Annali di Matematica (2017) 196:2213–2237
DOI 10.1007/s10231-017-0661-4

Classification of homogeneous minimal immersions from S^2 to $\mathbb{H}P^n$

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Received: 18 November 2016 / Accepted: 17 April 2017 / Published online: 22 April 2017

Abstract In this paper we determine all homogeneous minimal immersions of 2-spheres in quaternionic projective spaces $\mathbb{H}P^n$.

Keywords Homogeneous harmonic maps · Veronese surfaces · Classification · Quaternionic projective spaces

Mathematics Subject Classification 53C42 · 53C55

1 Introduction

In 1990 Ohnita (cf. [13]) gave a series of homogeneous minimal 2-spheres $\{\phi_{n,\alpha}\}$ of constant curvature in quaternionic projective spaces $\mathbb{H}P^n$ and conjectured that $\{\phi_{n,\alpha}\}$ exhaust all proper minimal isometric immersions of S^2 in $\mathbb{H}P^n$. Recently, we get a classification theorem of linearly full unramified conformal minimal immersions of constant curvature from S^2 to $\mathbb{H}P^2$ (cf. [8]), which verifies that in the case $n = 2$, $\{\phi_{n,\alpha}\}$ exhaust all linearly full unramified minimal isometric immersions of constant curvature from S^2 to $\mathbb{H}P^2$. In [10], we determine all conformal minimal immersions of 2-spheres in $\mathbb{H}P^n$ with parallel second fundamental form (implies that it is of constant curvature). In this paper, we determine all homogeneous (stronger than the condition of unramified with constant curvature) minimal 2-spheres in $\mathbb{H}P^n$ (see Theorem 5.5) and solve completely Ohnita's conjecture for n odd. Indeed, in this case, we find homogeneous minimal 2-spheres not in the series $\{\phi_{n,\alpha}\}$. When n is even, the

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series $\{\phi_{n,\alpha}\}$ gives all homogeneous minimal 2-spheres, but it is unknown whether there are any proper non-homogeneous minimal isometric immersions.

In this paper we mainly combine the methods of harmonic sequences and moving frames to study homogeneous harmonic maps from S^2 to $\mathbb{H}P^n$.

Our arrangement is as follows.

In the second section of this paper, firstly we give the definition of quaternionic projective space $\mathbb{H}P^n$ as the totally geodesic submanifold in $G(2, 2n+2)$, then we give some fundamental results concerning $G(k, N)$ from the viewpoint of harmonic sequences by moving frames, and at last we give some brief description of Veronese sequence and the rigidity theorem in $\mathbb{C}P^N$. In the third section, we simply introduce homogeneous harmonic maps from S^2 to $G(k, N)$ and give an important property. In the fourth section, we determine all reducible harmonic maps of constant curvature from S^2 to $\mathbb{H}P^n$, which represent the homogeneous ones completely. In the last section, we determine all irreducible homogeneous harmonic maps from S^2 to $\mathbb{H}P^n$ by mathematical induction. Finally we obtain the classification Theorem 5.5.

2 Preliminaries

For any $N = 1, 2, \ldots$, let $\langle , \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^N$ defined by $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$ where $z = (z_1, \ldots, z_N)^T, w = (w_1, \ldots, w_N)^T \in \mathbb{C}^N$ and $^-$ denotes complex conjugation. Let $\mathbb{H}$ denote the division ring of quaternions. Let j be a unit quaternion with $j^2 = -1$. Then we have an identification of $\mathbb{C}^2$ with $\mathbb{H}$ given by making $(a, b) \in \mathbb{C}^2$ correspond to $a + bj \in \mathbb{H}$; let $n \in \{1, 2, \ldots\}$, we have a corresponding identification of $\mathbb{C}^{2n+2}$ with $\mathbb{H}^{n+1}$. For any $a + bj \in \mathbb{H}$, the left multiplication by j is given by $j(a + bj) = -\overline{b} + \overline{a}j$; the conjugation is given by $\overline{a + bj} = \overline{a} - bj$; and the positive definite inner product is given by $\langle x, y\rangle_{\mathbb{H}} = Re(x\overline{y})$ for any $x, y \in \mathbb{H}$.

Let $\mathbf{J} : \mathbb{C}^{2n+2} \to \mathbb{C}^{2n+2}$ be the conjugate linear map given by left multiplication by j, i.e.,

$$\mathbf{J}(z_1, z_2, \ldots, z_{2n+1}, z_{2n+2})^T = (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n+2}, \bar{z}_{2n+1})^T.$$

Then $\mathbf{J}^2 = -id$ where id denotes the identity map on $\mathbb{C}^{2n+2}$. In fact, for any $v \in \mathbb{C}^{2n+2}$,

$$\mathbf{J}v = J_{n+1}\bar{v},$$

where $J_{n+1} = diag\underbrace{\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}}_{n+1}$.

Let $G(2, 2n+2)$ denote the Grassmann manifold of all complex 2-dimensional subspaces of $\mathbb{C}^{2n+2}$ with its standard Kähler structure. The quaternionic projective spaces $\mathbb{H}P^n$ are the set of all one-dimensional quaternionic subspaces of $\mathbb{H}^{n+1}$. Throughout the above we shall regard $\mathbb{H}P^n$ as the totally geodesic submanifold of $G(2, 2n + 2)$ given by

$$\mathbb{H}P^n = \{V \in G(2, 2n + 2) : \mathbf{J}V = V\}.$$

Let $Sp(n + 1) = \{g \in GL(n + 1; \mathbb{H}),\ g^*g = I_{n+1}\}$ be the symplectic isometry group of $\mathbb{H}P^n$. The explicit description is that the following diagram commutes:

$$\begin{array}{ccc} Sp(n+1) & \xrightarrow{i_1} & U(2n+2) \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ \mathbb{H}P^n & \xrightarrow{i_2} & G(2, 2n+2) \end{array}$$

where i_1, i_2 are inclusions and π_1, π_2 are projections, and $i_1(g) = E$, for $1 \le a, b \le n+1$

$$\begin{cases} E_{2b-1}^{2a-1} = A_b^a, & E_{2b}^{2a-1} = -\overline{B}_b^a, \\ E_{2b-1}^{2a} = B_b^a, & E_{2b}^{2a} = \overline{A}_b^a, \end{cases}$$

where $A = (A_b^a),\ B = (B_b^a) \in M_{n+1}(\mathbb{C}),\ g = A + \overline{B}j \in Sp(n+1)$;

$$\pi_1(g) = g \cdot \left[(1, 0, \ldots, 0)^T\right] \in \mathbb{H}P^n;$$

$$\pi_2(E) = E \cdot \left[\begin{pmatrix} 1, 0, 0, \ldots, 0 \\ 0, 1, 0, \ldots, 0 \end{pmatrix}^T\right] \in G(2, 2n+2);$$

$$i_2\left(\left[(z_1 + \overline{z}_2 j,\ \ldots,\ z_{2n+1} + \overline{z}_{2n+2} j)^T\right]\right) = \left[\begin{pmatrix} z_1,\ z_2,\ \ldots,\ z_{2n+1},\ z_{2n+2} \\ -\overline{z}_2,\ \overline{z}_1,\ \ldots,\ -\overline{z}_{2n+2},\ \overline{z}_{2n+1} \end{pmatrix}^T\right].$$

Here we consider $G(2, 2n+2)$ as the set of all Hermitian orthogonal projections from $\mathbb{C}^{2n+2}$ onto 2-dimensional complex subspaces, i.e.,

$$G(2, 2n+2) = \left\{\varphi \in M_{2n+2}(\mathbb{C}) | \varphi^2 = \varphi, \varphi^* = \varphi, \mathrm{tr}\varphi = 2\right\}.$$

Let $\tau : G(2, 2n+2) \to U(2n+2)$ be the Cartan embedding, which is defined by $\tau(\varphi) = 2\varphi - I \in U(2n+2)$. We take the bi-invariant metric $\mathrm{d}s^2_{U(2n+2)} = \frac{1}{8}\mathrm{tr}\omega\omega^*$ on $U(2n+2)$; then the metric on $G(2, 2n+2)$ induced by τ is given by

$$\mathrm{d}s^2_{G(2,2n+2)} = \frac{1}{2}\mathrm{tr}\mathrm{d}\varphi\mathrm{d}\varphi,$$

where ω is the Maurer–Cartan form of $U(2n+2)$.
Then the metric induced by i_2 is twice as much as the standard metric of constant Q-sectional curvature 4 on $\mathbb{H}P^n$.

Thus we regard the harmonic map from S^2 to $\mathbb{H}P^n$ as the one from S^2 to $G(2, 2n+2)$.

For any $g \in Sp(n+1)$, the action of g on $\mathbb{H}P^n$ induces an action of E on $\mathbb{C}P^{2n+1}$, where $E \in U(2n+2)$ which commutes with $\mathbf{J}$. Then

$$Sp(n+1) = \{E \in U(2n+2),\ E \circ \mathbf{J} = \mathbf{J} \circ E\} = \left\{E \in U(2n+2),\ E J_{n+1} E^T = J_{n+1}\right\}.$$

In the following, we deal with the symplectic isometry of $\mathbb{H}P^n$ through the corresponding symplectic isometry of $\mathbb{C}P^{2n+1}$.

Next, we simply introduce harmonic maps and harmonic sequences in $G(k, N)$ (cf. [4], [5]) and calculate some corresponding geometric quantities.

Let M be a simply connected domain in the unit sphere S^2 and let $(z, \overline{z})$ be complex coordinates on M. We take the metric $\mathrm{d}s^2_M = \mathrm{d}z\mathrm{d}\overline{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}.$$

We consider the complex Grassmann manifold $G(k, N)$ as the set of Hermitian orthogonal projections from $\mathbb{C}^N$ onto a k-dimensional subspace in $\mathbb{C}^N$. Then $\phi : S^2 \to G(k, N)$ is a Hermitian orthogonal projection onto a k-dimensional subbundle $\underline{\phi}$ of the trivial bundle $\underline{\mathbb{C}}^N = M \times \mathbb{C}^N$ given by setting the fiber of $\underline{\phi}$ at x, $\underline{\phi}_x$, equal to $\phi(x)$ for all $x \in M$. For any two orthogonal subbundles $\underline{\phi}$, $\underline{\psi}$ of $\underline{\mathbb{C}}^N$, define vector bundle morphisms over any coordinate chart, and $A'_{\phi,\psi}, A''_{\phi,\psi} : \underline{\phi} \to \underline{\psi}$ is called the ∂'- and ∂''-second fundamental forms of $\underline{\phi}$ in $\underline{\phi} \oplus \underline{\psi}$ by $A'_{\phi,\psi}(v) = \pi_\psi(\partial v)$, $A''_{\phi,\psi}(v) = \pi_\psi(\bar{\partial} v)$ for $v \in C^\infty(\underline{\phi})$. Here π_ψ denotes orthogonal projection onto $\underline{\psi}$ and $C^\infty(\underline{\phi})$ denotes the vector space of smooth sections of $\underline{\phi}$. Although these morphisms are only defined on coordinate charts, all the constructions we will do involve their images which are globally defined independent of choice of local coordinate. In particular $A'_\phi = A'_{\phi,\phi^\perp}$, $A''_\phi = A''_{\phi,\phi^\perp}$ are called the second fundamental forms of $\underline{\phi}$ in $\underline{\mathbb{C}}^N$.

Let $\phi : S^2 \to G(k, N)$ be a smooth harmonic map. Then from ϕ two harmonic sequences (cf. [4]) are derived as follows:

$$\underline{\phi} = \underline{\phi}_0 \xrightarrow{A'_{\phi_0}} \underline{\phi}_1 \xrightarrow{A'_{\phi_1}} \cdots \xrightarrow{A'_{\phi_{\alpha-1}}} \underline{\phi}_\alpha \xrightarrow{A'_{\phi_\alpha}} \cdots, \tag{2.1}$$

$$\underline{\phi} = \underline{\phi}_0 \xrightarrow{A''_{\phi_0}} \underline{\phi}_{-1} \xrightarrow{A''_{\phi_{-1}}} \cdots \xrightarrow{A''_{\phi_{-\alpha+1}}} \underline{\phi}_{-\alpha} \xrightarrow{A''_{\phi_{-\alpha}}} \cdots, \tag{2.2}$$

where $\underline{\phi}_\alpha = \underline{Im} A'_{\phi_{\alpha-1}}$ and $\underline{\phi}_{-\alpha} = \underline{Im} A''_{\phi_{-\alpha+1}}$ are harmonic subbundles of $\underline{\mathbb{C}}^N$ (i.e., represent harmonic maps), respectively, $\alpha = 1, 2, \ldots$.

We assume that ϕ is a linearly full harmonic map from S^2 to $G(k, N)$; here *linearly full* means that $\underline{\phi}$ cannot be contained in any proper trivial subbundle $\underline{\mathbb{C}}^m$ of $\underline{\mathbb{C}}^N$ ($m < N$). We know that several consecutive harmonic maps in (2.1) are not mutually orthogonal generally. So it is meaningful to define the *isotropy order* of ϕ (cf. [4], §3A) to be the greatest integer r such that $\underline{\phi}_i \perp \underline{\phi}_j$ $\forall i, j \in \mathbb{Z}$ with $0 < |i - j| \le r$; if $r = \infty$, then ϕ is said to be *strongly isotropic*. Now we consider a special harmonic sequence.

Suppose that $\phi : S^2 \to G(k, N)$ is a linearly full harmonic map having isotropy order at least 2. Then ϕ belongs to the following harmonic sequence:

$$0 \xleftarrow{A''_{\phi_{-\alpha_0}}} \underline{\phi}_{-\alpha_0} \xleftarrow{A''_{\phi_{-\alpha_0-1}}} \cdots \xleftarrow{A''_{\phi_{-1}}} \underline{\phi}_{-1} \xleftarrow{A''_{\phi}} \underline{\phi}_0 = \underline{\phi} \xrightarrow{A'_{\phi}} \underline{\phi}_1 \xrightarrow{A'_{\phi_1}} \cdots \xrightarrow{A'_{\phi_{\beta_0-1}}} \underline{\phi}_{\beta_0} \xrightarrow{A'_{\phi_{\beta_0}}} 0, \tag{2.3}$$

where for $\alpha = -\alpha_0 + 1, \ldots, \beta_0 - 1$, $\underline{\phi}_{\alpha-1}, \underline{\phi}_\alpha, \underline{\phi}_{\alpha+1}$ are mutually orthogonal.

