(T^*N\otimes S^2T^*N)$.*

(iii) *The E–L equations for the Lagrangian density Λ_β are the following:*

$$g^*(d_{M/N}\beta)\barwedge R^g+\mathrm{sym}_{12}\circ d^{\nabla^g}(\omega_{n-1}(g,\beta))=0,\tag{32}$$

where,

- *∇^g is the covariant differentiation with respect to the Levi-Civita connection of a section g of the bundle $p_M\colon M\to N$.*
- *The fibre differential $d_{M/N}\beta$ is understood to be a section of the vector bundle $(p_M)^*((S^2TN)\otimes\wedge^{n-2}T^*N\otimes\mathrm{End}(TN))$, taking the isomorphism $V^*(p_M)\cong(p_M)^*(S^2TN)$ into account.*
- *$g^*(d_{M/N}\beta)\barwedge R^g$ is the S^2TN-valued n-form on N defined by,*

$$\begin{aligned}&\left(g^*(d_{M/N}\beta)\barwedge R^g\right)(w_1,w_2,X_1,\dots,X_n)\\&=\textstyle\sum_{k<l}(-1)^{k+l+1}\cdot\\&\ \mathrm{trace}\left\{g^*(d_{M/N}\beta)\left(w_1,w_2,X_1,\dots,\widehat{X_k},\dots,\widehat{X_l},\dots,X_n\right)\circ R^g(X_k,X_l)\right\},\\&\forall X_1,\dots,X_n\in T_xN,\forall w_1,w_2\in T_x^*N.\end{aligned}$$

- *$\omega_{n-1}(g,\beta)$ is the $(TN\otimes TN)$-valued $(n-1)$-form on N given by,*

$$\omega_{n-1}(g,\beta)=\left((\phi_v^1)^{-1}\otimes\mathrm{id}_{TN}\otimes\phi_v^1\right)\left(d^{\nabla^g}(g^*\beta)^\sharp\right),$$

ϕ_v^1 being defined in the formula (16).
- *$\mathrm{sym}_{12}\colon\otimes^2TN\to S^2TN$ denotes the symmetrization operator.*

Proof (i) By comparing the formula (29) with the following:

$$L_{HE}(j_x^2 g) = \sum_{k<l} \rho(x) \left(\delta^{ik} g^{jl}(x) - \delta^{il} g^{jk}(x) \right) (R^g)^i_{jkl}(x),$$

we obtain (31) directly.
(ii) As a calculation shows, the matrix of b_{Λ_β} is given as follows:

$$\begin{aligned}
&\left(F_\beta\right)_{r\le s;i,a\le b,j} \\
&= \frac{\partial^2 L^0_\beta}{\partial y_{rs,i}\partial y_{ab,j}} - \frac{\partial L^{ij}_{ab}}{\partial y_{rs}} - \frac{\partial L^{ij}_{rs}}{\partial y_{ab}} \\
&= \frac{1}{2}\frac{1}{1+\delta_{ab}}\frac{1}{1+\delta_{rs}} \Big\{ -\left[(-1)^a \beta^{rs}_{at} y^{tb} + (-1)^b \beta^{rs}_{bt} y^{ta}\right] y^{ij} \\
&\quad + \left[(-1)^j \beta^{rs}_{jt} y^{tb} + (-1)^b \beta^{rs}_{bt} y^{tj}\right] y^{ia} + \left[(-1)^a \beta^{rs}_{at} y^{tj} + (-1)^j \beta^{rs}_{jt} y^{ta}\right] y^{ib} \\
&\quad + \left[(-1)^i \beta^{ab}_{it} y^{ts} + (-1)^s \beta^{ab}_{st} y^{ti}\right] y^{rj} + \left[(-1)^i \beta^{ab}_{it} y^{tr} + (-1)^r \beta^{ab}_{rt} y^{ti}\right] y^{sj} \\
&\quad - \left[(-1)^b \beta^{is}_{bt} y^{tj} + (-1)^j \beta^{is}_{jt} y^{tb}\right] y^{ra} - \left[(-1)^b \beta^{ir}_{bt} y^{tj} + (-1)^j \beta^{ir}_{jt} y^{tb}\right] y^{sa} \\
&\quad - \left[(-1)^a \beta^{is}_{at} y^{tj} + (-1)^j \beta^{is}_{jt} y^{ta}\right] y^{rb} - \left[(-1)^a \beta^{ir}_{at} y^{tj} + (-1)^j \beta^{ir}_{jt} y^{ta}\right] y^{sb} \\
&\quad - (-1)^a \beta^{ij}_{at} \left(y^{tr} y^{bs} + y^{ts} y^{br}\right) - (-1)^b \beta^{ij}_{bt} \left(y^{tr} y^{as} + y^{ts} y^{ar}\right) \\
&\quad - (-1)^r \beta^{ij}_{rt} \left(y^{ta} y^{bs} + y^{tb} y^{as}\right) - (-1)^s \beta^{ij}_{st} \left(y^{ta} y^{br} + y^{tb} y^{ar}\right) \\
&\quad + (1+\delta_{rs}) \left((-1)^a \frac{\partial \beta^{ij}_{at}}{\partial y_{rs}} y^{tb} + (-1)^b \frac{\partial \beta^{ij}_{bt}}{\partial y_{rs}} y^{ta} \right) \\
&\quad + (1+\delta_{ab}) \left((-1)^r \frac{\partial \beta^{ij}_{rt}}{\partial y_{ab}} y^{ts} + (-1)^s \frac{\partial \beta^{ij}_{st}}{\partial y_{ab}} y^{tr} \right) \Big\},
\end{aligned}$$

thus proving the statement.
(iii) The E–L equations for the Lagrangian density $\Lambda_\beta = L_\beta v$ are straightforwardly computed, thus obtaining,

$$\begin{aligned}
\mathcal{E}^{ab}(L_\beta) \circ j^2 g = {} & \frac{1}{2}(-1)^{k+l+1} \left(\frac{\partial \beta^j_{kl,i}}{\partial y_{ab}} \circ g \right) (R^g)^i_{jkl} \\
& - \frac{1}{1+\delta_{ab}} \left\{ \frac{\partial}{\partial x^r} \left[(-1)^a \Phi^{rb}_a + (-1)^b \Phi^{ra}_b \right] \right. \\
& \left. + (-1)^l \left[\Phi^{rb}_l (\Gamma^g)^a_{rl} + \Phi^{ra}_l (\Gamma^g)^b_{rl} \right] \right\},
\end{aligned}$$

for $1 \le a \le b \le n$, where

$$\Phi^{rb}_a = \sum_k (-1)^k \left(-\frac{\partial (\beta \circ g)^b_{ka,i}}{\partial x^k} + (\beta \circ g)^b_{ka,m} (\Gamma^g)^m_{ki} - (\beta \circ g)^m_{ka,i} (\Gamma^g)^b_{km} \right) g^{ri}.$$

Moreover, the following local expressions are deduced:

$$g^*(d_{M/N}\beta)\,\overline{\wedge}\,R^g = \frac{1}{2}(-1)^{k+l+1}\left(\frac{\partial\beta^j_{kl,i}}{\partial y_{ab}}\circ g\right)(R^g)^i_{jkl}\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^b}\otimes v,$$

$$\left(d^{\nabla^g}(g^*\beta)\right)^\sharp = \sum\nolimits_l \Phi^{ab}_l v_l\otimes\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^b},$$

from which the result follows. □

Corollary 3.3 *A flat metric g is a solution to equation (32) if and only if the form β in (28) satisfies the following equation:*

$$c^{23}_{12}\left[\left(\nabla^g\right)^2\left\{\mathrm{sym}_{14}\left(\tilde{\beta}^\sharp\circ g\right)\right\}\right] = 0,$$

where $c^{23}_{12}\colon \otimes^2T^*M\otimes^4TM\to\otimes^2TM$ denotes the contraction operator of the first covariant index with the second contravariant one, and the second covariant index with the third contravariant one.

Remark 3.5 The geometric construction of the form (31) is as follows: Given an arbitrary system $X_1,\dots,X_{n-2}\in T_xN$, we must define a skew-symmetric (with respect to g_x) endomorphism $\beta(g_x)(X_1,\dots,X_{n-2})\colon T_xN\to T_xN$.

If the given system is linearly dependent, then $\beta(g_x)(X_1,\dots,X_{n-2}) = 0$. We assume: i) the system $(X_1,\dots,X_{n-2})$ is linearly independent. Hence its orthogonal $\Pi = \langle X_1,\dots,X_{n-2}\rangle^\perp$ is a subspace of dimension 2 in T_xN; ii) the subspace $\langle X_1,\dots,X_{n-2}\rangle$ is not singular with respect to g_x. Hence

$$T_xN = \Pi\oplus\langle X_1,\dots,X_{n-2}\rangle,$$

and Π is also non-singular. Let $(n^+(\Pi),n^-(\Pi))\in\{(2,0),(1,1),(0,2)\}$ be its signature and let

$$\begin{pmatrix}\varepsilon_1(\Pi) & 0\\ 0 & \varepsilon_2(\Pi)\end{pmatrix},\quad (\varepsilon_1(\Pi),\varepsilon_2(\Pi))\in\{(1,1),(1,-1),(-1,-1)\},$$

be the matrix of g_x in an orthonormal basis (Y_1,Y_2) of Π, which, in addition, is assumed to satisfy the following: $v(X_1,\dots,X_{n-2},Y_1,Y_2)>0$. If $Z_j = b^i_jY_i$, $i,j=1,2$, is another orthonormal basis with $v(X_1,\dots,X_{n-2},Z_1,Z_2)>0$, then $\det(b^i_j)=1$. Hence (b^i_j) belongs to $SO(n^+(\Pi),n^-(\Pi))$, and the endomorphism $J^{g_x}_\Pi\colon\Pi\to\Pi$ given by $J^{g_x}_\Pi(Y_1)=\varepsilon_1(\Pi)Y_2$, $J^{g_x}_\Pi(Y_2)=-\varepsilon_2(\Pi)Y_1$, is independent of the basis chosen (as $SO(n^+(\Pi),n^-(\Pi))$ is commutative) and skew-symmetric. We define $\tilde{J}^{g_x}_\Pi\colon T_xN\to T_xN$ by setting, $\tilde{J}^{g_x}_\Pi|_\Pi = J^{g_x}_\Pi$, $\tilde{J}^{g_x}_\Pi|_{\langle X_1,\dots,X_{n-2}\rangle}=0$. Finally,

$$(\beta_{HE})(g_x)(X_1,\dots,X_{n-2}) = \det\left(g(Y_a,Y_b)\right)^2_{a,b=1}v_{g_x}(X_1,\dots,X_{n-2},Y_1,Y_2)\,\tilde{J}^{g_x}_\Pi.$$

Remark 3.6 The bilinear form b_{Λ_β} is identified to a section of the vector bundle $p^*_M\left((TN\otimes S^2TN)\otimes(TN\otimes S^2TN)\right)$, and the following formula holds:

$$b_{\Lambda_\beta} = \frac{1}{2}\,\mathrm{sym}_{45}\left(\mathrm{alt}_{46}\left[\mathrm{sym}_{12}(\hat{\beta})+\mathrm{alt}_{13}(\hat{\beta})-\hat{\beta}\right]\right)-\frac{1}{2}\left[\mathrm{sym}_{12}(\hat{\beta})+\mathrm{alt}_{13}(\hat{\beta})-\hat{\beta}\right]$$
$$+\frac{1}{2}\,\mathrm{sym}_{(12),(45)}\,d_{M/N}\left(\mathrm{sym}_{23}\left(\mathrm{sym}_{14}(\tilde{\beta}^\sharp)\right)\right),$$

where the operators alt_{ij}, sym_{ij}, $\mathrm{sym}_{(1,2)(4,5)} \colon \otimes^6 T_x N \to \otimes^6 T_x N$, $1 \le i < j \le 6$, are defined as follows:

$$\begin{aligned}
&\mathrm{alt}_{ij}(X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6) = \\
&X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6 - X_1 \otimes \cdots \otimes X_j \otimes \cdots \otimes X_i \otimes \cdots \otimes X_6, \\
&\mathrm{sym}_{ij}(X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6) = \\
&X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6 + X_1 \otimes \cdots \otimes X_j \otimes \cdots \otimes X_i \otimes \cdots \otimes X_6, \\
&\mathrm{sym}_{(1,2)(4,5)}(X_1 \otimes \cdots \otimes X_6) = \\
&X_1 \otimes \cdots \otimes X_6 + X_4 \otimes X_5 \otimes X_3 \otimes X_1 \otimes X_2 \otimes X_6, \\
&X_1, \ldots, X_6 \in T_x N,
\end{aligned}$$

and the contravariant 6-tensor $\hat{\beta}$ is given by,

$$\hat{\beta} = \mathrm{sym}_{15}\left[\mathrm{sym}_{23}\left(\mathrm{sym}_{14}(\tilde{\beta}^\sharp)\right) \otimes (g^\sharp)^\sharp\right] - \mathrm{sym}_{23}\left(\mathrm{sym}_{14}(\tilde{\beta}^\sharp)\right) \otimes (g^\sharp)^\sharp.$$

Remark 3.7 If $\beta = \beta_{HE}$ in Theorem 3.2-(iii), then the functions Φ_a^{rb} (appearing in the proof) vanish, and Eq. (32) reduce to Einstein's vacuum equations for arbitrary signature.

4 First-order equivalent Lagrangians

Theorem 4.1 *Let $\Lambda = Lv$ be a second-order Lagrangian density on $p \colon E \to N$ whose Poincaré–Cartan form projects onto $J^1 E$. We have*

- (i) *The H–C equations of the first-order Lagrangian $\bar{L}v$ given in (17) coincide locally with the H–C equations of Λ. Furthermore, if $\bar{L}'$ is another first-order Lagrangian fulfilling this property, then $\bar{L}'v - \bar{L}v = D\alpha_{n-1}$, where D denotes the horizontal exterior derivative and α_{n-1} is a p-horizontal $(n-1)$-form on E.*
- (ii) *The E–L equations of Λ, considered as a second-order partial differential system, satisfy the Helmholtz conditions.*
- (iii) *The E–L equations of the first-order Lagrangian $\bar{L}v$ above coincide with E–L equations of Λ.*
- (iv) *Let ϕ_v^1 be the isomorphism defined in (16) for $k = 1$ and let $w_L^{0,\sigma}$ be the TN-valued section on $J^1 E$ defined as in Proposition 1.2. The composite mapping $\phi_v^1 \circ w_L^{0,\sigma}$ can be viewed as a p^1-horizontal $(n-1)$-form on $J^1 E$ and the difference $\bar{L}_\sigma v = Lv - D(\phi_v^1 \circ w_L^{0,\sigma})$ determines a globally defined first-order Lagrangian which is variationally equivalent to Lv, but this is not canonically attached to Lv as it depends on the section σ.*

Proof (i) Locally, the Hamiltonian and the momenta associated to $\bar{L}$ are given, respectively, by [cf. formula (10) in Remark 2.1],

$$\bar{H} = \bar{L} - y_i^\alpha \frac{\partial \bar{L}}{\partial y_i^\alpha}, \quad \bar{p}_\alpha^i = \frac{\partial \bar{L}}{\partial y_i^\alpha}.$$

By comparing the H–C equations for $\bar{L}$ with the H–C equations for L given in (14), one obtains, $H = \bar{H}$ and $p_\alpha^i = \bar{p}_\alpha^i$. Hence

$$L_0 - y_i^\alpha L_\alpha^{i0} - \frac{\partial L^i}{\partial x^i} = \bar{L} - y_i^\alpha \frac{\partial \bar{L}}{\partial y_i^\alpha}, \tag{33}$$

$$L_\alpha^{i0} - \frac{\partial L^i}{\partial y^\alpha} = \frac{\partial \bar{L}}{\partial y_i^\alpha}. \tag{34}$$

Replacing (34) into (33), one concludes that $\bar{L}$ is given as in the formula (17). Moreover, if $\bar{L}'$ is the first-order Lagrangian associated to other primitive functions $L'^i = L^i + A^i$, $A^i \in C^\infty(E)$, according to Proposition 1.2, then $\bar{L}' = \bar{L} - D_i A^i$.
(ii) As a simple—although rather long—computation shows, the second-order differential operator $\mathcal{E}_\alpha(L)dy^\alpha \wedge v$ satisfies the equations (1.5a), (1.5b), and (1.5c) in [2]. In fact, by using the formulas (1), (2), and (8), the following equations are checked:

$$\text{(1.5a)}\ 0 = \frac{\partial \mathcal{E}_\alpha(L)}{\partial y^\sigma_{(ij)}} - \frac{\partial \mathcal{E}_\sigma(L)}{\partial y^\alpha_{(ij)}},$$

$$\text{(1.5b)}\ 0 = \frac{\partial \mathcal{E}_\alpha(L)}{\partial y^\sigma_i} + \frac{\partial \mathcal{E}_\sigma(L)}{\partial y^\alpha_i} - (1+\delta_{ij})D_j\left(\frac{\partial \mathcal{E}_\sigma(L)}{\partial y^\alpha_{(ij)}}\right),$$

$$\text{(1.5c)}\ 0 = \frac{\partial \mathcal{E}_\alpha(L)}{\partial y^\sigma} - \frac{\partial \mathcal{E}_\sigma(L)}{\partial y^\alpha} + D_i\left(\frac{\partial \mathcal{E}_\sigma(L)}{\partial y^\alpha_i}\right) - \textstyle\sum_{i\le j} D_i D_j\left(\frac{\partial \mathcal{E}_\sigma(L)}{\partial y^\alpha_{(ij)}}\right).$$

