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Characterizing spheres and Euclidean spaces by conformal vector fields

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Abstract It is well known that the Euclidean space $(R^n, \langle , \rangle)$, the n-sphere $S^n(c)$ of constant curvature c are examples of spaces admitting many conformal vector fields, and therefore conformal vector fields are used in obtaining characterizations of these spaces. In this paper, we use nontrivial conformal vector fields on a compact and connected Riemannian manifold to characterize the sphere $S^n(c)$. Also, we use a nontrivial conformal vector field on a complete and connected Riemannian manifold and find characterizations for a Euclidean space $(R^n, \langle , \rangle)$ and the sphere $S^n(c)$.

Keywords Conformal vector fields · Ricci curvature · Scalar curvature · Obata's theorem · Laplace operator

Mathematics Subject Classification 53C21 · 58J05 · 53C42 · 53A30

1 Introduction

Characterizations of spaces, the Euclidean space R^n, the Euclidean sphere S^n, and the complex projective space CP^n, are important topics in differential geometry and are considered by several authors (cf. [2–17,20,21]). In most of these characterizations, generally a conformal vector field plays an important role. Conformal vector fields are not only important in obtaining characterizations of spaces but also have an important role in the general theory of Relativity as well as in Mechanics.

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Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if its flow consists of conformal transformations or equivalently if there exists a smooth function f on M (called the potential function of the conformal vector field ξ) that satisfies $\pounds_\xi g = 2fg$, where $\pounds_\xi g$ is the Lie derivative of g with respect ξ. We say that ξ is a nontrivial conformal vector field if it is a non-Killing vector field (ξ is a Killing vector field if the potential function $f = 0$ or equivalently the flow of ξ consists of isometries of the Riemannian manifold). If in addition ξ is a closed vector field (or is a gradient of a smooth function), then ξ is said to be a closed conformal vector field (or a gradient conformal vector field). If ξ is a gradient conformal vector field with $\xi = \nabla\rho$ for a smooth function ρ on the Riemannian manifold (M, g), then we get the Poisson equation $\Delta\rho = nf$, thus the geometry of gradient conformal vector field on a Riemannian manifold is related to the Poisson equation on the Riemannian manifold (M, g). The role of differential equations in studying the geometry of a Riemannian manifold was initiated by the work of Obata (cf. [14–16]). A highly celebrated result of Obata is related to characterizing sphere $S^n(c)$ by second-order differential equations, and this result, consisting in a necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the n-sphere $S^n(c)$, is that there exists a non-constant smooth function f on M that satisfies the differential equation $H_f = -cfg$, where H_f is the Hessian of the smooth function f and c is a positive constant. Similarly, Tashiro [21] has shown that the Euclidean spaces R^n are characterized by the differential equation $H_f = cg$, (see also [18]).

There are many gradient conformal vector fields on the n-dimensional sphere $S^n(c)$ (cf. [1, 4,12–15]). If N is the unit normal vector field on $S^n(c)$, in the Euclidean space R^{n+1} with Euclidean metric $\langle , \rangle$, then for any nonzero constant vector field Z on the Euclidean space R^{n+1} its restriction to $S^n(c)$ can be expressed as $Z = \xi + fN$, where $f = \langle Z, N\rangle$ is a smooth function and ξ is a vector field on $S^n(c)$. Then it is straightforward to show that ξ is a gradient conformal vector field on $S^n(c)$ with potential function $-\sqrt{c}f$. This conformal vector field satisfies $\Delta\xi = -c\xi$. The Euclidean space $(R^n, \langle , \rangle)$ provides many examples of conformal vector fields, a trivial example being the position vector field ξ, which is a gradient conformal vector field. On a complex Euclidean space $(C^n, \langle , \rangle)$ (Euclidean complex space form) with standard complex structure J, the vector field $\xi = \psi + J\psi$, where ψ is the position vector field, is a conformal vector field that is not closed. Similarly, on the Euclidean space $(R^n, \langle , \rangle)$ with Euclidean coordinates $x^1, \ldots, x^n$, the vector field

$$\xi = \psi - \left\langle \psi, \frac{\partial}{\partial x^i} \right\rangle \frac{\partial}{\partial x^j} + \left\langle \psi, \frac{\partial}{\partial x^j} \right\rangle \frac{\partial}{\partial x^i},$$

where i, j are two fixed indices with $i \neq j$, is a conformal vector field that is, not closed. We note that this conformal vector field ξ on the Euclidean space $(R^n, \langle , \rangle)$ is being introduced for the first time. All these conformal vector fields on the Euclidean spaces satisfy $\Delta\xi = 0$.