Denote $k_\alpha = \operatorname{rank} \underline{\phi}_\alpha$ ($\alpha = -\alpha_0, \ldots, \beta_0$).

For the harmonic sequence (2.3) we choose the unit vectors $\{\ldots, e_{-1}, e_1, \ldots\}$ of $\underline{\mathbb{C}}^N$ such that $e_1, \ldots, e_{k_0}$ locally span subbundle $\underline{\phi}_0$, $e_{k_0+\ldots+k_{\alpha-1}+1}, \ldots, e_{k_0+\ldots+k_{\alpha-1}+k_\alpha}$ locally span subbundle $\underline{\phi}_\alpha$ ($\alpha = 1, \ldots, \beta_0$), $e_{-1}, \ldots, e_{-k_{-1}}$ locally span subbundle $\underline{\phi}_{-1}$, $e_{-k_{-1}-\ldots-k_{\alpha+1}-1}, \ldots, and e_{-k_{-1}-\ldots-k_{\alpha+1}-k_\alpha}$ locally span subbundle $\underline{\phi}_\alpha$ ($\alpha = -2, \ldots, -\alpha_0$). Let $W_0 = (e_1, \ldots, e_{k_0})$ be an $(N \times k_0)$-matrix, $W_\alpha = (e_{k_0+\ldots+k_{\alpha-1}+1}, \ldots, e_{k_0+\ldots+k_{\alpha-1}+k_\alpha})$ be an $(N \times k_\alpha)$-matrix for $\alpha = 1, \ldots, \alpha_0$ and let $W_{-1} = (e_{-1}, \ldots, e_{-k_{-1}})$ be an $(N \times k_{-1})$-matrix, $W_\alpha = (e_{-k_{-1}-\ldots-k_{\alpha+1}-1}, \ldots, e_{-k_{-1}-\ldots-k_{\alpha+1}-k_\alpha})$ be an $(N \times k_\alpha)$-matrix for $\alpha = -2, \ldots, -\alpha_0$. Since $\underline{\phi}_{\alpha-1}, \underline{\phi}_\alpha, \underline{\phi}_{\alpha+1}$ are mutually orthogonal, we can extend $\{W_{\alpha-1}, W_\alpha, W_{\alpha+1}\}$ into the local unitary frame with respect to ϕ_α. Then we have

$$\phi_\alpha = W_\alpha W_\alpha^*, \tag{2.4}$$

$$W_\alpha^* W_\alpha = I_{k_\alpha \times k_\alpha}, \quad W_\alpha^* W_{\alpha+1} = 0, \quad W_\alpha^* W_{\alpha-1} = 0. \tag{2.5}$$

By (2.5), a straightforward computation shows

$$\begin{cases} \partial W_\alpha = W_{\alpha+1}\Omega_\alpha + W_\alpha \Psi_\alpha, \\ \overline{\partial} W_\alpha = -W_{\alpha-1}\Omega^*_{\alpha-1} - W_\alpha \Psi^*_\alpha, \end{cases} \tag{2.6}$$

where Ω_α is a $(k_{\alpha+1} \times k_\alpha)$-matrix and Ψ_α is a $(k_\alpha \times k_\alpha)$-matrix for $\alpha = -\alpha_0+1, \ldots, \beta_0-1$.

It is very evident that integrability conditions for (2.6) are

$$\begin{aligned} \overline{\partial}\Omega_\alpha &= \Psi^*_{\alpha+1}\Omega_\alpha - \Omega_\alpha \Psi^*_\alpha, \\ \overline{\partial}\Psi_\alpha + \partial \Psi^*_\alpha &= \Omega^*_\alpha \Omega_\alpha + \Psi^*_\alpha \Psi_\alpha - \Omega_{\alpha-1}\Omega^*_{\alpha-1} - \Psi_\alpha \Psi^*_\alpha. \end{aligned}$$

From (2.6) we have $A'_{\phi_\alpha}(W_\alpha) = W_{\alpha+1}\Omega_\alpha$ and $A''_{\phi_{\alpha+1}}(W_{\alpha+1}) = -W_\alpha \Omega^*_\alpha$, which implies

$$\begin{aligned} \left|A'_{\phi_\alpha}\right|^2 &= \sup\left\{ \frac{\left|A'_{\phi_\alpha}(W_\alpha)\right|^2}{|W_\alpha|^2} : W_\alpha \in \underline{\phi}_\alpha \text{with} \quad W_\alpha \neq 0 \right\} \\ &= \sup\left\{ \frac{\mathrm{tr}(A'_{\phi_\alpha}(W_\alpha))(A'_{\phi_\alpha}(W_\alpha))^*}{\mathrm{tr} W_\alpha W^*_\alpha} : W_\alpha \in \underline{\phi}_\alpha \text{with} \quad W_\alpha \neq 0 \right\} \\ &= \mathrm{tr}(\Omega_\alpha \Omega^*_\alpha) \\ &= \left|A''_{\phi_{\alpha+1}}\right|^2. \end{aligned}$$

Set $L_\alpha = \mathrm{tr}(\Omega_\alpha \Omega^*_\alpha) = \left|A'_{\phi_\alpha}\right|^2 = \left|A''_{\phi_{\alpha+1}}\right|^2$. Then the metric induced by ϕ_α is given by

$$\mathrm{d}s^2_\alpha = (L_{\alpha-1} + L_\alpha)\mathrm{d}z\mathrm{d}\overline{z}. \tag{2.7}$$

The Laplacian $\triangle_\alpha$ and the curvature K_α of $\mathrm{d}s^2_\alpha$ are given by

$$\triangle_\alpha = \frac{4}{L_{\alpha-1}+L_\alpha}\partial\overline{\partial}, \quad K_\alpha = -\frac{2}{L_{\alpha-1}+L_\alpha}\partial\overline{\partial}\log(L_{\alpha-1}+L_\alpha). \tag{2.8}$$

In particular, let $\psi : S^2 \to \mathbb{C}P^N$ be a linearly full harmonic map. Eells and Wood's result (cf. [6]) shows that the following sequence in $\mathbb{C}P^N$ is uniquely determined by ψ

$$0 \xleftarrow{A''_0} \underline{\psi}^{(N)}_0 \xrightarrow{A'_0} \cdots \xrightarrow{A'_{i-1}} \underline{\psi} = \underline{\psi}^{(N)}_i \xrightarrow{A'_i} \cdots \xrightarrow{A'_{N-1}} \underline{\psi}^{(N)}_N \xrightarrow{A'_N} 0, \tag{2.9}$$

for some $i = 0, 1, \ldots, N$, and here A''_0, A'_j denote $A''_{\psi^{(N)}_0}$, $A'_{\psi^{(N)}_j}$, respectively, $(j = 0, \ldots, N)$.

Let $f^{(N)}_0$ be a holomorphic section of $\underline{\psi}^{(N)}_0$, i.e., $\overline{\partial} f^{(N)}_0 = 0$, and let $f^{(N)}_i$ be a local section of $\underline{\psi}^{(N)}_i$ such that

$$f^{(N)}_i = \psi^{(N)\perp}_{i-1}(\partial f^{(N)}_{i-1})$$

for $i = 1, \ldots, N$. Then we have some formulas as follows (cf. [3]):

$$\partial f^{(N)}_i = f^{(N)}_{i+1} + \partial \log|f^{(N)}_i|^2 f^{(N)}_i, \ i = 0, \ldots, N-1, \tag{2.10}$$

$$\overline{\partial} f^{(N)}_i = -l^{(N)}_{i-1} f^{(N)}_{i-1}, \ i = 1, \ldots, N, \tag{2.11}$$

$$\partial\overline{\partial}\log|f^{(N)}_i|^2 = l^{(N)}_i - l^{(N)}_{i-1}, \tag{2.12}$$

$$\partial\overline{\partial}\log l^{(N)}_i = l^{(N)}_{i+1} - 2l^{(N)}_i + l^{(N)}_{i-1}, \ i = 0, \ldots, N-1, \tag{2.13}$$

where $l^{(N)}_i = |f^{(N)}_{i+1}|^2/|f^{(N)}_i|^2$ for $i = 0, \ldots, N$, and $l^{(N)}_{-1} = l^{(N)}_N = 0$.

For convenience, we denote $\underline{f}_i^{(N)} = \underline{\psi}_i^{(N)}$ for $i = 0, 1, \ldots, N$.

In the following, we give a definition of the unramified harmonic map as follows:

Definition 2.1 If $\det(\Omega_\alpha \Omega_\alpha^*)\mathrm{d}z^{k_{\alpha+1}}\mathrm{d}\overline{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (2.3) for some $\alpha = -\alpha_0 + 1, \ldots, \beta_0 - 1$, we say that $\phi_\alpha : S^2 \to G(k_\alpha, N)$ is unramified. If $\det(\Omega_\alpha \Omega_\alpha^*)\mathrm{d}z^{k_{\alpha+1}}\mathrm{d}\overline{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (2.3) for each $\alpha = -\alpha_0 + 1, \ldots, \beta_0 - 1$, we say that the harmonic sequence (2.3) is totally unramified. In this case we also say that each map ϕ_α in (2.3) is totally unramified.

Here $\mathrm{d}z^{k_{\alpha+1}}\mathrm{d}\overline{z}^{k_{\alpha+1}} = \frac{1}{2}\left(\mathrm{d}z^{k_{\alpha+1}} \otimes \mathrm{d}\overline{z}^{k_{\alpha+1}} + \mathrm{d}\overline{z}^{k_{\alpha+1}} \otimes \mathrm{d}z^{k_{\alpha+1}}\right)$ and the quantity $\det(\Omega_\alpha \Omega_\alpha^*)\mathrm{d}z^{k_{\alpha+1}}\mathrm{d}\overline{z}^{k_{\alpha+1}}$ is independent of choice of local coordinate. In the case $k = 1$, the above definition is in accordance with that in §3 of [3].

Now recall ([4], §3A) that a harmonic map $\phi : S^2 \to G(k, N)$ in (2.1) (resp. (2.2)) is said to be ∂'-*irreducible* (resp. ∂''-*irreducible*) if rank $\underline{\phi}$=rank $\underline{\phi}_1$ (resp. rank $\underline{\phi}$=rank $\underline{\phi}_{-1}$) and ∂'-*reducible* (resp. ∂''-*reducible*) otherwise. We assume that ϕ_α in (2.3) is ∂'-irreducible, then $|\det \Omega_\alpha|^2 \mathrm{d}z^{k_\alpha}\mathrm{d}\overline{z}^{k_\alpha}$ is a well-defined invariant and has only isolated zeros on S^2. Under this condition, it can be checked that (cf. [11])

$$\partial\overline{\partial} \log |\det \Omega_\alpha|^2 = L_{\alpha-1} - 2L_\alpha + L_{\alpha+1}, \tag{2.14}$$

which is in accordance with (2.13) in the case $k = 1$. Furthermore if ϕ_α is ∂'-irreducible and unramified, then $|\det \Omega_\alpha|^2 \mathrm{d}z^{k_\alpha}\mathrm{d}\overline{z}^{k_\alpha}$ is a well-defined invariant and has no zeros on S^2. It follows from (2.14) that (cf. [11])

$$\delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1} = -2k_\alpha, \tag{2.15}$$

where $\delta_\alpha = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} L_\alpha \mathrm{d}\overline{z} \wedge \mathrm{d}z$.

At last, we review the rigidity theorem of conformal minimal immersions with constant curvature from S^2 to $\mathbb{C}P^N$.

The Veronese sequence. Let $f_i^{(N)} = \left(f_{i,0}, \ldots, f_{i,N}\right)^T$ for each $i = 0, \ldots, N$. Let $f_{i,p}$ be given for $i, p = 0, 1, \ldots, N$ as follows

$$f_{i,p} = \frac{i!}{(1 + z\overline{z})^i} \sqrt{\binom{N}{p}} z^{p-i} \sum_k (-1)^k \binom{p}{i-k} \binom{N-p}{k} (z\overline{z})^k. \tag{2.16}$$

Such a map $\phi_i^{(N)} = [f_i^{(N)}] : S^2 \to \mathbb{C}P^N$ is a conformal minimal immersion with constant curvature $\frac{4}{N+2i(N-i)}$ and constant Kähler angle $\theta_i^{(N)}$ given by

$$\left(\tan \frac{\theta_i^{(N)}}{2}\right)^2 = \frac{i(N-i+1)}{(i+1)(N-i)}.$$

Such a harmonic sequence $\phi_0^{(N)}, \ldots, \phi_N^{(N)} : S^2 \to \mathbb{C}P^N$ is called the *Veronese sequence*. We always denote it by $V_0^{(N)}, \ldots, V_N^{(N)} : S^2 \to \mathbb{C}P^N$.

Bolton et al. proved that (cf. [3]) if ψ is a linearly full conformal minimal 2-sphere of constant curvature immersed in $\mathbb{C}P^N$, then, up to a holomorphic isometry of $\mathbb{C}P^N$, ψ is an element of the Veronese sequence (i.e., a Veronese surface).