(iii) From the formula (17), it follows that the Lagrangian $\bar{L}$ can also be written as $\bar{L} = L - D_i L^i$, thus proving that L and $\bar{L}$ differ on a total divergence and hence $\mathcal{E}_\alpha(L) = \mathcal{E}_\alpha(\bar{L})$.
(iv) Locally, $w^{0,\sigma}_L = L^i_\sigma \partial/\partial x^i$; hence, $\phi^1_v \circ w^{0,\sigma}_L = (-1)^{i-1} L^i_\sigma v_i$, and consequently, $D(\phi^1_v \circ w^{0,\sigma}_L) = (D_i L^i_\sigma)v$. The result thus follows from $\bar{L}_\sigma = L - D_i L^i_\sigma$ in item (iii). □

Remark 4.1 As is known (e.g., see [16, (2.21)–(2.25)]), the Vainberg-Tonti Lagrangian L_{VT} attached to a second-order affine Lagrangian as in (4) is also affine, say $L_{VT} = (L_{VT})_0 + (L_{VT})_1$, with $(L_{VT})_1 = (L_{VT})^{ij}_\alpha y^\alpha_{(ij)}$. Then, as a computation shows, one has

$$L_{VT} - \bar{L} = -D_h\left(\int_0^1 y^\alpha \left(\frac{\partial \bar{L}}{\partial y^\alpha_h} \circ \chi_\lambda\right) \mathrm{d}\lambda\right),$$

where $\chi_\lambda(x^i, y^\alpha, y^\alpha_i) = (x^i, \lambda y^\alpha, \lambda y^\alpha_i)$, but it should be noted that the Vainberg-Tonti Lagrangian is of second order in the general case; e.g., if $L(x, y, \dot{y}, \ddot{y}) = L_1(x, y, \dot{y})\ddot{y} + L_0(x, y, \dot{y})$, then $L_{VT} = (L_{VT})_0 + (L_{VT})_1\ddot{y}$, with

$$\begin{aligned}(L_{VT})_1 = y\int_0^1 \Big\{&2\lambda\frac{\partial L_1}{\partial y}(x,\lambda y,\lambda\dot{y}) + \lambda\frac{\partial^2 L_1}{\partial x\partial\dot{y}}(x,\lambda y,\lambda\dot{y})\\ &+\lambda^2\dot{y}\frac{\partial^2 L_1}{\partial y\partial\dot{y}}(x,\lambda y,\lambda\dot{y}) - \lambda\frac{\partial^2 L_0}{\partial\dot{y}^2}(x,\lambda y,\lambda\dot{y})\Big\}\mathrm{d}\lambda,\\ (L_{VT})_0 = y\int_0^1 \Big\{&\frac{\partial L_0}{\partial y}(x,\lambda y,\lambda\dot{y}) + \frac{\partial^2 L_1}{\partial x^2}(x,\lambda y,\lambda\dot{y})\\ &+\lambda^2(\dot{y})^2\frac{\partial^2 L_1}{\partial y^2}(x,\lambda y,\lambda\dot{y}) + 2\lambda\dot{y}\frac{\partial^2 L_1}{\partial x\partial y}(x,\lambda y,\lambda\dot{y})\\ &-\frac{\partial^2 L_0}{\partial x\partial\dot{y}}(x,\lambda y,\lambda\dot{y}) - \lambda\dot{y}\frac{\partial^2 L_0}{\partial y\partial\dot{y}}(x,\lambda y,\lambda\dot{y})\Big\}\mathrm{d}\lambda.\end{aligned}$$

Therefore, L_{VT} is of second order, except when $(L_{VT})_1 = 0$, and this latter condition is seen to be equivalent to the following:

$$0 = 2\frac{\partial L_1}{\partial y} + \frac{\partial^2 L_1}{\partial x\partial\dot{y}} + \dot{y}\frac{\partial^2 L_1}{\partial y\partial\dot{y}} - \frac{\partial^2 L_0}{\partial\dot{y}^2}.$$

In the particular case of the bundle of metrics, there exists a more specific way to obtain a section σ of $p^1_0\colon M \to J^1 M$ than the procedure suggested in Remark 1.2, which depends on a linear connection only rather than a nonlinear connection; namely,

Lemma 4.2 *Let $p_M\colon M \to N$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n$, and let ∇ be a symmetric linear connection on N. For every $g_x \in (p_M)^{-1}(x)$, there exists a unique 1-jet of metric $j^1_x\tilde{g} \in J^1_x M$ such that 1) $\tilde{g}_x = g_x$, and 2) $(\nabla\tilde{g})_x = 0$. The mapping $\sigma^\nabla\colon M \to J^1M$ given by $\sigma^\nabla(g_x) = j^1_x\tilde{g}$ is a section of $p^1_0\colon J^1M \to M$.*

Proof If Γ^i_{jk} are the local symbols of ∇ in a coordinate system, then as a calculation shows, the condition (2)—assuming (1)—of the statement is equivalent to,

$$\frac{\partial\tilde{g}_{ij}}{\partial x^k}(x) = \Gamma^h_{ik}(x)g_{hj}(x) + \Gamma^h_{jk}(x)g_{hi}(x),$$

thus proving that σ^∇ makes sense. □

Proposition 4.3 (cf. [4, II]) *Let $p_M\colon M \to N$ be as in Lemma 4.2. For the H–E Lagrangian, the density $(\bar{L}_{HE})_{\sigma^\nabla}v$ introduced in Theorem 4.1-(iv) is given by, $(\bar{L}_{HE})_{\sigma^\nabla}(j^2_x g)v_x = c\left(\left(\mathrm{alt}_{23}\,(\nabla^g T^g)_x\right)^\sharp\right)\left(v_g\right)_x$, for all $j^2_x g \in J^2M$, where*

$$\mathrm{alt}_{23}\colon \otimes^3 T^*M \otimes TM \to \otimes^3 T^*M \otimes TM$$

denotes the alternating operator of the second and third covariant indices, and

$$\sharp\colon \otimes^3 T^*M \otimes TM \to \otimes^2 T^*M \otimes^2 TM$$

is the isomorphism induced by g, i.e.,

$$w_1 \otimes w_2 \otimes w_3 \otimes X \mapsto w_1 \otimes w_2 \otimes (w_3)^\sharp \otimes X,$$

and $c\colon \otimes^2 T^*M \otimes^2 TM \to \mathbb{R}$ is the total contraction of the first (resp. second) covariant index with the first (resp. second) contravariant one.

5 Symmetries and Noether invariants

Given fibred manifolds $p\colon E \to N$, $p'\colon E' \to N'$, every morphism $\Phi\colon E \to E'$ for which the associated map on the base manifolds $\phi\colon N \to N'$ is a diffeomorphism, induces a map

$$\Phi^{(r)}\colon J^rE \to J^rE',$$
$$\Phi^{(r)}(j^r_x s) = j^r_{\phi(x)}(\Phi \circ s \circ \phi^{-1}).$$

If Φ_t is the flow of a p-projectable vector field X, then $\Phi^{(r)}_t$ is the flow of a vector field $X^{(r)} \in \mathfrak{X}(J^rE)$, called the infinitesimal contact transformation of order r associated to the vector field X. The mapping $X \mapsto X^{(r)}$ is an injection of Lie algebras. For $r = 1, 2$, the general prolongation formulas read as follows:

$$X = u^i\frac{\partial}{\partial x^i} + v^\alpha\frac{\partial}{\partial y^\alpha},$$
$$u^i \in C^\infty(N),\, v^\alpha \in C^\infty(E),$$
$$X^{(1)} = u^i\frac{\partial}{\partial x^i} + v^\alpha\frac{\partial}{\partial y^\alpha} + v^\alpha_i\frac{\partial}{\partial y^\alpha_i},$$
$$v^\alpha_i = D_i\left(v^\alpha - u^h y^\alpha_h\right) + u^h y^\alpha_{(hi)},$$

$$X^{(2)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha} + \sum_{i\leq j} v_{ij}^\alpha \frac{\partial}{\partial y_{(ij)}^\alpha},$$
$$v_{ij}^\alpha = D_i D_j \left(v^\alpha - u^h y_h^\alpha \right) + u^h y_{(hij)}^\alpha.$$

Theorem 5.1 *Let $\Lambda = Lv$ be a second-order Lagrangian density on $p\colon E \to N$ with P–C form projectable onto J^1E. If X is a p-projectable vector field on E, then the P–C form of the second-order Lagrangian density $\Lambda' = L'v = L_{X^{(2)}}\Lambda$ also projects onto J^1E and the following formula holds:*

$$\Theta_{L_{X^{(2)}}\Lambda} = L_{X^{(1)}}\Theta_\Lambda.$$

Therefore, if $s\colon N \to E$ is an extremal for Λ and X is an infinitesimal symmetry (i.e., $L_{X^{(2)}}\Lambda = 0$), then the $(n-1)$-form $(j^1s)^* i_{X^{(1)}}\Theta$ is closed. The $(n-1)$-form $i_{X^{(1)}}\Theta$ is called the Noether invariant associated to X; also, see [34].

Proof We have $L' = X^{(2)}(L) + \operatorname{div}(X')L$, X' being the projection of X onto N and $\operatorname{div}(X')$ the divergence of X' with respect to v. According to Proposition 1.1, we must prove the existence of functions L'_0, $L_\alpha^{\prime ji} = L_\alpha^{\prime ij}$ on J^1E such that

$$L' = L_\alpha^{\prime ij} y_{(ij)}^\alpha + L'_0,$$
$$\frac{\partial L_\beta^{\prime ih}}{\partial y_a^\alpha} = \frac{\partial L_\alpha^{\prime ia}}{\partial y_h^\beta},\ a, h, i = 1, \dots, n,\ \alpha, \beta = 1, \dots, m.$$

As L satisfies such formulas by virtue of the hypothesis, and $X^{(2)}$ projects onto $X^{(1)}$, we have

$$\begin{aligned} L' &= \left[X^{(1)}\left(L_\alpha^{ab}\right) + \frac{\partial v^\beta}{\partial y^\alpha} L_\beta^{ab} - 2\frac{\partial u^a}{\partial x^r} L_\alpha^{br} + \operatorname{div}(X')L_\alpha^{ab} \right] y_{(ab)}^\alpha \\ &\quad + L_\alpha^{ij}\left(\frac{\partial^2 v^\alpha}{\partial x^i \partial x^j} + y_j^\beta \frac{\partial^2 v^\alpha}{\partial x^i \partial y^\beta} + y_j^\beta y_i^\gamma \frac{\partial^2 v^\alpha}{\partial y^\beta \partial y^\gamma} - \frac{\partial^2 u^h}{\partial x^i \partial x^j} y_h^\alpha \right) \\ &\quad + X^{(1)}(L_0) + \operatorname{div}(X')L_0. \end{aligned} \tag{35}$$

Hence

$$L_\alpha^{\prime ab} = X^{(1)}\left(L_\alpha^{ab}\right) + \frac{\partial v^\beta}{\partial y^\alpha} L_\beta^{ab} - 2\frac{\partial u^a}{\partial x^r} L_\alpha^{br} + \operatorname{div}(X')L_\alpha^{ab}, \tag{36}$$

$$\begin{aligned} L'_0 = L_\alpha^{ij}&\left(\frac{\partial^2 v^\alpha}{\partial x^i \partial x^j} + \frac{\partial^2 v^\alpha}{\partial x^i \partial y^\beta} y_j^\beta + \frac{\partial^2 v^\alpha}{\partial x^j \partial y^\beta} y_i^\beta + \frac{\partial^2 v^\alpha}{\partial y^\beta \partial y^\gamma} y_j^\beta y_i^\gamma - \frac{\partial^2 u^h}{\partial x^i \partial x^j} y_h^\alpha \right) \\ &+ X^{(1)}(L_0) + \operatorname{div}(X')L_0. \end{aligned} \tag{37}$$

From the formula (37), it also follows:

$$L'_0 = L_\alpha^{ij} v_{ij}^\alpha - \left(L_\beta^{ij} \frac{\partial v^\beta}{\partial y^\alpha} - 2\frac{\partial u^i}{\partial x^r} L_\alpha^{jr} \right) y_{(ij)}^\alpha + X^{(1)}(L_0) + \operatorname{div}(X')L_0.$$

Replacing $\frac{\partial (X^{(1)}(L_\alpha^{ij}))}{\partial y_h^\beta} = X^{(1)}\left(\frac{\partial L_\alpha^{ij}}{\partial y_h^\beta}\right) + \frac{\partial v^\gamma}{\partial y^\beta}\frac{\partial L_\alpha^{ij}}{\partial y_h^\gamma} - \frac{\partial u^h}{\partial x^a}\frac{\partial L_\alpha^{ij}}{\partial y_a^\beta}$ into the formula for $\frac{\partial L_\alpha^{\prime ij}}{\partial y_h^\beta}$, we obtain

$$\begin{aligned}\frac{\partial L'^{ij}_{\alpha}}{\partial y^{\beta}_{h}} &= X^{(1)}\left(\frac{\partial L^{ij}_{\alpha}}{\partial y^{\beta}_{h}}\right) + \frac{\partial v^{\gamma}}{\partial y^{\beta}}\frac{\partial L^{ij}_{\alpha}}{\partial y^{\gamma}_{h}} - \frac{\partial u^{h}}{\partial x^{a}}\frac{\partial L^{ij}_{\alpha}}{\partial y^{\beta}_{a}} + \frac{\partial v^{\gamma}}{\partial y^{\alpha}}\frac{\partial L^{ij}_{\gamma}}{\partial y^{\beta}_{h}}\\ &\quad - \frac{\partial u^{j}}{\partial x^{a}}\frac{\partial L^{ia}_{\alpha}}{\partial y^{\beta}_{h}} - \frac{\partial u^{i}}{\partial x^{a}}\frac{\partial L^{aj}_{\alpha}}{\partial y^{\beta}_{h}} + \operatorname{div}(X')\frac{\partial L^{ij}_{\alpha}}{\partial y^{\beta}_{h}},\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\partial L'^{ih}_{\beta}}{\partial y^{\alpha}_{j}} &= X^{(1)}\left(\frac{\partial L^{ih}_{\beta}}{\partial y^{\alpha}_{j}}\right) + \frac{\partial v^{\gamma}}{\partial y^{\alpha}}\frac{\partial L^{ih}_{\beta}}{\partial y^{\gamma}_{j}} - \frac{\partial u^{j}}{\partial x^{a}}\frac{\partial L^{ih}_{\beta}}{\partial y^{\alpha}_{a}} + \frac{\partial v^{\gamma}}{\partial y^{\beta}}\frac{\partial L^{ih}_{\gamma}}{\partial y^{\alpha}_{j}}\\ &\quad - \frac{\partial u^{h}}{\partial x^{a}}\frac{\partial L^{ia}_{\beta}}{\partial y^{\alpha}_{j}} - \frac{\partial u^{i}}{\partial x^{a}}\frac{\partial L^{ah}_{\beta}}{\partial y^{\alpha}_{j}} + \operatorname{div}(X')\frac{\partial L^{ih}_{\beta}}{\partial y^{\alpha}_{j}},\end{aligned}$$

and taking the formulas (5) into account, we can conclude that $\frac{\partial L'^{ij}_{\alpha}}{\partial y^{\beta}_{h}} = \frac{\partial L'^{ih}_{\beta}}{\partial y^{\alpha}_{j}}$. Moreover, from the formula [33, (8)] we know

$$\Theta_{\Lambda} = (-1)^{i-1}\left(L^{i0}_{\alpha}dy^{\alpha} + L^{ih}_{\alpha}dy^{\alpha}_{h}\right)\wedge v_{i} + \left(L - y^{\alpha}_{i}L^{i0}_{\alpha} - y^{\alpha}_{(hi)}L^{ih}_{\alpha}\right)v, \tag{38}$$

where

$$\begin{aligned}L - y^{\alpha}_{i}L^{i0}_{\alpha} - y^{\alpha}_{(hi)}L^{ih}_{\alpha} &= L_{0} - y^{\alpha}_{i}L^{i0}_{\alpha},\\ L^{ij}_{\alpha} &= \frac{1}{2-\delta_{ij}}\frac{\partial L}{\partial y^{\alpha}_{(ij)}},\\ L^{h0}_{\beta} &= \frac{\partial L_{0}}{\partial y^{\beta}_{h}} - \frac{\partial L^{hk}_{\beta}}{\partial x^{k}} - y^{\gamma}_{k}\frac{\partial L^{hk}_{\beta}}{\partial y^{\gamma}},\end{aligned}$$

the third equation again being a consequence of (5). Hence

$$\begin{aligned}L_{X^{(1)}}\Theta_{\Lambda} &= (-1)^{i-1}\left(X^{(1)}\left(L^{i0}_{\alpha}\right)dy^{\alpha} + X^{(1)}\left(L^{ih}_{\alpha}\right)dy^{\alpha}_{h}\right)\wedge v_{i}\\ &\quad + \left(L^{i0}_{\alpha}\frac{\partial v^{\alpha}}{\partial x^{i}} + L^{ih}_{\alpha}\frac{\partial v^{\alpha}_{h}}{\partial x^{i}}\right)v + (-1)^{i-1}\frac{\partial v^{\alpha}}{\partial y^{\beta}}L^{i0}_{\alpha}dy^{\beta}\wedge v_{i}\\ &\quad + (-1)^{i-1}L^{ih}_{\alpha}\left(\frac{\partial v^{\alpha}_{h}}{\partial y^{\beta}}dy^{\beta} + \frac{\partial v^{\alpha}_{h}}{\partial y^{\beta}_{j}}dy^{\beta}_{j}\right)\wedge v_{i}\\ &\quad + (-1)^{i-1}\left(L^{i0}_{\alpha}dy^{\alpha} + L^{ih}_{\alpha}dy^{\alpha}_{h}\right)\wedge L_{X'}(v_{i})\\ &\quad + \left[X^{(1)}\left(L_{0} - y^{\alpha}_{i}L^{i0}_{\alpha}\right)\right]v + \operatorname{div}(X')\left(L_{0} - y^{\alpha}_{i}L^{i0}_{\alpha}\right)v.\end{aligned}$$