Note that the scalar curvature of the Riemannian manifold being constant (or the manifold is an Einstein manifold) gives a good combination with the presence of a conformal vector field in studying the geometry of a Riemannian manifold, in particular, in getting the characterizations of spheres using conformal vector fields. However, if the scalar curvature of the Riemannian manifold is not a constant, then one faces difficulties and we do not find many results in the literature studying the geometry of Riemannian manifolds of non-constant scalar curvature admitting a conformal vector field.

The n-sphere $S^n(c)$ admits a conformal vector field that is also an eigenvector of the Laplace operator Δ with eigenvalue c; this raises the question "Is a compact and connected Riemannian manifold (M, g), which admits a nontrivial conformal vector field ξ satisfying $\Delta\xi = -\lambda\xi$ for a constant $\lambda > 0$ together with some restrictions on its Ricci curvatures, necessarily isometric to a n-sphere?" In this paper, we answer this question by proving that if the Ricci curvatures of a compact and connected n-dimensional Riemannian manifold (M, g) lie in the interval $[\lambda, n^{-1}(n-2)\lambda_1 + \lambda]$, where $\lambda > 0$ is a constant and λ_1 being the first nonzero eigenvalue of the Laplace operator Δ acting on smooth functions on M, and it admits a nontrivial conformal vector field ξ satisfying $\Delta\xi = -\lambda\xi$ and the scalar curvature S is constant along the integral curves of ξ, then M is isometric to $S^n(\lambda)$ (cf. Theorem 3.1).

Given a conformal vector field ξ on a Riemannian manifold (M, g) that is not closed, the differential of smooth 1-form η dual to ξ defines a skew symmetric $(1, 1)$ tensor field φ on M, we say the conformal vector field ξ is a null conformal vector field if ξ annihilates φ. We use a nontrivial null conformal vector field on a complete and connected Riemannian manifold (M, g) with some restrictions on the scalar curvature to find another characterization of the sphere $S^n(c)$ as well as of the Euclidean space $(R^n, \langle, \rangle)$ (cf. Theorem 3.2).

2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. Using Koszul's formula (cf. [1,3,17]), we immediately obtain the following for a vector field ξ on M

$$2g(\nabla_X\xi, Y) = \left(\pounds_\xi g\right)(X, Y) + \mathrm{d}\eta(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

where η is the 1-form dual to ξ that is, $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Define a skew symmetric tensor field φ of type $(1, 1)$ on M by

$$\mathrm{d}\eta(X, Y) = 2g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then for a conformal vector field ξ on a Riemannian manifold with potential function f, using above equation, we get

$$\nabla_X\xi = fX + \varphi X, \quad X \in \mathfrak{X}(M). \tag{2.1}$$

For a conformal vector field ξ, the skew symmetric tensor field φ in the above equation is called the *associate tensor field* of the conformal vector field ξ.

We shall denote by Δ the Laplace operator acting on smooth functions on M and by λ_1 the first nonzero eigenvalue of the Laplace operator Δ. For a smooth function h on the Riemannian manifold (M, g), we denote by ∇h the gradient of h and by A_h the Hessian operator $A_h\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $A_h(X) = \nabla_X\nabla h$ and the Hessian H_h by $H_h(X, Y) = g(A_hX, Y)$, $X, Y \in \mathfrak{X}(M)$. On an n-dimensional compact Riemannian manifold (M, g) that admits a conformal vector field ξ, using the skew symmetry of the tensor field φ and Eq. (2.1), we get $\operatorname{div}\xi = nf$ and consequently, we have

$$\int_M f = 0. \tag{2.2}$$

The above equation implies

$$\int_M \|\nabla f\|^2 \geq \lambda_1 \int_M f^2, \tag{2.3}$$

with equality holding if and only if $\Delta f = -\lambda_1 f$. Note that the smooth 2-form given by $g(\varphi X, Y)$ is closed and therefore, we have

$$g\left((\nabla\varphi)(X, Y), Z\right) + g\left((\nabla\varphi)(Y, Z), X\right) + g\left((\nabla\varphi)(Z, X), Y\right) = 0, \tag{2.4}$$

where covariant derivative $(\nabla\varphi)(X,Y)=\nabla_X\varphi Y-\varphi(\nabla_X Y),\ X,Y\in\mathfrak{X}(M)$. Moreover, if we compute the curvature tensor field $R(X,Y)\xi$ using Eq. (2.1), then we get

$$R(X,Y)\xi=X(f)Y-Y(f)X+(\nabla\varphi)(X,Y)-(\nabla\varphi)(Y,X).$$

Using the above identity in Eq. (2.4) and the skew symmetry of the tensor field φ, we get

$$g\left(R(X,Y)\xi+Y(f)X-X(f)Y,Z\right)+g\left((\nabla\varphi)(Z,X),Y\right)=0,$$

that is,

$$-g\left(R(Z,\xi)X,Y\right)+Y(f)g(X,Z)-X(f)g(Y,Z)+g\left((\nabla\varphi)(Z,X),Y\right)=0,$$

where we used the identity $R(X,Y;\xi,Z)=R(\xi,Z;X,Y)=-R(Z,\xi;X,Y)$. Hence, we get

$$(\nabla\varphi)(Z,X)=R(Z,\xi)X+X(f)Z-g(X,Z)\nabla f,$$

that is,

$$(\nabla\varphi)(X,Y)=R(X,\xi)Y+Y(f)X-g(X,Y)\nabla f,\quad X,Y\in\mathfrak{X}(M). \tag{2.5}$$