3 Homogeneous harmonic maps from S^2 to $G(k, N)$

An immersion $\phi : S^2 \to G(k, N)$ is said to be *homogeneous*, if for any two points $p, q \in S^2$ there exist an isometry σ of S^2 and a holomorphic isometry u of $G(k, N)$ such that $\sigma(p) = q$ and the following diagram communicates

$$\begin{array}{ccc} S^2 & \xrightarrow{\phi} & G(k, N) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle u} \\ S^2 & \xrightarrow{\phi} & G(k, N), \end{array}$$

i.e.,

$$\phi \circ \sigma = u \circ \phi. \tag{3.1}$$

Here we can identify σ (resp. u) with an element of $SU(2)$ (resp. $U(N)$) (cf. [7]). All such u form a subgroup G of $U(N)$ and G acts transitively on $\phi(S^2)$. It is known that such 2-spheres in $G(k, N)$ have constant curvature, but they are non-minimal in general. Let the complex coordinate z on $S^2 \simeq \mathbb{C}P^1$ be given by two complex variables z_0 and z_1, i.e., $z = [(z_0, z_1)^T] = [(1, \frac{z_1}{z_0})^T]$ (for computation convenience $z = \frac{z_1}{z_0}$) and let

$$SU(2) = \left\{ g = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}. \tag{3.2}$$

Then σ is given by

$$\sigma(z) = g^* z = [(\overline{a} z_0 + \overline{b} z_1, \ -b z_0 + a z_1)^T]. \tag{3.3}$$

In particular, if ϕ is harmonic and satisfies (3.1), then it is called a *homogeneous harmonic map* from S^2 to $G(k, N)$. In the following we give a property of homogeneous harmonic map.

Lemma 3.1 *Let $\phi : S^2 \to G(k, N)$ be a homogeneous harmonic map that belongs to the harmonic sequence (2.3), then for each $\alpha = -\alpha_0, \ldots, \beta_0$ the harmonic map $\phi_\alpha : S^2 \to G(k_\alpha, N)$ is homogeneous. This time the harmonic sequence (2.3) is said to be a homogeneous harmonic sequence.*

Proof Since ϕ_0 in (2.3) is homogeneous, it follows from (3.1) that

$$\phi_0 \circ \sigma = u \circ \phi_0. \tag{3.4}$$

Set $\underline{\phi}_0 = \operatorname{span}\{e_1, \ldots, e_{k_0}\}$, where $e_1, \ldots, e_{k_0}$ are unit orthogonal vectors of $\mathbb{C}^N$. Then from $\underline{\phi}_1 = A'_{\phi_0}(\underline{\phi}_0)$ we get $\underline{\phi}_1 = \operatorname{span}\{v_{k_0+1}, \ldots, v_{k_0+k_1}\}$; for each $i = 1, \ldots, k_0$, v_{k_0+i} is given by

$$v_{k_0+i} = \partial e_i - \sum_{s=1}^{k_0} \langle \partial e_i, e_s \rangle e_s, \tag{3.5}$$

since there may exist some i such that $v_{k_0+i} = 0$, we exclude all zero vectors, so here $k_1 \le k_0$.

Set $u \in U(N)$. By (3.5) a straightforward computation shows

$$\begin{aligned}\partial(ue_i) - \sum_{s=1}^{k_0} \langle \partial(ue_i), e_s\rangle e_s &= (\partial u e_i + u\partial e_i) - \sum_{s=1}^{k_0} \langle \partial u e_i + u\partial e_i, e_s\rangle e_s \\ &= u\left(\partial e_i - \sum_{s=1}^{k_0} \langle \partial e_i, e_s\rangle e_s\right) \\ &= u v_{k_0+i},\end{aligned} \tag{3.6}$$

which implies

$$A'_{\phi_0}(u \circ \underline{\phi}_0) = u \circ A'_{\phi_0}(\underline{\phi}_0) = u \circ \underline{\phi}_1. \tag{3.7}$$

On the other hand, from (3.3) $\phi_0 \circ \sigma = \phi_0(w)$, where $w = \frac{-b+az}{\bar{a}+\bar{b}z}$. Since $\frac{\partial \phi_0}{\partial z} = \frac{\partial \phi_0}{\partial w}\frac{\partial w}{\partial z}$, then

$$A'_{\phi_0}(\underline{\phi}_0 \circ \sigma) = A'_{\phi_0}(\underline{\phi}_0) \circ \sigma = \underline{\phi}_1 \circ \sigma. \tag{3.8}$$

It follows from (3.4), (3.7) and (3.8) that

$$\underline{\phi}_1 \circ \sigma = u \circ \underline{\phi}_1 \tag{3.9}$$

which verifies that ϕ_1 is homogeneous.

The other cases of ϕ_α are similar to the above. Thus we get the conclusion. □

In the case $k = 1$, Veronese sequence is the standard homogeneous harmonic sequence in $\mathbb{C}P^{N-1}$ (cf. [2]). In the case $k = 2$, if ϕ is a homogeneous harmonic map from S^2 to $\mathbb{H}P^n$, then u is an element of $Sp(n+1) \subset U(2n+2)$ in (3.1).

4 Reducible homogeneous harmonic maps from S^2 to $\mathbb{H}P^n$

Let $\phi : S^2 \to \mathbb{H}P^n$ be a linearly full harmonic map of isotropy order r. If ϕ has finite isotropy order, then $r = 2s$ for $1 \le s \le n$ by ([1], Proposition 3.2 and Lemma 3.10); if ϕ is strongly isotropic, then $r = \infty$. If $\phi_0 : S^2 \to \mathbb{H}P^n$ is a reducible linearly full harmonic map, then by ([1], Proposition 3.7) we know that ϕ_0 is a quaternionic mixed pair or a quaternionic Frenet pair.

Definition 4.1 ([1])

1. A map $\phi : M \to \mathbb{H}P^n$ is called a quaternionic mixed pair if $\underline{\phi} = \underline{f} \oplus \mathbf{J}\underline{f}$ where $f : M \to \mathbb{C}P^{2n+1}$ is holomorphic and $G'(f) \perp \mathbf{J}\underline{f}$. Here $G'(f) = \underline{Im A'_f}$.
2. A map $\phi : M \to \mathbb{H}P^n$ is called a quaternionic Frenet pair if $\underline{\phi} = G^{(r-1)}(h) \oplus G^{(r)}(h)$ for some integer r, with $1 \le r \le n+1$ and holomorphic map $h : M \to \mathbb{C}P^{2n+1}$ with $G^{(2r-1)}(h) = \mathbf{J}\underline{h}$. Here $G^{(0)}(h) = \underline{h}$, $G^{(i)}(h) = G'(G^{(i-1)}(h))$.

In the following we discuss these two types in the case of constant curvature.

If ϕ_0 is a linearly full quaternionic Frenet pair, then

$$\underline{\phi}_0 = \underline{f}_n^{(2n+1)} \oplus \underline{f}_{n+1}^{(2n+1)}, \tag{4.1}$$

where $\underline{f}_0^{(2n+1)}, \ldots, \underline{f}_{2n+1}^{(2n+1)} : S^2 \to \mathbb{C}P^{2n+1}$ is a harmonic sequence with the linearly full totally $\mathbf{J}$-isotropic map $\underline{f}_0^{(2n+1)}$.

Firstly we recall ([1], §3) that a full holomorphic map $\underline{f}_0^{(2n+1)} : S^2 \to CP^{2n+1}$ satisfying $\underline{f}_{2n+1}^{(2n+1)} = \mathbf{J}\underline{f}_0^{(2n+1)}$ is said to be *totally* $\mathbf{J}$*-isotropic*; this generates a harmonic sequence

$$0 \xleftarrow{A_0''} \underline{f}_0^{(2n+1)} \xrightarrow{A_0'} \cdots \xrightarrow{A_{n-1}'} \underline{f}_n^{(2n+1)} \xrightarrow{A_n'} \underline{f}_{n+1}^{(2n+1)} \xrightarrow{A_{n+1}'} \cdots \xrightarrow{A_{2n}'} \underline{f}_{2n+1}^{(2n+1)} \xrightarrow{A_{2n+1}'} 0.$$

From the harmonic sequence, $f_{2n+1-i}^{(2n+1)} = \mathbf{J}f_i^{(2n+1)}$ for all i so that:

$$l_j^{(2n+1)} = l_{2n+1-j}^{(2n+1)}, \tag{4.2}$$

and set $\mathbf{J}f_0^{(2n+1)} = \tau_{2n+1}f_{2n+1}^{(2n+1)}$, then

$$|\tau_{2n+1}|^2 = \frac{|f_0^{(2n+1)}|^2}{|f_{2n+1}^{(2n+1)}|^2}, \quad \mathbf{J}f_j^{(2n+1)} = \tau_{2n+1-j}f_{2n+1-j}^{(2n+1)}, \tag{4.3}$$

where $\tau_{2n+1-j} = (-1)^j\tau_{2n+1}\frac{|f_{2n+1}^{(2n+1)}|^2}{|f_{2n+1-j}^{(2n+1)}|^2}$ for each $j = 0, \ldots, 2n+1$.

Obviously ϕ_0 belongs to the following harmonic sequence (cf. [4])

$$0 \xleftarrow{A_0''} \underline{f}_0^{(2n+1)} \xleftarrow{A_1''} \cdots \xleftarrow{A_{n-1}''} \underline{f}_{n-1}^{(2n+1)} \xleftarrow{A_{\phi_0}''} \underline{\phi}_0 \xrightarrow{A_{\phi_0}'} \underline{f}_{n+2}^{(2n+1)} \xrightarrow{A_{n+2}'} \cdots \xrightarrow{A_{2n}'} \underline{f}_{2n+1}^{(2n+1)} \xrightarrow{A_{2n+1}'} 0. \tag{4.4}$$

Then we give the following proposition without proof:

Proposition 4.2 ([9], Proposition 3.1) *Let $\phi_0 : S^2 \to \mathbb{H}P^n$ be a linearly full quaternionic Frenet pair of constant curvature K_0. Denote the isotropy order of ϕ_0 by r. Then $K_0 = \frac{2}{n(n+2)}$, $r = \infty$, and up to a symplectic isometry of $\mathbb{H}P^n$,*

$$\underline{\phi}_0 = U\underline{V}_n^{(2n+1)} \oplus U\underline{V}_{n+1}^{(2n+1)}$$

for some $U \in G_{2n+2} \triangleq \left\{U \in U(2n+2),\ UW_{n+1}U^T = J_{n+1}\right\}$,

where $W_{n+1} = \text{antidiag}\underbrace{\left\{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \ldots, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\}}_{n+1}$.

Remark 4.3 Since both $\underline{V}_n^{(2n+1)}$ and $\underline{V}_{n+1}^{(2n+1)}$ are Veronese surfaces in $\mathbb{C}P^{2n+1}$. It is easy to check that such ϕ_0 is $SU(2)$-equivalent, so ϕ_0 is homogeneous. Thus all linearly full homogeneous quaternionic Frenet pairs are given by Proposition 4.2.

If ϕ_0 is a linearly full quaternionic mixed pair, then

$$\underline{\phi}_0 = \underline{f}_0^{(m)} \oplus \mathbf{J}\underline{f}_0^{(m)}, \tag{4.5}$$

where $\underline{f}_0^{(m)} : S^2 \to \mathbb{C}P^m \subseteq \mathbb{C}P^{2n+1}$ $(n \le m \le 2n+1)$ is holomorphic and $\underline{f}_1^{(m)} \perp \mathbf{J}\underline{f}_0^{(m)}$. Obviously ϕ_0 belongs to the following harmonic sequence

$$0 \xleftarrow{A_{Jm}''} \mathbf{J}\underline{f}_m^{(m)} \xleftarrow{A_{J(m-1)}''} \cdots \xleftarrow{A_{J1}''} \mathbf{J}\underline{f}_1^{(m)} \xleftarrow{A_{\phi^0}''} \underline{\phi}_0 \xrightarrow{A_{\phi^0}'} \underline{f}_1^{(m)} \xrightarrow{A_1'} \cdots \xrightarrow{A_{m-1}'} \underline{f}_m^{(m)} \xrightarrow{A_m'} 0, \tag{4.6}$$

where A_{Ji}'' represents $A''_{\mathbf{J}f_i^{(m)}}$ for each $i = 1, \ldots, m$.

So the induced metric by ϕ_0 is given by

$$\mathrm{d}s_0^2 = 2l_0^{(m)}\mathrm{d}z\mathrm{d}\bar{z}, \tag{4.7}$$

where $l_0^{(m)}\mathrm{d}z\mathrm{d}\bar{z}$ is the induced metric by the map $\underline{f}_0^{(m)}$.

Then we prove

Proposition 4.4 *Let $\phi_0 : S^2 \to \mathbb{H}P^n$ be a linearly full homogeneous quaternionic mixed pair. Denote the isotropy order and Gaussian curvature of ϕ_0 by r and K_0, respectively. Then up to a symplectic isometry of $\mathbb{H}P^n$, ϕ_0 belongs to one of the following minimal immersions.*

1. $\phi_0 = U\underline{V}_0^{(2n+1)} \oplus U\underline{V}_{2n+1}^{(2n+1)}$ *for some* $U \in G_{2n+2}$, *where* $K_0 = \frac{2}{2n+1}$, $r = 2n$;
2. $\phi_0 = \underline{V}_0^{(n)}$ *with* $K_0 = \frac{2}{n}$, $r = \infty$;
3. $n = 2t+1$ $(t > 0)$ *and* $\phi_0 = \left[\left(\phi_{0,0}, \ldots, \phi_{0,n}\right)^T\right]$, *for* $q_1 = 0, \ldots, t$, $q_2 = t+1, \ldots, n$, ϕ_{0,q_1}, ϕ_{0,q_2} *are given by*

$$\phi_{0,q_1} = \sqrt{\binom{n}{q_1}} z^{q_1} + (-1)^{q_1}\overline{\lambda}\sqrt{\binom{n}{n-q_1}}\overline{z}^{n-q_1} j,\ \phi_{0,q_2} = \sqrt{1-|\lambda|^2}\sqrt{\binom{n}{q_2}} z^{q_2},$$

where λ is a complex parameter satisfying $0 < |\lambda|^2 < 1$ and $K_0 = \frac{2}{n}$, $r = n-1$.

