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Harmonicity of the Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures

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Abstract In this paper, we describe the oriented Riemannian four-manifolds M for which the Atiyah–Hitchin–Singer or Eells–Salamon almost complex structure on the twistor space $\mathcal{Z}$ of M determines a harmonic map from $\mathcal{Z}$ into its twistor space.

Keywords Twistor spaces · Almost complex structures · Harmonic maps

Mathematics Subject Classification Primary 53C43; Secondary 58E20 · 53C28

1 Introduction

The twistor approach has been used for years for studying conformal geometry of four-manifolds by means of complex geometric methods, and in this way many important results have been obtained. Moreover, the twistor spaces endowed with the Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures are interesting geometric objects in their own right whose geometric properties have been studied by many authors. In this paper, we look at these structures from the point of view of variational theory. The motivation behind is the fact that if a Riemannian manifold admits an almost complex structure compatible with its

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metric, it possesses many such structures (cf., for example [6,9]). Thus, it is natural to seek criteria that distinguish some of these structures among all. One way to obtain such a criterion is to consider the compatible almost complex structures on a Riemannian manifold (N, h) as sections of its twistor bundle $\mathcal{Z}$. The smooth manifold $\mathcal{Z}$ admits a natural Riemannian metric h_1 such that the projection map $(\mathcal{Z}, h_1) \to (N, h)$ is a Riemannian submersion with totally geodesic fibres. From this point of view, Calabi and Gluck [4] have proposed to consider as "the best" those compatible almost complex structures J on (N, h) whose image $J(N)$ in $\mathcal{Z}$ is of minimal volume. They have proved that the standard almost Hermitian structure on the 6-sphere S^6, defined by means of the Cayley numbers, can be characterized by that property. Another criterion has been discussed by Wood [28,29] who has suggested to single out the structures J that are harmonic sections of the twistor bundle $\mathcal{Z}$, i.e. critical points of the energy functional under variations through sections of $\mathcal{Z}$. While the Kähler structures are absolute minima of the energy functional, there are many examples of non-Kähler structures, which are harmonic sections [28,29]. Sufficient conditions for a compatible almost complex structure to be a minimizer of the energy functional and examples of non-Kähler minimizers have been given by Bor et al. [3].

Forgetting the bundle structure of $\mathcal{Z}$, we can also consider compatible almost complex structures that are critical points of the energy functional under variations through all maps $N \to \mathcal{Z}$. These structures are genuine harmonic maps from (N, h) into $(\mathcal{Z}, h_1)$; we refer to [12] for basic facts about harmonic maps. The problem when a compatible almost complex structure on a four-dimensional Riemannian manifold is a harmonic map into its twistor space has been studied in [9] (see also [6]).

If the base manifold N is oriented, the twistor space $\mathcal{Z}$ has two connected components often called positive and negative twistor spaces of (N, h); their sections are compatible almost complex structures yielding the orientation and, respectively, the opposite orientation of N.

Setting $h_t = \pi^* h + t h^v$, $t > 0$, where $\pi : \mathcal{Z} \to N$ is the projection map and h^v is the metric of the fibre, define a 1-parameter family of Riemannian metrics on $\mathcal{Z}$ compatible with the almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on $\mathcal{Z}$ introduced, respectively, by Atiyah–Hitchin–Singer [1] and Eells–Salamon [13]. In [8] we have found geometric conditions on an oriented four-dimensional Riemannian manifold under which the almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on its negative twistor space $(\mathcal{Z}, h_t)$ are harmonic sections.

Theorem 1 *Let (M, g) be an oriented Riemannian 4-manifold and let $(\mathcal{Z}, h_t)$ be its negative twistor space. Then:*
(i) The Atiyah–Hitchin–Singer almost complex structure $\mathcal{J}_1$ on $(\mathcal{Z}, h_t)$ is a harmonic section if and only if (M, g) is a self-dual manifold.
(ii) The Eells–Salamon almost complex structure $\mathcal{J}_2$ on $(\mathcal{Z}, h_t)$ is a harmonic section if and only if (M, g) is a self-dual manifold with constant scalar curvature.

By a theorem of Atiyah–Hitchin–Singer [1], the self-duality of (M, g) is a necessary and sufficient condition for the integrability of the almost complex structure $\mathcal{J}_1$. In contrast, the almost complex structure $\mathcal{J}_2$ is never integrable by a result of Eells–Salamon [13] but it is very useful for constructing harmonic maps.

The aim of the present paper is to find the four-manifolds for which the almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are harmonic maps. More precisely, we prove the following

Theorem 2 *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures on the (negative) twistor space $(\mathcal{Z}, h_t)$ of an oriented Riemannian four-manifold (M, g). Each $\mathcal{J}_k$ (k=1 or 2) is a harmonic map if and only if (M, g) is either self-dual and*

Einstein, or is locally the product of an open interval in $\mathbb{R}$ *and a* 3*-dimensional Riemannian manifold of constant curvature.*

Note that any compact self-dual Einstein manifold with positive scalar curvature is isometric to the 4-sphere $\mathbb{S}^4$ or the complex projective space $\mathbb{CP}^2$ with their standard metrics [14,16] (see also [2, Theorem 13.30]). In the case of negative scalar curvature, a complete classification is not available yet and the only known compact examples are quotients of the unit ball in $\mathbb{C}^2$ with the metric of constant negative curvature or the Bergman metric. In contrast, there are many local examples of self-dual Einstein metrics with a prescribed sign of the scalar curvature (cf., e.g. [11,17,20–22,24,26]). Note also that every Riemannian manifold that locally is the product of an open interval in $\mathbb{R}$ and a 3-dimensional Riemannian manifold of constant curvature c is locally conformally flat with constant scalar curvature $6c$. It is not Einstein unless $c = 0$, i.e. Ricci flat.

The proof of Theorem 2 is based on an explicit formula for the second fundamental form $\widetilde{\nabla} J_*$ of a compatible almost complex structure J on a Riemannian manifold considered as a map from the manifold into its twistor space (Proposition 1). In particular, it follows from Theorem 1 mentioned above that if the vertical part of $Trace\widetilde{\nabla}\mathcal{J}_{k*}$ vanishes then the manifold (M, J) is self-dual. This simplifies the formulas for the values of the horizonal part of $Trace\widetilde{\nabla}\mathcal{J}_{k*}$ at vertical and horizontal vectors (Lemmas 1 and 2). Using these formulas, we show that the Ricci tensor of (M, g) is parallel and three of its eigenvalues coincide. Thus either (M, g) is Einstein or exactly three of the eigenvalues coincide. In the second case, a result in [10, Lemma 1] (essentially due to LeBrun and Apostolov) implies that the simple eigenvalue vanishes, thus (M, g) is locally the product of an interval in $\mathbb{R}$ and a 3-manifold of constant curvature.

Note also that if $(h_t, \mathcal{J}_1)$ is a Kähler structure, then $\mathcal{J}_1$ is a totally geodesic map. It is a result of Friedrich–Kurke [14] that $(h_t, \mathcal{J}_1)$ is Kähler exactly when the base manifold is self-dual and Einstein with positive scalar curvature $12/t$. The necessary and sufficient conditions for $\mathcal{J}_1$ and $\mathcal{J}_2$ to be totally geodesic maps will be discussed elsewhere.

2 Preliminaries

2.1 The manifold of compatible linear complex structures

Let V be a real vector space of even dimension $n = 2m$ endowed with an Euclidean metric g. Denote by $F(V)$ the set of all complex structures on V compatible with the metric g, i.e. g-orthogonal. This set has the structure of an imbedded submanifold of the vector space $\mathfrak{so}(V)$ of skew-symmetric endomorphisms of (V, g).

The group $O(V)$ of orthogonal transformations of (V, g) acts smoothly and transitively on the set $F(V)$ by conjugation. The isotropy subgroup at a fixed $J \in F(V)$ consists of the orthogonal transformations commuting with J. Therefore, $F(V)$ can be identified with the homogeneous space $O(2m)/U(m)$. In particular, $dim\, F(V) = m^2 - m$. Moreover, $F(V)$ has two connected components. If we fix an orientation on V, these components consist of all complex structures on V compatible with the metric g and inducing $\pm$ the orientation of V; each of them has the homogeneous representation $SO(2m)/U(m)$.

The tangent space of $F(V)$ at a point J consists of all endomorphisms $Q \in \mathfrak{so}(V)$ anti-commuting with J and we have the decomposition

$$\mathfrak{so}(V) = T_J F(V) \oplus \{S \in \mathfrak{so}(V) : \ SJ - JS = 0\}. \tag{1}$$

This decomposition is orthogonal with respect to the restriction to $F(V)$ of the metric $G(A, B) = -\frac{1}{n}TraceAB$ of $\mathfrak{so}(V)$ (the factor $1/n$ is chosen so that every $J \in F(V)$ to have unit norm). The metric G on $F(V)$ is compatible with the almost complex structure $\mathcal{J}$ defined by

$$\mathcal{J}Q = JQ \quad \text{for} \quad Q \in T_J F(V).$$

Let $J \in F(V)$ and let $e_1, \dots, e_{2m}$ be an orthonormal basis of V such that $Je_{2k-1} = e_{2k}$, $k = 1, \dots, m$. Define skew-symmetric endomorphisms $S_{a,b}$, $a, b = 1, \dots, 2m$, of V setting

$$S_{a,b}e_c = \sqrt{\frac{n}{2}}(\delta_{ac}e_b - \delta_{bc}e_a), \quad c = 1, \dots, 2m.$$

The maps $S_{a,b}$, $1 \le a < b \le 2m$, constitute a G-orthonormal basis of $\mathfrak{so}(V)$. Set

$$A_{r,s} = \frac{1}{\sqrt{2}}(S_{2r-1,2s-1} - S_{2r,2s}), \quad B_{r,s} = \frac{1}{\sqrt{2}}(S_{2r-1,2s} + S_{2r,2s-1}),$$
$$r = 1, \dots, m-1, \; s = r+1, \dots, m.$$

Then, $\{A_{r,s}, B_{r,s}\}$ is a G-orthonormal basis of $T_J F(V)$ with $B_{r,s} = \mathcal{J}A_{r,s}$.

Denote by D the Levi-Civita connection of the metric G on $F(V)$. Let X, Y be vector fields on $F(V)$ considered as $\mathfrak{so}(V)$-valued functions on $\mathfrak{so}(V)$. By the Koszul formula, for every $J \in F(V)$,

$$(D_X Y)_J = \frac{1}{2}(Y'(J)(X_J) + J \circ Y'(J)(X_J) \circ J) \tag{2}$$

where $Y'(J) \in Hom(\mathfrak{so}(V), \mathfrak{so}(V))$ is the derivative of the function $Y : \mathfrak{so}(V) \to \mathfrak{so}(V)$ at the point J. The latter formula easily implies that $(G, \mathcal{J})$ is a Kähler structure on $F(V)$. Note also that the metric G is Einstein with scalar curvature $\frac{m}{2}(m-1)(m^2 - m)$ (see, for example [5]).

2.2 The four-dimensional case

Suppose that $dim\, V = 4$. Then, as is well-known, each of the two connected components of $F(V)$ can be identified with the unit sphere S^2. It is often convenient to describe this identification in terms of the space $\Lambda^2 V$. The metric g of V induces a metric on $\Lambda^2 V$ given by

$$g(x_1 \wedge x_2, x_3 \wedge x_4) = \frac{1}{2}[g(x_1, x_3)g(x_2, x_4) - g(x_1, x_4)g(x_2, x_3)],$$

the factor $1/2$ being chosen in consistence with [7,8]. Consider the isomorphisms $\mathfrak{so}(V) \cong \Lambda^2 V$ sending $\varphi \in \mathfrak{so}(V)$ to the 2-vector $\varphi^\wedge$ for which

$$2g(\varphi^\wedge, x \wedge y) = g(\varphi x, y), \quad x, y \in V.$$

This isomorphism is an isometry with respect to the metric G on $\mathfrak{so}(V)$ and the metric g on $\Lambda^2 V$. Given $a \in \Lambda^2 V$, the skew-symmetric endomorphism of V corresponding to a under the inverse isomorphism will be denoted by K_a.