Recall that the Ricci operator Q is a symmetric $(1,1)$-tensor field that is defined by

$$g(QX,Y)=\mathrm{Ric}(X,Y),\ X,Y\in\mathfrak{X}(M),$$

where Ric is the Ricci tensor of the Riemannian manifold. Choosing a local orthonormal frame $\{e_1,\ldots,e_n\}$ on M, and using

$$Q(X)=\sum R(X,e_i)e_i$$

in Eq. (2.5), we compute

$$\sum(\nabla\varphi)(e_i,e_i)=-Q(\xi)-(n-1)\nabla f. \tag{2.6}$$

The scalar curvature of the Riemannian manifold (M,g) is the smooth function $S=TrQ$, we have the following for the gradient of the scalar curvature S (cf. [1,3])

$$\frac{1}{2}\nabla S=\sum(\nabla Q)(e_i,e_i). \tag{2.7}$$

Garcia-Rio and Al-Solamy [8] have initiated the study of the Laplace operator $\Delta\colon\mathfrak{X}(M)\to\mathfrak{X}(M)$, on a Riemannian manifold (M,g) defined by

$$\Delta X=\sum_{i=1}^{n}\left(\nabla_{e_i}\nabla_{e_i}X-\nabla_{\nabla_{e_i}e_i}X\right),$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on M. This operator is self-adjoint elliptic operator with respect to the inner product $\langle,\rangle$ on $\mathfrak{X}^C(M)$ the set of compactly supported vector fields in $\mathfrak{X}(M)$, defined by

$$\langle X,Y\rangle=\int_M g(X,Y),\quad X,Y\in\mathfrak{X}^C(M).$$

A vector field X is said to be an eigenvector of the Laplace operator Δ if there is a constant μ such that $\Delta X=-\mu X$. On a compact Riemannian manifold (M,g), using the properties of Δ with respect to the inner product $\langle,\rangle$, it is easy to conclude that the eigenvalue $\mu\geq 0$. For example consider the n-sphere $S^n(c)$ of constant curvature c as a hypersurface of the Euclidean space R^{n+1} with unit normal vector field N and take a constant vector field Z on

R^{n+1}, which can be expressed as $Z = \xi + fN$, where ξ is the tangential component of Z to $S^n(c)$ and $f = \langle Z, N\rangle$ is the smooth function on $S^n(c)$, $\langle , \rangle$ being the Euclidean metric on R^{n+1}. Then it is easy to show that $\Delta\xi = -c\xi$.

We state the following useful lemmas:

Lemma 2.1 [11] *Let ξ be a conformal vector field on an n-dimensional compact Riemannian manifold (M, g) with potential function f. Then*

$$\int_M g(\nabla f, \xi) = -n \int_M f^2$$

where ∇f is the gradient of the function f.

Lemma 2.2 [12] *Let (M, g) be a Riemannian manifold and f be a smooth function defined on M. Then the Hessian operator A_f satisfies*

$$\sum \left(\nabla A_f\right)(e_i, e_i) = Q(\nabla f) + \nabla(\Delta f),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame, Δ is the Laplace operator on M and $\left(\nabla A_f\right)(X, Y) = \nabla_X A_f(Y) - A_f(\nabla_X Y),\ X, Y \in \mathfrak{X}(M)$.

Lemma 2.3 (Bochner's Formula) [1,12] *Let (M, g) be a compact Riemannian manifold and f be a smooth function defined on M. Then*

$$\int_M \left((\mathrm{Ric}(\nabla f, \nabla f) + \left\|A_f\right\|^2 - (\Delta f)^2\right) = 0.$$

3 Characterization of the spheres and Euclidean spaces

Given a conformal vector field ξ on a Riemannian manifold (M, g) with potential function f, and dual 1-form η we say that ξ is a *null* conformal vector field if

$$\mathrm{d}\eta(\xi, X) = 0, \quad X \in \mathfrak{X}(M),$$

that is, if ξ is a null conformal vector field, then we have $\varphi(\xi) = 0$. In this section, we study the geometry of an n-dimensional Riemannian manifold (M, g) that admits a nontrivial conformal vector field ξ satisfying $\Delta\xi = -\lambda\xi$ for a constant $\lambda > 0$ as well as the geometry of Riemannian manifold (M, g) that admits a nontrivial null conformal vector field. If the conformal vector field ξ satisfies $\Delta\xi = -\lambda\xi$, then using Eq. (2.1), we get

$$\Delta\xi = \nabla f + \sum (\nabla\varphi)(e_i, e_i) = -\lambda\xi, \tag{3.1}$$

which in view of Eq. (2.6), gives

$$Q(\xi) = \lambda\xi - (n-2)\nabla f. \tag{3.2}$$

Now we prove the following result.