Proof Let $\phi_0 : S^2 \to \mathbb{H}P^n$ be a linearly full quaternionic mixed pair. Since ϕ_0 is homogeneous, we know that there exists a matrix $u \in Sp(n+1) \subset SU(2n+2)$ such that

$$u \circ f_0^{(m)} = f_0^{(m)} \circ \sigma. \tag{4.8}$$

By (4.7) we get that $K_0 = \frac{2}{m}$, and up to a holomorphic isometry of CP^{2n+1}, $\underline{f}_0^{(m)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2\backslash\{pt\}$ so that $f_0^{(m)} = UV_0^{(m)}$, where $U \in U(2n+2)$ and $V_0^{(m)}$ has the standard expression given in (2.16) (adding zeros to $V_0^{(m)}$ such that $V_0^{(m)} \in \mathbb{C}^{2n+2}$). A straightforward computation shows

$$V_0^{(m)} \circ \sigma(z_0,\ z_1) = \begin{pmatrix} D_{m+1} & * \\ O & * \end{pmatrix} V_0^{(m)}(z_0,\ z_1), \tag{4.9}$$

where

$$D_{m+1} = \left(d_{ij}\right)_{0\le i,j\le m},\ d_{ij} = \frac{\sqrt{\binom{m}{i}}}{\sqrt{\binom{m}{j}}}\binom{m-i}{k}\binom{i}{j-k}\overline{a}^{m-i-k}\overline{b}^{k}(-b)^{i-j+k}a^{j-k}. \tag{4.10}$$

Then we have

$$\overline{U}^T uU V_0^{(m)}(z_0,\ z_1) = \begin{pmatrix} D_{m+1} & D_2 \\ O & D_4 \end{pmatrix} V_0^{(m)}(z_0,\ z_1). \tag{4.11}$$

By differentiating with respect to z_0 in the above formula, the matrices $\overline{U}^T uU$ and $\begin{pmatrix} D_{m+1} & D_2 \\ O & D_4 \end{pmatrix}$ have the same effect on all derivatives of $V_0^{(m)}(z_0,\ z_1)$. Generally, $V_0^{(m)}(z_0,\ z_1)$ is not full in $\mathbb{C}^{2n+2}$, so the two above matrixes are not identical. But the matrix $\overline{U}^T uU$ is still the type of the following matrix, i.e.,

$$\overline{U}^T uU = \begin{pmatrix} D_{m+1} & * \\ O & * \end{pmatrix}. \tag{4.12}$$

Since $U \in U(2n+2)$ and $u \in Sp(n+1) \subset SU(2n+2)$, we get

$$\overline{U}^T uU = \begin{pmatrix} D_{m+1} & O \\ O & C_{2n+1-m} \end{pmatrix}, \tag{4.13}$$

where $D_{m+1} \in U(m+1)$ and $C_{2n+1-m} \in U(2n+1-m)$.

Set $U^T J_{n+1} U = \overline{W}$. Then

$$W^T = -W,\ W^* W = I, \tag{4.14}$$

where I is the identity matrix.
Since $u \in Sp(n+1)$, from (4.13) we have

$$\begin{pmatrix} \overline{D_{m+1}} & O \\ O & \overline{C_{2n+1-m}} \end{pmatrix} \overline{W} = \overline{W} \begin{pmatrix} D_{m+1} & O \\ O & C_{2n+1-m} \end{pmatrix}. \tag{4.15}$$

Set $\overline{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$ with $W_{11} = (w_{kl})_{0\le k,l\le m}$. Then (4.10) and (4.15) yield

$$\begin{cases} w_{m-j,j} = (-1)^{m-j} w_{0m},\ j = 0,\ldots,m \\ w_{kl} = 0,\ k+l \neq m \end{cases} \tag{4.16}$$

Case I . r is finite. It follows from the harmonic sequence (4.6) that $m \ge r+1$. Thus for each $1 \le \alpha \le r$, we have $\underline{f}^{(m)}_{\alpha} \perp \mathbf{J}\underline{f}^{(m)}_{0}$, which are equivalent to the following equations

$$tr\left(V^{(m)}_{\alpha} V^{(m)^T}_{0} U^T J_{n+1} U\right) = 0 \Leftrightarrow tr\left(V^{(m)}_{\alpha} V^{(m)^T}_{0} \overline{W}\right) = 0,\ \alpha = 1,\ldots,r \tag{4.17}$$

hold.

Set $V^{(m)}_{\alpha} V^{(m)^T}_{0} = \begin{pmatrix} V_{11} & O \\ O & O \end{pmatrix}$ with $V_{11} = \left(v_{ij}\right)_{0\le i,j\le m}$, then

$$v_{ij} = \frac{\alpha!}{(1+z\overline{z})^{\alpha}} \sqrt{\binom{m}{i}\binom{m}{j}} \sum_{k} (-1)^k \binom{i}{\alpha-k} \binom{m-i}{k} z^{i+j+k-\alpha} \overline{z}^{k}. \tag{4.18}$$

Since r is finite, we know $w_{0m} \neq 0$. From (4.16) and (4.18), we get

$$tr\left(V^{(m)}_{m-1} V^{(m)^T}_{0} \overline{W}\right) = 0,\ tr\left(V^{(m)}_{m} V^{(m)^T}_{0} \overline{W}\right) \neq 0, \tag{4.19}$$

which implies that $\underline{f}^{(m)}_{m-1}$ is perpendicular to $\mathbf{J}\underline{f}^{(m)}_{0}$, but $\underline{f}^{(m)}_{m}$ is not perpendicular to $\mathbf{J}\underline{f}^{(m)}_{0}$ by (4.17).
Hence we have

$$m = r+1. \tag{4.20}$$

Let $r = 2s$. It follows from (4.16) that

$$W_{11} = w_{m0} W_{s+1}. \tag{4.21}$$

Then the corresponding $\overline{W}$ is given by

$$\overline{W} = \begin{pmatrix} w_{m0} W_{s+1} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{4.22}$$

where W_{21} is a $(2n+1-m)\times(m+1)$-matrix.

Define a set

$$G_W \triangleq \left\{ U \in U(2n+2),\ UWU^T = J_{n+1} \right\}.$$

For a given W, the following can be easily checked

(i) $\forall\, A \in Sp(n+1),\ U \in G_W$, we have that $AU \in G_W$;
(ii) $\forall\, U, V \in G_W,\ \exists\, A = UV^* \in Sp(n+1)\ s.t.\ U = AV$.

In the following, in order to get the explicit expression of ϕ_0, we discuss W, respectively, by the two cases that n is an even or odd.

Case I.1. n is an even, i.e., $n = 2t$ $(t > 0)$.

In this case, since $m > n$, i.e., $m + 1 > 2n + 1 - m$, then from (4.14) and (4.22) we know $W_{21} = O$, i.e.,

$$W = \begin{pmatrix} W_{s+1} & O \\ O & W_{22} \end{pmatrix}, \tag{4.23}$$

where $W_{22} \in U(2n + 1 - m)$.

Now we claim that $m = 2n + 1$ $(s = n)$. Otherwise if $m < 2n + 1$, then $s < n$, and for any given W, we can choose proper $U \in G_W$ such that $\mathbf{J}UV_0^{(m)} = -UV_m^{(m)}$, and then up to $Sp(n + 1)$,

$$\underline{\phi}_0 = U\underline{V}_0^{(m)} \oplus U\underline{V}_m^{(m)}.$$

Obviously ϕ_0 has image in $\mathbb{H}P^s$, so it is not linearly full. It contradicts our assumption that ϕ_0 is linearly full.

Furthermore we get $K_0 = \frac{2}{2n+1}$, $W = W_{n+1}$ and $G_W = G_{2n+2}$. Then the proof of Proposition 4.2(see ([9], Proposition 3.1)) gives that up to $Sp(n + 1)$,

$$\underline{\phi}_0 = U\underline{V}_0^{(2n+1)} \oplus U\underline{V}_{2n+1}^{(2n+1)},$$

where $U \in G_{2n+2}$.

Obviously it belongs to case (1) with $K_0 = \frac{2}{2n+1}$, $r = 2n$.

Case I.2. n is an odd, i.e., $n = 2t + 1$ $(t > 0)$.

In this case, if $m > n$, then by the same discussion as

Case I.1, we conclude that $m = 2n + 1$, and up to $Sp(n + 1)$, the corresponding ϕ_0 belongs to case (1) with $K_0 = \frac{2}{2n+1}$, $r = 2n$.

If $m = n$, then the corresponding W_{21} in (4.22) is a $(n + 1) \times (n + 1)$-matrix. If $|w_{n0}|^2 = 0$, then $W_{11} = O$, which implies $f_m^{(m)} \perp \mathbf{J}f_0^{(m)}$, so ϕ_0 is strongly isotropic. It contradicts our assumption that the isotropy order r is finite. If $|w_{n0}|^2 = 1$, then the corresponding ϕ_0 is not linearly full.

Now we assume $0 < |w_{n0}|^2 < 1$. Denote w_{n0} by λ. Then we get the type of $\overline{W} = U^T J_{n+1} U \in U(2n + 2)$ as follows:

$$\overline{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & (-1)^n\lambda & w_{1,n+2} & \cdots & w_{1,2n+2} \\ 0 & 0 & 0 & (-1)^{n-1}\lambda & 0 & w_{2,n+2} & \cdots & w_{2,2n+2} \\ 0 & 0 & \iddots & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & (-1)^1\lambda & 0 & 0 & 0 & w_{n,n+2} & \cdots & w_{n,2n+2} \\ \lambda & 0 & 0 & 0 & 0 & w_{n+1,n+2} & \cdots & w_{n+1,2n+2} \\ -w_{1,n+2} & -w_{2,n+2} & \cdots & -w_{n,n+2} & -w_{n+1,n+2} & 0 & \cdots & w_{n+2,2n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{1,2n+2} & -w_{2,2n+2} & \cdots & -w_{n,2n+2} & -w_{n+1,2n+2} & -w_{n+2,2n+2} & \cdots & 0 \end{pmatrix}, \tag{4.24}$$

where $0 < |\lambda|^2 < 1$.

From $\overline{W}U^T = U^T J_{n+1}$, the corresponding $U = \left[e_1, e_2, \ldots, e_{2n+1}, e_{2n+2}\right]^T$ satisfy

$$e_{2\alpha} = \overline{W}e_{2\alpha-1}, \quad \alpha = 1, \ldots, n + 1, \tag{4.25}$$

where e_i are unit column vectors in $\mathbb{C}^{2n+2}$.

Generally, suppose $\{e_1, e_2, \ldots, e_{2\alpha-3}, e_{2\alpha-2} = \overline{W}\overline{e}_{2\alpha-3}\}$ $(\alpha \geq 2)$ are mutually orthogonal, we choose a unit column vector $e_{2\alpha-1} \in \mathbb{C}^{2n+2}$ such that $\{e_1, e_2, \ldots, e_{2\alpha-3}, e_{2\alpha-2}, e_{2\alpha-1}\}$ are mutually orthogonal. Set $e_{2\alpha} = \overline{W}\overline{e}_{2\alpha-1}$, then

$$\langle e_{2\alpha},\ e_{2\alpha-1}\rangle = e_{2\alpha-1}^T W^T e_{2\alpha-1} = -tr(e_{2\alpha-1}e_{2\alpha-1}^T W) = 0,$$

and for any $2 \leq \beta \leq \alpha$,

$$\begin{aligned}\langle e_{2\alpha},\ e_{2\beta-3}\rangle &= e_{2\alpha-1}^T W^T e_{2\beta-3} = -e_{2\alpha-1}^T W e_{2\beta-3} = -e_{2\alpha-1}^T \overline{e}_{2\beta-2}\\ &= -\langle e_{2\alpha-1},\ e_{2\beta-2}\rangle = 0,\end{aligned}$$

$$\langle e_{2\alpha},\ e_{2\beta-2}\rangle = e_{2\alpha-1}^T W^T \overline{W}\overline{e}_{2\beta-3} = e_{2\alpha-1}^T \overline{e}_{2\beta-3} = \langle e_{2\alpha-1},\ e_{2\beta-3}\rangle = 0.$$

Thus $\{e_1, e_2, \ldots, e_{2\alpha-3}, e_{2\alpha-2}, e_{2\alpha-1}, e_{2\alpha}\}$ are mutually orthogonal.

Without loss of generality, in this case for $p = 0, \ldots, t$ we choose

$$e_{2p+1} = \left(\underbrace{0, \ldots, 0}_{p}, 1, 0, \ldots, 0\right)^T, \tag{4.26}$$

then the corresponding e_{2p+2} is given by

$$e_{2p+2} = \overline{W}\overline{e}_{2p+1} = \left(\underbrace{0, \ldots, 0}_{n-p}, (-1)^p\lambda, 0, \ldots, 0, -w_{p+1,n+2}, \ldots, -w_{p+1,2n+2}\right)^T. \tag{4.27}$$

Observing (4.26) and (4.27), we find $\{e_1, e_2, \ldots, e_{2t+1}, e_{2t+2}\}$ are mutually orthogonal. Next for $q = t+1, \ldots, n$ we successively choose

$$\begin{aligned}&e_{2q+1}\\ &= \left(\underbrace{0, \ldots, 0}_{q}, \sqrt{1-|\lambda|^2}, 0, \ldots, 0, (-1)^{n-q}\frac{\overline{\lambda}w_{n+1-q,n+2}}{\sqrt{1-|\lambda|^2}}, \ldots, (-1)^{n-q}\frac{\overline{\lambda}w_{n+1-q,2n+2}}{\sqrt{1-|\lambda|^2}}\right)^T\end{aligned} \tag{4.28}$$

then the corresponding e_{2q+2} is given by

$$e_{2q+2} = \left(\underbrace{0, \ldots, 0}_{n+1}, *\right)^T. \tag{4.29}$$

From (4.26)-(4.29) we obtain the type of the corresponding U; furthermore, we have

$$UV_0^{(n)} = \left(\ldots, \sqrt{\binom{n}{p}}z^p, (-1)^p\lambda\sqrt{\binom{n}{n-p}}z^{n-p}, \ldots, \sqrt{1-|\lambda|^2}\sqrt{\binom{n}{q}}z^q, 0, \ldots\right)^T, \tag{4.30}$$

where $p = 0, \ldots, t$ and $q = t+1, \ldots, n$.
So the corresponding ϕ_0 belongs to case (3) with $K_0 = \frac{2}{n}$, $r = n-1$.