Expanding the right-hand side above, we obtain

$$\begin{aligned}L_{X^{(1)}}\Theta_{\Lambda} &= (-1)^{i-1}\left(X^{(1)}\left(L^{i0}_{\alpha}\right) + \frac{\partial v^{\beta}}{\partial y^{\alpha}}L^{i0}_{\beta} + \frac{\partial v^{\beta}_{h}}{\partial y^{\alpha}}L^{ih}_{\beta}\right)dy^{\alpha}\wedge v_{i}\\ &\quad + (-1)^{i-1}\left(X^{(1)}\left(L^{ih}_{\alpha}\right) + \frac{\partial v^{\beta}_{j}}{\partial y^{\alpha}_{h}}L^{ij}_{\beta}\right)dy^{\alpha}_{h}\wedge v_{i}\\ &\quad + (-1)^{i-1}\left(L^{i0}_{\alpha}dy^{\alpha} + L^{ih}_{\alpha}dy^{\alpha}_{h}\right)\wedge L_{X'}(v_{i})\end{aligned}$$

$$+ \left(X^{(1)} \left(L_0 - y^\alpha_i L^{i0}_\alpha \right) + L^{i0}_\alpha \frac{\partial v^\alpha}{\partial x^i} + L^{ih}_\alpha \frac{\partial v^\alpha_h}{\partial x^i} \right) v$$
$$+ \operatorname{div}(X') \left(L_0 - y^\alpha_i L^{i0}_\alpha \right) v.$$

Moreover, by applying the formula (38) to the density Λ' we have

$$\Theta_{\Lambda'} = (-1)^{i-1} \left(L'^{i0}_\alpha dy^\alpha + L'^{ih}_\alpha dy^\alpha_h \right) \wedge v_i + \left(L'_0 - y^\alpha_i L'^{i0}_\alpha \right) v.$$

We first compute L'^{i0}_α. From (2), (35), (36), and (37), we deduce

$$L'^{i0}_\alpha = X^{(1)} \left(L^{i0}_\alpha \right) + \operatorname{div}(X') L^{i0}_\alpha + \frac{\partial v^\beta}{\partial y^\alpha} L^{i0}_\beta - \frac{\partial u^i}{\partial x^r} L^{r0}_\alpha + \frac{\partial v^\beta_h}{\partial y^\alpha} L^{hi}_\beta .$$

Furthermore,

$$L'^{ij}_\alpha = X^{(1)}(L^{ij}_\alpha) + \operatorname{div}(X') L^{ij}_\alpha + A^{ij}_\alpha,$$
$$L'_0 = X^{(1)}(L_0) + \operatorname{div}(X') L_0 + T^\beta_{hk} L^{hk}_\beta,$$

with

$$A^{ij}_\alpha = \frac{\partial v^\beta}{\partial y^\alpha} L^{ij}_\beta - \frac{\partial u^i}{\partial x^r} L^{rj}_\alpha - \frac{\partial u^j}{\partial x^r} L^{ri}_\alpha,$$
$$T^\beta_{hk} = \frac{\partial^2 v^\beta}{\partial x^h \partial x^k} + \frac{\partial^2 v^\beta}{\partial y^\gamma \partial x^k} y^\gamma_h + \frac{\partial^2 v^\beta}{\partial y^\gamma \partial x^h} y^\gamma_k + \frac{\partial^2 v^\beta}{\partial y^\gamma \partial y^\sigma} y^\gamma_h y^\sigma_k - \frac{\partial^2 u^r}{\partial x^h \partial x^k} y^\beta_r .$$

Hence

$$L'_0 - y^\alpha_i L'^{i0}_\alpha = X^{(1)}(L_0 - y^\alpha_i L^{i0}_\alpha) + \operatorname{div}(X') \left(L_0 - y^\alpha_i L^{i0}_\alpha \right) + \frac{\partial v^\alpha}{\partial x^i} L^{i0}_\alpha + \frac{\partial v^\beta_h}{\partial x^k} L^{hk}_\beta ,$$

and we obtain

$$\Theta_{\Lambda'} = (-1)^{i-1} \left(X^{(1)} \left(L^{i0}_\alpha \right) + \frac{\partial v^\beta}{\partial y^\alpha} L^{i0}_\beta + \frac{\partial v^\beta_h}{\partial y^\alpha} L^{hi}_\beta \right) dy^\alpha \wedge v_i$$
$$+ (-1)^{i-1} \left(\operatorname{div}(X') L^{i0}_\alpha - \frac{\partial u^i}{\partial x^r} L^{r0}_\alpha \right) dy^\alpha \wedge v_i$$
$$+ (-1)^{i-1} \left(X^{(1)}(L^{ih}_\alpha) + \frac{\partial v^\beta}{\partial y^\alpha} L^{ih}_\beta - \frac{\partial u^h}{\partial x^r} L^{ri}_\alpha \right) dy^\alpha_h \wedge v_i$$
$$- (-1)^{i-1} \frac{\partial u^i}{\partial x^r} L^{rh}_\alpha dy^\alpha_h \wedge v_i + (-1)^{i-1} \operatorname{div}(X') L^{ih}_\alpha dy^\alpha_h \wedge v_i$$
$$+ \left(X^{(1)}(L_0 - y^\alpha_i L^{i0}_\alpha) + \operatorname{div}(X') \left(L_0 - y^\alpha_i L^{i0}_\alpha \right) + \frac{\partial v^\alpha}{\partial x^i} L^{i0}_\alpha + \frac{\partial v^\beta_h}{\partial x^k} L^{hk}_\beta \right) v.$$

By using the formula $L_{X'}(v_i) = \operatorname{div}(X') v_i + \sum_{h=1}^n (-1)^{h-i-1} \frac{\partial u^h}{\partial x^i} v_h$, we can thus conclude that $\Theta_{\Lambda'} = L_{X^{(1)}} \Theta_\Lambda$. Finally, if $L_{X^{(2)}} \Lambda = 0$, then $\Theta_{L_{X^{(2)}} \Lambda} = 0$ and by virtue of the formula in the first part of the statement we deduce $L_{X^{(1)}} \Theta_\Lambda = 0$. Hence $(j^1 s)^*(d i_{X^{(1)}} \Theta) + (j^1 s)^*(i_{X^{(1)}} d\Theta) = 0$, and we can conclude recalling that the second term in the left-hand side vanishes, as follows from the H–C equations in Theorem 2.1. □

6 Symmetries of the H–E Lagrangian density

Example 6.1 In the particular case of the bundle of pseudo-Riemannian metrics of a given signature $p_M : M \to N$ (cf. Sect. 3.1), the natural lift of a vector field $X' = u^i \frac{\partial}{\partial x^i}$ in $\mathfrak{X}(N)$ is given as follows (cf. [32, section 2.2]):

$$X'_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M),$$

and from the geometric properties of the scalar curvature the H–E Lagrangian density Λ_{HE} admits X'_M as an infinitesimal symmetry for every $X' \in \mathfrak{X}(N)$. Let us compute its Noether invariant $(j^1 g)^* i_{(X'_M)^{(1)}} \Theta_{HE}$ along an Einstein metric g. From the formulas

$$\begin{aligned}
\left(X'_M\right)^{(1)} &= u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{hi} \right) \frac{\partial}{\partial y_{ij}} \\
&\quad - \sum_{i \le j} \left(\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} + \frac{\partial u^h}{\partial x^i} y_{hj,k} + \frac{\partial u^h}{\partial x^j} y_{hi,k} + \frac{\partial u^h}{\partial x^k} y_{ij,h} \right) \frac{\partial}{\partial y_{ij,k}}, \\
\Theta_{HE} &= (-1)^{i-1} \sum\nolimits_{k \le l} \left((L_{HE})^{i0}_{kl} \, dy_{kl} + (L_{HE})^{ih}_{kl} \, dy_{kl,h} \right) \wedge v_i \\
&\quad + \left((L_{HE})_0 - \sum\nolimits_{k \le l} y_{kl,i} \, (L_{HE})^{i0}_{kl} \right) v,
\end{aligned}$$

where $(L_{HE})^{i0}_{kl}$, $(L_{HE})^{ih}_{kl}$, and $(L_{HE})_0$ are given in (2), (21), and (20), respectively, by using a normal coordinate system $(x^i)_{i=1}^n$ centred at $x \in N$ we eventually obtain

$$\begin{aligned}
\left((j^1 g)^* \left(i_{(X'_M)^{(1)}} \Theta_{HE} \right) \right)_x = (-1)^i \Bigg\{ &\left(\varepsilon_h \frac{\partial^2 u^i}{\partial x^h \partial x^h} - \varepsilon_i \frac{\partial^2 u^h}{\partial x^i \partial x^h} \right)(x) \\
&+ \left(\varepsilon_i \varepsilon_k \frac{\partial^2 g_{ik}}{\partial x^k \partial x^j} - \varepsilon_i \varepsilon_k \frac{\partial^2 g_{kk}}{\partial x^i \partial x^j} \right)(x) u^j(x) \\
&- \left(\varepsilon_j \varepsilon_k \frac{\partial^2 g_{kj}}{\partial x^j \partial x^k} - \varepsilon_h \varepsilon_k \frac{\partial^2 g_{kk}}{\partial x^h \partial x^h} \right)(x) u^i(x) \Bigg\} (v_i)_x.
\end{aligned}$$

By composing the tensor $\left((\nabla^g)^2 X' \right)_x$ and the isomorphism induced by the metric, $g^\sharp_x \otimes \mathrm{id} : T^*_x N \otimes T^*_x N \otimes T_x N \to T_x N \otimes T^*_x N \otimes T_x N$, we have

$$\left(\left(\nabla^g\right)^2 X' \right)^\sharp_x = \varepsilon_a \left(\frac{\partial}{\partial x^a} \right)_x \otimes \left(dx^h \right)_x \otimes \left(\frac{\partial^2 u^c}{\partial x^a \partial x^h} + u^b \frac{\partial \Gamma^c_{hb}}{\partial x^a} \right)(x) \left(\frac{\partial}{\partial x^c} \right)_x .$$

Contracting the first contravariant index and the first covariant one, it follows:

$$c^1_1 \left(\left(\nabla^g\right)^2 X' \right)^\sharp_x = \varepsilon_h \left(\frac{\partial^2 u^i}{\partial x^h \partial x^h} + u^b \frac{\partial \Gamma^i_{hb}}{\partial x^h} \right)(x) \left(\frac{\partial}{\partial x^i} \right)_x ,$$

and contracting $c^1_1 \left((\nabla^g)^2 X' \right)^\sharp_x$ and the volume form,

$$i_{c^1_1 \left((\nabla^g)^2 X' \right)^\sharp_x} v_x = (-1)^{i-1} \varepsilon_h \left(\frac{\partial^2 u^i}{\partial x^h \partial x^h} + u^b \frac{\partial \Gamma^i_{hb}}{\partial x^h} \right)(x) (v_i)_x .$$

Similarly, contracting the second contravariant index in $\left((\nabla^g)^2 X'\right)^\sharp_x$ and the first covariant one, it follows:

$$c_1^2\left(\left(\nabla^g\right)^2 X'\right)^\sharp_x = \varepsilon_i\left(\frac{\partial^2 u^h}{\partial x^i \partial x^h} + u^b \frac{\partial \Gamma^h_{hb}}{\partial x^i}\right)(x)\left(\frac{\partial}{\partial x^i}\right)_x,$$

and also,

$$i_{c_1^2((\nabla^g)^2 X')^\sharp_x} v_x = (-1)^{i-1}\varepsilon_i\left(\frac{\partial^2 u^h}{\partial x^i \partial x^h} + u^b \frac{\partial \Gamma^h_{hb}}{\partial x^i}\right)(x)(v_i)_x.$$

Finally,

$$\left(j^1 g\right)^*\left(i_{(X'_M)^{(1)}}\Theta_{HE}\right) = i_{c_1^2((\nabla^g)X')^\sharp}(v) - i_{c_1^1((\nabla^g)X')^\sharp}(v).$$

Theorem 6.1 *For $n = \dim N \geq 3$ the vector fields of the form X'_M, where $X' \in \mathfrak{X}(N)$, are the only infinitesimal symmetries of the Lagrangian density Λ_{HE}.*

Proof Let $p_M : M \to N$ be the bundle of pseudo-Riemannian metrics of a given signature. If X is an infinitesimal symmetry of Λ_{HE} and X' is its p_M-projection onto N, then $X - X'_M$ is a p_M-vertical symmetry of Λ_{HE}. Hence, the statement is equivalent to saying that the only p_M-vertical symmetry X of the H–E Lagrangian is the null vector field.

Let $p : E \to N$ be a submersion. If $X = V^\alpha \frac{\partial}{\partial y^\alpha}$, $V^\alpha \in C^\infty(E)$ is an infinitesimal symmetry of a second-order Lagrangian L with P–C form projectable onto J^1E, then $X^{(2)}(L) = 0$, where

$$X^{(2)} = V^\alpha \frac{\partial}{\partial y^\alpha} + D_i(V^\alpha)\frac{\partial}{\partial y^\alpha_i} + \sum\nolimits_{h\leq i} D_h D_i(V^\alpha)\frac{\partial}{\partial y^\alpha_{hi}}.$$

As

$$\begin{aligned}
D_i(V^\alpha) &= \frac{\partial V^\alpha}{\partial x^i} + y^\rho_i \frac{\partial V^\alpha}{\partial y^\rho},\\
D_h D_i(V^\alpha) &= \frac{\partial^2 V^\alpha}{\partial x^h \partial x^i} + y^\beta_h \frac{\partial^2 V^\alpha}{\partial x^i \partial y^\beta} + y^\beta_{hi}\frac{\partial V^\alpha}{\partial y^\beta} + y^\beta_i\left(\frac{\partial^2 V^\alpha}{\partial x^h \partial y^\beta} + y^\gamma_h \frac{\partial^2 V^\alpha}{\partial y^\beta \partial y^\gamma}\right),
\end{aligned}$$

it follows:

$$\begin{aligned}
X^{(2)}(L) = {} & V^\alpha \frac{\partial L_0}{\partial y^\alpha} + \left(\frac{\partial V^\alpha}{\partial x^i} + y^\rho_i \frac{\partial V^\alpha}{\partial y^\rho}\right)\frac{\partial L_0}{\partial y^\alpha_i}\\
& + \left(V^\alpha \frac{\partial L^{jk}_\beta}{\partial y^\alpha} + \left(\frac{\partial V^\alpha}{\partial x^i} + y^\rho_i \frac{\partial V^\alpha}{\partial y^\rho}\right)\frac{\partial L^{jk}_\beta}{\partial y^\alpha_i} + L^{jk}_\alpha \frac{\partial V^\alpha}{\partial y^\beta}\right) y^\beta_{jk}\\
& + \sum_{h\leq i} L^{hi}_\alpha\left(\frac{\partial^2 V^\alpha}{\partial x^h \partial x^i} + y^\beta_h \frac{\partial^2 V^\alpha}{\partial x^i \partial y^\beta} + y^\beta_i\left(\frac{\partial^2 V^\alpha}{\partial x^h \partial y^\beta} + y^\gamma_h \frac{\partial^2 V^\alpha}{\partial y^\beta \partial y^\gamma}\right)\right).
\end{aligned}$$

Hence, the coefficient of y^β_{jk} must vanish and we obtain the following system of partial differential equations:

$$0 = V^{\alpha}\frac{\partial L^{jk}_{\beta}}{\partial y^{\alpha}} + \left(\frac{\partial V^{\alpha}}{\partial x^{i}} + y^{\rho}_{i}\frac{\partial V^{\alpha}}{\partial y^{\rho}}\right)\frac{\partial L^{jk}_{\beta}}{\partial y^{\alpha}_{i}} + L^{jk}_{\alpha}\frac{\partial V^{\alpha}}{\partial y^{\beta}},$$
$$0 = V^{\alpha}\frac{\partial L_{0}}{\partial y^{\alpha}} + \left(\frac{\partial V^{\alpha}}{\partial x^{i}} + y^{\rho}_{i}\frac{\partial V^{\alpha}}{\partial y^{\rho}}\right)\frac{\partial L_{0}}{\partial y^{\alpha}_{i}}$$
$$+ \sum\nolimits_{h\leq i} L^{hi}_{\alpha}\left(\frac{\partial^{2}V^{\alpha}}{\partial x^{h}\partial x^{i}} + y^{\beta}_{h}\frac{\partial^{2}V^{\alpha}}{\partial x^{i}\partial y^{\beta}} + y^{\beta}_{i}\left(\frac{\partial^{2}V^{\alpha}}{\partial x^{h}\partial y^{\beta}} + y^{\gamma}_{h}\frac{\partial^{2}V^{\alpha}}{\partial y^{\beta}\partial y^{\gamma}}\right)\right).$$