Fix an orientation on V and denote by $F_\pm(V)$ the set of complex structures on V compatible with the metric g and inducing $\pm$ the orientation of V. The Hodge star operator defines an endomorphism $*$ of $\Lambda^2 V$ with $*^2 = Id$. Hence, we have the decomposition

$$\Lambda^2 V = \Lambda^2_- V \oplus \Lambda^2_+ V$$

where $\Lambda^2_{\pm}V$ are the subspaces of $\Lambda^2 V$ corresponding to the (± 1)-eigenvalues of the operator $*$. Let (e_1, e_2, e_3, e_4) be an oriented orthonormal basis of V. Set

$$s_1^{\pm} = e_1 \wedge e_2 \pm e_3 \wedge e_4, \quad s_2^{\pm} = e_1 \wedge e_3 \pm e_4 \wedge e_2, \quad s_3^{\pm} = e_1 \wedge e_4 \pm e_2 \wedge e_3. \tag{3}$$

Then, $(s_1^{\pm}, s_2^{\pm}, s_3^{\pm})$ is an orthonormal basis of $\Lambda^2_{\pm}V$. Note that this basis defines an orientation on $\Lambda^2_{\pm}V$, which does not depend on the choice of the basis (e_1, e_2, e_3, e_4) (see, for example, [6]). We call this orientation "canonical".

It is easy to see that the isomorphism $\varphi \to \varphi^{\wedge}$ identifies $F_{\pm}(V)$ with the unit sphere $S(\Lambda^2_{\pm}V)$ of the Euclidean vector space $(\Lambda^2_{\pm}V, g)$. Under this isomorphism, if $J \in F_{\pm}(V)$, the tangent space $T_J F(V) = T_J F_{\pm}(V)$ is identified with the orthogonal complement $(\mathbb{R}J^{\wedge})^{\perp}$ of the space $\mathbb{R}J^{\wedge}$ in $\Lambda^2_{\pm}V$.

Consider the 3-dimensional Euclidean space $(\Lambda^2_{\pm}V, g)$ with its canonical orientation and denote by $\times$ the usual vector-cross product in it. Then, if $a, b \in \Lambda^2_{\pm}V$, the isomorphism $\Lambda^2 V \cong \mathfrak{so}(V)$ sends $a \times b$ to $\pm\frac{1}{2}[K_a, K_b]$. Thus, if $J \in F_{\pm}(V)$ and $Q \in T_J F(V) = T_J F_{\pm}(V)$, we have

$$(\mathcal{J}Q)^{\wedge} = \pm(J^{\wedge} \times Q^{\wedge}). \tag{4}$$

2.3 The twistor space of an even-dimensional Riemannian manifold

Let (N, g) be a Riemannian manifold of dimension $n = 2m$. Denote by $\pi : \mathcal{Z} \to N$ the bundle over N whose fibre at every point $p \in N$ consists of all compatible complex structures on the Euclidean vector space $(T_p N, g_p)$. This is the associated bundle

$$\mathcal{Z} = O(N) \times_{O(n)} F(\mathbb{R}^n)$$

where $O(N)$ is the principal bundle of orthonormal frames on N and $F(\mathbb{R}^n)$ is the manifold of complex structures on $\mathbb{R}^n$ compatible with its standard metric. The manifold $\mathcal{Z}$ is called the twistor space of (N, g).

The Levi-Civita connection of (N, g) gives rise to a splitting $\mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of any bundle associated with $O(N)$ into vertical and horizontal parts. This allows one to define a natural 1-parameter family of Riemannian metrics h_t, $t > 0$, on the manifold $\mathcal{Z}$ sometimes called "the canonical variation of the metric of N" [2, Chapter 9 G]. For every $J \in \mathcal{Z}$, the horizontal subspace $\mathcal{H}_J$ of $T_J\mathcal{Z}$ is isomorphic via the differential π_{*J} to the tangent space $T_{\pi(J)}N$ and the metric h_t on $\mathcal{H}_J$ is the lift of the metric g on $T_{\pi(J)}N$, $h_t|\mathcal{H}_J = \pi^* g$. The vertical subspace $\mathcal{V}_J$ of $T_J\mathcal{Z}$ is the tangent space at J to the fibre of the bundle $\mathcal{Z}$ through J and $h_t|\mathcal{V}_J$ is defined as t times the metric G of this fibre. Finally, the horizontal space $\mathcal{H}_J$ and the vertical space $\mathcal{V}_J$ are declared to be orthogonal. Then, by the Vilms theorem [27], the projection $\pi : (\mathcal{Z}, h_t) \to (N, g)$ is a Riemannian submersion with totally geodesic fibres (this can also be proved directly).

The manifold $\mathcal{Z}$ admits two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ defined in the case $dim\, N = 4$ by Atiyah–Hitchin–Singer [1] and Eells–Salamon [13], respectively. On a vertical space $\mathcal{V}_J$, $\mathcal{J}_1$ is defined to be the complex structure $\mathcal{J}_J$ of the fibre through J, while $\mathcal{J}_2$ is defined as the conjugate complex structure, i.e. $\mathcal{J}_2|\mathcal{V}_J = -\mathcal{J}_J$. On a horizontal space $\mathcal{H}_J$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are both defined to be the lift to $\mathcal{H}_J$ of the endomorphism J of $T_{\pi(J)}N$. The almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible with each metric h_t.

Consider $\mathcal{Z}$ as a submanifold of the bundle

$$\pi : A(TN) = O(N) \times_{O(n)} so(n) \to N$$

of skew-symmetric endomorphisms of TN. The inclusion of $\mathcal{Z}$ into $A(TN)$ is fibre-preserving and, for every $J \in \mathcal{Z}$, the horizontal subspace $\mathcal{H}_J$ of $T_J\mathcal{Z}$ coincides with the horizontal subspace of $T_J A(TN)$ since the inclusion of $F(\mathbb{R}^n)$ into $so(n)$ is $O(n)$-equivariant.

The Levi-Civita connection of (N, g) determines a connection on the bundle $A(TN)$, both denoted by ∇, and the corresponding curvatures are related by

$$(R(X,Y)\varphi)(Z) = R(X,Y)\varphi(Z) - \varphi(R(X,Y)Z)$$

for $\varphi \in A(TN)$, $X, Y, Z \in TN$. The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TN$ defined by

$$g(\mathcal{R}(X\wedge Y), Z\wedge T) = g(R(X,Y)Z, T), \quad X, Y, Z, T \in TN.$$

Let us note that we adopt the following definition for the curvature tensor $R : R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

Let $(U, x_1, \ldots, x_n)$ be a local coordinate system of N and $E_1, \ldots, E_n$ an orthonormal frame of TN on U. Define sections S_{ij}, $1 \le i, j \le n$, of $A(TN)$ by the formula

$$S_{ij}E_l = \sqrt{\frac{n}{2}}(\delta_{il}E_j - \delta_{lj}E_i), \quad l = 1, \ldots, n. \tag{5}$$

Then, S_{ij}, $i < j$, form an orthonormal frame of $A(TN)$ with respect to the metric $G(a,b) = -\dfrac{1}{n} Trace(a \circ b)$; $a, b \in A(TN)$. Set

$$\tilde{x}_i(a) = x_i \circ \pi(a), \;\; y_{jl}(a) = \sqrt{\frac{2}{n}}G(a, S_{jl}), \quad j < l,$$

for $a \in A(TN)$. Then, $(\tilde{x}_i, y_{jl})$ is a local coordinate system of the manifold $A(TN)$. Setting $y_{lk} = -y_{kl}$ for $l \ge k$, we have $aE_j = \sum_{l=1}^n y_{jl}E_l$, j=1,…,n.

For each vector field

$$X = \sum_{i=1}^n X^i \tfrac{\partial}{\partial x_i}$$

on U, the horizontal lift X^h on $\pi^{-1}(U)$ is given by

$$X^h = \sum_{i=1}^n (X^i \circ \pi)\tfrac{\partial}{\partial \tilde{x}_i} - \sum_{j<l}\sum_{p<q} y_{pq}G(\nabla_X S_{pq}, S_{jl}) \circ \pi \tfrac{\partial}{\partial y_{jl}}. \tag{6}$$

Let $a \in A(TN)$ and $p = \pi(a)$. Then, (6) implies that, under the standard identification $T_aA(TN) \cong A(T_pN)$ ($=$ the skew-symmetric endomorphisms of (T_pN, g_p)), we have

$$[X^h, Y^h]_a = [X,Y]^h_a + R(X,Y)a. \tag{7}$$

Farther we shall often make use of the isomorphism $A(TN) \cong \Lambda^2 TN$ that assigns to each $a \in A(T_pN)$ the 2-vector $a^\wedge$ for which

$$2g(a^\wedge, X \wedge Y) = g(aX, Y), \quad X, Y \in T_pN,$$

the metric on $\Lambda^2 TN$ being defined by

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = \frac{1}{2}[g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)].$$

Lemma 1 ([5]) *For every $a, b \in A(T_pN)$ and $X, Y \in T_pN$, we have*

$$G(R(X, Y)a, b) = \frac{2}{n} g(R([a, b]^{\wedge})X, Y). \tag{8}$$

Proof Let $E_1, \dots, E_n$ be an orthonormal basis of T_pN. Then,

$$[a, b] = \frac{1}{2} \sum_{i,j=1}^{n} g([a, b]E_i, E_j)E_i \wedge E_j.$$

Therefore,

$$\begin{aligned}
&g(R([a, b]^{\wedge})X, Y)\\
&= \frac{1}{2} \sum_{i,j=1}^{n} g(R(X, Y)E_i, E_j)[g(abE_i, E_j) + g(aE_i, bE_j)]\\
&= \frac{1}{2} \sum_{i=1}^{n} g(R(X, Y)E_i, abE_i)\\
&\quad + \frac{1}{2} \sum_{i,j,k=1}^{n} g(R(X, Y)E_i, E_j)g(E_i, aE_k)g(E_j, bE_k)\\
&= -\frac{1}{2} \sum_{i=1}^{n} g(a(R(X, Y)E_i), bE_i) + \frac{1}{2} \sum_{k=1}^{n} g(R(X, Y)aE_k, bE_k)\\
&= \frac{n}{2} G(R(X, Y)a, b).
\end{aligned}$$

□

For every $J \in \mathcal{Z}$, we identify the vertical space $\mathcal{V}_J$ with the subspace of $A(T_{\pi(J)}N)$ of skew-symmetric endomorphisms anti-commuting with J. Then, for every section K of the twistor space $\mathcal{Z}$ near a point $p \in N$ and every $X \in T_pN$, the endomorphism $\nabla_X K$ of T_pN belongs to the vertical space $\mathcal{V}_{K(p)}$.

Lemma 1 implies that

$$h_t(R(X, Y)J, V) = \frac{2t}{n} g(R([J, V]^{\wedge})X, Y) = \frac{4t}{n} g(R((J \circ V)^{\wedge})X, Y). \tag{9}$$

Denote by D the Levi-Civita connection of $(\mathcal{Z}, h_t)$.

Lemma 2 ([5,7]) *If X, Y are vector fields on N and V is a vertical vector field on $\mathcal{Z}$, then*

$$(D_{X^h} Y^h)_J = (\nabla_X Y)^h_J + \frac{1}{2} R_p(X \wedge Y)J \tag{10}$$

$$(D_V X^h)_J = \mathcal{H}(D_{X^h} V)_J = -\frac{2t}{n} \left(R_p\left((J \circ V_J)^{\wedge}\right) X\right)^h_J \tag{11}$$

where $J \in \mathcal{Z}$, $p = \pi(J)$, and $\mathcal{H}$ means "the horizontal component".

Proof Identity (10) follows from the Koszul formula for the Levi-Civita connection and (7).