Case II. $r = \infty$. In this case, it follows from ([9], Proposition 3.2) that up to $Sp(n+1)$,

$$\phi_0 = U\underline{V}_0^{(n)} \oplus \mathbf{J}U\underline{V}_0^{(n)}$$

with $K_0 = \frac{2}{n}$ for some $U \in G_{2n+2}$.
For convenience, we choose

$$U = \begin{pmatrix} 1\,0\cdots 0 & 0 & 0 & \cdots & 0 & 0 \\ 0\,0\cdots 0 & 0 & 0 & \cdots & 0 & (-1)^0 \\ 0\,1\cdots 0 & 0 & 0 & \cdots & 0 & 0 \\ 0\,0\cdots 0 & 0 & 0 & \cdots & (-1)^1 & 0 \\ \vdots\ \vdots\ \ddots\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0\,0\cdots 0 & 0 & (-1)^{n-1} & \cdots & 0 & 0 \\ 0\,0\cdots 1 & 0 & 0 & \cdots & 0 & 0 \\ 0\,0\cdots 0 & (-1)^n & 0 & \cdots & 0 & 0 \end{pmatrix} \in G_{2n+2}.$$

A straightforward calculation shows

$$UV_0^n = \left[(1, 0, \sqrt{2}z, 0, \ldots, z^n, 0)^T\right],$$
$$\mathbf{J}UV_0^n = \left[(0, 1, 0, \sqrt{2}\overline{z}, \ldots, 0, \overline{z}^n)^T\right],$$

which implies

$$\phi_0 = \left[(1, \sqrt{2}z, \ldots, z^n)^T\right] : S^2 \to \mathbb{C}P^n \hookrightarrow \mathbb{H}P^n.$$

Hence, in this case, up to $Sp(n+1)$, ϕ_0 is the composition of $V_0^{(n)}$ with the totally geodesic inclusion of $\mathbb{C}P^n$ in $\mathbb{H}P^n$ and belongs to case (2) with $K_0 = \frac{2}{n}$, $r = \infty$.

In summary we get the conclusion. □

Remark 4.5 We conjecture that all linearly full quaternionic mixed pairs of constant curvature are given by Proposition 4.4.

5 Irreducible homogeneous harmonic maps from S^2 to $\mathbb{H}P^n$

Now, we consider the irreducible harmonic maps $\phi : S^2 \to \mathbb{H}P^n$ of isotropy order r. In the following we consider the two cases of finite isotropy order or strongly isotropic.

At first we consider the case of finite isotropy order, i.e., $r = 2s$ $(s = 1, \ldots, n-1)$. Here we exclude the case of $s = n$, as if $r = 2n$, then the corresponding ϕ is reducible by ([1], Lemma 3.10). We state the following lemma by ([1], Theorem 4.7):

Lemma 5.1 ([1]) *Let $\phi : S^2 \to \mathbb{H}P^n$ be an irreducible harmonic map of finite isotropy order r, where $r = 2s$ $(s = 1, \ldots, n-1)$. Then there is a unique sequence of harmonic maps $\phi^i : S^2 \to G(2, 2n+2)$ $(i = 0, 1, \ldots, 2l)$ where $l = 1, \ldots, n-s$ such that*

(i) $\underline{\phi}^0 = \underline{f}_0^{(m)} \oplus \mathbf{J}\underline{f}_0^{(m)}$ for some holomorphic map $\underline{f}_0^{(m)} : S^2 \to \mathbb{C}P^m \subseteq \mathbb{C}P^{2n+1}$ $(n \le m \le 2n+1)$ satisfying $\underline{f}_p^{(m)} \perp \mathbf{J}\underline{f}_0^{(m)}$ for $1 \le p \le 2l+2s$ but $\underline{f}_p^{(m)} \not\perp \mathbf{J}\underline{f}_0^{(m)}$ for $p = 2l+2s+1$, that is, ϕ^0 is a quaternionic mixed pair of isotropy order $2l+2s$;
(ii) $\phi^{2l} = \phi$;
(iii) for $k = 0, 1, \ldots, l-1$, ϕ^{2k+1} is obtained from ϕ^{2k} by forward replacement of some holomorphic subbundle of $\underline{\phi}^{2k}$ not equal to the image of the first ∂'-return map of ϕ^{2k}. Then the inverse of this transformation is given by ([1],Proposition 4.6(a)) and is given by backward replacement of the image of the first ∂''-return map;

(iv) for $k = 0, 1, \ldots, l-1$, ϕ^{2k+2} is obtained from ϕ^{2k+1} by backward replacement of $\underline{\beta}_k^{\perp} \cap \underline{\phi}^{2k+1}$ where $\underline{\beta}_k$ is the unique holomorphic subbundle of $\underline{\phi}^{2k+1}$ not equal to the image of the first ∂'-return map of ϕ^{2k+1} such that $\underline{Im}(A'_{\phi^{2k+1}}|_{\underline{\beta}_k}) \perp \mathbf{J}\underline{\beta}_k$. In fact, it is the operation in ([1],Proposition 4.5(b)) with $\underline{\beta}_k^{\perp} \cap \underline{\phi}^{2k+1}$ equal to the antiholomorphic subbundle β in ([1],Proposition 4.5(b)). Then the inverse of this transformation is given by ([1],Proposition 4.5(a)) and is given by forward replacement of the image of the first ∂'-return map.

Furthermore,

(v) for $k = 0, 1, \ldots, l-1$, $\phi^{2k} : S^2 \to \mathbb{H}P^n$ is a quaternionic harmonic map;

(vi) for $k = 0, 1, \ldots, l-1$, $\phi^{2k+1} : S^2 \to G(2, 2n+2)$ satisfies $G''(\phi^{2k+1}) = \mathbf{J}\phi^{2k+1}$;

(vii) for $i = 0, 1, \ldots, 2l$, the isotropy order of ϕ^i is $2l + 2s - i$.

Let $\phi : S^2 \to \mathbb{H}P^n$ be a linearly full irreducible homogeneous harmonic map of finite isotropy order. By Lemma 5.1 we know the construction of harmonic maps is reversible. Then for $i = 0, 1, \ldots, 2l-1$ the corresponding harmonic map ϕ^i is obtained by the direct sum of two line bundles of the harmonic maps in the harmonic sequence of ϕ^{i+1}. They are all homogeneous. Moreover each harmonic map in the harmonic sequence generated by ϕ^i is also homogeneous by Lemma 3.1. In the following we use Lemma 5.1 to determine all irreducible homogeneous harmonic maps ϕ of finite isotropy order.

In (i) of Lemma 5.1 ϕ^0 belongs to the harmonic sequence as follows:

$$0 \xleftarrow{A''_{Jm}} \mathbf{J}\underline{f}_m^{(m)} \xleftarrow{A''_{J(m-1)}} \cdots \xleftarrow{A''_{J1}} \mathbf{J}\underline{f}_1^{(m)} \xleftarrow{A''_{\phi^0}} \underline{\phi}^0 \xrightarrow{A'_{\phi^0}} \underline{f}_1^{(m)} \xrightarrow{A'_1} \cdots \xrightarrow{A'_{m-1}} \underline{f}_m^{(m)} \xrightarrow{A'_m} 0, \quad (5.1)$$

where $\underline{\phi}^0 = \underline{f}_0^{(m)} \oplus \mathbf{J}\underline{f}_0^{(m)}$. In fact we find $m = 2n+1$ ($2l + 2s = 2n$) or $m = n = 2t+1$ ($2l + 2s = n-1$) by Proposition 4.4. At this time the isotropy order of ϕ^0 is $m-1$.

By (iii) of Lemma 5.1 and (5.1), there exists a local section $V_0 = x_0 f_0^{(m)} + \mathbf{J}f_0^{(m)}$ such that $\underline{V}_0$ is an antiholomorphic subbundle of $\underline{\phi}^0$, and ϕ^1 is obtained from ϕ^0 by forward replacement of $\underline{V}_0^{\perp}$, i.e.,

$$\underline{\phi}^1 = \underline{V}_0 \oplus \underline{f}_1^{(m)}, \quad (5.2)$$

where x_0 is a smooth function on S^2 expect at some isolated points, and here $\underline{V}_0$ denotes the line bundle generated by V_0.

Since $\underline{V}_0$ is an antiholomorphic subbundle of $\underline{\phi}^0$, then we get $\pi_{\phi^0}(\partial V_0) \in \underline{V}_0$, which implies the equation

$$\partial x_0 + x_0 \partial \log |f_0^{(2n+1)}|^2 = 0 \quad (5.3)$$

holds.

Then ϕ^1 with isotropy order $m-2$ belongs to the harmonic sequence as follows:

$$0 \xleftarrow{A''_{Jm}} \mathbf{J}\underline{f}_m^{(m)} \xleftarrow{A''_{J(m-1)}} \cdots \xleftarrow{A''_{J2}} \mathbf{J}\underline{f}_2^{(m)} \xleftarrow{A''_{\phi_0^1}} \underline{\phi}_0^1 \xrightarrow{A'_{\phi_0^1}} \underline{\phi}_1^1 \xrightarrow{A'_{\phi_1^1}} \underline{f}_2^{(m)} \xrightarrow{A'_2} \cdots \xrightarrow{A'_{m-1}} \underline{f}_m^{(m)} \xrightarrow{A'_m} 0, \quad (5.4)$$

where $\underline{\phi}_0^1 = \mathbf{J}\underline{\phi}^1$ and $\underline{\phi}_1^1 = \underline{\phi}^1$.

Since ϕ_0^1 is homogeneous, it has constant curvature. Next we compute the geometric quantities of ϕ_0^1 to determine its explicit expression. We choose the following orthogonal unit vectors in $\mathbb{C}^{2n+2}$

$$e_1 = \frac{\mathbf{J}V_0}{|V_0|}, e_2 = \frac{\mathbf{J}f_1^{(m)}}{|f_1^{(m)}|}, e_3 = \frac{V_0}{|V_0|}, e_4 = \frac{f_1^{(m)}}{|f_1^{(m)}|}, e_5 = \frac{f_2^{(m)}}{|f_2^{(m)}|}, e_{-1} = \frac{\mathbf{J}f_2^{(m)}}{|f_2^{(m)}|}, \quad (5.5)$$

and extend them into the local unitary frame with respect to ϕ_0^1.
Set $W_0 = (e_1, e_2)$, $W_1 = (e_3, e_4)$, $W_2 = (e_5)$, $W_{-1} = (e_{-1})$, then by (2.6) we get

$$\Omega_0 = \begin{pmatrix} \lambda_0 & t_0 \\ \mu_0 & 0 \end{pmatrix}, \ \Omega_1 = \begin{pmatrix} 0 & \frac{|f_2^{(m)}|}{|f_1^{(m)}|} \end{pmatrix}, \ \Omega_{-1} = \begin{pmatrix} 0 \\ -\frac{|f_2^{(m)}|}{|f_1^{(m)}|} \end{pmatrix}, \tag{5.6}$$

where $\lambda_0 = \frac{\partial \bar{x}_0 - \bar{x}_0 \partial \log |f_0^{(m)}|^2}{|x_0|^2+1}$, $\mu_0 = t_0 = -\frac{|f_1^{(m)}|}{|V_0|}$ with $|V_0|^2 = (|x_0|^2+1)|f_0^{(m)}|^2$.
A straightforward computation shows

$$|det\Omega_0|^2 \mathrm{d}z^2 \mathrm{d}\bar{z}^2 = \frac{1}{(|x_0|^2+1)^2} l_0^{(m)} l_0^{(m)} \mathrm{d}z^2 \mathrm{d}\bar{z}^2, \tag{5.7}$$

$$L_0 = \lambda_0 \bar{\lambda}_0 + \mu_0 \bar{\mu}_0 + t_0 \bar{t}_0, \tag{5.8}$$

$$L_1 = L_{-1} = l_1^{(m)}. \tag{5.9}$$

We claim that $\frac{1}{(|x_0|^2+1)^2}$ has no zeros on S^2. Otherwise if $z = z_0$ is a zero point of $\frac{1}{(|x_0|^2+1)^2}$, then rank $\Omega_0|_{z=z_0} = 1$ by (5.6), which implies that ϕ_0^1 is not homogeneous. It is a contradiction.
So $|det\Omega_0|^2 \mathrm{d}z^2 \mathrm{d}\bar{z}^2 \neq 0$ everywhere on S^2. It follows from (2.15) that

$$\delta_1 - 2\delta_0 + \delta_{-1} = -4, \tag{5.10}$$

where $\delta_\alpha = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} L_\alpha d\bar{z} \wedge \mathrm{d}z$ $(\alpha = -1, 0, 1)$.
Since all harmonic maps in the harmonic sequence (5.4) have constant curvature, then we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that the induced metric $\mathrm{d}s^2_{\phi_0^1} = (L_0 + L_{-1})\mathrm{d}z\mathrm{d}\bar{z}$ by ϕ_0^1 is given by

$$\mathrm{d}s^2_{\phi_0^1} = \frac{4m-2}{(1+z\bar{z})^2} \mathrm{d}z\mathrm{d}\bar{z}, \tag{5.11}$$

and from ([3], §3) we get

$$l_i^{(m)} = \frac{(i+1)(m-i)}{(1+z\bar{z})^2}, \quad \delta_i^{(m)} = (i+1)(m-i), \quad i = 0, \ldots, m-1. \tag{5.12}$$