In the case of the H–E Lagrangian, we obtain

$$\text{[i]}\quad 0 = \sum_{a\leq b}\left[\frac{\partial\,(L_{HE})^{jk}_{st}}{\partial y_{ab}}V^{ab} + (L_{HE})^{jk}_{ab}\frac{\partial V^{ab}}{\partial y_{st}}\right],\quad j\leq k, s\leq t,$$
$$\text{[ii]}\quad 0 = \sum_{a\leq b}\left\{\frac{\partial\,(L_{HE})_{0}}{\partial y_{ab}}V^{ab} + \left(\frac{\partial V^{ab}}{\partial x^{i}} + \sum_{u\leq v} y_{uv,i}\frac{\partial V^{ab}}{\partial y_{uv}}\right)\frac{\partial\,(L_{HE})_{0}}{\partial y_{ab,i}}\right.$$
$$+ \sum_{h\leq i}(L_{HE})^{hi}_{ab}\left[\frac{\partial^{2}V^{ab}}{\partial x^{h}\partial x^{i}} + \sum_{s\leq t} y_{st,h}\frac{\partial^{2}V^{ab}}{\partial x^{i}\partial y_{st}}\right.$$
$$\left.\left. + \sum_{s\leq t} y_{st,i}\left(\frac{\partial^{2}V^{ab}}{\partial x^{h}\partial y_{st}} + \sum_{u\leq v} y_{uv,h}\frac{\partial^{2}V^{ab}}{\partial y_{st}\partial y_{uv}}\right)\right]\right\}, \tag{39}$$

as $\frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{ab,i}} = 0$, by virtue of (21), with $V = \sum_{a\leq b} V^{ab}\frac{\partial}{\partial y_{ab}}$, $V^{ab}\in C^{\infty}(M)$. Collecting the terms of degrees 2, 1, and 0 in the variables $y_{ab,c}$, $a\leq b$, on the right-hand side of [ii]-(39), it breaks into the following equations:

$$0 = \sum_{a\leq b}\left\{\frac{\rho}{2}\left[\frac{\partial A^{kl,i;rs,j}_{0}}{\partial y_{ab}}V^{ab} + \left(A^{ab,i;rs,j}_{0} + A^{rs,i;ab,j}_{0}\right)\frac{\partial V^{ab}}{\partial y_{kl}}\right],\right.$$
$$\left. + \frac{1}{2-\delta_{ij}}(L_{HE})^{ij}_{ab}\frac{\partial^{2}V^{ab}}{\partial y_{rs}\partial y_{kl}}\right\}$$
$$r\leq s, k\leq l; i, j, k, l, r, s = 1,\ldots,n,$$
$$0 = \frac{2}{2-\delta_{ij}}(L_{HE})^{ij}_{ab}\frac{\partial^{2}V^{ab}}{\partial x^{i}\partial y_{rs}} + \frac{\rho}{2}\left(A^{ab,i;rs,j}_{0} + A^{rs,j;ab,i}_{0}\right)\frac{\partial V^{ab}}{\partial x^{i}},$$
$$r\leq s; j, r, s = 1,\ldots,n,$$
$$0 = \sum_{h\leq i}\sum_{a\leq b}(L_{HE})^{hi}_{ab}\frac{\partial^{2}V^{ab}}{\partial x^{h}\partial x^{i}},$$

where we have used the notations below,

$$(L_{HE})_{0} = \frac{\rho}{2}\sum\nolimits_{r\leq s}\sum\nolimits_{k\leq l} A^{kl,i;rs,j}_{0}\, y_{kl,i}\, y_{rs,j},$$
$$A^{kl,i;rs,j}_{0} = \sum_{r\leq s}\sum_{k\leq l}\frac{1}{(1+\delta_{kl})(1+\delta_{rs})}\left(2y^{rs}\left(y^{ki}y^{jl} + y^{li}y^{jk}\right) - 2y^{kl}y^{sr}y^{ji}\right.$$
$$\left. + 2y^{kl}\left(y^{jr}y^{si} + y^{js}y^{ri}\right) + 3y^{ij}\left(y^{kr}y^{ls} + y^{ks}y^{lr}\right)\right)$$

$$- y^{ir}\left(y^{ks}y^{jl} + y^{ls}y^{jk}\right) - y^{is}\left(y^{kr}y^{jl} + y^{lr}y^{jk}\right)$$
$$-2y^{ki}\left(y^{sl}y^{jr} + y^{rl}y^{js}\right) - 2y^{li}\left(y^{sk}y^{jr} + y^{rk}y^{js}\right)\bigg).$$

Moreover, as a calculation shows, we have

$$\det\left((L_{HE})^{ij}_{rs}\right)^{1\leq i\leq j\leq n}_{1\leq r\leq s\leq n} = -(n-1)\rho^{\frac{1}{2}(n+1)(n+4)},$$

where ρ is defined in (19). If $\Lambda = \left(\Lambda^{jk}_{ab}\right)^{1\leq j\leq k\leq n}_{1\leq a\leq b\leq n}$ is the inverse matrix of $\left((L_{HE})^{jk}_{ab}\right)^{1\leq j\leq k\leq n}_{1\leq a\leq b\leq n}$, then from (39)-[i] for $h \leq i$, it follows:

$$\frac{\partial V^{ab}}{\partial y_{st}} = -\sum\nolimits_{c\leq d}\sum\nolimits_{p\leq q}\Lambda^{ab}_{pq}\frac{\partial (L_{HE})^{pq}_{st}}{\partial y_{cd}}V^{cd}, \quad a\leq b, s\leq t, \tag{40}$$

and by imposing the integrability conditions to these equations, we obtain

$$\begin{aligned}0 = \sum_{a\leq b}\sum_{j\leq k}\Bigg[\Bigg(&\frac{\partial\Lambda^{hi}_{jk}}{\partial y_{uv}}\frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{ab}} - \frac{\partial\Lambda^{hi}_{jk}}{\partial y_{st}}\frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{ab}}\\ &+\Lambda^{hi}_{jk}\left\{\frac{\partial^2 (L_{HE})^{jk}_{st}}{\partial y_{ab}\partial y_{uv}} - \frac{\partial^2 (L_{HE})^{jk}_{uv}}{\partial y_{ab}\partial y_{st}}\right\}\Bigg)V^{ab}\\ &+\Lambda^{hi}_{jk}\left\{\frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{ab}}\frac{\partial V^{ab}}{\partial y_{uv}} - \frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{ab}}\frac{\partial V^{ab}}{\partial y_{st}}\right\}\Bigg]\end{aligned}$$

and substituting (40) in the previous equation, we eventually have

$$\begin{aligned}0 = \sum_{c\leq d}\sum_{j\leq k}\Bigg[&\frac{\partial\Lambda^{hi}_{jk}}{\partial y_{uv}}\frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{cd}} - \frac{\partial\Lambda^{hi}_{jk}}{\partial y_{st}}\frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{cd}}\\ &+\Lambda^{hi}_{jk}\left(\frac{\partial^2 (L_{HE})^{jk}_{st}}{\partial y_{cd}\partial y_{uv}} - \frac{\partial^2 (L_{HE})^{jk}_{uv}}{\partial y_{cd}\partial y_{st}}\right)\\ &+\Lambda^{hi}_{jk}\sum_{a\leq b}\sum_{p\leq q}\Lambda^{ab}_{pq}\left(\frac{\partial (L_{HE})^{pq}_{st}}{\partial y_{cd}}\frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{ab}} - \frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{ab}}\frac{\partial (L_{HE})^{pq}_{uv}}{\partial y_{cd}}\right)\Bigg]V^{cd}.\end{aligned} \tag{41}$$

Furthermore, from $\frac{\partial\Lambda}{\partial y_{pq}} = -\Lambda\cdot\frac{\partial L}{\partial y_{pq}}\cdot\Lambda$, it follows:

$$\frac{\partial\Lambda^{hi}_{jk}}{\partial y_{uv}} = -\sum_{\zeta\leq\eta}\sum_{\rho\leq\sigma}\Lambda^{hi}_{\zeta\eta}\frac{\partial (L_{HE})^{\zeta\eta}_{\rho\sigma}}{\partial y_{uv}}\Lambda^{\rho\sigma}_{jk}.$$

Hence

$$\frac{\partial \Lambda^{hi}_{jk}}{\partial y_{uv}} \frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{cd}} - \frac{\partial \Lambda^{hi}_{jk}}{\partial y_{st}} \frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{cd}}$$
$$= \sum_{\zeta \le \eta} \sum_{\rho \le \sigma} \Lambda^{hi}_{\zeta\eta} \left(-\frac{\partial (L_{HE})^{\zeta\eta}_{\rho\sigma}}{\partial y_{uv}} \Lambda^{\rho\sigma}_{jk} \frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{cd}} + \frac{\partial (L_{HE})^{\zeta\eta}_{\rho\sigma}}{\partial y_{st}} \Lambda^{\rho\sigma}_{jk} \frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{cd}} \right),$$

and letting

$$\begin{aligned} \Phi^{jk}_{st,uv,cd} = & \frac{\partial^2 (L_{HE})^{jk}_{st}}{\partial y_{cd} \partial y_{uv}} - \frac{\partial^2 (L_{HE})^{jk}_{uv}}{\partial y_{cd} \partial y_{st}} \\ & + \sum_{a \le b} \sum_{p \le q} \Lambda^{ab}_{pq} \left(\left(\frac{\partial (L_{HE})^{jk}_{ab}}{\partial y_{st}} - \frac{\partial (L_{HE})^{jk}_{st}}{\partial y_{ab}} \right) \frac{\partial (L_{HE})^{pq}_{uv}}{\partial y_{cd}} \right. \\ & \left. + \left(\frac{\partial (L_{HE})^{jk}_{uv}}{\partial y_{ab}} - \frac{\partial (L_{HE})^{jk}_{ab}}{\partial y_{uv}} \right) \frac{\partial (L_{HE})^{pq}_{st}}{\partial y_{cd}} \right) \end{aligned}$$

equation (41) transform into the following:

$$0 = \sum_{c \le d} \left(\Lambda \cdot \Phi_{st,uv} \right)^{hi}_{cd} V^{cd}, \quad h \le i, s \le t, u \le v,$$

where $\Phi_{st,uv}$ is the matrix $\left(\Phi_{st,uv}\right)^{jk}_{cd} = \Phi^{jk}_{st,uv,cd}$, for every $s \le t, u \le v$. As dim $\Phi_{11,23} \neq 0$ for $n = 3$ and det $\Phi_{12,34} \neq 0$ for $n \ge 4$, it follows $V^{cd} = 0$. □

Remark 6.1 For $n = 2$ the H–E Lagrangian density is known to be a conformally invariant 2-form; hence, Λ_{HE} admits—in this dimension—the Liouville vector field as a vertical infinitesimal symmetry.

7 Jacobi fields and presymplectic structure

Let $V(p) \subset TE$ be the subbundle of p-vertical tangent vectors for the submersion $p\colon E \to N$. The infinitesimal variation of a one-parameter variation S_t of a section $s\colon N \to E$ is the p-vertical vector field along s, $X \in \Gamma(N, s^*V(p))$, defined by the formula, $X_x =$ tangent vector at $t = 0$ to the curve $t \mapsto S_t(x)$, $\forall x \in N$. On a fibred coordinate system (x^i, y^α), we have

$$X_x = \frac{\partial (y^\alpha \circ S)}{\partial t}(0, x) \left(\frac{\partial}{\partial y^\alpha} \right)_{s(x)}, \quad \forall x \in N. \tag{42}$$

Let $\mathcal{S}$ be the sheaf of extremals of a second-order Lagrangian density $\Lambda = Lv$ on $p\colon E \to N$ whose Poincaré–Cartan form projects onto J^1E: For every open subset $U \subseteq N$, we denote by $\mathcal{S}(U)$ the set of solutions to the Euler–Lagrange equations of Λ, which are defined on U. As is well known [12,15,37] in the Hamiltonian formalism extremals can be characterized as the solutions to the Hamilton–Cartan equation; that is, s is an extremal if and only if $(j^1 s)^*(i_Y d\Theta_\Lambda) = 0$ for all $Y \in \mathfrak{X}(J^1E)$. Jacobi fields are the solutions to the linearized Hamilton–Cartan equation. Precisely, a Jacobi field along an extremal $s \in \mathcal{S}(U)$ is a p-vertical vector field defined along s, $X \in \Gamma(U, s^*V(p))$, satisfying the Jacobi equation $(j^1 s)^*(i_Y L_{X^{(1)}} d\Theta_\Lambda) = 0$, $\forall Y \in \mathfrak{X}(J^1(p^{-1}U))$, where $X^{(1)}$ is the first-order infinitesimal contact transformation on J^1E associated to X (e.g., see [27,29,37]). If $X_x = V^\alpha(x) \left(\frac{\partial}{\partial y^\alpha} \right)_{s(x)}$, then

$$\left(X^{(1)}\right)_{j_x^1 s} = V^\alpha(x)\left(\frac{\partial}{\partial y^\alpha}\right)_{j_x^1 s} + \frac{\partial V^\alpha}{\partial x^j}(x)\left(\frac{\partial}{\partial y_j^\alpha}\right)_{j_x^1 s}.$$

In fact, it is readily checked that if S_t is a one-parameter variation of s and S_t is an extremal for every t, then the infinitesimal variation X of S_t [see (42)] satisfies the Jacobi equation. Hence we think of the Jacobi fields along s as being the tangent space at s to the "manifold" $\mathcal{S}(U)$ of extremals and accordingly we denote it by $T_s\mathcal{S}(U)$. Let $s\colon N \to E$ be an extremal of a Lagrangian density Λ defined on J^1E.

In a fibred coordinate system (x^i, y^α), a vector field $X \in \Gamma(U, s^*V(p))$ along an extremal s is a Jacobi field if and only if $(j^1s)^*(i_{\partial/\partial y^\alpha} L_{X^{(1)}} d\Theta_\Lambda) = 0$, for $1 \leq \alpha \leq m$ (see [30, section 3.5]). By using the formulas (11), (12), and (13), we obtain

$$\begin{aligned} L_{X^{(1)}} d\Theta_\Lambda &= L_{X^{(1)}}\left\{(-1)^{i-1} dp_\alpha^i \wedge dy^\alpha \wedge v_i + dH \wedge v\right\} \\ &= (-1)^{i-1} d\left(X^{(1)} p_\alpha^i\right) \wedge dy^\alpha \wedge v_i \\ &\quad + (-1)^{i-1} dp_\alpha^i \wedge dV^\alpha \wedge v_i + d\left(X^{(1)} H\right) \wedge v. \end{aligned}$$

Hence

$$\begin{aligned} i_{\partial/\partial y^\alpha} L_{X^{(1)}} d\Theta_\Lambda &= (-1)^{i-1}\frac{\partial X^{(1)}(p_\beta^i)}{\partial y^\alpha} dy^\beta \wedge v_i - \frac{\partial\left(X^{(1)}(p_\alpha^i)\right)}{\partial x^i} v \\ &\quad - (-1)^{i-1}\frac{\partial\left(X^{(1)}(p_\alpha^i)\right)}{\partial y^\beta} dy^\beta \wedge v_i - (-1)^{i-1}\frac{\partial\left(X^{(1)}(p_\alpha^i)\right)}{\partial y_j^\beta} dy_j^\beta \wedge v_i \\ &\quad + \frac{\partial p_\beta^i}{\partial y^\alpha}\frac{\partial V^\beta}{\partial x^i} v + \frac{\partial X^{(1)}(H)}{\partial y^\alpha} v, \end{aligned}$$

and finally,

$$\begin{aligned} 0 &= \frac{\partial s^\beta}{\partial x^i}\left\{\left(\frac{\partial X^{(1)}(p_\beta^i)}{\partial y^\alpha} \circ j^1 s\right) - \left(\frac{\partial\left(X^{(1)}(p_\alpha^i)\right)}{\partial y^\beta} \circ j^1 s\right)\right\} \\ &\quad + \left(\frac{\partial X^{(1)}(H)}{\partial y^\alpha} \circ j^1 s\right) - \left(\frac{\partial\left(X^{(1)}(p_\alpha^i)\right)}{\partial x^i} \circ j^1 s\right) \\ &\quad - \left(\frac{\partial\left(X^{(1)}(p_\alpha^i)\right)}{\partial y_j^\beta} \circ j^1 s\right)\frac{\partial^2 s^\beta}{\partial x^i \partial x^j} + \frac{\partial V^\beta}{\partial x^i}\left(\frac{\partial p_\beta^i}{\partial y^\alpha} \circ j^1 s\right). \end{aligned}$$

Expanding,

$$\begin{aligned} \frac{\partial^2 V^\gamma}{\partial x^i \partial x^j}\left(\frac{\partial p_\alpha^i}{\partial y_j^\gamma} \circ j^1 s\right) &= V^\gamma\left\{\frac{\partial s^\beta}{\partial x^i}\left(\frac{\partial^2 p_\beta^i}{\partial y^\alpha \partial y^\gamma} \circ j^1 s - \frac{\partial^2 p_\alpha^i}{\partial y^\beta \partial y^\gamma} \circ j^1 s\right)\right. \\ &\quad + \frac{\partial^2 H}{\partial y^\alpha \partial y^\gamma} \circ j^1 s - \frac{\partial^2 s^\beta}{\partial x^i \partial x^j}\left(\frac{\partial^2 p_\alpha^i}{\partial y^\gamma \partial y_j^\beta} \circ j^1 s\right) \\ &\quad \left. - \frac{\partial^2 p_\alpha^i}{\partial x^i \partial y^\gamma} \circ j^1 s\right\} \\ &\quad + \frac{\partial V^\gamma}{\partial x^h}\left\{\frac{\partial s^\beta}{\partial x^i}\left(\frac{\partial^2 p_\beta^i}{\partial y^\alpha \partial y_h^\gamma} \circ j^1 s - \frac{\partial^2 p_\alpha^i}{\partial y^\beta \partial y_h^\gamma} \circ j^1 s\right)\right. \end{aligned}$$

$$-\frac{\partial^2 s^\beta}{\partial x^i \partial x^j}\left(\frac{\partial^2 p_\alpha^i}{\partial y_j^\beta \partial y_h^\gamma}\circ j^1 s\right)+\left(\frac{\partial p_\gamma^h}{\partial y^\alpha}-\frac{\partial p_\alpha^h}{\partial y^\gamma}\right)\circ j^1 s$$
$$\left.-\frac{\partial^2 p_\alpha^i}{\partial x^i \partial y_h^\gamma}\circ j^1 s+\frac{\partial^2 H}{\partial y^\alpha \partial y_h^\gamma}\circ j^1 s\right\},\quad 1\le\alpha\le m. \tag{43}$$