Let W be a vertical vector field on $\mathcal{Z}$. Then,

$$h_t\left(D_V X^h, W\right) = -h_t\left(X^h, D_V W\right) = 0$$

since the fibres are totally geodesic submanifolds, so $D_V W$ is a vertical vector field. Therefore, $D_V X^h$ is a horizontal vector field. Moreover, $[V, X^h]$ is a vertical vector field, hence $D_V X^h = \mathcal{H} D_{X^h} V$. Thus,

$$h_t\left(D_V X^h, Y^h\right) = h_t\left(D_{X^h} V, Y^h\right) = -h_t\left(V, D_{X^h} Y^h\right).$$

Now (11) follows from (10) and (9). □

3 The second fundamental form of an almost Hermitian structure as a map into the twistor space

Now let J be an almost complex structure on the manifold N compatible with the metric g. Then, J can be considered as a section of the bundle $\pi : \mathcal{Z} \to N$. Thus, we have a map $J : (N, g) \to (\mathcal{Z}, h_t)$ between Riemannian manifolds. Let $J^*T\mathcal{Z} \to N$ be the pull-back of the bundle $T\mathcal{Z} \to \mathcal{Z}$ under the map $J : N \to \mathcal{Z}$. Then, we can consider the differential $J_* : TN \to T\mathcal{Z}$ as a section of the bundle $Hom(TN, J^*T\mathcal{Z}) \to N$. Denote by $\widetilde{D}$ the connection on $J^*T\mathcal{Z}$ induced by the Levi-Civita connection D on $T\mathcal{Z}$. The Levi Civita connection ∇ on TN and the connection $\widetilde{D}$ on $J^*T\mathcal{Z}$ induce a connection $\widetilde{\nabla}$ on the bundle $Hom(TN, J^*T\mathcal{Z})$. Recall that the second fundamental form of the map J is, by definition,

$$\widetilde{\nabla} J_*$$

The map $J : (N, g) \to (\mathcal{Z}, h_t)$ is harmonic if and only if

$$Trace_g \widetilde{\nabla} J_* = 0.$$

Recall also that the map $J : (N, g) \to (\mathcal{Z}, h_t)$ is totally geodesic exactly when $\widetilde{\nabla} J_* = 0$.

Any (local) section a of the bundle $A(TN)$ determines a (local) vertical vector field $\widetilde{a}$ defined by

$$\widetilde{a}_I = \frac{1}{2}(a(p) + I \circ a(p) \circ I), \quad p = \pi(I).$$

Thus, if $aE_j = \sum_{l=1}^n a_{jl} E_l$,

$$\widetilde{a} = \sum_{j<l} \widetilde{a}_{jl} \frac{\partial}{\partial y_{jl}}$$

where

$$\widetilde{a}_{jl} = \frac{1}{2}\left[a_{jl} \circ \pi + \sum_{r,s=1}^{n} y_{jr}(a_{rs} \circ \pi) y_{sl}\right]$$

The next lemma is "folklore".

Lemma 3 *If $I \in \mathcal{Z}$ and X is a vector field on a neighbourhood of the point $p = \pi(I)$, then*

$$[X^h, \widetilde{a}]_I = (\widetilde{\nabla_X a})_I.$$

Proof Take an orthonormal frame $E_1, \ldots, E_n$ of TN near the point p such that $\nabla E_i|_p = 0$, $i = 1, \ldots, n$. Let $(\tilde{x}_i, y_{jl})$, $1 \leq j < l \leq n$, be the local coordinates of $A(TN)$ defined by means of a local coordinate system x of N at p and the frame $E_1, \ldots, E_n$. Then, by (6),

$$\left[X^h, \frac{\partial}{\partial y_{jl}}\right]_I = 0, \quad j, l = 1, \ldots, n, \quad X^h = \sum_{i=1}^n X^i(p) \left(\frac{\partial}{\partial \tilde{x}_i}\right)_I.$$

It follows that

$$\left[X^h, \widetilde{a}\right]_I = \frac{1}{2}\left[X_p(a_{jl}) + \sum_{k,m=1}^{n} y_{jk}(I)X_p(a_{km})y_{ml}(I)\right] = \left(\widetilde{\nabla_X a}\right)_I$$

since

$$\left(\nabla_{X_p} a\right)(E_i) = \sum_{l=1}^{n} X_p(a_{jl})(E_l)_p.$$

□

Remark 1 For every $I \in \mathcal{Z}$, we can find local sections $a_1, \ldots, a_{m^2-m}$ of $A(TN)$ whose values at $p = \pi(I)$ constitute a basis of the vertical space $\mathcal{V}_I$ and such that $\nabla a_\alpha|_p = 0, \alpha = 1, \ldots, m^2 - m$. Let $\widetilde{a}_\alpha$ be the vertical vector fields determined by the sections a_α. Lemma 3 and the Koszul formula for the Levi-Civita connection imply that $h_t(D_{\widetilde{a}_\alpha}\widetilde{a}_\beta, X^h)_I = 0$ for every $X \in T_pN$. Therefore, for every vertical vector fields U and V, the covariant derivative $(D_U V)_I$ at I is a vertical vector. It follows that the fibres of the twistor bundle are totally geodesic submanifolds.

Let $I \in \mathcal{Z}$ and let $U, V \in \mathcal{V}_I$. Take sections a and b of $A(TN)$ such that $a(p) = U$, $b(p) = V$ for $p = \pi(I)$. Let $\widetilde{a}$ and $\widetilde{b}$ be the vertical vector fields determine by the sections a and b. Taking into account the fact that the fibre of $\mathcal{Z}$ through the point I is a totally geodesic submanifold and applying formula (2) we get

$$(D_{\widetilde{a}}\widetilde{b})_I = \frac{1}{4}[UVI + IVU + I(UVI + IVU)I] = 0. \tag{12}$$

Lemma 4 *For every $p \in N$, there exists a h_t-orthonormal frame of vertical vector fields $\{V_\alpha : \alpha = 1, \ldots, m^2 - m\}$ such that*

(1) $(D_{V_\alpha} V_\beta)_{J(p)} = 0, \quad \alpha, \beta = 1, \ldots, m^2 - m.$

(2) *If X is a vector field near the point p, $[X^h, V_\alpha]_{J(p)} = 0$.*

(3) $\nabla_{X_p}(V_\alpha \circ J) \perp \mathcal{V}_{J(p)}$

Proof Let $E_1, \ldots, E_n$ be an orthonormal frame of TN in a neighbourhood N of p such that $J(E_{2k-1})_p = (E_{2k})_p$, $k = 1, \ldots, m$, and $\nabla E_l|_p = 0$, $l = 1, \ldots, n$. Define sections S_{ij},$1 \le i, j \le n$ by (5) and, as in Sect. 2, set

$$A_{r,s} = \tfrac{1}{\sqrt{2}}(S_{2r-1,2s-1} - S_{2r,2s}), \quad B_{r,s} = \tfrac{1}{\sqrt{2}}(S_{2r-1,2s} + S_{2r,2s-1}),$$
$$r = 1, \ldots, m-1, \; s = r+1, \ldots, m.$$

Then, $\{(A_{r,s})_p, (B_{r,s})_p\}$ is a G-orthonormal basis of the vertical space $\mathcal{V}_{J(p)}$ such that $(B_{r,s})_p = J(A_{r,s})_p$. Note also that $\nabla A_{r,s}|_p = \nabla B_{r,s}|_p = 0$. Let $\widetilde{A}_{r,s}$ and $\widetilde{B}_{r,s}$ be the vertical vector fields on $\mathcal{Z}$ determined by the sections $A_{r,s}$ and $B_{r,s}$ of $A(TN)$. These vector fields constitute a frame of the vertical bundle $\mathcal{V}$ in a neighbourhood of the point $J(p)$.

Consider $\widetilde{A}_{r,s} \circ J$ as a section of $A(TN)$. Then, if $X \in T_pN$, we have

$$\begin{aligned}
\nabla_{X_p}\left(\widetilde{A}_{r,s} \circ J\right) &= \frac{1}{2}\left\{\left(\nabla_{X_p} J\right) \circ (A_{r,s})_p \circ J_p + J_p \circ (A_{r,s}) \circ \left(\nabla_{X_p} J\right)\right\} \\
&= \frac{1}{2}\left\{-\nabla_{X_p} J \circ J_p \circ (A_{r,s})_p + J_p \circ (A_{r,s}) \circ \left(\nabla_{X_p} J\right)\right\} \\
&= \frac{1}{2}\left[(B_{r,s})_p, \nabla_{X_p} J\right]
\end{aligned}$$

The endomorphisms $(B_{r,s})_p$ and $\nabla_{X_p} J$ of $T_p N$ belong to $\mathcal{V}_{J(p)}$, so they anti-commute with $J(p)$, hence their commutator commutes with $J(p)$. Therefore, in view of (1), the commutator $[(B_{r,s})_p, \nabla_{X_p} J]$ is G-orthogonal to the vertical space at $J(p)$. Thus

$$\nabla_{X_p}\left(\widetilde{A}_{r,s} \circ J\right) \perp \mathcal{V}_{J(p)}$$

and similarly $\nabla_{X_p}(\widetilde{B}_{r,s} \circ J) \perp \mathcal{V}_{J(p)}$.

It is convenient to denote the elements of the frame $\{\widetilde{A}_{r,s}, \widetilde{B}_{r,s}\}$ by $\{\widetilde{V}_1, \dots, \widetilde{V}_{m^2-m}\}$. In this way, we have a frame of vertical vector fields near the point $J(p)$ with the property (3) of the lemma. Properties (1) and (2) are also satisfied by this frame according to (12) and Lemma 3, respectively. In particular,

$$\left(\widetilde{V}_\gamma\right)_{J(p)}\left(h_t\left(\widetilde{V}_\alpha, \widetilde{V}_\beta\right)\right) = 0, \quad \alpha, \beta, \gamma = 1, \dots, m^2 - m.$$

Note also that, in view of (11),

$$\mathcal{V}\left(D_{X^h}\widetilde{V}_\alpha\right)_{J(p)} = \left[X^h, \widetilde{V}_\alpha\right]_{J(p)} = 0,$$

hence

$$X^h_{J(p)}\left(h_t\left(\widetilde{V}_\alpha, \widetilde{V}_\beta\right)\right) = 0.$$

Now it is clear that the h_t-orthonormal frame $\{V_1, \dots, V_{m^2-m}\}$ obtained from $\{\widetilde{V}_1, \dots, \widetilde{V}_{m^2-m}\}$ by the Gram-Schmidt process has the properties stated in the lemma. □

Proposition 1 *For every $X, Y \in T_p N$, $p \in N$,*

$$\begin{aligned}\widetilde{\nabla} J_*(X, Y) = &\frac{1}{2}\mathcal{V}\left(\nabla^2_{XY} J + \nabla^2_{YX} J\right)\\ &-\frac{2t}{n}\left[\left(R\left((J \circ \nabla_X J)^\wedge\right) Y\right)^h_{J(p)} + \left(R\left((J \circ \nabla_Y J)^\wedge\right) X\right)^h_{J(p)}\right]\end{aligned}$$

where $\nabla^2_{XY} J = \nabla_X \nabla_Y J - \nabla_{\nabla_X Y} J$ is the second covariant derivative of J.

Proof Extend X and Y to vector fields in a neighbourhood of the point p. Let $V_1, \dots, V_{m^2-m}$ be a h_t-orthonormal frame of vertical vector fields with the properties (1) - (3) stated in Lemma 4.