It follows from (5.9) and (5.12) that

$$L_{-1} = \frac{\delta_{-1}}{(1+z\bar{z})^2}. \tag{5.13}$$

Analyzing (5.11) and (5.13), we find for $i = -1, 0, 1$,

$$L_i = \frac{\delta_i}{(1+z\bar{z})^2}. \tag{5.14}$$

Using (5.7), (5.9), (5.10), (5.12), (5.14) and (2.14), we obtain

$$\partial\bar{\partial} \log(|x_0|^2 + 1) = 0. \tag{5.15}$$

It follows from (5.7) that $|x_0|^2 + 1$ is globally defined on $\mathbb{C}$ and has a positive constant limit c_1 as $z \to \infty$. Thus from (5.15)

$$|x_0|^2 = c_1 - 1. \tag{5.16}$$

In view of (5.3) we have $\overline{\partial}(\overline{x}_0|f_0^{(m)}|^2)=0$, which implies

$$\overline{x}_0=\frac{h(z)}{|f_0^{(m)}|^2}=\frac{h(z)}{(1+z\overline{z})^m}, \tag{5.17}$$

where $h(z)$ is a holomorphic function on $\mathbb{C}$.
It follows from (5.16) and (5.17) that

$$|h(z)|^2=(c_1-1)(1+z\overline{z})^{2m}. \tag{5.18}$$

Now we claim that $c_1=1$. Otherwise if $c_1\neq 1$, then the holomorphic function $h(z)$ includes the factor such as $(1+z\overline{z})$, which is impossible. So $x_0=0$ and

$$\underline{\phi}^1=\mathbf{J}\underline{f}_0^{(m)}\oplus\underline{f}_1^{(m)}. \tag{5.19}$$

By (iv) of Lemma 5.1, (5.4) and (5.19), there exists a local section $V=y_1f_1^{(m)}+\mathbf{J}f_0^{(m)}$ such that $\underline{\beta}^{\perp}\cap\underline{\phi}^1=\underline{V}$ is an antiholomorphic subbundle of $\underline{\phi}^1$, and ϕ^2 is obtained from ϕ^1 by backward replacement of $\underline{V}$, i.e.,

$$\underline{\phi}^2=\underline{X}\oplus\mathbf{J}\underline{X}, \tag{5.20}$$

where $X=\frac{1}{|f_1^{(m)}|^2}f_1^{(m)}-\frac{\overline{y}_1}{|f_0^{(m)}|^2}\mathbf{J}f_0^{(m)}$ and y_1 is a smooth function on S^2 expect at some isolated points.
By the properties of the harmonic sequence (5.4), we know $\pi_{(\phi^1)^{\perp}}(\overline{\partial}V)=\mathbf{J}\underline{X}$, which implies $y_1=0$ by a straightforward computation. So

$$\underline{\phi}^2=\underline{f}_1^{(m)}\oplus\mathbf{J}\underline{f}_1^{(m)}. \tag{5.21}$$

Then ϕ^2 with isotropy order $m-3$ belongs to the harmonic sequence as follows:

$$0\xleftarrow{A''_{Jm}}\cdots\xleftarrow{A''_{J3}}\mathbf{J}\underline{f}_3^{(m)}\xleftarrow{A''_{\phi^2_{-1}}}\underline{\phi}^2_{-1}\xleftarrow{A''_{\phi^2_0}}\underline{\phi}^2_0\xrightarrow{A'_{\phi^2_0}}\underline{\phi}^2_1\xrightarrow{A'_{\phi^2_1}}\underline{f}_3^{(m)}\xrightarrow{A'_3}\cdots\xrightarrow{A'_m}0, \tag{5.22}$$

where $\underline{\phi}^2_0=\underline{\phi}^2$, $\underline{\phi}^2_1=\mathbf{J}\underline{f}_0^{(m)}\oplus\underline{f}_2^{(m)}$ and $\underline{\phi}^2_{-1}=\mathbf{J}\underline{\phi}^2_1$.
Now we prove

Proposition 5.2 *Let $\phi:S^2\to\mathbb{H}P^n$ be a linearly full irreducible homogeneous harmonic map of finite isotropy order r with constant curvature K. Then up to a symplectic isometry of $\mathbb{H}P^n$, ϕ is one of the following:*

(1) For some $p=1,\ldots,n-1$, $\phi=U\underline{V}_p^{(2n+1)}\oplus U\underline{V}_{2n+1-p}^{(2n+1)}$ with some $U\in G_{2n+2}$, where $K=\frac{2}{2p(2n+1-p)+2n+1}$, $r=2n-2p$;

(2) $n=2t+1$ $(t>0)$ and for some $p=1,\ldots,t-1$, $\phi=\left[\left(\phi_{p,0},\ldots,\phi_{p,n}\right)^T\right]$, for $q_1=0,\ldots,t$, $q_2=t+1,\ldots,n$, ϕ_{p,q_1}, ϕ_{p,q_2} are given by

$$\phi_{p,q_1}=\sqrt{\binom{n}{q_1}}\sum_k(-1)^k\binom{q_1}{p-k}\binom{n-q_1}{k}\left(z^{q_1+k-p}\overline{z}^k+(-1)^{q_1+p}\overline{\lambda}z^{p-k}\overline{z}^{n-q_1-k}j\right),$$

$$\phi_{p,q_2}=\sqrt{1-|\lambda|^2}\sqrt{\binom{n}{q_2}}z^{q_2-p}\sum_k(-1)^k\binom{q_2}{p-k}\binom{n-q_2}{k}(z\overline{z})^k,$$

where λ is a complex parameter satisfying $0<|\lambda|^2<1$ and $K=\frac{2}{2p(n-p)+n}$, $r=n-1-2p$.

Proof In Lemma 5.1 we add the condition of homogeneous. For any $0 \leq k \leq \frac{m-3}{2}$ we prove

$$\underline{\phi}^{2k+1} = \mathbf{J}\underline{f}_k^{(m)} \oplus \underline{f}_{k+1}^{(m)},\ \underline{\phi}^{2k+2} = \underline{f}_{k+1}^{(m)} \oplus \mathbf{J}\underline{f}_{k+1}^{(m)} \tag{5.23}$$

hold by induction on k. If $k = 0$, then

$$\underline{\phi}^{1} = \mathbf{J}\underline{f}_0^{(m)} \oplus \underline{f}_1^{(m)},\ \underline{\phi}^{2} = \underline{f}_1^{(m)} \oplus \mathbf{J}\underline{f}_1^{(m)} \tag{5.24}$$

hold by (5.19) and (5.21).

Assume now that the assertion is correct for $0, \ldots, k-1$. Consider the case of k ($k < \frac{m-3}{2}$). By induction hypotheses we have

$$\underline{\phi}^{2k} = \underline{f}_k^{(m)} \oplus \mathbf{J}\underline{f}_k^{(m)}. \tag{5.25}$$

Then ϕ^{2k} with isotropy order $m-1-2k$ (≥ 4) belongs to the harmonic sequence as follows:

$$0 \xleftarrow{A''_{Jm}} \cdots \xleftarrow{A''_{\phi_{-1}^{2k}}} \underline{\phi}_{-1}^{2k} \xleftarrow{A''_{\phi_0^{2k}}} \underline{\phi}_0^{2k} \xrightarrow{A'_{\phi_0^{2k}}} \underline{\phi}_1^{2k} \xrightarrow{A'_{\phi_1^{2k}}} \cdots \xrightarrow{A'_{\phi_{k-1}^{2k}}} \underline{\phi}_k^{2k} \xrightarrow{A'_{\phi_k^{2k}}} \underline{f}_{-2k+1}^{(m)} \xrightarrow{A'_{2k+1}} \cdots \xrightarrow{A'_m} 0, \tag{5.26}$$

where $\underline{\phi}_0^{2k} = \underline{\phi}^{2k}$, for $\alpha = 1, \ldots, k$, $\underline{\phi}_\alpha^{2k} = \mathbf{J}\underline{f}_{k-\alpha}^{(m)} \oplus \underline{f}_{k+\alpha}^{(m)}$ and $\underline{\phi}_{-\alpha}^{2k} = \mathbf{J}\underline{\phi}_\alpha^{2k}$.

By (iii) of Lemma 5.1 and (5.26), there exists a local section $V_0^{2k} = x_k f_k^{(m)} + \mathbf{J}f_k^{(m)}$ such that $\underline{V}_0^{2k}$ is an antiholomorphic subbundle of $\underline{\phi}^{2k}$, and ϕ^{2k+1} is obtained from ϕ^{2k} by forward replacement of $(\underline{V}_0^{2k})^\perp \cap \underline{\phi}^{2k} = \mathbf{J}\underline{V}_0^{2k}$, i.e.,

$$\underline{\phi}^{2k+1} = \underline{V}_0^{2k} \oplus \underline{V}_1^{2k}, \tag{5.27}$$

where $\underline{V}_1^{2k} = \pi_{(\phi^{2k})^\perp}(\partial\mathbf{J}\underline{V}_0^{2k})$, i.e., $V_1^{2k} = \overline{x}_k l_{k-1}^{(m)}\mathbf{J}f_{k-1}^{(m)} + f_{k+1}^{(m)}$, x_k is a smooth function on S^2 expect at some isolated points, and here $\underline{V}_0^{2k}$ denotes a line bundle consists of V_0^{2k}. Since $\underline{V}_0^{2k}$ is an antiholomorphic subbundle of $\underline{\phi}^{2k}$, then we get $\pi_{\phi^{2k}}(\partial V_0^{2k}) \in \underline{V}_0^{2k}$, which implies the equation

$$\partial x_k + x_k \partial \log |f_k^{(m)}|^2 = 0 \tag{5.28}$$

holds.

In fact ϕ^{2k+1} with isotropy order $m-2-2k$ (≥ 3) belongs to the harmonic sequence as follows:

$$0 \xleftarrow{A''_{Jm}} \cdots \xleftarrow{A''_{J2}} \underline{\phi}_{-1}^{2k+1} \xleftarrow{A''_{\phi_0^1}} \underline{\phi}_0^{2k+1} \xrightarrow{A'_{\phi_0^1}} \underline{\phi}_1^{2k+1} \xrightarrow{A'_2} \cdots \xrightarrow{A'_{\phi_0^1}} \underline{\phi}_{k+1}^{2k+1} \xrightarrow{A'_{\phi_1^1}} \underline{f}_{2k+2}^{(m)} \xrightarrow{A'_2} \cdots \xrightarrow{A'_m} 0, \tag{5.29}$$

where $\underline{\phi}_0^{2k+1} = \mathbf{J}\underline{\phi}^{2k+1}$, $\underline{\phi}_1^{2k+1} = \underline{\phi}^{2k+1}$, and for $\alpha = 1, \ldots, k$, $\underline{\phi}_{\alpha+1}^{2k+1} = \underline{W}_\alpha^{2k} \oplus \underline{V}_{\alpha+1}^{2k}$, $\underline{\phi}_{-\alpha}^{2k+1} = \mathbf{J}\underline{\phi}_{\alpha+1}^{2k+1}$ with $\underline{W}_\alpha^{2k} = (\underline{V}_\alpha^{2k})^\perp \cap \underline{\phi}_\alpha^{2k}$, $\underline{V}_{\alpha+1}^{2k} = \pi_{(\phi_\alpha^{2k})^\perp}(\partial V_\alpha^{2k})$, but $\underline{V}_{k+1}^{2k} = \underline{f}_{2k+1}^{(m)}$. Since ϕ^{2k+1} is homogeneous, it has constant curvature; moreover, all the harmonic maps in the harmonic sequence (5.29) are homogeneous and have constant curvature. Next we compute the geometric quantities of the corresponding harmonic maps to determine their explicit expressions. We choose the following unit vectors in $\mathbb{C}^{2n+2}$

$$\begin{cases} e_1 = \frac{\mathbf{J}V_1^{2k}}{|V_1^{2k}|}, e_2 = \frac{\mathbf{J}V_0^{2k}}{|V_0^{2k}|}, e_3 = \frac{V_0^{2k}}{|V_0^{2k}|}, e_4 = \frac{V_1^{2k}}{|V_1^{2k}|}, \\ e_{2\alpha+3} = \frac{W_\alpha^{2k}}{|W_\alpha^{2k}|}, e_{2\alpha+4} = \frac{V_{\alpha+1}^{2k}}{|V_{\alpha+1}^{2k}|}, \text{for } \alpha = 1, \ldots, k, \\ e_{2k+5} = \frac{f_{2k+1}^{(m)}}{|f_{2k+1}^{(m)}|}. \end{cases} \tag{5.30}$$

Set $W_\alpha = (e_{2\alpha+1}, e_{2\alpha+2})$, for $\alpha = 0, 1, \ldots, k+1$, and $W_{k+2} = e_{2k+5}$, $W_{-1} = \mathbf{J}W_2$. Since for each $\alpha = 0, 1, \ldots, k$ the harmonic map ϕ_α^{2k+1} is irreducible and the isotropy order ≥ 3, then $W_{\alpha-1}, W_\alpha, W_{\alpha+1}$ are orthogonal and it is suitable to extend them into the local unitary frame with respect to ϕ_α^{2k+1}. By (2.6) we get

$$\Omega_0^{2k+1} = \begin{pmatrix} \lambda_0 & t_0 \\ 0 & \mu_0 \end{pmatrix}, \tag{5.31}$$

where $\lambda_0 = \mu_0 = -\frac{|x_k|^2 l_{k-1}^{(m)} |f_k^{(m)}|^2 + |f_{k+1}^{(m)}|^2}{|V_0^{2k}||V_1^{2k}|}$, $t_0 = \frac{\partial \overline{x}_k - \overline{x}_k \partial \log |f_k^{(m)}|^2}{|x_k|^2+1}$ with $|V_0^{2k}|^2 = (|x_k|^2 + 1)|f_k^{(m)}|^2$ and $|V_1^{2k}|^2 = |x_k|^2 l_{k-1}^{(m)} |f_k^{(m)}|^2 + |f_{k+1}^{(m)}|^2$.
A straightforward computation shows

$$|det\Omega_0^{2k+1}|^2 \mathrm{d}z^2 \mathrm{d}\overline{z}^2 = \left(\frac{|x_k|^2 l_{k-1}^{(m)} + l_k^{(m)}}{|x_k|^2 + 1} \right)^2 \mathrm{d}z^2 \mathrm{d}\overline{z}^2, \tag{5.32}$$

$$L_1^{2k+1} = L_{-1}^{2k+1}, \tag{5.33}$$

$$L_{k+1}^{2k+1} = l_{2k+1}^{(m)}. \tag{5.34}$$

For each $\alpha = 0, 1, \ldots, k$ the harmonic map ϕ_α^{2k+1} is irreducible and homogeneous, so $|det\Omega_\alpha|^2 \mathrm{d}z^2 \mathrm{d}\overline{z}^2 \neq 0$ everywhere on S^2. It follows from (2.15) that

$$\delta_{\alpha-1}^{2k+1} - 2\delta_\alpha^{2k+1} + \delta_{\alpha+1}^{2k+1} = -4, \tag{5.35}$$

where $\delta_\alpha^{2k+1} = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} L_\alpha^{2k+1} d\overline{z} \wedge \mathrm{d}z$ $(\alpha = 0, 1, \ldots, k)$.
Since all the harmonic maps in the harmonic sequence (5.29) have constant curvature, then we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ such that for $\alpha = 0, 1, \ldots, k+1$ the induced metric $\mathrm{d}s^2_{\phi_\alpha^{2k+1}} = (L_{\alpha-1}^{2k+1} + L_\alpha^{2k+1}) \mathrm{d}z \mathrm{d}\overline{z}$ by ϕ_α^{2k+1} is given by

$$\mathrm{d}s^2_{\phi_\alpha^{2k+1}} = \frac{\delta_{\alpha-1}^{2k+1} + \delta_\alpha^{2k+1}}{(1 + z\overline{z})^2} \mathrm{d}z \mathrm{d}\overline{z}, \tag{5.36}$$

and from ([3], §3) we get

$$l_i^{(m)} = \frac{(i+1)(m-i)}{(1+z\overline{z})^2}, \quad \delta_i^{(m)} = (i+1)(m-i), \quad i = 0, \ldots, m-1. \tag{5.37}$$