Remark 7.1 In the case of the H–E Lagrangian density, Greek indices of the general case transform into a pair of non-decreasing Latin indices: $\alpha=(a,b)$, $1\le a\le b\le n$, and a Jacobi vector field along g can locally be written as follows:

$$\begin{aligned}X_x &= V^{ab}(x)\left(\frac{\partial}{\partial y_{ab}}\right)_{g(x)}\\ &= V^{ab}(x)(dx^a)_x\otimes(dx^b)_x,\quad \forall x\in N,\end{aligned}$$

with $V^{ab}=V^{ba}$ for $a>b$. Moreover, in this case, the general equations (43) for Jacobi fields can also be written as follows:

$$\begin{aligned}0 &= \frac{1}{2}\left[\left(\delta_{a\nu}\delta_{j\mu}+\delta_{a\mu}\delta_{\nu j}\right)g^{ib}-g^{ij}\delta_{a\nu}\delta_{b\mu}-g^{ab}\delta_{i\nu}\delta_{j\mu}\right]\frac{\partial^2 V^{ab}}{\partial x^i\partial x^j}\\
&+\left\{\frac{1}{2}g^{ab}\left(\Gamma^g\right)^i_{\mu\nu}-g^{ib}\left(\Gamma^g\right)^a_{\mu\nu}+\frac{\delta_{a\nu}\delta_{i\mu}+\delta_{a\mu}\delta_{i\nu}}{2}\left[g^{\sigma b}\left(\Gamma^g\right)^\lambda_{\lambda\sigma}-g^{\lambda\sigma}\frac{\partial g_{\sigma\beta}}{\partial x^\lambda}g^{b\beta}\right]\right.\\
&-\frac{\delta_{a\mu}\delta_{b\nu}}{2}\left[g^{\sigma i}\left(\Gamma^g\right)^\lambda_{\lambda\sigma}-g^{\lambda\sigma}\frac{\partial g_{\sigma\beta}}{\partial x^\lambda}g^{i\beta}\right]+\frac{\delta_{i\nu}}{2}g^{\lambda a}\left(\Gamma^g\right)^b_{\mu\lambda}+\frac{\delta_{i\mu}}{2}g^{\lambda a}\left(\Gamma^g\right)^b_{\lambda\nu}\\
&\left.+\frac{\delta_{b\nu}}{2}\left[g^{\lambda i}\left(\Gamma^g\right)^a_{\mu\lambda}-g^{\lambda a}\left(\Gamma^g\right)^i_{\mu\lambda}\right]+\frac{\delta_{b\mu}}{2}\left[g^{\lambda i}\left(\Gamma^g\right)^a_{\nu\lambda}-g^{\lambda a}\left(\Gamma^g\right)^i_{\nu\lambda}\right]\right\}\frac{\partial V^{ab}}{\partial x^i}\\
&+g^{\lambda b}\left\{(R^g)^a_{\mu\nu\lambda}+g^{ar}g_{t\sigma}\left(\left(\Gamma^g\right)^t_{r\nu}\left(\Gamma^{g_0}\right)^\sigma_{\mu\lambda}-\left(\Gamma^g\right)^t_{r\lambda}\left(\Gamma^g\right)^\sigma_{\mu\nu}\right)\right.\\
&\left.+\left(\Gamma^g\right)^a_{\nu\sigma}\left(\Gamma^g\right)^\sigma_{\mu\lambda}-\left(\Gamma^g\right)^a_{\lambda\sigma}\left(\Gamma^g\right)^\sigma_{\mu\nu}-\left(\Gamma^g\right)^\sigma_{\sigma\lambda}\left(\Gamma^g\right)^a_{\mu\nu}+\left(\Gamma^g\right)^a_{\mu\sigma}\left(\Gamma^g\right)^\sigma_{\nu\lambda}\right\}V^{ab},\\
&1\le\mu\le\nu\le n,\end{aligned}\tag{44}$$

where Γ^g denotes the Levi-Civita connection of g, and R^g its curvature tensor.

Example 7.1 If $N=(\mathbb{R}/2\pi\mathbb{Z})^4$ is a 4-dimensional torus with Lorentzian metric $g=\varepsilon_i(dx^i)^2$, $\varepsilon_1=-1$, $\varepsilon_2=\varepsilon_3=\varepsilon_4=+1$, then Eq. (44) of the Jacobi fields along g are as follows:

$$\sum\nolimits_{B=1}^{10}P_B^A(D)U^B=0,\quad 1\le A\le 10, \tag{45}$$

where

$$U^1=V^{11},\ U^2=V^{12},\ U^3=V^{13},\ U^4=V^{14},\ U^5=V^{22},$$
$$U^6=V^{23},\ U^7=V^{24},\ U^8=V^{33},\ U^9=V^{34},\ U^{10}=V^{44},$$

$$\begin{array}{lll}
P_1^1=\frac{-1}{2}\sum_{i=2}^4(D^i)^2, & P_2^1=D^1D^2, & P_3^1=D^1D^3,\\
P_4^1=D^1D^4, & P_5^1=\frac{-1}{2}(D^1)^2, & P_6^1=0,\\
P_7^1=0, & P_8^1=\frac{-1}{2}(D^1)^2, & P_9^1=0,\\
P_{10}^1=\frac{-1}{2}(D^1)^2, & P_1^2=0, & P_2^2=\frac{-1}{2}(D^3)^2-\frac{1}{2}(D^4)^2,\\
P_3^2=\frac{1}{2}D^2D^3, & P_4^2=\frac{1}{2}D^2D^4, & P_5^2=0,\\
P_6^2=\frac{1}{2}D^1D^3, & P_7^2=\frac{1}{2}D^1D^4, & P_8^2=\frac{-1}{2}D^1D^2,
\end{array}$$

$$
\begin{array}{lll}
P_9^2 = 0, & P_{10}^2 = \frac{-1}{2} D^1 D^2, & P_1^3 = 0, \\
P_2^3 = \frac{1}{2} D^2 D^3, & P_3^3 = \frac{-1}{2} (D^2)^2 - \frac{1}{2} (D^4)^2, & P_4^3 = \frac{1}{2} D^3 D^4, \\
P_5^3 = \frac{-1}{2} D^1 D^3, & P_6^3 = \frac{1}{2} D^1 D^2, & P_7^3 = 0, \\
P_8^3 = 0, & P_9^3 = \frac{1}{2} D^1 D^4, & P_{10}^3 = \frac{-1}{2} D^1 D^3, \\
P_1^4 = 0, & P_2^4 = \frac{1}{2} D^2 D^4, & P_3^4 = \frac{1}{2} D^3 D^4, \\
P_4^4 = \frac{-1}{2} (D^2)^2 - \frac{1}{2} (D^3)^2, & P_5^4 = \frac{-1}{2} D^1 D^4, & P_6^4 = 0, \\
P_7^4 = \frac{1}{2} D^1 D^2, & P_8^4 = \frac{-1}{2} D^1 D^4, & P_9^4 = \frac{1}{2} D^1 D^3, \\
P_{10}^4 = 0, & P_1^5 = \frac{1}{2} D^2 D^2 & P_2^5 = -D^1 D^2 \\
P_3^5 = 0, & P_4^5 = 0, & P_5^5 = \frac{1}{2} (D^1)^2 - \frac{1}{2} \sum_{i=3}^4 (D^i)^2, \\
P_6^5 = D^2 D^3, & P_7^5 = D^2 D^4, & P_8^5 = \frac{-1}{2} D^2 D^2, \\
P_9^5 = 0, & P_{10}^5 = \frac{-1}{2} D^2 D^2, & P_1^6 = \frac{1}{2} D^2 D^3,
\end{array}
$$

$$
\begin{array}{lll}
P_2^6 = \frac{-1}{2} D^1 D^3, & P_3^6 = \frac{-1}{2} D^1 D^2, & P_4^6 = 0, \\
P_5^6 = 0, & P_6^6 = \frac{1}{2} (D^1)^2 - \frac{1}{2} (D^4)^2, & P_7^6 = \frac{1}{2} D^3 D^4, \\
P_8^6 = 0, & P_9^6 = \frac{1}{2} D^2 D^4, & P_{10}^6 = \frac{-1}{2} D^2 D^3, \\
P_1^7 = \frac{1}{2} D^2 D^4, & P_2^7 = \frac{-1}{2} D^1 D^4, & P_3^7 = 0, \\
P_4^7 = \frac{-1}{2} D^1 D^2, & P_5^7 = 0, & P_6^7 = \frac{1}{2} D^3 D^4, \\
P_7^7 = \frac{1}{2} (D^1)^2 - \frac{1}{2} (D^3)^2, & P_8^7 = \frac{-1}{2} D^2 D^4, & P_9^7 = \frac{1}{2} D^2 D^3, \\
P_{10}^7 = 0, & P_1^8 = \frac{1}{2} (D^3)^2, & P_2^8 = 0, \\
P_3^8 = -D^1 D^3, & P_4^8 = 0, & P_5^8 = -\frac{1}{2} (D^3)^2, \\
P_6^8 = D^2 D^3, & P_7^8 = 0, & P_8^8 = \frac{1}{2} (D^1)^2 - \frac{1}{2} (D^2)^2 - \frac{1}{2} (D^4)^2, \\
P_9^8 = D^3 D^4, & P_{10}^8 = \frac{-1}{2} (D^3)^2, & P_1^9 = \frac{1}{2} D^3 D^4, \\
P_2^9 = 0, & P_3^9 = \frac{-1}{2} D^1 D^4, & P_4^9 = \frac{-1}{2} D^1 D^3, \\
P_5^9 = \frac{-1}{2} D^3 D^4, & P_6^9 = \frac{1}{2} D^2 D^4, & P_7^9 = \frac{1}{2} D^2 D^3, \\
P_8^9 = 0, & P_9^9 = \frac{1}{2} (D^1)^2 - \frac{1}{2} (D^2)^2, & P_{10}^9 = 0, \\
P_1^{10} = \frac{1}{2} (D^4)^2, & P_2^{10} = 0, & P_3^{10} = 0, \\
P_4^{10} = -D^1 D^4, & P_5^{10} = \frac{-1}{2} (D^4)^2, & P_6^{10} = 0, \\
P_7^{10} = D^2 D^4, & P_8^{10} = -\frac{1}{2} (D^4)^2, & P_9^{10} = D^3 D^4, \\
P_{10}^{10} = \frac{1}{2} (D^1)^2 - \frac{1}{2} \sum_{i=2}^3 (D^i)^2, & D^i = \frac{\partial}{\partial x^i}, & 1 \leq i \leq 4.
\end{array}
$$

We can obtain the global solutions to Jacobi equations (45) by expanding in Fourier series; namely,

$$
U^A = \sum\nolimits_{(k_1,\ldots,k_4) \in \mathbb{Z}^4} U^A_{k_1,\ldots,k_4} \exp(i k_j x^j), \quad U^A_{k_1,\ldots,k_4} \in \mathbb{C},
$$

so that $\frac{\partial^2 U^A}{\partial x^r \partial x^s} = -\sum_{(k_1,\dots,k_4)\in\mathbb{Z}^4} k_r k_s U^A_{k_1,\dots,k_4} \exp(ik_j x^j)$ and the Jacobi equations (45) transform into the following:

$$
\begin{aligned}
0 &= \frac{1}{2}\left((k_2)^2+(k_3)^2+(k_4)^2\right)U^1_{k_1,\dots,k_4} - k_1k_2U^2_{k_1,\dots,k_4} - k_1k_3U^3_{k_1,\dots,k_4}\\
&\quad - k_1k_4U^4_{k_1,\dots,k_4} + \frac{1}{2}(k_1)^2\left(U^5_{k_1,\dots,k_4}+U^8_{k_1,\dots,k_4}+U^{10}_{k_1,\dots,k_4}\right),\\
0 &= -\left((k_3)^2+(k_4)^2\right)U^2_{k_1,\dots,k_4} + k_2k_3U^3_{k_1,\dots,k_4} + k_2k_4U^4_{k_1,\dots,k_4}\\
&\quad + k_1k_3U^6_{k_1,\dots,k_4} + k_1k_4U^7_{k_1,\dots,k_4} - k_1k_2\left(U^8_{k_1,\dots,k_4}+U^{10}_{k_1,\dots,k_4}\right),\\
0 &= k_2k_3U^2_{k_1,\dots,k_4} - \left((k_2)^2+(k_4)^2\right)U^3_{k_1,\dots,k_4} + k_3k_4U^4_{k_1,\dots,k_4}\\
&\quad + k_1k_2U^6_{k_1,\dots,k_4} - k_1k_3\left(U^5_{k_1,\dots,k_4}+U^{10}_{k_1,\dots,k_4}\right) + k_1k_4U^9_{k_1,\dots,k_4},\\
0 &= k_2k_4U^2_{k_1,\dots,k_4} + k_3k_4U^3_{k_1,\dots,k_4} - \left((k_2)^2+(k_3)^2\right)U^4_{k_1,\dots,k_4}\\
&\quad - k_1k_4\left(U^5_{k_1,\dots,k_4}+U^8_{k_1,\dots,k_4}\right) + k_1k_2U^7_{k_1,\dots,k_4} + k_1k_3U^9_{k_1,\dots,k_4},\\
0 &= \frac{1}{2}(k_2)^2U^1_{k_1,\dots,k_4} - k_1k_2U^2_{k_1,\dots,k_4} + \frac{1}{2}\left((k_1)^2-(k_3)^2-(k_4)^2\right)U^5_{k_1,\dots,k_4}\\
&\quad + k_2k_3U^6_{k_1,\dots,k_4} + k_2k_4U^7_{k_1,\dots,k_4} - \frac{1}{2}(k_2)^2\left(U^8_{k_1,\dots,k_4}+U^{10}_{k_1,\dots,k_4}\right),\\
0 &= k_2k_3U^1_{k_1,\dots,k_4} - k_1k_3U^2_{k_1,\dots,k_4} - k_1k_2U^3_{k_1,\dots,k_4}\\
&\quad + \left((k_1)^2-(k_4)^2\right)U^6_{k_1,\dots,k_4} + k_3k_4U^7_{k_1,\dots,k_4} + k_2k_4U^9_{k_1,\dots,k_4} - k_2k_3U^{10}_{k_1,\dots,k_4},\\
0 &= k_2k_4U^1_{k_1,\dots,k_4} - k_1k_4U^2_{k_1,\dots,k_4} - k_1k_2U^4_{k_1,\dots,k_4}\\
&\quad + k_3k_4U^6_{k_1,\dots,k_4} + \left((k_1)^2-(k_3)^2\right)U^7_{k_1,\dots,k_4} - k_2k_4U^8_{k_1,\dots,k_4} + k_2k_3U^9_{k_1,\dots,k_4},\\
0 &= \frac{1}{2}k_3^2\left(U^1_{k_1,\dots,k_4}-U^5_{k_1,\dots,k_4}\right) - k_1k_3U^3_{k_1,\dots,k_4} + k_2k_3U^6_{k_1,\dots,k_4}\\
&\quad + \frac{1}{2}\left((k_1)^2-(k_2)^2-(k_4)^2\right)U^8_{k_1,\dots,k_4} + k_3k_4U^9_{k_1,\dots,k_4} - \frac{1}{2}(k_3)^2U^{10}_{k_1,\dots,k_4},\\
0 &= k_3k_4U^1_{k_1,\dots,k_4} - k_1k_4U^3_{k_1,\dots,k_4} - k_1k_3U^4_{k_1,\dots,k_4}\\
&\quad - k_3k_4U^5_{k_1,\dots,k_4} + k_2k_4U^6_{k_1,\dots,k_4} + k_2k_3U^7_{k_1,\dots,k_4} + \left((k_1)^2-(k_2)^2\right)U^9_{k_1,\dots,k_4},\\
0 &= \frac{1}{2}(k_4)^2\left(U^1_{k_1,\dots,k_4}-U^5_{k_1,\dots,k_4}-U^8_{k_1,\dots,k_4}\right) - k_1k_4U^4_{k_1,\dots,k_4}\\
&\quad + k_2k_4U^7_{k_1,\dots,k_4} + k_3k_4U^9_{k_1,\dots,k_4} + \frac{1}{2}\left((k_1)^2-(k_2)^2-(k_3)^2\right)U^{10}_{k_1,\dots,k_4},
\end{aligned}
$$

for every system $(k_1,\dots,k_4)\in\mathbb{Z}^4$. Solving these equations for $k_4\neq 0$, we obtain

$$
\begin{aligned}
U^1_{k_1,\dots,k_4} &= 2\frac{k_1}{k_4}U^4_{k_1,\dots,k_4} - \frac{k_1^2}{k_4^2}U^{10}_{k_1,\dots,k_4},\\
U^2_{k_1,\dots,k_4} &= \frac{k_2}{k_4}U^4_{k_1,\dots,k_4} + \frac{k_1}{k_4}U^7_{k_1,\dots,k_4} - \frac{k_1k_2}{k_4^2}U^{10}_{k_1,\dots,k_4},\\
U^3_{k_1,\dots,k_4} &= \frac{k_3}{k_4}U^4_{k_1,\dots,k_4} + \frac{k_1}{k_4}U^9_{k_1,\dots,k_4} - \frac{k_1k_3}{k_4^2}U^{10}_{k_1,\dots,k_4},\\
U^5_{k_1,\dots,k_4} &= 2\frac{k_2}{k_4}U^7_{k_1,\dots,k_4} - \frac{k_2^2}{k_4^2}U^{10}_{k_1,\dots,k_4},
\end{aligned}
$$

$$U^6_{k_1,\dots,k_4} = \frac{k_3}{k_4}U^7_{k_1,\dots,k_4} + \frac{k_2}{k_4}U^9_{k_1,\dots,k_4} - \frac{k_2k_3}{k_4^2}U^{10}_{k_1,\dots,k_4},$$

$$U^8_{k_1,\dots,k_4} = 2\frac{k_3}{k_4}U^9_{k_1,\dots,k_4} - \frac{k_3^2}{k_4^2}U^{10}_{k_1,\dots,k_4},$$

and the unknowns $U^4_{k_1,\dots,k_4}$, $U^7_{k_1,\dots,k_4}$, $U^9_{k_1,\dots,k_4}$ and $U^{10}_{k_1,\dots,k_4}$ remain undetermined.