We have

$$J_* \circ Y = Y^h \circ J + \nabla_Y J = Y^h \circ J + \sum_{\alpha=1}^{m^2-m} h_t\left(\nabla_Y J, V_\alpha \circ J\right)\left(V_\alpha \circ J\right),$$

hence

$$\begin{aligned}\widetilde{D}_X\left(J_* \circ Y\right) = &\left(D_{J_* X} Y^h\right) \circ J + \sum_{\alpha=1}^{m^2-m} h_t\left(\nabla_Y J, V_\alpha\right)\left(D_{J_* X} V_\alpha\right) \circ J\\ &+ t\sum_{\alpha=1}^{m^2-m} G\left(\nabla_X \nabla_Y J, V_\alpha \circ J\right)\left(V_\alpha \circ J\right)\end{aligned}$$

This, in view of Lemma 2, implies

$$\begin{aligned}\widetilde{D}_{X_p}(J_*\circ Y) &= (\nabla_X Y)^h_{J(p)} + \frac{1}{2}R(X\wedge Y)J(p) - \frac{2t}{n}\left(R\left((J\circ\nabla_X J)^\wedge\right)Y\right)^h_{J(p)}\\ &\quad + t\sum_{\alpha=1}^{m^2-m} G\left(\nabla_{X_p}\nabla_Y J, V_\alpha\circ J\right)_p V_\alpha(J(p))\\ &\quad - \frac{2t}{n}\left(R\left((J\circ\nabla_Y J)^\wedge\right)X\right)^h_{J(p)}\\ &= \left(\nabla_{X_p}Y\right)^h_{J(p)} + \frac{1}{2}\mathcal{V}\left(\nabla_{X_p}\nabla_Y J + \nabla_{Y_p}\nabla_X J\right) + \frac{1}{2}\nabla_{[X,Y]_p}J\\ &\quad - \frac{2t}{n}\left[R\left((J\circ\nabla_X J)^\wedge\right)Y\right)^h_{J(p)} + \left(R\left((J\circ\nabla_Y J)^\wedge\right)X\right)^h_{J(p)}\Big].\end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{\nabla}J_*(X,Y) &= \widetilde{D}_{X_p}(J_*\circ Y) - (\nabla_X Y)^h_\sigma - \nabla_{\nabla_{X_p}Y}J\\ &= \frac{1}{2}\mathcal{V}\left(\nabla_{X_p}\nabla_Y J - \nabla_{\nabla_{X_p}Y}J + \nabla_{Y_p}\nabla_X J - \nabla_{\nabla_{Y_p}X}J\right)\\ &\quad - \frac{2t}{n}\left[R\left((J\circ\nabla_X J)^\wedge\right)Y\right)^h_{J(p)} + \left(R\left((J\circ\nabla_Y J)^\wedge\right)X\right)^h_{J(p)}\Big].\end{aligned}$$

□

Corollary 1 *If (N,g,J) is Kähler, the map $J:(N,g)\to(\mathcal{Z},h_t)$ is a totally geodesic isometric imbedding.*

Remark 2 By a result of Wood [28,29], J is a harmonic almost complex structure, i.e. a harmonic section of the twistor space $(\mathcal{Z},h_t)\to(N,g)$ if and only if $[J,\nabla^*\nabla J]=0$ where $\nabla^*\nabla$ is the rough Laplacian. This, in view of the decomposition (1), is equivalent to the condition that the vertical part of $\nabla^*\nabla J = -Trace\nabla^2 J$ vanishes. Thus, by Proposition 1, J is a harmonic section if and only if

$$\mathcal{V}Trace\widetilde{\nabla}J_* = 0.$$

4 The Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures as harmonic sections

Let (M,g) be an oriented Riemannian manifold of dimension four. The twistor space of such a manifold has two connected components, which can be identified with the unit sphere subbundles $\mathcal{Z}_\pm$ of the bundles $\Lambda^2_\pm TM\to M$, the eigensubbundles of the bundle $\pi:\Lambda^2TM\to M$ corresponding to the eigenvalues ±1 of the Hodge star operator. The sections of $\mathcal{Z}_\pm$ are the almost complex structures on M compatible with the metric and $\pm$-orientation of M. The spaces $\mathcal{Z}_+$ and $\mathcal{Z}_-$ are called the "positive" and the "negative" twistor space of (M,g).

The Levi-Civita connection ∇ of M preserves the bundles $\Lambda^2_\pm TM$, so it induces a metric connection on each of them denoted again by ∇. The horizontal distribution of $\Lambda^2_\pm TM$ with respect to ∇ is tangent to the twistor space $\mathcal{Z}_\pm$. Thus, we have the decomposition $T\mathcal{Z}_\pm = \mathcal{H}\oplus\mathcal{V}$ of the tangent bundle of $\mathcal{Z}_\pm$ into horizontal and vertical components. The vertical space $\mathcal{V}_\tau=\{V\in T_\tau\mathcal{Z}_\pm : \pi_*V=0\}$ at a point $\tau\in\mathcal{Z}$ is the tangent space to the fibre of $\mathcal{Z}_\pm$ through τ. Considering $T_\tau\mathcal{Z}_\pm$ as a subspace of $T_\tau(\Lambda^2_\pm TM)$ (as we shall always do), $\mathcal{V}_\tau$ is the orthogonal complement of τ in $\Lambda^2_\pm T_{\pi(\tau)}M$.

Given $a \in \Lambda^2 TM$, define, as in Sec. 2.1, an endomorphism K_a of $T_{\pi(a)}M$ by

$$g(K_a X, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_{\pi(a)}M.$$

For $\sigma \in \mathcal{Z}_\pm$, K_σ is a complex structure on the vector space $T_{\pi(\sigma)}M$ compatible with the metric and $\pm$ the orientation.

Denote by $\times$ the vector-cross product in the 3-dimensional oriented Euclidean space $(\Lambda^2_\pm T_p M, g_p)$, $p \in M$.

It is easy to show that if $a, b \in \Lambda^2_\pm TM$

$$K_a \circ K_b = -g(a,b)Id \pm K_{a\times b}. \tag{13}$$

This identity implies that for every vertical vector $V \in \mathcal{V}_\sigma$ and every $X, Y \in T_{\pi(\sigma)}M$

$$g(V, X \wedge K_\sigma Y) = g(V, K_\sigma X \wedge Y) = g(\sigma \times V, X \wedge Y). \tag{14}$$

Note also that, in view of (4), the Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ at a point $\sigma \in \mathcal{Z}_\pm$ can be written as

$$\begin{gathered}\mathcal{J}_k V = \pm(-1)^{k+1}\sigma \times V \ \text{ for } \ V \in \mathcal{V}_\sigma,\\ \mathcal{J}_k X^h_\sigma = K_\sigma X \ \text{ for } \ X \in T_{\pi(\sigma)}M,\\ k = 1, 2.\end{gathered}$$

Denote by $\mathcal{B} : \Lambda^2 TM \to \Lambda^2 TM$ the endomorphism corresponding to the traceless Ricci tensor. If s denotes the scalar curvature of (M, g) and $\rho : TM \to TM$ the Ricci operator, $g(\rho(X), Y) = Ricci(X, Y)$, we have

$$\mathcal{B}(X \wedge Y) = \rho(X) \wedge Y + X \wedge \rho(Y) - \frac{s}{2} X \wedge Y.$$

Let $\mathcal{W} : \Lambda^2 TM \to \Lambda^2 TM$ be the endomorphism corresponding the Weyl conformal tensor. Denote the restriction of $\mathcal{W}$ to $\Lambda^2_\pm TM$ by $\mathcal{W}_\pm$, so $\mathcal{W}_\pm$ sends $\Lambda^2_\pm TM$ to $\Lambda^2_\pm TM$ and vanishes on $\Lambda^2_\mp TM$.

It is well-known that the curvature operator decomposes as (see, e.g. [2, Chapter 1 H])

$$\mathcal{R} = \frac{s}{6} Id + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-.$$

Note that this differ by the factor $1/2$ from [2] because of the factor $1/2$ in our definition of the induced metric on $\Lambda^2 TM$.

The Riemannian manifold (M, g) is Einstein exactly when $\mathcal{B} = 0$. It is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$). By a well-known result of Atiyah–Hitchin–Singer [1], the almost complex structure $\mathcal{J}_1$ on $\mathcal{Z}_-$ (resp. $\mathcal{Z}_+$) is integrable (i.e. comes from a complex structure) if and only if (M, g) is self-dual (resp. anti-self-dual). On the other hand, the almost complex structure $\mathcal{J}_2$ is never integrable by a result of Eells–Salamon [13] but nevertheless it is very useful in harmonic map theory.

Convention. In what follows the negative twistor space $\mathcal{Z}_-$ will be called simply "the twistor space" and will be denoted by $\mathcal{Z}$.

Changing the orientation of M interchanges the roles of $\Lambda^2_+ TM$ and $\Lambda^2_- TM$, respectively, of $\mathcal{Z}_+$ and $\mathcal{Z}_-$. But note that the Fubini-Study metric on $\mathbb{CP}^2$ is self-dual and not anti-self-dual, so the structure $\mathcal{J}_1$ on the negative twistor space $\mathcal{Z}_-$ is integrable while on $\mathcal{Z}_+$ it is not. This is one of the reasons to prefer $\mathcal{Z}_-$ rather than $\mathcal{Z}_+$.

Remark 2, Proposition 1 and Theorem 1 imply

Corollary 2 *(i)* $\mathcal{V}Trace\widetilde{\nabla}\mathcal{J}_{1*} = 0$ *if and only if* (M, g) *is self-dual.*
(ii) $\mathcal{V}Trace\widetilde{\nabla}\mathcal{J}_{2*} = 0$ *if and only if* (M, g) *is self-dual and with constant scalar curvature.*

5 The Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures as harmonic maps

In this section, we prove Theorem 2, which is the main result of the paper.

Note first that the almost complex structure $\mathcal{J}_k$, $k = 1$ or 2, is a harmonic map if and only if $\mathcal{V}Trace\widetilde{\nabla}\mathcal{J}_{k*} = 0$ and $\mathcal{H}Trace\widetilde{\nabla}\mathcal{J}_{k*} = 0$. By Corollary 2 if the vertical part of $Trace\widetilde{\nabla}\mathcal{J}_{k*}$ vanishes, then the manifold (M, g) is self-dual. According to Proposition 1 $\mathcal{H}Trace\widetilde{\nabla}\mathcal{J}_{k*} = 0$, $k = 1, 2$, if and only if for every $\sigma \in \mathcal{Z}$ and every $F \in T_\sigma\mathcal{Z}$

$$Trace_{h_t}\left\{T_\sigma\mathcal{Z} \ni A \to h_t(R_{\mathcal{Z}}((\mathcal{J}_k \circ D_A\mathcal{J}_k)^\wedge)A), F)\right\} = 0.$$

Set for brevity

$$Tr_k(F) = Trace_{h_t}\left\{T_\sigma\mathcal{Z} \ni A \to h_t(R_{\mathcal{Z}}((\mathcal{J}_k \circ D_A\mathcal{J}_k)^\wedge)A), F)\right\}.$$

The next two technical lemmas, giving explicit formulas for $Tr_k(F)$ in the self-dual case, will be proved in the next section.

Lemma 5 *Suppose that* (M, g) *is self-dual. Then, if* $\sigma \in \mathcal{Z}$ *and* $U \in \mathcal{V}_\sigma$,

$$Tr_k(U) = \frac{t}{4}g(\mathcal{B}(U), \mathcal{B}(\sigma)), \quad k = 1, 2.$$

Lemma 6 *Suppose that* (M, g) *is self-dual. Then, if* $X \in T_pM$, $p = \pi(\sigma)$,

$$\begin{aligned}Tr_k\left(X^h_\sigma\right) = &\left[1 + (-1)^k\right]\frac{s(p)}{144}X(s) + \frac{1}{12}\left(\frac{ts(p)}{6} - 2\right)X(s)\\ &+ Trace_{h_t}\left\{\mathcal{V}_\sigma \ni V \to \left[\frac{t}{8}g\left((\nabla_X\mathcal{B})(V), \mathcal{B}(V)\right)\right.\right.\\ &\left.\left.+(-1)^{k+1}\frac{ts(p)}{24}g\left(\delta\mathcal{B}(K_VX), V\right)\right]\right\}.\end{aligned}$$

Proof of Theorem 2 Suppose that $\mathcal{J}_1$ or $\mathcal{J}_2$ is a harmonic map. By Corollary 2, (M, g) is self-dual or self-dual with constant scalar curvature. Moreover, $Tr_k(U) = 0$ for every vertical vector U and $Tr_k(X^h) = 0$ for every horizontal vector X^h, $k = 1$ or $k = 2$. Note that in both cases the first term in the expression for $Tr_k(X^h)$ given in Lemma 6 vanishes:

$$[1 + (-1)^k]\frac{s(p)}{144}X(s) = 0, \quad k = 1, 2.$$

By Lemma 5, for every $p \in M$ and every orthonormal basis v_1, v_2, v_3 of $\Lambda^2_-T_pM$, $g(\mathcal{B}(v_i), \mathcal{B}(v_j)) = 0, i, j = 1, 2, 3, i \neq j$. This implies $g(\mathcal{B}(v_i), \mathcal{B}(v_i)) = g(\mathcal{B}(v_j), \mathcal{B}(v_j))$, $i \neq j$. It follows that the function $\mathcal{Z}_p \ni \sigma \to ||\mathcal{B}(\sigma)||^2$ is constant on the fibre $\mathcal{Z}_p$ of $\mathcal{Z}$ at p. Thus, we have a smooth function f on M such that $f(p) = ||\mathcal{B}(\sigma)||^2$ for every $\sigma \in \mathcal{Z}_p$. It follows that

$$X(f) = 2g((\nabla_X\mathcal{B})(\sigma), \mathcal{B}(\sigma)) \tag{15}$$

for every tangent vector $X \in T_pM$. □

Let $E_1, \dots, E_4$ be an oriented orthonormal basis of T_pM consisting of eigenvectors of ρ. Denote by $\lambda_1, \dots, \lambda_4$ the corresponding eigenvalues. We have $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = s$ and

$$\mathcal{B}(X \wedge Y) = \rho(X) \wedge Y + X \wedge \rho(Y) - \frac{s}{2} X \wedge Y. \tag{16}$$