It follows from (5.34) and (5.37) that

$$L_{k+1}^{2k+1} = \frac{\delta_{k+1}^{2k+1}}{(1+z\overline{z})^2}. \tag{5.38}$$

Analyzing (5.35), (5.36) and (5.38), we find for $\alpha = -1, 0, 1, \ldots, k$,

$$L_\alpha^{2k+1} = \frac{\delta_\alpha^{2k+1}}{(1+z\overline{z})^2}. \tag{5.39}$$

Using (5.32), (5.33), (5.35), (5.37), (5.39)and (2.14), we obtain

$$\partial\overline{\partial} \log \left[\frac{k(m-k+1)|x_k|^2 + (k+1)(m-k)}{|x_k|^2 + 1} \right] = 0. \tag{5.40}$$

Since $\frac{k(m-k+1)|x_k|^2+(k+1)(m-k)}{|x_k|^2+1}$ is globally defined on $\mathbb{C}$ and has a positive constant limit c_1 as $z \to \infty$. Thus from (5.40)

$$\frac{k(m-k+1)|x_k|^2+(k+1)(m-k)}{|x_k|^2+1} = c_1. \tag{5.41}$$

It follows from $k(m-k+1) \neq (k+1)(m-k)$ that

$$|x_k|^2 = c, \tag{5.42}$$

where c is a constant.
In view of (5.28) we have $\overline{\partial}(\overline{x}_k |f_k^{(m)}|^2) = 0$, which implies

$$\overline{x}_k = \frac{h_k(z)}{|f_k^{(m)}|^2} = \frac{h_k(z)}{(1+z\overline{z})^{m-2k}}, \tag{5.43}$$

where $h_k(z)$ is a holomorphic function on $\mathbb{C}$.
It follows from (5.42) and (5.43) that

$$|h_k(z)|^2 = c(1+z\overline{z})^{2m-4k}. \tag{5.44}$$

Now we claim that $c = 0$. Otherwise if $c \neq 0$, then the holomorphic function $h_k(z)$ includes the factor such as $(1+z\overline{z})$, which is impossible. So $x_k = 0$ and

$$\underline{\phi}^{2k+1} = \mathbf{J}\underline{f}_k^{(m)} \oplus \underline{f}_{k+1}^{(m)}. \tag{5.45}$$

By (iv) of Lemma 5.1, (5.29) and (5.45), there exists a local section $V^{2k+1} = y_{k+1}f_{k+1}^{(m)} + \mathbf{J}f_k^{(m)}$ such that $\underline{\beta}^{\perp} \cap \underline{\phi}^{2k+1} = \underline{V}^{2k+1}$ is an antiholomorphic subbundle of $\underline{\phi}^{2k+1}$, and ϕ^{2k+2} is obtained from ϕ^{2k+1} by backward replacement of $\underline{V}^{2k+1}$, i.e.,

$$\underline{\phi}^{2k+2} = \underline{X}^{2k+1} \oplus \mathbf{J}\underline{X}^{2k+1}, \tag{5.46}$$

where $X^{2k+1} = \frac{1}{|f_{k+1}^{(m)}|^2} f_{k+1}^{(m)} - \frac{\overline{y}_{k+1}}{|f_k^{(m)}|^2}\mathbf{J}f_k^{(m)}$ and y_{k+1} is a smooth function on S^2 expect at some isolated points.
By the properties of the harmonic sequence (5.29) we know $\pi_{(\phi^{2k+1})^{\perp}}(\overline{\partial}V^{2k+1}) = \mathbf{J}\underline{X}^{2k+1}$, which implies $y_{k+1} = 0$ by a straightforward computation. So

$$\underline{\phi}^{2k+2} = \underline{f}_{k+1}^{(m)} \oplus \mathbf{J}\underline{f}_{k+1}^{(m)}, \tag{5.47}$$

and the isotropy order of ϕ^{2k+2} is $m-3-2k$ (≥ 2).

Hence it verifies (5.23) by (5.45) and (5.47).

In fact in the harmonic sequence (5.1), ϕ^0 only belongs to case (1) or (3) of Proposition 4.4 by Proposition 4.4. Thus the conclusion follows from (5.23). □

At last we consider the case of strongly isotropic. Let $\phi : S^2 \to \mathbb{H}P^n$ be a linearly full irreducible strongly isotropic harmonic map, then ϕ must belong to the following harmonic sequence:

$$0 \longleftarrow \cdots \xleftarrow{A''_{\phi^0_{-\alpha}}} \underline{\phi}^0_{-\alpha} \xleftarrow{A''_{\phi^0_{-(\alpha-1)}}} \cdots \xleftarrow{A''_{\phi^0_0}} \underline{\phi}^0_0 = \underline{\phi} \xrightarrow{A'_{\phi^0_0}} \cdots \xrightarrow{A'_{\phi^0_{\alpha-1}}} \underline{\phi}^0_{\alpha} \xrightarrow{A'_{\phi^0_{\alpha}}} \underline{\phi}^0_{\alpha+1} \xrightarrow{A'_{\phi^0_{\alpha+1}}} \cdots \longrightarrow 0, \tag{5.48}$$

where for $\beta = 1, \ldots, \underline{\phi}^0_{-\beta} = \mathbf{J}\underline{\phi}^0_{\beta}$ and for $\beta = 1, \ldots, \alpha$, rank $\underline{\phi}^0_{\beta} = 2$; for $\beta = \alpha+1, \ldots,$ rank $\underline{\phi}^0_{\beta} = 0$ or 1.
We give a corresponding Lemma as follows.

Lemma 5.3 *Let $\phi : S^2 \to \mathbb{H}P^n$ be a linearly full irreducible strongly isotropic harmonic map, then from ϕ by $2\alpha-1$ steps of proper forward or backward replacements, we get the harmonic map $\phi^{2\alpha-1} : S^2 \to G(2, 2n+2)$, which is strongly isotropic and belongs to the following harmonic sequence:*

$$0 \xleftarrow{A''_{Jm}} \cdots \xleftarrow{A''_{Js}} \mathbf{J}\underline{f}_s^{(m)} \xleftarrow{A''_{\phi_{-1}^{2\alpha-1}}} \underline{\phi}_{-1}^{2\alpha-1} \xleftarrow{A''_{\phi_0^{2\alpha-1}}} \underline{\phi}_0^{2\alpha-1} \xrightarrow{A'_{\phi_0^{2\alpha-1}}} \underline{f}_s^{(m)} \xrightarrow{A'_s} \cdots \xrightarrow{A'_{m-1}} \underline{f}_m^{(m)} \xrightarrow{A'_m} 0, \tag{5.49}$$

where $\underline{\phi}_0^{2\alpha-1} = \underline{\phi}^{2\alpha-1}$, $\underline{\phi}_{-1}^{2\alpha-1} = \mathbf{J}\underline{\phi}^{2\alpha-1}$ with $f_i^{(m)}, \mathbf{J}f_i^{(m)} : S^2 \to \mathbb{C}P^m \subseteq \mathbb{C}P^{2n+1}$ and A''_{Ji} denotes $A''_{\mathbf{J}f_i^{(m)}}$ $(i = 0, \ldots, m)$.

Furthermore, let $\underline{\beta}$ be the holomorphic line subbundle of $\underline{\phi}^{2\alpha-1}$ defined by $\underline{Ker}\, A'_{\phi^{2\alpha-1}}$. Let $\phi^{2\alpha}$ be obtained from $\phi^{2\alpha-1}$ by backward replacement of $\underline{\alpha} = \underline{\beta}^{\perp} \cap \underline{\phi}^{2\alpha-1}$. Then $\phi^{2\alpha}$ is quaternionic. Moreover, $\phi^{2\alpha}$ is a quaternionic mixed pair or a quaternionic Frenet pair.

Proof In the harmonic sequence (5.48), if rank $\underline{\phi}_{\alpha+1}^0 = 1$, let $\underline{\gamma}$ be the holomorphic line subbundle of $\underline{\phi}_{\alpha}^0$ defined by $\underline{Ker}\, A'_{\phi_\alpha^0}$, then $\phi_{\alpha-1}^1$ obtained from ϕ_α^0 by backward replacement of $\underline{\gamma}^{\perp} \cap \underline{\phi}_{\alpha}^0$ belongs to the following harmonic sequences

$$0 \longleftarrow \cdots \xleftarrow{A''_{\phi_{-\alpha}^1}} \underline{\phi}_{-\alpha}^1 \xleftarrow{A''_{\phi_{-(\alpha-1)}^1}} \cdots \xleftarrow{A''_{\phi_0^1}} \underline{\phi}_0^1 \xrightarrow{A'_{\phi_0^1}} \cdots \xrightarrow{A'_{\phi_{\alpha-2}^1}} \underline{\phi}_{\alpha-1}^1 \xrightarrow{A'_{\phi_{\alpha-1}^1}} \underline{\phi}_{\alpha}^1 \xrightarrow{A'_{\phi_\alpha^1}} \cdots \longrightarrow 0, \tag{5.50}$$

where for $\beta = 1, \ldots, \underline{\phi}_{-\beta}^1 = \mathbf{J}\underline{\phi}_{\beta-1}^1$ and for $\beta = 0, \ldots, \alpha-1$, rank $\underline{\phi}_{\beta}^1 = 2$; for $\beta = \alpha, \ldots,$ rank $\underline{\phi}_{\beta}^1 = 1$.

Reusing the above methods we will get the wanted harmonic sequence (5.49) from (5.50) by $2\alpha-2$ steps of backward replacements.

If rank $\underline{\phi}_{\alpha+1}^0 = 0$, let $\underline{\gamma}$ be any holomorphic line subbundle of $\underline{\phi}_0^0$, then $\underline{\phi}_0^0 = \underline{\gamma} \oplus \mathbf{J}\underline{\gamma}$; using ([1], Lemma 4.1) we get a new harmonic sequence which is just (5.50). Similarly we can get (5.49).

The rest follows from ([8], Lemma 3.3). □

Let $\phi : S^2 \to \mathbb{H}P^n$ be an irreducible strongly isotropic homogeneous harmonic map, then from Lemma 5.3 we know the corresponding harmonic map $\phi^{2\alpha}$ is also homogeneous and strongly isotropic. Through Proposition 4.2 and 4.4 we get $\phi^{2\alpha}$ is given by

$$\underline{\phi}^{2\alpha} = \underline{f}_n^{(2n+1)} \oplus \underline{f}_{n+1}^{(2n+1)}, \text{ or } \underline{\phi}^{2\alpha} = \underline{f}_0^{(n)} \oplus \mathbf{J}\underline{f}_0^{(n)}, \tag{5.51}$$

where $f_n^{(2n+1)} = UV_n^{(2n+1)}$ and $f_0^{(n)} = UV_0^{(n)}$ with $U \in G_{2n+2}$.

In the following we discuss the above two cases, respectively, to prove the following Proposition.

Proposition 5.4 *Let $\phi : S^2 \to \mathbb{H}P^n$ be a linearly full irreducible strongly isotropic homogeneous harmonic map with constant curvature K. Then up to a symplectic isometry of $\mathbb{H}P^n$, for some $p = 1, \ldots, \left[\frac{n}{2}\right]$, ϕ is given by*

$$\underline{\phi} = \underline{V}_p^{(n)}$$

with $K = \frac{2}{2p(n-p)+n}$. Here $\left[\frac{n}{2}\right]$ denotes the maximal integer which is not more than $\frac{n}{2}$.