If $k_3 \neq 0$ but $k_4 = 0$, then the solutions to the previous equations are

$$U^1_{k_1,\dots,k_4} = 2\frac{k_1}{k_3}U^3_{k_1,\dots,k_4} - \frac{k_1^2}{k_3^2}U^8_{k_1,\dots,k_4},$$

$$U^2_{k_1,\dots,k_4} = \frac{k_2}{k_3}U^3_{k_1,\dots,k_4} + \frac{k_1}{k_3}U^6_{k_1,\dots,k_4} - \frac{k_1k_2}{k_3^2}U^8_{k_1,\dots,k_4},$$

$$U^4_{k_1,\dots,k_4} = \frac{k_1}{k_3}U^9_{k_1,\dots,k_4},$$

$$U^5_{k_1,\dots,k_4} = 2\frac{k_2}{k_3}U^6_{k_1,\dots,k_4} - \frac{k_2^2}{k_3^2}U^8_{k_1,\dots,k_4},$$

$$U^7_{k_1,\dots,k_4} = \frac{k_2}{k_3}U^9_{k_1,\dots,k_4},$$

$$U^{10}_{k_1,\dots,k_4} = 0,$$

the unknowns $U^3_{k_1,\dots,k_4}$, $U^6_{k_1,\dots,k_4}$, $U^8_{k_1,\dots,k_4}$ and $U^9_{k_1,\dots,k_4}$ remain undetermined. If $k_3 = k_4 = 0$ but $k_2 \neq 0$, then

$$U^1_{k_1,\dots,k_4} = 2\frac{k_1}{k_2}U^2_{k_1,\dots,k_4} - \frac{k_1^2}{k_2^2}U^5_{k_1,\dots,k_4},$$

$$U^3_{k_1,\dots,k_4} = \frac{k_1}{k_2}U^6_{k_1,\dots,k_4},$$

$$U^4_{k_1,\dots,k_4} = \frac{k_1}{k_2}U^7_{k_1,\dots,k_4},$$

$$U^8_{k_1,\dots,k_4} = U^9_{k_1,\dots,k_4} = U^{10}_{k_1,\dots,k_4} = 0$$

the unknowns $U^2_{k_1,\dots,k_4}$, and $U^A_{k_1,\dots,k_4}$, $5 \leq A \leq 7$ remain undetermined. Finally, if $k_2 = k_3 = k_4 = 0$, then $U^A_{k_1,\dots,k_4} = 0$, $5 \leq A \leq 10$, and the unknowns $U^A_{k_1,\dots,k_4}$, $1 \leq A \leq 4$ remain undetermined.

Therefore

$$\begin{aligned} U^1 = &\sum_{k_4\neq 0}\left(2\frac{k_1}{k_4}U^4_{k_1,\dots,k_4} - \frac{k_1^2}{k_4^2}U^{10}_{k_1,\dots,k_4}\right)\exp(ik_jx^j) \\ &+ \sum_{k_3\neq 0, k_4=0}\left(2\frac{k_1}{k_3}U^3_{k_1,\dots,k_4} - \frac{k_1^2}{k_3^2}U^8_{k_1,\dots,k_4}\right)\exp(ik_jx^j) \\ &+ \sum_{k_3=k_4=0, k_2\neq 0}\left(2\frac{k_1}{k_2}U^2_{k_1,\dots,k_4} - \frac{k_1^2}{k_2^2}U^5_{k_1,\dots,k_4}\right)\exp(ik_jx^j) \\ &+ U^1_{k_1000}\exp(ik_1x^1) \end{aligned}$$

$$U^2 = \sum_{k_4\neq 0}\left(\frac{k_2}{k_4}U^4_{k_1,\dots,k_4} + \frac{k_1}{k_4}U^7_{k_1,\dots,k_4} - \frac{k_1k_2}{k_4^2}U^{10}_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum_{k_3\neq 0,k_4=0}\left(\frac{k_2}{k_3}U^3_{k_1,\dots,k_4} + \frac{k_1}{k_3}U^6_{k_1,\dots,k_4} - \frac{k_1k_2}{k_3^2}U^8_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum_{k_3=k_4=0,k_2\neq 0} U^2_{k_1,\dots,k_4}\exp(ik_jx^j) + U^2_{k_1000}\exp(ik_jx^j),$$

$$U^3 = \sum_{k_4\neq 0}\left(\frac{k_3}{k_4}U^4_{k_1,\dots,k_4} + \frac{k_1}{k_4}U^9_{k_1,\dots,k_4} - \frac{k_1k_3}{k_4^2}U^{10}_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum_{k_3\neq 0,k_4=0} U^3_{k_1,\dots,k_4}\exp(ik_jx^j)$$
$$+ \sum_{k_3=k_4=0,k_2\neq 0} \frac{k_1}{k_2}U^6_{k_1,\dots,k_4}\exp(ik_jx^j) + U^3_{k_1000}\exp(ik_1x^1),$$

$$U^4 = \sum_{k_4\neq 0} U^4_{k_1,\dots,k_4}\exp(ik_jx^j) + \sum_{k_3\neq 0,k_4=0}\frac{k_1}{k_3}U^9_{k_1,\dots,k_4}\exp(ik_jx^j)$$
$$+ \sum\nolimits_{k_3=k_4=0,k_2\neq 0}\frac{k_1}{k_2}U^7_{k_1,\dots,k_4}\exp(ik_jx^j) + U^4_{k_1000}\exp(ik_1x^1),$$

$$U^5 = \sum_{k_4\neq 0}\left(2\frac{k_2}{k_4}U^7_{k_1,\dots,k_4} - \frac{k_2^2}{k_4^2}U^{10}_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum_{k_3\neq 0,k_4=0}\left(2\frac{k_2}{k_3}U^6_{k_1,\dots,k_4} - \frac{k_2^2}{k_3^2}U^8_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum\nolimits_{k_3=k_4=0,k_2\neq 0} U^5_{k_1,\dots,k_4}\exp(ik_jx^j),$$

$$U^6 = \sum_{k_4\neq 0}\left(\frac{k_3}{k_4}U^7_{k_1,\dots,k_4} + \frac{k_2}{k_4}U^9_{k_1,\dots,k_4} - \frac{k_2k_3}{k_4^2}U^{10}_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum_{k_3\neq 0,k_4=0} U^6_{k_1,\dots,k_4}\exp(ik_jx^j)$$
$$+ \sum\nolimits_{k_3=k_4=0,k_2\neq 0} U^6_{k_1,\dots,k_4}\exp(ik_jx^j),$$

$$U^7 = \sum_{k_4\neq 0} U^7_{k_1,\dots,k_4}\exp(ik_jx^j) + \sum_{k_4=0,k_3\neq 0}\frac{k_2}{k_3}U^9_{k_1,\dots,k_4}\exp(ik_jx^j)$$
$$+ \sum_{k_3=k_4=0,k_2\neq 0} U^7_{k_1,\dots,k_4}\exp(ik_jx^j),$$

$$U^8 = \sum_{k_4\neq 0}\left(2\frac{k_3}{k_4}U^9_{k_1,\dots,k_4} - \frac{k_3^2}{k_4^2}U^{10}_{k_1,\dots,k_4}\right)\exp(ik_jx^j)$$
$$+ \sum_{k_4=0,k_3\neq 0} U^8_{k_1,\dots,k_4}\exp(ik_jx^j),$$

$$U^9 = \sum_{k_4 \neq 0} U^9_{k_1,\dots,k_4} \exp(ik_j x^j) + \sum_{k_4=0, k_3 \neq 0} U^9_{k_1,\dots,k_4} \exp(ik_j x^j),$$
$$U^{10} = \sum_{k_4 \neq 0} U^{10}_{k_1,\dots,k_4} \exp(ik_j x^j).$$

Hence we obtain

$$\begin{aligned}
\sum_{A=1}^{10} U^A E_A = {} & U^1_{k_1 000} \exp(ik_1 x^1) E_1 + \sum_{k_3=k_4=0, k_2 \neq 0} U^2_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ 2\frac{k_1}{k_2} E_1 + E_2 \right\} \\
& + U^2_{k_1 000} \exp(ik_1 x^1) E_2 + \sum_{k_3 \neq 0, k_4=0} U^3_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ 2\frac{k_1}{k_3} E_1 + \frac{k_2}{k_3} E_2 \right\} \\
& + \sum_{k_3 \neq 0, k_4=0} U^3_{k_1,\dots,k_4} \exp(ik_j x^j) E_3 + U^3_{k_1 000} \exp(ik_1 x^1) E_3 \\
& + \sum_{k_4 \neq 0} U^4_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ 2\frac{k_1}{k_4} E_1 + \frac{k_2}{k_4} E_2 + \frac{k_3}{k_4} E_3 \right\} \\
& + \sum_{k_4 \neq 0} U^4_{k_1,\dots,k_4} \exp(ik_j x^j) E_4 + U^4_{k_1 000} \exp(ik_1 x^1) E_4 \\
& + \sum_{k_3=k_4=0, k_2 \neq 0} U^5_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ -\frac{k_1^2}{k_2^2} E_1 + E_5 \right\} \\
& + \sum_{k_3 \neq 0, k_4=0} U^6_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1}{k_3} E_2 + 2\frac{k_2}{k_3} E_5 + E_6 \right\} \\
& + \sum_{k_3=k_4=0, k_2 \neq 0} U^6_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1}{k_2} E_3 + E_6 \right\} \\
& + \sum_{k_4 \neq 0} U^7_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1}{k_4} E_2 + 2\frac{k_2}{k_4} E_5 + \frac{k_3}{k_4} E_6 + E_7 \right\} \\
& + \sum_{k_3=k_4=0, k_2 \neq 0} U^7_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1}{k_2} E_4 + E_7 \right\} \\
& + \sum_{k_3 \neq 0, k_4=0} U^8_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ -\frac{k_1^2}{k_3^2} E_1 - \frac{k_1 k_2}{k_3^2} E_2 - \frac{k_2^2}{k_3^2} E_5 + E_8 \right\} \\
& + \sum_{k_4 \neq 0} U^9_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1}{k_4} E_3 + \frac{k_2}{k_4} E_6 + 2\frac{k_3}{k_4} E_8 + E_9 \right\} \\
& + \sum_{k_3 \neq 0, k_4=0} U^9_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1}{k_3} E_4 + \frac{k_2}{k_3} E_7 + E_9 \right\} \\
& - \sum_{k_4 \neq 0} U^{10}_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ \frac{k_1 k_2}{k_4^2} E_2 + \frac{k_1 k_3}{k_4^2} E_3 \right\} \\
& + \sum_{k_4 \neq 0} U^{10}_{k_1,\dots,k_4} \exp(ik_j x^j) \left\{ -\frac{k_1^2}{k_4^2} E_1 - \frac{k_2^2}{k_4^2} E_5 - \frac{k_2 k_3}{k_4^2} E_6 - \frac{k_3^2}{k_4^2} E_8 + E_{10} \right\},
\end{aligned}$$

where

$$\begin{array}{l} E_1 = \frac{\partial}{\partial y_{11}},\ E_2 = \frac{\partial}{\partial y_{12}},\ E_3 = \frac{\partial}{\partial y_{13}},\ E_4 = \frac{\partial}{\partial y_{14}},\ E_5 = \frac{\partial}{\partial y_{22}}, \\ E_6 = \frac{\partial}{\partial y_{23}},\ E_7 = \frac{\partial}{\partial y_{24}},\ E_8 = \frac{\partial}{\partial y_{33}},\ E_9 = \frac{\partial}{\partial y_{34}},\ E_{10} = \frac{\partial}{\partial y_{44}}. \end{array} \tag{46}$$

and the following vector fields

$$X_1^k = \exp(ik_1x^1)\frac{\partial}{\partial y_{11}},$$

$$X_2^k = \exp(i\left(k_1x^1 + k_2x^2\right))\left\{2\frac{k_1}{k_2}\frac{\partial}{\partial y_{11}} + \frac{\partial}{\partial y_{12}}\right\}, \quad k_2 \neq 0,$$

$$X_3^k = \exp(ik_1x^1)\frac{\partial}{\partial y_{12}},$$

$$X_4^k = \exp(i\left(k_1x^1 + k_2x^2 + k_3x^3\right))\left\{2\frac{k_1}{k_3}\frac{\partial}{\partial y_{11}} + \frac{k_2}{k_3}\frac{\partial}{\partial y_{12}}\right\}, \quad k_3 \neq 0,$$

$$X_5^k = \exp(i\left(k_1x^1 + k_2x^2 + k_3x^3\right))\frac{\partial}{\partial y_{13}}, \quad k_3 \neq 0,$$

$$X_6^k = \exp(ik_1x^1)\frac{\partial}{\partial y_{13}},$$

$$X_7^k = \exp(ik_jx^j)\left\{2\frac{k_1}{k_4}\frac{\partial}{\partial y_{11}} + \frac{k_2}{k_4}\frac{\partial}{\partial y_{12}} + \frac{k_3}{k_4}\frac{\partial}{\partial y_{13}}\right\}, \quad k_4 \neq 0,$$

$$X_8^k = \exp(ik_jx^j)\frac{\partial}{\partial y_{14}}, \quad k_4 \neq 0,$$

$$X_9^k = \exp(ik_1x^1)\frac{\partial}{\partial y_{14}},$$

$$X_{10}^k = \exp(i\left(k_1x^1 + k_2x^2\right))\left\{-\frac{k_1^2}{k_2^2}\frac{\partial}{\partial y_{11}} + \frac{\partial}{\partial y_{22}}\right\}, \quad k_2 \neq 0,$$

$$X_{11}^k = \exp(i\left(k_1x^1 + k_2x^2 + k_3x^3\right))\left\{\frac{k_1}{k_3}\frac{\partial}{\partial y_{12}} + 2\frac{k_2}{k_3}\frac{\partial}{\partial y_{22}} + \frac{\partial}{\partial y_{23}}\right\}, \quad k_3 \neq 0,$$

$$X_{12}^k = \exp(i\left(k_1x^1 + k_2x^2\right))\left\{\frac{k_1}{k_2}\frac{\partial}{\partial y_{13}} + \frac{\partial}{\partial y_{23}}\right\}, \quad k_2 \neq 0,$$

$$X_{13}^k = \exp(ik_jx^j)\left\{\frac{k_1}{k_4}\frac{\partial}{\partial y_{12}} + 2\frac{k_2}{k_4}\frac{\partial}{\partial y_{22}} + \frac{k_3}{k_4}\frac{\partial}{\partial y_{23}} + \frac{\partial}{\partial y_{24}}\right\}, \quad k_4 \neq 0,$$

$$X_{14}^k = \exp(i\left(k_1x^1 + k_2x^2\right))\left\{\frac{k_1}{k_2}\frac{\partial}{\partial y_{14}} + \frac{\partial}{\partial y_{24}}\right\}, \quad k_2 \neq 0,$$

$$X_{15}^k = \exp(i\left(k_1x^1 + k_2x^2 + k_3x^3\right))\left\{\frac{k_1^2}{k_3^2}\frac{\partial}{\partial y_{11}} + \frac{k_1k_2}{k_3^2}\frac{\partial}{\partial y_{12}} + \frac{k_2^2}{k_3^2}\frac{\partial}{\partial y_{22}} - \frac{\partial}{\partial y_{33}}\right\}, \quad k_3 \neq 0,$$

$$X_{16}^k = \exp(ik_jx^j)\left\{\frac{k_1}{k_4}\frac{\partial}{\partial y_{13}} + \frac{k_2}{k_4}\frac{\partial}{\partial y_{23}} + 2\frac{k_3}{k_4}\frac{\partial}{\partial y_{33}} + \frac{\partial}{\partial y_{34}}\right\}, \quad k_4 \neq 0,$$

$$X_{17}^k = \exp(i\left(k_1x^1 + k_2x^2 + k_3x^3\right))\left\{\frac{k_1}{k_3}\frac{\partial}{\partial y_{14}} + \frac{k_2}{k_3}\frac{\partial}{\partial y_{24}} + \frac{\partial}{\partial y_{34}}\right\}, \quad k_3 \neq 0,$$

$$X_{18}^k = \exp(ik_jx^j)\left\{\frac{k_1k_2}{k_4^2}\frac{\partial}{\partial y_{12}} + \frac{k_1k_3}{k_4^2}\frac{\partial}{\partial y_{13}}\right\}, \quad k_4 \neq 0,$$

$$X^k_{19}=\exp(ik_jx^j)\left\{\frac{k_1^2}{k_4^2}\frac{\partial}{\partial y_{11}}+\frac{k_2^2}{k_4^2}\frac{\partial}{\partial y_{22}}+\frac{k_2k_3}{k_4^2}\frac{\partial}{\partial y_{23}}+\frac{k_3^2}{k_4^2}\frac{\partial}{\partial y_{33}}-\frac{\partial}{\partial y_{44}}\right\},\quad k_4\neq 0,$$

with $k\in\mathbb{Z}^4$, span $T_g\mathcal{S}((\mathbb{R}/2\pi\mathbb{Z})^4)$ topologically.

Let Λ be a Lagrangian density on an arbitrary fibred manifold $p\colon E\to N$ and let Θ_Λ be the P–C form associated to Λ. Let $X,Y\in T_s\mathcal{S}(N)$ be Jacobi vector fields defined along an extremal $s\in\mathcal{S}(N)$ for the Lagrangian density Λ. Then, $d[(j^1s)^*(i_{Y^{(1)}}i_{X^{(1)}}d\Theta_\Lambda)]=0$ (e.g., see [12]); i.e., the $(n-1)$-form $i_{Y^{(1)}}i_{X^{(1)}}d\Theta_\Lambda$ is closed along j^1s.

The alternate bilinear mapping taking values in the space $Z^{n-1}(N)$ of closed $(n-1)$-forms, defined by

$$\begin{array}{l}(\omega_2)_s\colon T_s\mathcal{S}(N)\times T_s\mathcal{S}(N)\to Z^{n-1}(N),\\ (\omega_2)_s(X,Y)=(j^1s)^*\left(i_{Y^{(1)}}i_{X^{(1)}}d\Theta_\Lambda\right)\end{array}$$

is called the presymplectic structure associated to Λ.