Define s_i^+ and $s_i = s_i^-$, $i = 1, 2, 3$, as in (3) by means of the basis $E_1, \dots, E_4$. Then,

$$\mathcal{B}(s_1) = \left(\lambda_1 + \lambda_2 - \frac{s}{2}\right) s_1^+, \quad \mathcal{B}(s_2) = \left(\lambda_1 + \lambda_3 - \frac{s}{2}\right) s_2^+,$$
$$\mathcal{B}(s_3) = \left(\lambda_1 + \lambda_4 - \frac{s}{2}\right) s_3^+.$$

Therefore, $||\mathcal{B}(\cdot)||^2 = const$ on the fibre $\mathcal{Z}_p$ if and only if

$$\left|\lambda_1 + \lambda_2 - \frac{s}{2}\right| = \left|\lambda_1 + \lambda_3 - \frac{s}{2}\right| = \left|\lambda_1 + \lambda_4 - \frac{s}{2}\right|,$$

i.e. if and only if, at every point $p \in M$, three eigenvalues of ρ coincide.

Moreover,

$$3f(p) = ||\mathcal{B}(s_1)||^2 + ||\mathcal{B}(s_2)||^2 + ||\mathcal{B}(s_3)||^2 = ||\rho||^2 - \frac{s^2(p)}{4}.$$

and, by (15), it follows that

$$Trace_{h_t}\{\mathcal{V}_\sigma \ni V \to g((\nabla_X \mathcal{B})(V), \mathcal{B}(V))\} = \frac{1}{3} X\left(||\rho||^2\right) - \frac{s(p)X(s)}{6}.$$

Fix a tangent vector $X \in T_pM$ and denote by P the symmetric bilinear form on $\Lambda^2_- T_pM$ corresponding to the quadratic form

$$P(a, a) = \frac{ts(p)}{24} g(\delta\mathcal{B}(K_a X), a). \tag{17}$$

Set

$$\psi = -\left(\frac{ts(p)}{144} + \frac{1}{6}\right) X(s) + \frac{t}{24} X(||\rho||^2).$$

Then, for every $\sigma \in \mathcal{Z}_p$ and every $V \in \mathcal{V}_\sigma$ with $||V||_g = 1$ we have

$$Tr_k(X^h_\sigma) = (-1)^{k+1}\left[\frac{1}{t} P(V, V) + \frac{1}{t} P(\sigma \times V, \sigma \times V)\right] + \psi. \tag{18}$$

Let $\{s_1, s_2, s_3\}$ be an orthonormal basis of $\Lambda^2_- T_pM$. Take

$$\sigma = \frac{1}{\sqrt{y_1^2 + y_2^2 + y_3^2}}(y_1 s_1 + y_2 s_2 + y_3 s_3)$$

for $(y_1, y_2, y_3) \in \mathbb{R}^3$ with $y_1 \neq 0$. Set

$$V = \frac{1}{\sqrt{y_1^2 + y_2^2}}(-y_2 s_1 + y_1 s_2).$$

Then,

$$\sigma \times V = \frac{1}{\sqrt{\left(y_1^2 + y_2^2\right)\left(y_1^2 + y_2^2 + y_3^2\right)}}\left(-y_1 y_3 s_1 - y_2 y_3 s_2 + \left(y_1^2 + y_2^2\right) s_3\right).$$

Now varying (y_1, y_2, y_3) we see from (18) that the identity $Tr_k(X^h_\sigma) = 0$ implies

$$P(s_i, s_j) = 0, \quad (-1)^{k+1}\frac{1}{t}\left[P(s_i, s_i) + P\left(s_j, s_j\right)\right] + \psi = 0, \quad i, j = 1, 2, 3,\ i \neq j.$$

Since $P(s_i, s_j) = 0$, $i \neq j$, for every orthonormal basis, we have $P(s_i, s_i) = P(s_j, s_j)$.

Suppose that $s(p) \neq 0$. Then, by the latter identity,

$$g\left(\delta\mathcal{B}\left(K_{s_i}X\right), s_i\right) = g\left(\delta\mathcal{B}\left(K_{s_j}X\right), s_j\right), \quad i, j = 1, 2, 3.$$

Take an oriented orthonormal basis $E_1, \ldots, E_4$ of T_pM and, using it, define $s_i = s_i^-$, $i = 1, 2, 3$. Then, $g(\delta\mathcal{B}(K_{s_1}X), s_1) = g(\delta\mathcal{B}(K_{s_2}X), s_2)$ for every $X \in T_pM$. This, in view of (13), gives

$$-g(\delta\mathcal{B}(X), s_1) = g(\delta\mathcal{B}(K_{s_3}X), s_2), \quad X \in T_pM.$$

Applying the latter identity for the basis E_3, E_4, E_1, E_2, we get

$$g(\delta\mathcal{B}(X), s_1) = g(\delta\mathcal{B}(K_{s_3}X), s_2).$$

Hence, $g(\delta\mathcal{B}(X), s_1) = 0$. Similarly, $g(\delta\mathcal{B}(X), s_2) = (\delta\mathcal{B}(X), s_3) = 0$. Therefore, for every $X \in T_pM$ and $a \in \Lambda^2_- T_pM$

$$g(\delta\mathcal{B}(X), a) = 0.$$

Then, by (17), $P(a, a) = 0$ for every $a \in \Lambda^2_- T_pM$. Thus, we see from (18) that the condition $Tr_k(X^h_\sigma) = 0$ for every $\sigma \in \mathcal{Z}$, $X \in T_{\pi(\sigma)}M$ is equivalent to the identities

$$g(\delta\mathcal{B}(X), \sigma) = 0, \quad \psi = 0.$$

Identity (16) implies that for every $X \in T_pM$ and every orthonormal basis $E_1, .., E_4$ of T_pM

$$\delta\mathcal{B}(X) = \delta\rho \wedge X - \sum_{m=1}^{4}\left[E_m \wedge (\nabla_{E_m}\rho)(X) - \tfrac{1}{2}E_m(s)E_m \wedge X\right].$$

Therefore, the identity $g(\delta\mathcal{B}(X), \sigma) = 0$ is equivalent to

$$g\left(\delta\rho, K_\sigma X\right) + \sum_{m=1}^{4} g\left(\left(\nabla_{E_m}\rho\right)(X), K_\sigma E_m\right) + \tfrac{1}{2}(K_\sigma X)(s) = 0.$$

This is equivalent to

$$\sum_{m=1}^{4} g((\nabla_{E_m}\rho)(K_\sigma E_m), X) = 0 \tag{19}$$

since $g(\delta\rho, Z) = -\frac{1}{2}Z(s)$ by the second Bianchi identity and the Ricci operator ρ is g-symmetric. Let $r(X, Y)$ be the Ricci tensor and set

$$dr(X, Y, Z) = (\nabla_Y r)(Z, X) - (\nabla_Z r)(Y, X).$$

Thus,

$$dr(X, Y, Z) = g((\nabla_Y\rho)(Z), X) - g((\nabla_Z\rho)(Y), X).$$

The left-hand side of (19) clearly does not depend on the choice of the basis $(E_1, \ldots, E_4)$. So, take an oriented orthonormal basis $(E_1, \ldots, E_4)$ such that $E_2 = K_\sigma E_1$ and $E_4 = -K_\sigma E_3$.

Then,

$$dr(X, E_1, E_2) - dr(X, E_3, E_4) = \sum_{m=1}^{4} g\left(\left(\nabla_{E_m}\rho\right)(K_\sigma E_m), X\right).$$

Denote by W_- the 4-tensor corresponding to the operator $\mathcal{W}_-$,

$$W_-(X, Y, Z, T) = g(\mathcal{W}_-(X \wedge Y), Z \wedge T).$$

Then, the second Bianchi identity implies

$$dr(X, E_1, E_2) - dr(X, E_3, E_4) = -2[\delta W_-(X, E_1, E_2) - \delta W_-(X, E_3, E_4)].$$

Since (M, g) is self-dual, we see from the latter identity that identity (19) is always satisfied. The above identity shows also that

$$dr(X, \sigma) = 0, \quad \sigma \in \mathcal{Z},\ X \in T_{\pi(\sigma)}M. \tag{20}$$

Let $\lambda_1(p) \leq \lambda_2(p) \leq \lambda_3(p) \leq \lambda_4(p)$ be the eigenvalues of the symmetric operator $\rho_p : T_pM \to T_pM$ in the ascending order. It is well-known that the functions $\lambda_1, \ldots, \lambda_4$ are continuous (see, e.g. [18, Chapter Two, § 5.7] or [25, Chapter I, § 3]). We have seen that, at every point of M, at least three eigenvalues of the operator ρ coincide. The set U of points at which exactly three eigenvalues coincide is open by the continuity of $\lambda_1, \ldots, \lambda_4$. For every $p \in U$ denote the simple eigenvalue of ρ by $\lambda(p)$ and the triple eigenvalue by $\mu(p)$, so the spectrum of ρ is (λ, μ, μ, μ) with $\lambda(p) \neq \mu(p)$ for every $p \in U$. As is well-known, the implicit function theorem implies that the function λ is smooth. It is also well-known that, in a neighbourhood of every point p of U, there is a (smooth) unit vector field E_1 which is an eigenvector of ρ corresponding to λ. (for a proof see [19, Chapter 9, Theorem 7]). Fix $p \in U$ and choose local vector fields E_2, E_3, E_4 such that (E_1, E_2, E_3, E_4) is an oriented orthonormal frame. Let α be the dual 1-form to E_1, $\alpha(X) = g(E_1, X)$. Then,

$$r(X, Y) = (\lambda - \mu)\alpha(X)\alpha(Y) - \mu g(X, Y)$$

in a neighbourhood of p. Note that the function $\mu = \frac{1}{3}(s - \lambda)$ is also smooth. Hence, the identity $\delta r = -\frac{1}{2}ds$ reads as

$$\begin{aligned} &-E_1(\lambda - \mu)\alpha(X) - X(\mu) + (\lambda - \mu)\left[\delta\alpha.\alpha(X) - \left(\nabla_{E_1}\alpha\right)(X)\right] \\ &\quad = -\tfrac{1}{2}\left[X(\lambda) + 3X(\mu)\right], \quad X \in TU. \end{aligned} \tag{21}$$

Let $s_i = s_i^-$, $i = 1, 2, 3$, be defined by means of $E_1, \ldots, E_4$. Taking into account that $(\nabla_X\alpha)(E_1) = 0$, we easily see that the identities $dr(E_k, s_1) = 0$, $k = 1, 2, 3, 4$, give

$$\begin{gathered} (\lambda - \mu)\left[\left(\nabla_{E_1}\alpha\right)(E_2) - \left(\nabla_{E_3}\alpha\right)(E_4) + \left(\nabla_{E_4}\alpha\right)(E_3)\right] - E_2(\lambda) = 0 \\ (\lambda - \mu)\left(\nabla_{E_2}\alpha\right)(E_2) - E_1(\mu) = 0, \quad (\lambda - \mu)\left(\nabla_{E_2}\alpha\right)(E_3) - E_4(\mu) = 0 \\ (\lambda - \mu)\left(\nabla_{E_2}\alpha\right)(E_4) + E_3(\mu) = 0. \end{gathered} \tag{22}$$