Proof **Case I.** $\underline{\phi}^{2\alpha} = \underline{f}_n^{(2n+1)} \oplus \underline{f}_{n+1}^{(2n+1)}$. Obviously $\phi^{2\alpha}$ belongs to the harmonic sequence (4.4). In (5.49) we choose a local section $V = x_n f_n^{(2n+1)} + f_{n+1}^{(2n+1)}$ such that

$$\underline{\phi}^{2\alpha-1} = \underline{V} \oplus \underline{f}_{n+2}^{(2n+1)}, \tag{5.52}$$

where x_n is a smooth function on S^2 expect at some isolated points and $x_n \neq 0$.
This time $\phi^{2\alpha-1}$ belongs to the harmonic sequence as follows:

$$0 \xleftarrow{A''_m} \cdots \xleftarrow{A''_{n-2}} \underline{f}_{n-2}^{(2n+1)} \xleftarrow{A''_{\phi_0^{2\alpha-1}}} \underline{\phi}_0^{2\alpha-1} \xrightarrow{A'_{\phi_0^{2\alpha-1}}} \underline{\phi}_1^{2\alpha-1} \xrightarrow{A'_{\phi_1^{2\alpha-1}}} \underline{f}_{n+3}^{(2n+1)} \xrightarrow{A'_{n+3}} \cdots \xrightarrow{A'_{2n+1}} 0, \tag{5.53}$$

where $\underline{\phi}_0^{2\alpha-1} = \mathbf{J}\underline{\phi}^{2\alpha-1}$ and $\underline{\phi}_1^{2\alpha-1} = \underline{\phi}^{2\alpha-1}$.
Since $A'_{\phi^{2\alpha-1}}(\underline{\phi}^{2\alpha-1}) = \underline{f}_{n+3}^{(2n+1)}$, then we get by (5.52)

$$\pi_{(\phi^{2\alpha-1})^\perp}(\partial V) = \underline{f}_{n+3}^{(2n+1)}, \tag{5.54}$$

which is equivalent to the equation

$$\partial x_n - x_n^2 + x_n \partial \log l_n^{(2n+1)} = 0, \tag{5.55}$$

hold. From (5.53) we choose a local unitary frame with respect to $\phi_0^{2\alpha-1}$ in $\mathbb{C}^{2n+2}$ as follows:

$$\begin{cases} e_1 = \frac{\mathbf{J}V}{|V|}, \quad e_2 = \frac{\mathbf{J}f_{n+2}^{(2n+1)}}{|f_{n+2}^{(2n+1)}|}, \quad e_3 = \frac{V}{|V|}, \quad e_4 = \frac{f_{n+2}^{(2n+1)}}{|f_{n+2}^{(2n+1)}|}, \\ e_{2\alpha-1} = \frac{f_{n+\alpha}^{(2n+1)}}{|f_{n+\alpha}^{(2n+1)}|}, e_{2\alpha} = \frac{f_{n+1-\alpha}^{(2n+1)}}{|f_{n+1-\alpha}^{(2n+1)}|}, \text{for } \alpha = 3, \ldots, n+1. \end{cases} \tag{5.56}$$

Set $W_0 = (e_1, e_2)$, $W_1 = (e_3, e_4)$, $W_2 = (e_5)$, $W_{-1} = (e_6)$, then by (2.6) we get

$$\Omega_0 = \begin{pmatrix} \lambda_0 & t_0 \\ \mu_0 & 0 \end{pmatrix}, \ \Omega_1 = \begin{pmatrix} 0 & \frac{|f_{n+3}^{(2n+1)}|}{|f_{n+2}^{(2n+1)}|} \end{pmatrix}, \ \Omega_{-1} = \begin{pmatrix} 0 \\ (-1)^{n-1}\frac{|f_{n-1}^{(2n+1)}|}{|f_{n-2}^{(2n+1)}|} \end{pmatrix}, \tag{5.57}$$

where $\lambda_0 = \frac{\partial \overline{x}_n - l_n^{(2n+1)}\tau_{n+1}|f_{n+1}^{(2n+1)}|^2}{|V|^2}$, $t_0 = -\frac{\overline{x}_n l_{n+1}^{(2n+1)}\tau_n |f_n^{(2n+1)}|^2}{|f_{n+2}^{(2n+1)}||V|}$, $\mu_0 = -\frac{\overline{x}_n l_{n-1}^{(2n+1)}\tau_{n+2}|f_{n+2}^{(2n+1)}|}{|V|}$
with $|V|^2 = (|x_n|^2 + l_n^{(2n+1)})|f_n^{(2n+1)}|^2$ and τ_i $(i = n, n+1, n+2)$ is given by (4.3).
A straightforward computation shows

$$|det\Omega_0|^2 \mathrm{d}z^2 \mathrm{d}\overline{z}^2 = \frac{|x_n|^4}{(|x_n|^2 + l_n^{(2n+1)})^2} l_{n+1}^{(2n+1)} l_{n-1}^{(2n+1)} \mathrm{d}z^2 \mathrm{d}\overline{z}^2, \tag{5.58}$$

$$L_1 = L_{-1} = l_{n+2}^{(2n+1)}. \tag{5.59}$$

Since $\phi_0^{2\alpha-1}$ is irreducible and homogeneous, so $|det\Omega_0|^2 \mathrm{d}z^2 \mathrm{d}\overline{z}^2 \neq 0$ everywhere on S^2. It follows from (2.15) that

$$\delta_{-1} - 2\delta_0 + \delta_1 = -4, \tag{5.60}$$

where $\delta_i = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} L_i d\overline{z} \wedge \mathrm{d}z$ $(i = -1, 0, 1)$.
Since all the harmonic maps in the harmonic sequence (5.53) have constant curvature, then we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ such that the induced metric $\mathrm{d}s^2_{\phi_0^{2\alpha-1}} = (L_{-1} + L_0)\mathrm{d}z\mathrm{d}\overline{z}$ by $\phi_0^{2\alpha-1}$ is given by

$$\mathrm{d}s^2_{\phi_0^{2\alpha-1}} = \frac{\delta_{-1} + \delta_0}{(1 + z\overline{z})^2} \mathrm{d}z\mathrm{d}\overline{z}, \tag{5.61}$$

and from ([3], §3) we get

$$l_i^{(2n+1)}=\frac{(i+1)(2n+1-i)}{(1+z\bar{z})^2},\quad \delta_i^{(2n+1)}=(i+1)(2n+1-i),\quad i=0,\ldots,2n. \tag{5.62}$$

It follows from (5.59) and (5.62) that

$$L_{-1}=\frac{\delta_{-1}}{(1+z\bar{z})^2}. \tag{5.63}$$

Analyzing (5.61) and (5.63) we find for $i=-1,0,1$,

$$L_i=\frac{\delta_i}{(1+z\bar{z})^2}. \tag{5.64}$$

Using (5.58), (5.59), (5.60), (5.62), (5.64) and (2.14), we obtain

$$\partial\bar{\partial}\log\left[\frac{|x_n|^2}{|x_n|^2+l_n^{(2n+1)}}\right]=0. \tag{5.65}$$

Since $\frac{|x_n|^2}{|x_n|^2+l_n^{(2n+1)}}$ is globally defined on $\mathbb{C}$ and has a positive constant limit c_1 as $z\to\infty$. Thus from (5.65)

$$\frac{|x_n|^2}{|x_n|^2+l_n^{(2n+1)}}=c_1. \tag{5.66}$$

It follows from (5.62) and (5.66) that

$$x_n=\frac{c}{1+z\bar{z}}, \tag{5.67}$$

where c is a constant.
In view of (5.55) and (5.67) we have $c=0$. Then $x_n=0$, which contradicts the fact that $x_n\neq 0$. So this case does not occur.

Case II. $\underline{\phi}^{2\alpha}=\underline{f}_0^{(n)}\oplus\mathbf{J}\underline{f}_0^{(n)}$. It follows from the proof of Proposition 5.2 that for $k=1,\ldots,n-1$, $\underline{\phi}^{2\alpha-2k}$ is given by

$$\underline{\phi}^{2\alpha-2k}=\underline{f}_k^{(n)}\oplus\mathbf{J}\underline{f}_k^{(n)}, \tag{5.68}$$

where $f_k^{(n)}=UV_k^{(n)}$ with $U\in G_{2n+2}$ and the corresponding constant curvature $K=\frac{2}{2k(n-k)+n}$.
Hence we get the conclusion. □

By Proposition 4.2, 4.4, 5.2 and 5.4, we obtain a classification of homogeneous minimal 2-spheres in quaternionic projective space $\mathbb{H}P^n$ as follows:

Theorem 5.5 *Let $\phi:S^2\to\mathbb{H}P^n$ be a linearly full homogeneous harmonic map of isotropy order r with constant curvature K. Then up to a symplectic isometry of $\mathbb{H}P^n$, ϕ is one of the following:*

(1) For some $p=0,1,\ldots,n$, ϕ is given by

$$\underline{\phi}=U\underline{V}_p^{(2n+1)}\oplus U\underline{V}_{2n+1-p}^{(2n+1)},\ U\in G_{2n+2}$$

with $K=\frac{2}{2p(2n+1-p)+2n+1}$ and $r=2n-2p$ $\left(K=\frac{2}{n(n+2)},\ r=+\infty \text{ when } p=n\right)$;

(2) *For some* $p = 0, 1, \ldots, \left[\frac{n}{2}\right]$, ϕ *is given by*

$$\underline{\phi} = \underline{V}_p^{(n)}$$

with $K = \frac{2}{2p(n-p)+n}$ *and* $r = +\infty$;

(3) $n = 2t + 1$ $(t > 0)$ *and for some* $p = 0, 1, \ldots, t-1$, $\phi = \left[\left(\phi_{p,0}, \ldots, \phi_{p,n}\right)^T\right]$, *for* $q_1 = 0, \ldots, t$, $q_2 = t+1, \ldots, n$, ϕ_{p,q_1}, ϕ_{p,q_2} *are given by*

$$\phi_{p,q_1} = \sqrt{\binom{n}{q_1}} \sum_k (-1)^k \binom{q_1}{p-k} \binom{n-q_1}{k} \left(z^{q_1+k-p}\overline{z}^k + (-1)^{q_1+p}\overline{\lambda} z^{p-k}\overline{z}^{n-q_1-k} j\right),$$

$$\phi_{p,q_2} = \sqrt{1-|\lambda|^2}\sqrt{\binom{n}{q_2}} z^{q_2-p} \sum_k (-1)^k \binom{q_2}{p-k} \binom{n-q_2}{k} (z\overline{z})^k,$$

where λ *is a complex parameter satisfying* $0 < |\lambda|^2 < 1$ *and* $K = \frac{2}{2p(n-p)+n}$, $r = n - 1 - 2p$.

Remark 5.6 (a) Case (1) is just the series of $SU(2)$-equivariant minimal 2-spheres in $\mathbb{H}P^n$ given by ([13], Proposition 7.1).

(b) Case (2) is contained in totally geodesic submanifold $\mathbb{C}P^n \subset \mathbb{H}P^n$. In this case, since there exists a $Sp(n)$ matrix which transforms $\underline{V}_p^{(n)}$ into $\underline{V}_{n-p}^{(n)}$, then $\underline{V}_p^{(n)}$ and $\underline{V}_{n-p}^{(n)}$ are congruent in $\mathbb{H}P^n$ (but they are not congruent in $\mathbb{C}P^n$). So here we omit the cases of $p > \left[\frac{n}{2}\right]$.

(c) Case (3) is not totally geodesic and exists only when n is odd. In this case, for some $p = t, \ldots, 2t\ (= n-1)$, ϕ is given by

$$\underline{\phi} = \underline{f}_p^{(n)} \oplus \mathbf{J}\underline{f}_p^{(n)},$$

where $f_p^{(n)} = UV_p^{(n)}$ and U is given by (4.26)–(4.29).
A straightforward computation shows that these ϕ are homogeneous but non-minimal in $\mathbb{H}P^n$. So we obtain the conclusion that a homogeneous 2-sphere in $\mathbb{H}P^n$ may be non-minimal.

Theorem 5.5 gives all homogeneous minimal 2-spheres in $\mathbb{H}P^n$ and shows that they contain those given by ([13], Proposition 7.1.), even more than those in the case of n is odd. Of course a natural problem that how to decide all homogeneous (non-minimal) 2-sphere in $\mathbb{H}P^n$ deserves further consideration.

Acknowledgements This project is supported by the NSFC (Nos. 11501548, 11401481, 11471308, 11626250) and RDF Program of XJTLU (Grant No. RDF13-01-14). The authors thank the help of Professor Xiaoxiang Jiao and the referee's comments.

References

1. Bahy-El-Dien, A., Wood, J.C.: The explicit construction of all harmonic two-spheres in quaternionic projective spaces. Proc. Lond. Math. Soc. **62**, 202–224 (1991)
2. Bando, S., Ohnita, Y.: Minimal 2-spheres with constant curvature in $\mathbb{C}P^n$. J. Math. Soc. Jpn. **39**(3), 477–487 (1987)
3. Bolton, J., Jensen, G.R., Rigoli, M., Woodward, L.M.: On conformal minimal immersion of S^2 into $\mathbb{C}P^N$. Math. Ann. **279**, 599–620 (1988)

4. Burstall, F.E., Wood, J.C.: The construction of harmonic maps into complex Grassmannians. J. Differ. Geom. **23**, 255–297 (1986)
5. Chern, S.S., Wolfson, J.: Harmonic maps of the two-sphere into a complex Grassmann manifold II. Ann. Math. **125**, 301–335 (1987)
6. Eells, J., Wood, J.C.: Harmonic maps of surfaces to complex projective spaces. Adv. Math. **49**, 217–263 (1983)
7. Fei, J., Jiao, X.X., Xiao, L., Xu, X.W.: On the classification of homogeneous 2-spheres in complex Grassmannians. Osaka J. Math. **50**, 135–152 (2013)
8. He, L., Jiao, X.X.: Classification of conformal minimal immersions of constant curvature from S^2 to $\mathbb{H}P^2$. Math. Ann. **359**, 663–694 (2014)
9. He, L., Jiao, X.X.: On conformal minimal immersions of constant curvature from S^2 to $\mathbb{H}P^n$. Math. Z. **280**, 851–871 (2015)
10. He, L., Jiao, X.X.: On conformal minimal immersions of S^2 in $\mathbb{H}P^n$ with parallel second fundamental form. Ann. Mat. Pur. Appl. **194**, 1301–1317 (2015)
11. Jiao, X.X.: Pseudo-holomorphic curves of constant curvature in complex Grassmannians. Isr. J. Math. **163**(1), 45–63 (2008)
12. Jiao, X.X.: On minimal two-spheres immersed in complex Grassmann manifolds with parallel second fundamental form. Monatsh. Math. **168**, 381–401 (2012)
13. Ohnita, Y.: Homogeneous harmonic maps into complex projective spaces. Tokyo J. Math. **13**(1), 87–116 (1990)
14. Peng, C., Xu, X.: Classification of minimal homogeneous two-spheres in the complex Grassmann manifold G(2, n). J. Math. Pures Appl. **103**(2), 374–399 (2015)