Theorem 7.1 *Let s be an extremal of a second-order Lagrangian density $\Lambda=Lv$ on $p\colon E\to N$ with Poincaré–Cartan form projectable onto J^1E. Assume that the variational problem defined by Λ is regular in the sense of Proposition 2.2. For every $x\in N$, let $R^2_x(\Lambda)\subseteq J^2_x(s^*V(p))$ be the vector subspace of 2-jets j^2_xX of p-vertical vector fields along s that satisfy the Jacobi equations (43) at x. If the natural projection $p^2_1\colon R^2_x(\Lambda)\to J^1_x(s^*V(p))$ is surjective for every $x\in N$, then the radical of the valued 2-form $(\omega_2)_s$ vanishes.*

Proof According to (13), we have $d\Theta_\Lambda=(-1)^{i-1}dp^i_\alpha\wedge dy^\alpha\wedge v_i+dH\wedge v$. If $X^{(1)}=V^\sigma\frac{\partial}{\partial y^\sigma}+\frac{\partial V^\sigma}{\partial x^j}\frac{\partial}{\partial y^\sigma_j}$, $Y^{(1)}=W^\sigma\frac{\partial}{\partial y^\sigma}+\frac{\partial W^\sigma}{\partial x^j}\frac{\partial}{\partial y^\sigma_j}$, with $V^\sigma,W^\sigma\in C^\infty(N)$, then

$$\begin{aligned}(\omega_2)_s(X,Y)=(-1)^{i-1}\Bigg\{&\left(V^\sigma W^\alpha-V^\alpha W^\sigma\right)\left(\frac{\partial p^i_\alpha}{\partial y^\sigma}\circ j^1s\right)\\ &+\left(\frac{\partial V^\sigma}{\partial x^j}W^\alpha-V^\alpha\frac{\partial W^\sigma}{\partial x^j}\right)\left(\frac{\partial p^i_\alpha}{\partial y^\sigma_j}\circ j^1s\right)\Bigg\}\Bigg|_{j^1s}v_i.\end{aligned}$$

If we assume the vector field X belongs to $\mathrm{rad}(\omega_2)_s$, then by evaluating at x the equation $(\omega_2)_s(X,Y)=0$, $\forall Y\in T_s\mathcal{S}(N)$, we obtain

$$\begin{aligned}0=&\left[V^\sigma(x)W^\alpha(x)-V^\alpha(x)W^\sigma(x)\right]\frac{\partial p^i_\alpha}{\partial y^\sigma}(j^1_xs)\\ &+\left[\frac{\partial V^\sigma}{\partial x^j}(x)W^\alpha(x)-V^\alpha(x)\frac{\partial W^\sigma}{\partial x^j}(x)\right]\frac{\partial p^i_\alpha}{\partial y^\sigma_j}(j^1_xs),\quad 1\le i\le n.\end{aligned}\tag{47}$$

The assumption in the statement implies that given arbitrary values for $W^\beta(x)$ and $\frac{\partial W^\beta}{\partial x^h}(x)$, there exists an element $j^2_xY\in R^2_x(\Lambda)$ projecting under the natural mapping $p^2_1\colon R^2_x(\Lambda)\to J^1_x(s^*V(p))$ onto the 1-jet at x the coordinates of which coincide with these values. Accordingly, the coefficients of $W^\beta(x)$ and $\frac{\partial W^\beta}{\partial x^h}(x)$ in (47) must vanish, i.e.

$$\begin{aligned}&0=V^\alpha\left(\frac{\partial p^i_\beta}{\partial y^\alpha}\circ j^1s\right)-V^\alpha\left(\frac{\partial p^i_\alpha}{\partial y^\beta}\circ j^1s\right)+\frac{\partial V^\alpha}{\partial x^j}\left(\frac{\partial p^i_\beta}{\partial y^\alpha_j}\circ j^1s\right),\\ &\quad 1\le i\le n,\ 1\le\beta\le m,\\ &0=V^\alpha\left(\frac{\partial p^i_\alpha}{\partial y^\beta_h}\circ j^1s\right),\qquad h,i=1,\dots,n,\ 1\le\beta\le m,\end{aligned}\tag{48}$$

as the point x is arbitrary. Hence the formulas (48) are the equations for the radical of $(\omega_2)_s$. If we set

$$V = (V^1, \dots, V^m), \quad O_M = (0, \dots, 0), \quad \Upsilon = \left(\frac{\partial p^i_\alpha}{\partial y^\beta_h} \right)^{1 \le i \le n, 1 \le \alpha \le m}_{1 \le h \le n, 1 \le \beta \le m},$$

then the second group of equations in (48) can matricially be written as

$$\underbrace{(V, \dots, V)}_{n \text{ times}} \cdot \left(\Upsilon \circ j^1 s\right) = \underbrace{(O_M, \dots, O_M)}_{n \text{ times}}.$$

If the variational problem defined by the density Λ is regular in the sense of Proposition 2.2, then $\det \Upsilon \neq 0$. Hence $V = 0$. □

Criterion 7.1 Next, we give a criterion in order to ensure that the condition of Theorem 7.1 holds. According to (18) we have $p^i_\alpha = \frac{\partial \bar{L}}{\partial y^\alpha_i}$, where $\bar{L}$ is the first-order Lagrangian defined by (17), see also Theorem 4.1. As is known, the Hessian metric of $\bar{L}$ is the section of the vector bundle $S^2 V^*(p^1_0)$ locally given by, $\mathrm{Hess}(\bar{L}) = \frac{\partial^2 \bar{L}}{\partial y^\alpha_i \partial y^\beta_j} d^1_0 y^\alpha_i \otimes d^1_0 y^\beta_j$. As mentioned in Sect. 2, there is a canonical isomorphism

$$\begin{aligned} &I \colon (p^1_0)^* \left(p^*(T^* N) \otimes V(p)\right) \to V(p^1_0), \\ &I\left(j^1_x s, (dx^i)_x \otimes \left(\tfrac{\partial}{\partial y^\alpha}\right)_{s(x)}\right) = \left(\tfrac{\partial}{\partial y^\alpha_i}\right)_{j^1_x s}, \end{aligned}$$

and dually,

$$\begin{aligned} &I^* \colon V^*(p^1_0) \to (p^1_0)^* \left(p^*(TN) \otimes V^*(p)\right), \\ &I^* \left(j^1_x s, \left(\tfrac{\partial}{\partial x^i}\right)_x \otimes (dy^\alpha)_{s(x)}\right) = \left(d^1_0 y^\alpha_i\right)_{j^1_x s}. \end{aligned}$$

Hence the Hessian metric can be viewed as a symmetric bilinear form

$$\mathrm{Hess}(\bar{L})_{j^1_x s} \colon V_{j^1_x s}(p^1_0) \times V_{j^1_x s}(p^1_0) \cong \left[(T^*_x N) \otimes V_{s(x)}(p)\right] \times \left[(T^*_x N) \otimes V_{s(x)}(p)\right] \to \mathbb{R},$$

and we can define a linear map as follows:

$$\begin{aligned} &\mathrm{Hess}(\bar{L})^\natural_{j^1_x s} \colon (T^*_x N) \otimes (T^*_x N) \otimes V_{s(x)}(p) \to V^*_{s(x)}(p), \\ &\mathrm{Hess}(\bar{L})^\natural_{j^1_x s} (w_1, w_2, X_1)(X_2) = \mathrm{Hess}(\bar{L})_{j^1_x s} (w_1 \otimes X_1, w_2 \otimes X_2), \\ &\forall w_1, w_2 \in T^*_x N, \quad \forall X_1, X_2 \in V_{s(x)}(p). \end{aligned}$$

The matrix of $\mathrm{Hess}(\bar{L})^\natural_{j^1_x s}\Big|_{S^2(T^*_x N) \otimes V_{s(x)}(p)}$ is $\Upsilon^\natural = \left(\frac{1}{1+\delta_{ij}} \left[\frac{\partial p^i_\alpha}{\partial y^\gamma_j} + \frac{\partial p^j_\alpha}{\partial y^\gamma_i} \right] (j^1_x s) \right)^\alpha_{\gamma, i \le j}$ in the standard basis. Moreover, letting $v^\gamma_{ij} = \frac{\partial^2 V^\gamma}{\partial x^i \partial x^j}(x)$, and denoting by E^α the right-hand side of the formula (43), this formula, evaluated at x, reads as follows: $v^\gamma_{ij} \frac{\partial p^i_\alpha}{\partial y^\gamma_j}(j^1_x s) = E^\alpha(x)$, which is a linear system with m equations in the $\frac{m}{2} n(n+1)$ unknowns v^γ_{ij}, $1 \le i \le j \le n$, and the matrix of this system is precisely $\Upsilon^\natural$. Consequently, if $\mathrm{Hess}(\bar{L})^\natural_{j^1_x s}$ is assumed to be surjective, then the previous system is compatible.

Corollary 7.2 *The radical of the valued 2-form $(\omega_2)_g$ corresponding to the H–E Lagrangian density along an arbitrary extremal metric g, vanishes.*

Proof According to Theorem 7.1, in order to prove the corollary above, we need only to verify that the projection $p_1^2\colon R_x^2(\Lambda) \to J_x^1(s^*V(p))$ is surjective for every $x \in N$. By considering a system of normal coordinates for the metric g at the point x, and letting $v^{ab} = V^{ab}(x)$, $v_{ij}^{ab} = \frac{\partial^2 V^{ab}}{\partial x^i \partial x^j}(x)$, equation (44) evaluated at x are written as follows:

$$
\begin{aligned}
0 &= \frac{1}{2}\left[\varepsilon_i\left(\delta_{a\nu}\delta_{j\mu} + \delta_{a\mu}\delta_{\nu j}\right)\delta^{ib} - \varepsilon_i\delta^{ij}\delta_{a\nu}\delta_{b\mu} - \varepsilon_a\delta^{ab}\delta_{i\nu}\delta_{j\mu}\right] v_{ij}^{ab} \\
&\quad + \varepsilon_b (R^g)^a_{\mu\nu b}(x) v^{ab} \\
&= \frac{\varepsilon_i}{2}\left(v_{i\mu}^{\nu i} + v_{i\nu}^{\mu i} - v_{\mu\nu}^{ii} - v_{ii}^{\mu\nu}\right) + \varepsilon_b (R^g)^a_{\mu\nu b}(x) v^{ab}, \\
&\quad 1 \leq \mu \leq \nu \leq n,
\end{aligned}
$$

which is a system with $\frac{1}{2}n(n+1)$ equations in the $\frac{1}{4}n^2(n+1)^2$ unknowns $v_{\mu\nu}^{ij}$, $1 \leq i \leq j \leq n$, $1 \leq \mu \leq \nu \leq n$, with $v_{\mu\nu}^{ij} = v_{\nu\mu}^{ij} = v_{\mu\nu}^{ji}$, and where the scalars v^{ab}, $1 \leq a \leq b \leq n$, can take arbitrary values. A particular solution to this system is obtained by letting,

$$
\left.
\begin{array}{l}
\text{(i) } \varepsilon_i v_{\mu\nu}^{ii} = \varepsilon_i v_{ii}^{\mu\nu} = 0 \\
\text{(ii) } \varepsilon_i v_{i\mu}^{\nu i} = \varepsilon_i v_{i\nu}^{\mu i} = -\varepsilon_b (R^g)^a_{\mu\nu b}(x) v^{ab}
\end{array}
\right\}, \quad 1 \leq \mu \leq \nu \leq n. \tag{49}
$$

Equation (49)-(i) hold by setting $v_{\mu\nu}^{ii} = v_{ii}^{\mu\nu}$, $\forall i, \mu, \nu = 1, \ldots, n, \ \mu \leq \nu$, while equation (49)-(ii) hold by setting

$$
\left.
\begin{array}{ll}
v_{i\nu}^{\mu i} = v_{\mu i}^{\nu i} = 0, & 2 \leq i \leq n \\
v_{1\nu}^{1\mu} = v_{1\mu}^{1\nu} = -\varepsilon_1 \varepsilon_b (R^g)^a_{\mu\nu b}(x) v^{ab}, &
\end{array}
\right\}, \quad 1 \leq \mu \leq \nu \leq n.
$$

□

Example 7.2 Below, we compute the presymplectic structure associated to Example 7.1; i.e., we compute $(\omega_2)_g$ for the H–E Lagrangian density when $N = (\mathbb{R}/2\pi\mathbb{Z})^4$ and $g = \varepsilon_i (dx^i)^2$, $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = +1$ by using the basis X_h^k, $1 \leq h \leq 8$, $k \in \mathbb{Z}^4$ of that example. We follow some ideas in [39, Section 7] for our particular case.

According to the previous notations and calculations, we have

$$
\begin{aligned}
(\omega_2)_g(X, Y) = (-1)^{i-1} &\left\{ \left(V^{kl}W^{ab} - V^{ab}W^{kl}\right)\left(\frac{\partial p^i_{ab}}{\partial y_{kl}} \circ j^1 g\right) \right. \\
&\left.\left. + \left(\frac{\partial V^{kl}}{\partial x^j} W^{ab} - V^{ab}\frac{\partial W^{kl}}{\partial x^j}\right)\left(\frac{\partial p^i_{ab}}{\partial y_{kl,j}} \circ j^1 g\right)\right\}\right|_{j^1 g} v_i,
\end{aligned}
$$

$$
X^{(1)} = V^{ab}\frac{\partial}{\partial y_{ab}} + \frac{\partial V^{ab}}{\partial x^j}\frac{\partial}{\partial y_{ab,j}}, \quad Y^{(1)} = W^{ab}\frac{\partial}{\partial y_{ab}} + \frac{\partial W^{ab}}{\partial x^j}\frac{\partial}{\partial y_{ab,j}},
$$

and from the formulas (22), (23) it follows:

$$
\frac{\partial p^i_{kl}}{\partial y_{uv}} \circ j^1 g = 0.
$$

Therefore

$$
(\omega_2)_g(X, Y) = (-1)^{i-1}\omega_2^i(X, Y) v_i,
$$

where

$$
\omega_2^i(X, Y) = \sum_{k \leq l}\sum_{a \leq b}\left(\frac{\partial p^i_{ab}}{\partial y_{kl,j}} \circ j^1 g\right)\left(\frac{\partial V^{kl}}{\partial x^j} W^{ab} - V^{ab}\frac{\partial W^{kl}}{\partial x^j}\right),
$$