The identities obtained from the latter ones by cycle permutations of E_2, E_3, E_4 also hold as a consequence of the identities $dr(E_k, s_2) = 0$ and $dr(E_k, s_3) = 0$. Thus

$$(\lambda - \mu)\left(\nabla_{E_j}\alpha\right)(E_j) = E_1(\mu), \quad j = 2, 3, 4. \tag{23}$$

Hence,

$$(\lambda - \mu)\delta\alpha = -3E_1(\mu) \tag{24}$$

Moreover, we have

$$\left(\nabla_{E_3}\alpha\right)(E_4) = E_2(\mu), \quad \left(\nabla_{E_4}\alpha\right)(E_3) = -E_2(\mu)$$

and the first identity of (22) gives

$$(\lambda-\mu)\left(\nabla_{E_1}\alpha\right)(E_2)=E_2(\lambda)+2E_2(\mu).$$

On the other hand, identity (21) implies

$$(\lambda-\mu)\left(\nabla_{E_1}\alpha\right)(E_2)=\frac{1}{2}E_2(\lambda)+\frac{1}{2}E_2(\mu).$$

It follows that

$$0=\frac{1}{2}E_2(\lambda)+\frac{3}{2}E_2(\mu)=\frac{1}{2}E_2(s),$$

so $E_2(s)=0$. Similarly, $E_3(s)=0$ and $E_4(s)=0$. Identity (21) for $X=E_1$ together with (24) implies $0=E_1(\lambda)+3E_1(\mu)=E_1(s)$. It follows that the scalar curvature s is locally constant on U. Then, identity $\psi=0$ implies that $||\rho||^2$ is locally constant. Thus in a neighbourhood of every point $p\in U$, we have $\lambda+3\mu=a$ and $\lambda^2+3\mu^2=b^2$ where a and b are some constants. It follows that

$$\mu=\frac{1}{12}\left(3a\pm\sqrt{12b^2-3a^2}\right).$$

Note that $12b^2-3a^2\neq 0$ since otherwise we would have $\mu=\frac{1}{4}a$, hence $\lambda=a-3\mu=\frac{1}{4}a=\mu$, a contradiction. Since μ is continuous, we see that μ is constant, hence λ is also constant. Then, by (23), $(\nabla_{E_j}\alpha)(E_j)=0$ for $j=2,3,4$ and the first equation of (22) gives $(\nabla_{E_1}\alpha)(E_2)=0$. Similarly $(\nabla_{E_1}\alpha)(E_3)=(\nabla_{E_1}\alpha)(E_4)=0$. Thus $(\nabla_X\alpha)(E_j)=0$ for every X and $j=2,3,4$. This and the obvious identity $(\nabla_X\alpha)(E_1)=0$ imply that the 1-form α is parallel. It follows that the restriction of the Ricci tensor to U is parallel.

In the interior of the closed set $M\setminus U$ the eigenvalues of the Ricci tensor coincide, hence the metric g is Einstein on this open set. Therefore, the scalar curvature s is locally constant on $Int\,(M\setminus U)$ and the Ricci tensor is parallel on it. Thus the Ricci tensor is parallel on the open set $U\cup Int\,(M\setminus U)=M\setminus bU$, where bU stands for the boundary of U. Since $M\setminus bU$ is dense in M it follows the Ricci tensor is parallel on M. This implies that the eigenvalues $\lambda_1\leq\cdots\leq\lambda_4$ of the Ricci tensor are constant. Thus either M is Einstein or exactly three of the eigenvalues coincide. Since (M,g) is self-dual, in the second case the simple eigenvalue λ vanishes by [10, Lemma 1]. Therefore, M is locally the product of an interval in $\mathbb{R}$ and a 3-dimensional manifold of constant curvature.

Conversely, suppose that (M,g) is self-dual and either Einstein or locally is the product of an interval and a manifold of constant curvature. Then, at least three of the eigenvalues of the Ricci tensor coincide which, as we have seen, imply that $||\mathcal{B}(\cdot)||^2=const$ on every fibre of $\mathcal{Z}$. It follows that $g(\mathcal{B}(\sigma),\mathcal{B}(\tau))=0$ for every $\sigma,\tau\in\mathcal{Z}$ with $g(\sigma,\tau)=0$. Therefore, $Tr_k(U)=0$ for every vertical vector U, $k=1,2$, by Lemma 5. Moreover, $Tr_k(X^h)=0$ by Lemma 6 since the scalar curvature is constant and $\nabla\mathcal{B}=0$.

Remark 3 According to Theorems 1 and 2, the conditions under which $\mathcal{J}_1$ or $\mathcal{J}_2$ is a harmonic section or a harmonic map do not depend on the parameter t of the metric h_t. Taking certain special values of t, we can obtain a metric h_t with nice properties (cf., for example, [7,10,23]).

6 Proofs of Lemmas 5 and 6

Denote by $R_{\mathcal{Z}}$ the curvature tensor of the Riemannian manifold $(\mathcal{Z},h_t)$.

Let $\Omega_{k,t}(A,B)=h_t(\mathcal{J}_k A,B)$ be the fundamental 2-form of the almost Hermitian manifold $(\mathcal{Z},h_t,\mathcal{J}_k)$, $k=1,2$. Then, for $A,B,C\in T_\sigma\mathcal{Z}$,

$$h_t(\mathcal{J}_k\circ D_A\mathcal{J}_k)^\wedge,B\wedge C)=-\frac{1}{2}h_t((D_A\mathcal{J}_k)(B),\mathcal{J}_kC)=-\frac{1}{2}(D_A\Omega_{k,t})(B,\mathcal{J}_kC).$$

Lemma 7 ([23]) *Let $\sigma\in\mathcal{Z}$ and $X,Y\in T_{\pi(\sigma)}M$, $V\in\mathcal{V}_\sigma$. Then,*

$$\left(D_{X^h_\sigma}\Omega_{k,t}\right)\left(Y^h_\sigma,V\right)=\frac{t}{2}\left[(-1)^k g\left(\mathcal{R}(V),X\wedge Y\right)-g\left(\mathcal{R}\left(\sigma\times V\right),X\wedge K_\sigma Y\right)\right].$$
$$\left(D_V\Omega_{k,t}\right)\left(X^h_\sigma,Y^h_\sigma\right)=\frac{t}{2}g\left(\mathcal{R}\left(\sigma\times V\right),X\wedge K_\sigma Y+K_\sigma X\wedge Y\right)+2g(V,X\wedge Y).$$

Moreover, $(D_A\Omega_{k,t})(B,C)=0$ when A,B,C are three horizontal vectors at σ or at least two of them are vertical.

Corollary 3 *Let $\sigma\in\mathcal{Z}$, $X\in T_{\pi(\sigma)}M$, $U\in\mathcal{V}_\sigma$. If $E_1,\dots,E_4$ is an orthonormal basis of $T_{\pi(\sigma)}M$ and V_1,V_2 is a h_t-orthonormal basis of $\mathcal{V}_\sigma$,*

$$\begin{aligned}\left(\mathcal{J}_k\circ D_{X^h_\sigma}\mathcal{J}_k\right)^\wedge=&-\frac{1}{2}\sum_{i=1}^{4}\sum_{l=1}^{2}\left[g\left(\mathcal{R}\left(\sigma\times V_l\right),X\wedge E_i\right)\right.\\&\left.+(-1)^k g\left(\mathcal{R}(V_l),X\wedge K_\sigma E_i\right)\right]\left(E_i^h\right)_\sigma\wedge V_l\\(\mathcal{J}_k\circ D_U\mathcal{J}_k)^\wedge=&\sum_{1\le i<j\le 4}\left[\frac{t}{2}g\left(\mathcal{R}\left(\sigma\times U\right),E_i\wedge E_j-K_\sigma E_i\wedge K_\sigma E_j\right)\right.\\&\left.-2g\left(U,E_i\wedge K_\sigma E_j\right)\right]\left(E_i^h\right)_\sigma\wedge\left(E_j^h\right)_\sigma.\end{aligned}$$

The sectional curvature of the Riemannian manifold $(\mathcal{Z},h_t)$ can be computed in terms of the curvature of the base manifold M by means of the following formula.

Proposition 2 ([7]) *Let $E,F\in T_\sigma\mathcal{Z}$ and $X=\pi_*E$, $Y=\pi_*F$, $V=\mathcal{V}E$, $W=\mathcal{V}F$. Then,*

$$\begin{aligned}h_t\left(R_{\mathcal{Z}}(E,F)E,F\right)=&\,g\left(R(X,Y)X,Y\right)\\&-tg\left(\left(\nabla_X\mathcal{R}\right)(X\wedge Y),\sigma\times W\right)+tg\left(\left(\nabla_Y\mathcal{R}\right)(X\wedge Y),\sigma\times V\right)\\&-3tg\left(\mathcal{R}(\sigma),X\wedge Y\right)g\left(\sigma\times V,W\right)\\&-t^2g\left(R\left(\sigma\times V\right)X,R\left(\sigma\times W\right)Y\right)\\&+\frac{t^2}{4}||R\left(\sigma\times W\right)X+R\left(\sigma\times V\right)Y||^2\\&-\frac{3t}{4}||R(X,Y)\sigma||^2+t\left(||V||^2||W||^2-g(V,W)^2\right).\end{aligned}$$

Using this formula, the well-known expression of the Levi-Civita curvature tensor by means of sectional curvatures and differential Bianchi identity one gets the following.

Corollary 4 *Let $\sigma\in\mathcal{Z}$, $X,Y,Z,T\in T_{\pi(\sigma)}M$, and $U,V,W\in\mathcal{V}_\sigma$. Then,*

$$\begin{aligned}h_t\left(R_{\mathcal{Z}}\left(X^h,Y^h\right)Z^h,T^h\right)_\sigma=&\,g\left(R(X,Y)Z,T\right)\\-\frac{3t}{12}&\left[2g\left(R(X,Y)\sigma,R(Z,T)\sigma\right)-g\left(R(X,T)\sigma,R(Y,Z)\sigma\right)\right.\\+g&\left.\left(R(X,Z)\sigma,R(Y,T)\sigma\right)\right].\end{aligned}$$

$$h_t\left(R_{\mathcal{Z}}(X^h,Y^h)Z^h,U\right)_\sigma=-\frac{t}{2}g\left(\nabla_Z\mathcal{R}(X\wedge Y),\sigma\times U\right).$$
$$h_t\left(R_{\mathcal{Z}}(X^h,U)Y^h,V\right)_\sigma=\frac{t^2}{4}g\left(R\left(\sigma\times V\right)X,R\left(\sigma\times U\right)Y\right)$$
$$+\frac{t}{2}g\left(\mathcal{R}(\sigma),X\wedge Y\right)g\left(\sigma\times V,U\right).$$
$$h_t\left(R_{\mathcal{Z}}(X^h,Y^h)U,V\right)_\sigma=\frac{t^2}{4}\left[g\left(R\left(\sigma\times V\right)X,R\left(\sigma\times U\right)Y\right)\right.$$
$$\left.-g\left(R\left(\sigma\times U\right)X,R\left(\sigma\times V\right)Y\right)\right]$$
$$+tg\left(\mathcal{R}(\sigma),X\wedge Y\right)g\left(\sigma\times V,U\right).$$
$$h_t\left(R_{\mathcal{Z}}(X^h,U)V,W\right)=0.$$

We have stated in Lemma 5 that if (M,g) is self-dual,

$$Tr_k(U)=\frac{t}{4}g(\mathcal{B}(U),\mathcal{B}(\sigma))\text{ for every }U\in\mathcal{V}_\sigma,\ \sigma\in\mathcal{Z}.$$

Proof of Lemma 5. Let $E_1,\dots,E_4$ be an orthonormal basis of T_pM, $p=\pi(\sigma)$, such that $E_2=K_\sigma E_1$, $E_4=-K_\sigma E_3$. Define $s_1=s_1^-,s_2=s_2^-,s_3=s_3^-$ via (3) by means of $E_1,\dots,E_4$, so that $\sigma=s_1$ and $\mathcal{V}_\sigma=span\{s_2,s_3\}$. Thus $V_1=\frac{1}{\sqrt{t}}s_2$, $V_2=\frac{1}{\sqrt{t}}s_3$ is a h_t-orthonormal basis of $\mathcal{V}_\sigma$. □

By Corollary 3, for every $U\in\mathcal{V}_\sigma$

$$\begin{aligned}Tr_k(U)=&-\frac{1}{2}\sum_{i,j=1}^{4}\sum_{l=1}^{2}\left[g\left(\mathcal{R}\left(\sigma\times V_l\right),E_j\wedge E_i\right)+(-1)^k g(\mathcal{R}(V_l),E_j\wedge K_\sigma E_i\right]\\&\times h_t\left(R_{\mathcal{Z}}\left(E_i^h,V_l\right)E_j^h,U\right)\\&+\frac{t}{2}\sum_{l=1}^{2}\{g\left(\mathcal{R}(\sigma\times V_l),s_2\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_1^h,E_3^h\right)V_l,U\right)-h_t\left(R_{\mathcal{Z}}\left(E_4^h,E_2^h\right)V_l,U\right)\right]\\&+g\left(\mathcal{R}\left(\sigma\times V_l\right),s_3\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_1^h,E_4^h\right)V_l,U\right)-h_t\left(R_{\mathcal{Z}}\left(E_2^h,E_3^h\right)V_l,U\right)\right]\}\\&-2\sum_{1\le i<j\le 4}\sum_{l=1}^{2}g\left(V_l,E_i\wedge K_\sigma E_j\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h,E_j^h\right)V_l,U\right)\end{aligned}\tag{25}$$

We show first that

$$\begin{aligned}&\sum_{i,j=1}^{4}\sum_{l=1}^{2}[g\left(\mathcal{R}\left(\sigma\times V_l\right),E_j\wedge E_i\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h,V_l\right)E_j^h,U\right)\\&=-\frac{t}{2}Trace_{h_t}\left\{\mathcal{V}_\sigma\ni V\to g\left(\mathcal{R}\left(\sigma\times V\right),\mathcal{R}(\sigma)\right)g\left(\sigma\times U,V\right)\right\}.\end{aligned}\tag{26}$$

In order to prove this identity, we note that if $F\in T_\sigma\mathcal{Z}$, $V\in\mathcal{V}_\sigma$ and $a\in\Lambda^2T_{\pi(\sigma)}M$, the algebraic Bianchi identity implies

$$\begin{aligned}&\sum_{i,j=1}^{4}g\left(a,E_j\wedge E_i\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h,V\right)E_j^h,F\right)\\&=-\frac{1}{2}\sum_{i,j=1}^{4}g\left(a,E_i\wedge E_j\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h,E_j^h\right)V,F\right).\end{aligned}\tag{27}$$

Using the latter identity and Corollary 4 we obtain

$$\sum_{i,j=1}^{4}\sum_{l=1}^{2}[g\left(\mathcal{R}(\sigma\times V_l)\,,E_j\wedge E_i\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h,V_l\right)E_j^h,U\right)$$
$$=-\frac{t}{2}\sum_{l=1}^{2}g\left(\mathcal{R}\left(\sigma\times V_l\right),\mathcal{R}(\sigma)\right)g\left(\sigma\times U,V_l\right)$$
$$=-\frac{t}{2}Trace_{h_t}\left\{\mathcal{V}_\sigma\ni V\to g\left(\mathcal{R}\left(\sigma\times V\right),\mathcal{R}(\sigma)\right)g\left(\sigma\times U,V\right)\right\}.$$

Next, we claim that

$$\sum_{i,j=1}^{4}\sum_{l=1}^{2}g(\mathcal{R}(V_l),E_j\wedge K_\sigma E_i)h_t(R_{\mathcal{Z}}(E_i^h,V_l)E_j^h,U)=0. \tag{28}$$

For every $V\in\mathcal{V}_\sigma$, we have

$$\sum_{i,j=1}^{4}g\left(\mathcal{R}(V),E_j\wedge K_\sigma E_i\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h,V\right)E_j^h,U\right)$$
$$=g\left(\mathcal{R}(V),E_1\wedge E_2\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_1^h,V\right)E_1^h,U\right)+h_t\left(R_{\mathcal{Z}}\left(E_2^h,V\right)E_2^h,U\right)\right]$$
$$-g\left(\mathcal{R}(V),E_3\wedge E_4\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_3^h,V\right)E_3^h,U\right)+h_t\left(R_{\mathcal{Z}}\left(E_4^h,V\right)E_4^h,U\right)\right]$$
$$+g\left(\mathcal{R}(V),E_1\wedge E_3\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_4^h,V\right)E_1^h,U\right)+h_t\left(R_{\mathcal{Z}}\left(E_2^h,V\right)E_3^h,U\right)\right]$$
$$+g\left(\mathcal{R}(V),E_1\wedge E_4\right)\left[-h_t\left(R_{\mathcal{Z}}\left(E_3^h,V\right)E_1^h,U\right)+h_t\left(R_{\mathcal{Z}}\left(E_2^h,V\right)E_4^h,U\right)\right]$$
$$+g\left(\mathcal{R}(V),E_2\wedge E_3\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_4^h,V\right)E_2^h,U\right)-h_t\left(R_{\mathcal{Z}}\left(E_1^h,V\right)E_3^h,U\right)\right]$$
$$+g\left(\mathcal{R}(V),E_4\wedge E_2\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_3^h,V\right)E_2^h,U\right)+h_t\left(R_{\mathcal{Z}}\left(E_1^h,V\right)E_4^h,U\right)\right] \tag{29}$$

Corollary 4 implies that

$$h_t\left(R_{\mathcal{Z}}\left(E_4^h,V\right)E_1^h,U\right)+h_t\left(R_{\mathcal{Z}}\left(E_2^h,V\right)E_3^h,U\right)$$
$$=\frac{t^2}{4}\left[g\left(R\left(\sigma\times U\right)E_4,E_2\right)g\left(\mathcal{R}\left(\sigma\times V\right),s_1\right)\right.$$
$$\left.+g\left(R\left(\sigma\times V\right)E_1,E_3\right)g\left(\mathcal{R}\left(\sigma\times U\right),s_1\right)\right]$$
$$-\frac{t}{2}g\left(\mathcal{R}(\sigma),s_3\right)g\left(\sigma\times U,V\right)$$

Since (M,g) is self-dual, for every $\tau\in\Lambda^2_-T_{\pi(\sigma)}M$,

$$\mathcal{R}(\tau)=\frac{s}{6}\tau+\mathcal{B}(\tau)$$

where $\mathcal{B}(\tau)\in\Lambda^2_+T_{\pi(\sigma)}M$. Therefore,

$$g(\mathcal{R}(\sigma\times V),s_1)=g(\mathcal{R}(\sigma\times V),\sigma)=0$$

and

$$g(\mathcal{R}(\sigma\times U),s_1)=0,\quad g(\mathcal{R}(\sigma),s_3)=0.$$

Thus

$$h_t\left(R_{\mathcal{Z}}\left(E_4^h, V\right)E_1^h, U\right) + h_t\left(R_{\mathcal{Z}}\left(E_2^h, V\right)E_3^h, U\right) = 0. \tag{30}$$

Similarly

$$\begin{aligned}
-h_t\left(R_{\mathcal{Z}}\left(E_3^h, V\right)E_1^h, U\right) + h_t\left(R_{\mathcal{Z}}\left(E_2^h, V\right)E_4^h, U\right) &= 0\\
h_t\left(R_{\mathcal{Z}}\left(E_4^h, V\right)E_2^h, U\right) - h_t\left(R_{\mathcal{Z}}\left(E_1^h, V\right)E_3^h, U\right) &= 0\\
h_t\left(R_{\mathcal{Z}}\left(E_3^h, V\right)E_2^h, U\right) + h_t\left(R_{\mathcal{Z}}\left(E_1^h, V\right)E_4^h, U\right) &= 0.
\end{aligned} \tag{31}$$

Moreover, a straightforward computation gives

$$\begin{aligned}
&\sum_{l=1}^{2}\left\{g\left(\mathcal{R}(V_l), E_1\wedge E_2\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_1^h, V_l\right)E_1^h, U\right) + h_t\left(R_{\mathcal{Z}}\left(E_2^h, V_l\right)E_2^h, U\right)\right]\right.\\
&\left.-g\left(\mathcal{R}(V_l), E_3\wedge E_4\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_3^h, V_l\right)E_3^h, U\right) + h_t\left(R_{\mathcal{Z}}\left(E_4^h, V_l\right)E_4^h, U\right)\right]\right\}\\
&= \frac{t^2}{8}\sum_{l=1}^{2} g\left(\mathcal{R}(V_l), s_1^+\right) g\left(\mathcal{R}\left(\sigma\times U\right), s_1\right) g\left(\mathcal{B}\left(\sigma\times V_l\right), s_1^+\right) = 0.
\end{aligned}$$

In view of (29), the latter identity, (30) and (31) imply (28).

Using the algebraic Bianchi identity, we see from (31) that

$$\begin{aligned}
h_t\left(R_{\mathcal{Z}}\left(E_1^h, E_3^h\right)V, U\right) - h_t\left(R_{\mathcal{Z}}\left(E_4^h, E_2^h\right)V, U\right) &= 0\\
h_t\left(R_{\mathcal{Z}}\left(E_1^h, E_4^h\right)V, U\right) - h_t\left(R_{\mathcal{Z}}\left(E_2^h, E_3^h\right)V, U\right) &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{l=1}^{2}\left\{g\left(\mathcal{R}\left(\sigma\times V_l\right), s_2\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_1^h, E_3^h\right)V_l, U\right) - h_t\left(R_{\mathcal{Z}}\left(E_4^h, E_2^h\right)V_l, U\right)\right]\right.\\
&\left.+g\left(\mathcal{R}\left(\sigma\times V_l\right), s_3\right)\left[h_t\left(R_{\mathcal{Z}}\left(E_1^h, E_4^h\right)V_l, U\right) - h_t\left(R_{\mathcal{Z}}\left(E_2^h, E_3^h\right)V_l, U\right)\right]\right\} = 0.
\end{aligned} \tag{32}$$

Using (14) and Corollary 4, we get

$$\begin{aligned}
&\sum_{1\le i<j\le 4} g\left(V, E_i\wedge K_\sigma E_j\right) h_t\left(R_{\mathcal{Z}}\left(E_i^h, E_j^h\right)V, U\right)\\
&= \frac{t^2}{4}\sum_{1\le i<j\le 4} g\left(\sigma\times V, E_i\wedge E_j\right)\left[g\left(R\left(\sigma\times U\right)E_i, R\left(\sigma\times V\right)E_j\right)\right.\\
&\left.-g(R\left(\sigma\times V\right)E_i, R\left(\sigma\times U\right)E_j\right]\\
&= \frac{t^2}{8}\sum_{i=1}^{4} g\left(R\left(\sigma\times U\right)E_i, R\left(\sigma\times V\right)K_{\sigma\times V}E_i\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{1\le i<j\le 4}\sum_{l=1}^{2} g(V_l, E_i\wedge K_\sigma E_j) h_t(R_{\mathcal{Z}}(E_i^h, E_j^h)V_l, U)\\
&= \frac{t}{8}\sum_{i,k=1}^{4}[g(R(\sigma\times U)E_i, E_k) g(R(s_3)K_{s_3}E_i, E_k)\\
&+g(R(\sigma\times U)E_i, E_k) g(R(s_2)K_{s_2}E_i, E_k)]
\end{aligned}$$

$$= \frac{t}{8}[-g(\mathcal{R}(\sigma \times U), \sigma)g(\mathcal{R}(s_3), s_2) + g(\mathcal{R}(\sigma \times U), s_2)g(\mathcal{R}(s_3), s_1)$$
$$+g(\mathcal{R}(\sigma \times U), \sigma)g(\mathcal{R}(s_2), s_3) - g(\mathcal{R}(\sigma \times U), s_3)g(\mathcal{R}(s_2), s_1)]$$

This, by virtue of the self-duality of (M, g), gives

$$\sum_{1\le i<j\le 4} \sum_{l=1}^{2} g(V_l, E_i \wedge K_\sigma E_j) h_t\left(R_{\mathcal{Z}}\left(E_i^h, E_j^h\right) V_l, U\right) = 0. \tag{33}$$

Identities (25),(26),(28), (32) and (33) imply

$$Tr_k(U) = \frac{t}{4} Trace_{h_t}\{\mathcal{V}_\sigma \ni V \to g(\mathcal{R}(\sigma \times V), \mathcal{R}(\sigma))\, g(\sigma \times U, V)\},\ k = 1, 2.$$

Now the lemma follows from the latter identity since $g(\mathcal{R}(\tau), \mathcal{R}(\sigma)) = g(\mathcal{B}(\tau), \mathcal{B}(\sigma))$ for every τ, σ with $\tau \perp \sigma$.