and, as a computation shows, the scalar differential forms ω_2^i are given by

$$
\begin{aligned}
\omega_2^1 = {} & \frac{1}{2}\left(\frac{\partial W^{13}}{\partial x^3}+\frac{\partial W^{14}}{\partial x^4}+\frac{\partial W^{12}}{\partial x^2}\right)V^{11}-\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^2}+\frac{\partial W^{33}}{\partial x^2}+\frac{\partial W^{22}}{\partial x^2}+\frac{\partial W^{44}}{\partial x^2}\right)V^{12}\\
& -\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^3}+\frac{\partial W^{22}}{\partial x^3}+\frac{\partial W^{33}}{\partial x^3}+\frac{\partial W^{44}}{\partial x^3}\right)V^{13}-\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^4}+\frac{\partial W^{33}}{\partial x^4}+\frac{\partial W^{22}}{\partial x^4}+\frac{\partial W^{44}}{\partial x^4}\right)V^{14}\\
& +\left(\frac{\partial W^{12}}{\partial x^3}+\frac{\partial W^{13}}{\partial x^2}-\frac{\partial W^{23}}{\partial x^1}\right)V^{23}+\frac{1}{2}\left(\frac{\partial W^{33}}{\partial x^1}+\frac{\partial W^{44}}{\partial x^1}+\frac{\partial W^{12}}{\partial x^2}-\frac{\partial W^{13}}{\partial x^3}-\frac{\partial W^{14}}{\partial x^4}\right)V^{22}\\
& +\left(\frac{\partial W^{12}}{\partial x^4}+\frac{\partial W^{14}}{\partial x^2}-\frac{\partial W^{24}}{\partial x^1}\right)V^{24}\\
& +\frac{1}{2}\left(\frac{\partial W^{44}}{\partial x^1}+\frac{\partial W^{22}}{\partial x^1}-\frac{\partial W^{12}}{\partial x^2}+\frac{\partial W^{13}}{\partial x^3}-\frac{\partial W^{14}}{\partial x^4}\right)V^{33}\\
& +\left(\frac{\partial W^{13}}{\partial x^4}+\frac{\partial W^{14}}{\partial x^3}-\frac{\partial W^{34}}{\partial x^1}\right)V^{34}+\frac{1}{2}\left(\frac{\partial W^{33}}{\partial x^1}+\frac{\partial W^{22}}{\partial x^1}-\frac{\partial W^{12}}{\partial x^2}-\frac{\partial W^{13}}{\partial x^3}+\frac{\partial W^{14}}{\partial x^4}\right)V^{44}\\
& -\frac{1}{2}\left(\frac{\partial V^{13}}{\partial x^3}+\frac{\partial V^{12}}{\partial x^2}+\frac{\partial V^{14}}{\partial x^4}\right)W^{11}+\frac{1}{2}\left(\frac{\partial V^{22}}{\partial x^2}+\frac{\partial V^{44}}{\partial x^2}+\frac{\partial V^{11}}{\partial x^2}+\frac{\partial V^{33}}{\partial x^2}\right)W^{12}\\
& +\frac{1}{2}\left(\frac{\partial V^{11}}{\partial x^3}+\frac{\partial V^{33}}{\partial x^3}+\frac{\partial V^{44}}{\partial x^3}+\frac{\partial V^{22}}{\partial x^3}\right)W^{13}+\frac{1}{2}\left(\frac{\partial V^{22}}{\partial x^4}+\frac{\partial V^{33}}{\partial x^4}+\frac{\partial V^{11}}{\partial x^4}+\frac{\partial V^{44}}{\partial x^4}\right)W^{14}\\
& +\left(\frac{\partial V^{23}}{\partial x^1}-\frac{\partial V^{13}}{\partial x^2}-\frac{\partial V^{12}}{\partial x^3}\right)W^{23}-\frac{1}{2}\left(\frac{\partial V^{33}}{\partial x^1}+\frac{\partial V^{44}}{\partial x^1}+\frac{\partial V^{12}}{\partial x^2}-\frac{\partial V^{13}}{\partial x^3}-\frac{\partial V^{14}}{\partial x^4}\right)W^{22}\\
& +\left(\frac{\partial V^{24}}{\partial x^1}-\frac{\partial V^{14}}{\partial x^2}-\frac{\partial V^{12}}{\partial x^4}\right)W^{24}-\frac{1}{2}\left(\frac{\partial V^{22}}{\partial x^1}+\frac{\partial V^{44}}{\partial x^1}-\frac{\partial V^{12}}{\partial x^2}+\frac{\partial V^{13}}{\partial x^3}-\frac{\partial V^{14}}{\partial x^4}\right)W^{33}\\
& +\left(\frac{\partial V^{34}}{\partial x^1}-\frac{\partial V^{14}}{\partial x^3}-\frac{\partial V^{13}}{\partial x^4}\right)W^{34}-\frac{1}{2}\left(\frac{\partial V^{22}}{\partial x^1}+\frac{\partial V^{33}}{\partial x^1}-\frac{\partial V^{12}}{\partial x^2}-\frac{\partial V^{13}}{\partial x^3}+\frac{\partial V^{14}}{\partial x^4}\right)W^{44},\\
\omega_2^2 = {} & \frac{1}{2}\left(-\frac{\partial W^{12}}{\partial x^1}-\frac{\partial W^{23}}{\partial x^3}+\frac{\partial W^{44}}{\partial x^2}+\frac{\partial W^{33}}{\partial x^2}-\frac{\partial W^{24}}{\partial x^4}\right)V^{11}\\
& +\frac{1}{2}\left(\frac{\partial V^{12}}{\partial x^1}+\frac{\partial V^{23}}{\partial x^3}-\frac{\partial V^{44}}{\partial x^2}-\frac{\partial V^{33}}{\partial x^2}+\frac{\partial V^{24}}{\partial x^4}\right)W^{11}\\
& +\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^1}+\frac{\partial W^{22}}{\partial x^1}-\frac{\partial W^{33}}{\partial x^1}-\frac{\partial W^{44}}{\partial x^1}\right)V^{12}-\frac{1}{2}\left(\frac{\partial V^{11}}{\partial x^1}+\frac{\partial V^{22}}{\partial x^1}-\frac{\partial V^{33}}{\partial x^1}-\frac{\partial V^{44}}{\partial x^1}\right)W^{12}\\
& +\left(\frac{\partial W^{12}}{\partial x^3}+\frac{\partial W^{23}}{\partial x^1}-\frac{\partial W^{13}}{\partial x^2}\right)V^{13}-\left(\frac{\partial V^{12}}{\partial x^3}+\frac{\partial V^{23}}{\partial x^1}-\frac{\partial V^{13}}{\partial x^2}\right)W^{13}\\
& +\left(\frac{\partial W^{24}}{\partial x^1}-\frac{\partial W^{14}}{\partial x^2}+\frac{\partial W^{12}}{\partial x^4}\right)V^{14}-\left(\frac{\partial V^{24}}{\partial x^1}-\frac{\partial V^{14}}{\partial x^2}+\frac{\partial V^{12}}{\partial x^4}\right)W^{14}\\
& +\frac{1}{2}\left(-\frac{\partial W^{12}}{\partial x^1}+\frac{\partial W^{24}}{\partial x^4}+\frac{\partial W^{23}}{\partial x^3}\right)V^{22}-\frac{1}{2}\left(-\frac{\partial V^{12}}{\partial x^1}+\frac{\partial V^{24}}{\partial x^4}+\frac{\partial V^{23}}{\partial x^3}\right)W^{22}\\
& -\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^3}+\frac{\partial W^{22}}{\partial x^3}-\frac{\partial W^{33}}{\partial x^3}-\frac{\partial W^{44}}{\partial x^3}\right)V^{23}+\frac{1}{2}\left(\frac{\partial V^{11}}{\partial x^3}+\frac{\partial V^{22}}{\partial x^3}-\frac{\partial V^{33}}{\partial x^3}-\frac{\partial V^{44}}{\partial x^3}\right)W^{23}\\
& -\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^4}+\frac{\partial W^{22}}{\partial x^4}-\frac{\partial W^{33}}{\partial x^4}-\frac{\partial W^{44}}{\partial x^4}\right)V^{24}+\frac{1}{2}\left(\frac{\partial V^{11}}{\partial x^4}+\frac{\partial V^{22}}{\partial x^4}-\frac{\partial V^{33}}{\partial x^4}-\frac{\partial V^{44}}{\partial x^4}\right)W^{24}\\
& +\frac{1}{2}\left(-\frac{\partial W^{23}}{\partial x^3}-\frac{\partial W^{12}}{\partial x^1}+\frac{\partial W^{24}}{\partial x^4}-\frac{\partial W^{44}}{\partial x^2}+\frac{\partial W^{11}}{\partial x^2}\right)V^{33}\\
& -\frac{1}{2}\left(-\frac{\partial V^{23}}{\partial x^3}-\frac{\partial V^{12}}{\partial x^1}+\frac{\partial V^{24}}{\partial x^4}-\frac{\partial V^{44}}{\partial x^2}+\frac{\partial V^{11}}{\partial x^2}\right)W^{33}\\
& +\left(-\frac{\partial W^{23}}{\partial x^4}-\frac{\partial W^{24}}{\partial x^3}+\frac{\partial W^{34}}{\partial x^2}\right)V^{34}-\left(-\frac{\partial V^{23}}{\partial x^4}-\frac{\partial V^{24}}{\partial x^3}+\frac{\partial V^{34}}{\partial x^2}\right)W^{34}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left(-\frac{\partial W^{24}}{\partial x^3}-\frac{\partial W^{12}}{\partial x^1}-\frac{\partial W^{33}}{\partial x^2}+\frac{\partial W^{23}}{\partial x^3}+\frac{\partial W^{11}}{\partial x^2}\right)V^{44}\\
&-\frac{1}{2}\left(-\frac{\partial V^{24}}{\partial x^3}-\frac{\partial V^{12}}{\partial x^1}-\frac{\partial V^{33}}{\partial x^2}+\frac{\partial V^{23}}{\partial x^3}+\frac{\partial V^{11}}{\partial x^2}\right)W^{44},\\
\omega_2^3=&\ \frac{1}{2}\left(-\frac{\partial W^{23}}{\partial x^2}+\frac{\partial W^{22}}{\partial x^3}+\frac{\partial W^{44}}{\partial x^3}-\frac{\partial W^{13}}{\partial x^1}-\frac{\partial W^{34}}{\partial x^4}\right)V^{11}\\
&-\frac{1}{2}\left(-\frac{\partial V^{23}}{\partial x^2}-\frac{\partial V^{22}}{\partial x^3}+\frac{\partial V^{44}}{\partial x^3}-\frac{\partial V^{13}}{\partial x^1}-\frac{\partial V^{34}}{\partial x^4}\right)W^{11}\\
&+\left(\frac{\partial W^{13}}{\partial x^2}-\frac{\partial W^{12}}{\partial x^3}+\frac{\partial W^{23}}{\partial x^1}\right)V^{12}-\left(\frac{\partial V^{13}}{\partial x^2}-\frac{\partial V^{12}}{\partial x^3}+\frac{\partial V^{23}}{\partial x^1}\right)W^{12}\\
&+\frac{1}{2}\left(\frac{\partial W^{33}}{\partial x^1}-\frac{\partial W^{44}}{\partial x^1}-\frac{\partial W^{22}}{\partial x^1}+\frac{\partial W^{11}}{\partial x^1}\right)V^{13}-\frac{1}{2}\left(\frac{\partial V^{33}}{\partial x^1}-\frac{\partial V^{44}}{\partial x^1}-\frac{\partial V^{22}}{\partial x^1}+\frac{\partial V^{11}}{\partial x^1}\right)W^{13}\\
&-\left(\frac{\partial W^{14}}{\partial x^3}+\frac{\partial W^{13}}{\partial x^4}+\frac{\partial W^{34}}{\partial x^1}\right)V^{14}+\left(\frac{\partial V^{14}}{\partial x^3}+\frac{\partial V^{13}}{\partial x^4}+\frac{\partial V^{34}}{\partial x^1}\right)W^{14}\\
&+\frac{1}{2}\left(\frac{\partial W^{34}}{\partial x^4}-\frac{\partial W^{23}}{\partial x^2}+\frac{\partial W^{11}}{\partial x^3}-\frac{\partial W^{13}}{\partial x^1}-\frac{\partial W^{44}}{\partial x^3}\right)V^{22}\\
&-\frac{1}{2}\left(\frac{\partial V^{34}}{\partial x^4}-\frac{\partial V^{23}}{\partial x^2}+\frac{\partial V^{11}}{\partial x^3}-\frac{\partial V^{13}}{\partial x^1}-\frac{\partial V^{44}}{\partial x^3}\right)W^{22}\\
&+\frac{1}{2}\left(\frac{\partial W^{22}}{\partial x^2}-\frac{\partial W^{33}}{\partial x^2}+\frac{\partial W^{44}}{\partial x^2}-\frac{\partial W^{11}}{\partial x^2}\right)V^{23}-\frac{1}{2}\left(\frac{\partial V^{22}}{\partial x^2}-\frac{\partial V^{33}}{\partial x^2}+\frac{\partial V^{44}}{\partial x^2}-\frac{\partial V^{11}}{\partial x^2}\right)W^{23}\\
&+\left(-\frac{\partial W^{34}}{\partial x^2}+\frac{\partial W^{24}}{\partial x^3}-\frac{\partial W^{23}}{\partial x^4}\right)V^{24}-\left(-\frac{\partial V^{34}}{\partial x^2}+\frac{\partial V^{24}}{\partial x^3}-\frac{\partial V^{23}}{\partial x^4}\right)W^{24}\\
&+\frac{1}{2}\left(\frac{\partial W^{23}}{\partial x^2}-\frac{\partial W^{13}}{\partial x^1}+\frac{\partial W^{34}}{\partial x^4}\right)V^{33}-\frac{1}{2}\left(\frac{\partial V^{23}}{\partial x^2}-\frac{\partial V^{13}}{\partial x^1}+\frac{\partial V^{34}}{\partial x^4}\right)W^{33}\\
&+\frac{1}{2}\left(-\frac{\partial W^{33}}{\partial x^4}+\frac{\partial W^{44}}{\partial x^4}+\frac{\partial W^{22}}{\partial x^4}-\frac{\partial W^{11}}{\partial x^4}\right)V^{34}-\frac{1}{2}\left(-\frac{\partial V^{33}}{\partial x^4}+\frac{\partial V^{44}}{\partial x^4}+\frac{\partial V^{22}}{\partial x^4}-\frac{\partial V^{11}}{\partial x^4}\right)W^{34}\\
&+\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^3}-\frac{\partial W^{13}}{\partial x^1}-\frac{\partial W^{34}}{\partial x^4}-\frac{\partial W^{22}}{\partial x^3}+\frac{\partial W^{23}}{\partial x^2}\right)V^{44}\\
&-\frac{1}{2}\left(\frac{\partial V^{11}}{\partial x^3}-\frac{\partial V^{13}}{\partial x^1}-\frac{\partial V^{34}}{\partial x^4}-\frac{\partial V^{22}}{\partial x^3}+\frac{\partial V^{23}}{\partial x^2}\right)W^{44},\\
\omega_2^4=&\ \frac{1}{2}\left(\frac{\partial W^{33}}{\partial x^4}-\frac{\partial W^{34}}{\partial x^3}+\frac{\partial W^{22}}{\partial x^4}-\frac{\partial W^{24}}{\partial x^2}-\frac{\partial W^{14}}{\partial x^1}\right)V^{11}\\
&-\frac{1}{2}\left(\frac{\partial V^{33}}{\partial x^4}-\frac{\partial V^{34}}{\partial x^3}+\frac{\partial V^{22}}{\partial x^4}-\frac{\partial V^{24}}{\partial x^2}-\frac{\partial V^{14}}{\partial x^1}\right)W^{11}\\
&+\left(-\frac{\partial W^{12}}{\partial x^4}+\frac{\partial W^{24}}{\partial x^1}+\frac{\partial W^{14}}{\partial x^2}\right)V^{12}-\left(-\frac{\partial V^{12}}{\partial x^4}+\frac{\partial V^{24}}{\partial x^1}+\frac{\partial V^{14}}{\partial x^2}\right)W^{12}\\
&+\left(-\frac{\partial W^{13}}{\partial x^4}+\frac{\partial W^{34}}{\partial x^1}+\frac{\partial W^{14}}{\partial x^3}\right)V^{13}-\left(-\frac{\partial V^{13}}{\partial x^4}+\frac{\partial V^{34}}{\partial x^1}+\frac{\partial V^{14}}{\partial x^3}\right)W^{13}\\
&+\frac{1}{2}\left(\frac{\partial W^{11}}{\partial x^1}-\frac{\partial W^{22}}{\partial x^1}-\frac{\partial W^{33}}{\partial x^1}+\frac{\partial W^{44}}{\partial x^1}\right)V^{14}-\frac{1}{2}\left(\frac{\partial V^{11}}{\partial x^1}-\frac{\partial V^{22}}{\partial x^1}-\frac{\partial V^{33}}{\partial x^1}+\frac{\partial V^{44}}{\partial x^1}\right)W^{14}\\
&+\frac{1}{2}\left(-\frac{\partial W^{24}}{\partial x^4}+\frac{\partial W^{34}}{\partial x^3}-\frac{\partial W^{14}}{\partial x^1}-\frac{\partial W^{33}}{\partial x^4}+\frac{\partial W^{11}}{\partial x^4}\right)V^{22}\\
&-\frac{1}{2}\left(-\frac{\partial V^{24}}{\partial x^4}+\frac{\partial V^{34}}{\partial x^3}-\frac{\partial V^{14}}{\partial x^1}-\frac{\partial V^{33}}{\partial x^4}+\frac{\partial V^{11}}{\partial x^4}\right)W^{22}\\
&+\left(-\frac{\partial W^{34}}{\partial x^2}-\frac{\partial W^{24}}{\partial x^3}+\frac{\partial W^{23}}{\partial x^4}\right)V^{23}-\left(-\frac{\partial V^{34}}{\partial x^2}-\frac{\partial V^{24}}{\partial x^3}+\frac{\partial V^{23}}{\partial x^4}\right)W^{23}
\end{aligned}
$$

$$+\frac{1}{2}\left(-\frac{\partial W^{11}}{\partial x^2}+\frac{\partial W^{22}}{\partial x^2}+\frac{\partial W^{33}}{\partial x^2}-\frac{\partial W^{44}}{\partial x^2}\right)V^{24}-\frac{1}{2}\left(-\frac{\partial V^{11}}{\partial x^2}+\frac{\partial V^{22}}{\partial x^2}+\frac{\partial V^{33}}{\partial x^2}-\frac{\partial V^{44}}{\partial x^2}\right)W^{24}$$
$$+\frac{1}{2}\left(-\frac{\partial W^{14}}{\partial x^1}-\frac{\partial W^{34}}{\partial x^3}+\frac{\partial W^{24}}{\partial x^2}+\frac{\partial W^{11}}{\partial x^4}-\frac{\partial W^{22}}{\partial x^4}\right)V^{33}$$
$$-\frac{1}{2}\left(-\frac{\partial V^{14}}{\partial x^1}-\frac{\partial V^{34}}{\partial x^3}+\frac{\partial V^{24}}{\partial x^2}+\frac{\partial V^{11}}{\partial x^4}-\frac{\partial V^{22}}{\partial x^4}\right)W^{33}$$
$$+\frac{1}{2}\left(\frac{\partial W^{22}}{\partial x^3}+\frac{\partial W^{33}}{\partial x^3}-\frac{\partial W^{11}}{\partial x^3}-\frac{\partial W^{44}}{\partial x^3}\right)V^{34}-\frac{1}{2}\left(\frac{\partial V^{22}}{\partial x^3}+\frac{\partial V^{33}}{\partial x^3}-\frac{\partial V^{11}}{\partial x^3}-\frac{\partial V^{44}}{\partial x^3}\right)W^{34}$$
$$+\frac{1}{2}\left(-\frac{\partial W^{14}}{\partial x^1}+\frac{\partial W^{34}}{\partial x^3}+\frac{\partial W^{24}}{\partial x^2}\right)V^{44}-\frac{1}{2}\left(-\frac{\partial V^{14}}{\partial x^1}+\frac{\partial V^{34}}{\partial x^3}+\frac{\partial V^{24}}{\partial x^2}\right)W^{44}.$$

Remark 7.2 The cohomology class $[\omega_2(X_a^k, X_b^l)]$ of the closed 3-form omega $\omega_2(X_a^k, X_b^l)$ may be non-trivial for certain particular values of k, l, a, b; for example: For $a = 13, b = 18$ (hence $k_4 \neq 0, l_4 \neq 0$), we have

$$\omega_2^i(X_{13}^k, X_{18}^l) = c_{13,18}^i \exp\left(i\left(k_j + l_j\right)x^j\right),$$
$$c_{13,18}^1 = -\frac{il_1}{k_4 l_4^2}\left(l_2\left(l_2k_2 + 2l_3k_3 + l_4k_4\right) + k_2\left(-l_3^2 + l_2k_2 + l_3k_3\right)\right),$$
$$c_{13,18}^2 = \frac{il_1}{k_4 l_4^2}\left(\left(l_1 + k_1\right)l_2k_2 + 2l_3k_1k_3\right),$$
$$c_{13,18}^3 = -\frac{il_1l_3}{k_4 l_4^2}k_2\left(-l_1 + k_1\right),$$
$$c_{13,18}^4 = \frac{il_1l_2k_1}{k_4 l_4}.$$

Hence, for $k_i = -l_i$, $1 \leq i \leq 4$, and $l_2 = 0$ we have

$$[\omega_2(X_{13}^k, X_{18}^l)] = \omega_2^2(X_{13}^k, X_{18}^l)[v_2] = -2\frac{l_1^2 l_3^2}{l_4^3}[v_2].$$

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