Recall that, according to Lemma 6, if (M, g) is self-dual

$$Tr_k(X_\sigma^h) = \left[1 + (-1)^k\right] \frac{s(p)}{144} X(s) + \frac{1}{12}\left(\frac{ts(p)}{6} - 2\right) X(s)$$
$$+ Trace_{h_t}\left\{\mathcal{V}_\sigma \ni V \to \left[\frac{t}{8} g((\nabla_X \mathcal{B})(V), \mathcal{B}(V))\right.\right.$$
$$\left.\left. +(-1)^{k+1} \frac{ts(p)}{24} g(\delta\mathcal{B}(K_V X), V)\right]\right\}.$$

for $X \in T_{\pi(\sigma)}, \sigma \in \mathcal{Z}$.

Proof of Lemma 6. Let $s_1 = s_1^-, s_2 = s_2^-, s_3 = s_3^-$ be the basis of $\Lambda_-^2 T_p M$, $p = \pi(\sigma)$, defined by means of an oriented orthonormal basis $E_1, \ldots, E_4$ of $T_p M$ such that $E_2 = K_\sigma E_1$, $E_4 = -K_\sigma E_3$. Set $V_1 = \frac{1}{\sqrt{t}} s_2$, $V_2 = \frac{1}{\sqrt{t}} s_3$. □

Then, by Corollary 3,

$$Tr_k(X_\sigma^h) = -\frac{1}{2}\sum_{i,j=1}^{4}\sum_{l=1}^{2}\left[g\left(\mathcal{R}(\sigma \times V_l), E_j \wedge E_i\right) + (-1)^k g(\mathcal{R}(V_l), E_j \wedge K_\sigma E_i\right]$$
$$\times h_t\left(R_{\mathcal{Z}}\left(E_i^h, V_l\right) E_j^h, X^h\right)$$
$$+\sum_{i<j}\sum_{l=1}^{2}\left[\frac{t}{2} g\left(\mathcal{R}(\sigma \times V_l), E_i \wedge E_j - K_\sigma E_i \wedge K_\sigma E_j\right) - 2g\left(V_l, E_i \wedge K_\sigma E_j\right)\right]$$
$$\times h_t\left(R_{\mathcal{Z}}\left(E_i^h, E_j^h\right) V_l, X^h\right).$$

Identity (27) and Corollary 4 imply

$$\sum_{i,j=1}^{4}\sum_{l=1}^{2} g\left(\mathcal{R}(\sigma \times V_l), E_j \wedge E_i\right) h_t\left(R_{\mathcal{Z}}\left(E_i^h, V_l\right) E_j^h, X^h\right)_\sigma$$
$$= -\frac{t}{4}\sum_{l=1}^{2} g\left(\frac{1}{6} X(s)\sigma \times V_l + (\nabla_X \mathcal{B})(\sigma \times V_l), \frac{1}{6} s(p)\sigma \times V_l + \mathcal{B}\right)(\sigma \times V_l)$$
$$= -\frac{s(p)}{72} X(s) - \frac{t}{4} Trace_{h_t}\{\mathcal{V}_\sigma \ni V \to g((\nabla_X \mathcal{B})(V), \mathcal{B}(V))\},$$

where the latter identity follows from the fact that $g((\nabla_X\mathcal{B})(a), b) = 0$ for every $a, b \in \Lambda^2_- T_pM$ (since the operator $\mathcal{B}$ sends $\Lambda^2_- TM$ into $\Lambda^2_+ TM$, and the connection ∇ preserves the bundles $\Lambda^2_\pm TM$).

Taking into account identity (14) and the fact that

$$E_i \wedge E_j - K_\sigma E_i \wedge K_\sigma E_j \in \Lambda^2_- T_{\pi(\sigma)}M,$$

we have

$$\begin{aligned}
&\sum_{i<j}\sum_{l=1}^{2}\left[\frac{t}{2}g\left(\mathcal{R}\left(\sigma\times V_l\right), E_i\wedge E_j - K_\sigma E_i\wedge K_\sigma E_j\right) - 2g\left(V_l, E_i\wedge K_\sigma E_j\right)\right] \\
&\quad\times h_t\left(R_{\mathcal{Z}}\left(E_i^h, E_j^h\right)V_l, X^h\right)_\sigma \\
&= \left(\frac{ts(p)}{6} - 2\right)\sum_{i<j}\sum_{l=1}^{2}g\left(\sigma\times V_l, E_i\wedge E_j\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h, E_j^h\right)V_l, X^h\right) \\
&= \frac{t}{4}\left(\frac{ts(p)}{6} - 2\right)\sum_{l=1}^{2}g\left(\left(\nabla_X\mathcal{R}\right)\left(\sigma\times V_l\right), \sigma\times V_l\right) \\
&= \frac{1}{4}\left(\frac{ts(p)}{6} - 2\right)\frac{X(s)}{3}.
\end{aligned}$$

Thus

$$\begin{aligned}
Tr_k\left(X^h_\sigma\right) &= (-1)^{k+1}\frac{1}{2}\sum_{i,j=1}^{4}\sum_{l=1}^{2}g(\mathcal{R}(V_l), E_j\wedge K_\sigma E_i h_t\left(R_{\mathcal{Z}}\left(E_i^h, V_l\right)E_j^h, X^h\right)_\sigma \\
&\quad + \frac{s(p)}{144}X(s) + \frac{1}{12}\left(\frac{ts(p)}{6} - 2\right)X(s) \\
&\quad + Trace_{h_t}\left\{\mathcal{V}_\sigma \ni V \to \frac{t}{8}g\left(\left(\nabla_X\mathcal{B}\right)(V), \mathcal{B}(V)\right)\right\}
\end{aligned}$$

In order to compute the first summand in the right-hand side of the latter identity, it is convenient to set $C_{ilj} = h_t(R_{\mathcal{Z}}(E_i^h, V_l)E_j^h, X^h)_\sigma$. Then,

$$\begin{aligned}
&\sum_{i,j=1}^{4}\sum_{l=1} g\left(\mathcal{R}(V_l), E_j\wedge K_\sigma E_i\right)h_t\left(R_{\mathcal{Z}}\left(E_i^h, V_l\right)E_j^h, X^h\right)_\sigma \\
&= \frac{1}{2}\sum_{l=1}^{2}\left[g\left(\mathcal{R}(V_l), s_1^+ + s_1\right)C_{1l1} - g\left(\mathcal{R}(V_l), s_3^+ - s_3\right)C_{1l3} + g\left(\mathcal{R}(V_l), s_2^+ - s_2\right)C_{1l4}\right. \\
&\quad + g\left(\mathcal{R}(V_l), s_1^+ + s_1\right)C_{2l2} + g\left(\mathcal{R}(V), s_2^+ + s_2\right)C_{2l3} + g\left(\mathcal{R}(V), s_3^+ + s_3\right)C_{2l4} \\
&\quad - g\left(\mathcal{R}(V), s_3^+ + s_3\right)C_{3l1} + g\left(\mathcal{R}(V), s_2^+ - s_2\right)C_{3l2} - g\left(\mathcal{R}(V), s_1^+ - s_1\right)C_{3l3} \\
&\quad \left. g\left(\mathcal{R}(V), s_2^+ + s_2\right)C_{4l1} + g\left(\mathcal{R}(V), s_3^+ - s_3\right)C_{4l2} - g\left(\mathcal{R}(V), s_1^+ - s_1\right)C_{4l4}\right] \\
&= \frac{s(p)}{12\sqrt{t}}\left[\left(-C_{114} + C_{213} - C_{312} + C_{411}\right) + \left(C_{123} + C_{224} - C_{321} - C_{422}\right)\right] \\
&\quad + \frac{1}{2}\sum_{l=1}^{2}\left[g\left(\mathcal{B}(V_l), s_1^+\right)\left(C_{1l1} + C_{2l2} - C_{3l3} - C_{4l4}\right)\right. \\
&\quad + g\left(\mathcal{B}(V_l), s_2^+\right)\left(C_{1l4} + C_{2l3} + C_{3l2} + C_{4l1}\right) \\
&\quad \left. + g\left(\mathcal{B}(V_l), s_3^+\right)\left(-C_{1l3} + C_{2l4} - C_{3l1} + C_{4l2}\right)\right].
\end{aligned}$$

By Corollary 4

$$-C_{124}+C_{223}-C_{322}+C_{421}$$
$$=\frac{\sqrt{t}}{2}\sum_{i=1}^{4}\left[-\tfrac{1}{12}E_i(s)g(E_i,X)+g\left(\left(\nabla_{E_i}\mathcal{B}\right)\left(K_{s_3}E_i\wedge X\right),s_3\right)\right].$$

For every $i=1,\ldots,4$, $K_{s_3}E_i\wedge X+E_i\wedge K_{s_3}X\in\Lambda^2_-T_pM$. Hence,

$$g\left(\left(\nabla_{E_i}\mathcal{B}\right)\left(K_{s_3}E_i\wedge X+E_i\wedge K_{s_3}X\right),s_3\right)=0.$$

It follows that

$$-C_{124}+C_{223}-C_{322}+C_{421}=\frac{\sqrt{t}}{2}\left[-\frac{1}{12}X(s)+g(\delta\mathcal{B}(K_{s_3}X),s_3)\right].$$

Similarly

$$C_{133}+C_{234}-C_{331}-C_{432}$$
$$=\frac{\sqrt{t}}{2}\left[-\frac{1}{12}X(s)+g(\delta\mathcal{B}(K_{s_2}X),s_2)\right].$$

Hence,

$$(-C_{124}+C_{223}-C_{322}+C_{421})+(C_{133}+C_{234}-C_{331}-C_{432})$$
$$=\frac{\sqrt{t}}{2}[-\frac{1}{6}X(s)+tTrace_{h_t}\{\mathcal{V}_\sigma\ni V\to g(\delta\mathcal{B}(K_VX),V)\}.$$

Set for short

$$\begin{aligned}\Sigma(E_1,\ldots,E_4)=\sum_{l=1}^{2}&\left[g\left(\mathcal{B}(V_l),s_1^+\right)(C_{1l1}+C_{2l2}-C_{3l3}-C_{4l4})\right.\\&+g\left(\mathcal{B}(V_l),s_2^+\right)(C_{1l4}+C_{2l3}+C_{3l2}+C_{4l1})\\&\left.+g\left(\mathcal{B}(V_l),s_3^+\right)(-C_{1l3}+C_{2l4}-C_{3l1}+C_{4l2})\right].\end{aligned}$$

Under this notation, we have

$$\begin{aligned}Tr_k(X^h_\sigma)=&\left[1+(-1)^k\right]\frac{s(p)}{144}X(s)+\frac{1}{12}\left(\frac{ts(p)}{6}-2\right)X(s)\\&+Trace_{h_t}\left\{\mathcal{V}_\sigma\ni V\to\left[\frac{t}{8}g\left(\left(\nabla_X\mathcal{B}\right)(V),\mathcal{B}(V)\right)\right.\right.\\&\left.\left.+(-1)^{k+1}\frac{ts(p)}{24}g\left(\delta\mathcal{B}(K_VX),V\right)\right]\right\}.\\&+(-1)^{k+1}\frac{1}{2}\Sigma(E_1,\ldots,E_4).\end{aligned}$$

In particular, the sum $\Sigma(E_1,\ldots,E_4)$ does not depend on the choice of the oriented orthonormal basis $E_1,\ldots,E_4$ (clearly it does not depend on the choice of the h_t-orthonormal basis V_1,V_2 of $\mathcal{V}_\sigma$ as well). Since

$$\Sigma(E_3,E_4,E_1,E_2)=-\Sigma(E_1,E_2,E_3,E_4),$$

it follows that

$$\Sigma(E_1,E_2,E_3,E_4)=0.$$

This proves the lemma.

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