Theorem 3.1 *An n-dimensional $(n \geq 3)$ compact and connected Riemannian manifold (M, g) with scalar curvature S, first nonzero eigenvalue λ_1 and Ricci curvatures lying in the interval $[\lambda, n^{-1}(n-2)\lambda_1 + \lambda]$ for a constant $\lambda > 0$ admits a nontrivial conformal vector field ξ such that the scalar curvature is constant along the integral curves of ξ, satisfying $\Delta\xi = -\lambda\xi$, if and only if it is isometric to the sphere $S^n(\lambda)$.*

Proof Suppose (M, g) is an n-dimensional compact connected Riemannian manifold that admits a nontrivial conformal vector field ξ with potential function f satisfying $\Delta\xi = -\lambda\xi$ and $\xi(S) = 0$. Then taking the divergence in Eq. (3.2), we get

$$fS = n\lambda f - (n-2)\Delta f,$$

which gives

$$\Delta f = -\frac{S - n\lambda}{n-2} f. \tag{3.3}$$

Integrating by parts the above equation multiplied by f and using the inequality (2.3), we arrive at

$$\frac{1}{n-2}\int_M ((n-2)\lambda_1 + n\lambda - S)\, f^2 \leq 0.$$

Since the Ricci curvatures of M lie in the interval $[\lambda, n^{-1}(n-2)\lambda_1 + \lambda]$, we have $S \leq (n-2)\lambda_1 + n\lambda$ and thus the above inequality gives

$$f^2((n-2)\lambda_1 + n\lambda - S) = 0.$$

However, on connected M, the choice $f = 0$ will give that the conformal vector field ξ is Killing, which is contrary to our assumption that ξ is a nontrivial conformal vector field. Hence the above equation gives that $S = (n-2)\lambda_1 + n\lambda =$ a constant, and Eq. (3.2) gives

$$\Delta f = -\lambda_1 f \quad \text{and consequently,} \int_M \|\nabla f\|^2 = \lambda_1 \int_M f^2 \tag{3.4}$$

Next, on using Eqs. (2.6) and (3.1), we have

$$\operatorname{div}(\varphi(\nabla f)) = -g\left(\nabla f, \sum (\nabla\varphi)(e_i, e_i)\right) = \|\nabla f\|^2 + \lambda\xi(f),$$

which on integration together with Lemma 2.1, gives

$$\int_M \|\nabla f\|^2 = n\lambda \int_M f^2.$$

Comparing the above equation with Eq. (3.4), we get $\lambda_1 = n\lambda$ and consequently, that the Ricci curvatures satisfy $\mathrm{Ric} \leq (n-1)\lambda$. Also using Eqs. (2.1) and (3.1), we compute

$$\operatorname{div}(\varphi(\xi)) = -\|\varphi\|^2 - g\left(\xi, \sum(\nabla\varphi)(e_i, e_i)\right) = -\|\varphi\|^2 + \lambda\|\xi\|^2 + \xi(f),$$

which on integrating and using Lemma 2.1, gives

$$\int_M \|\varphi\|^2 = \int_M \left(\lambda\|\xi\|^2 - nf^2\right). \tag{3.5}$$

Also, using Eq. (2.1), $\Delta f = -\lambda_1 f$ and the Lemma (2.2), we compute

$$\operatorname{div}\left(A_f\xi\right) = f\Delta f + g\left(\xi, \sum(\nabla A_f)(e_i, e_i)\right) = f\Delta f + \mathrm{Ric}(\nabla f, \xi) - \lambda_1\xi(f),$$

which on integrating by parts gives

$$\int_M \mathrm{Ric}(\nabla f, \xi) = \int_M \left(\|\nabla f\|^2 - n\lambda_1 f^2\right) = -n(n-1)\lambda\int_M f^2, \tag{3.6}$$

where we used Eq. (3.4) and $\lambda_1 = n\lambda$. Also, we use Eqs. (3.4), (3.5) and Lemma 2.1, to compute

$$\int_M \|\nabla f + \lambda\xi\|^2 = \int_M \left(\|\nabla f\|^2 + \lambda^2 \|\xi\|^2 + 2\lambda\xi(f)\right)$$
$$= \int_M \left(n\lambda f^2 + \lambda^2 \|\xi\|^2 - 2n\lambda f^2\right) = \lambda \int_M \|\varphi\|^2 \tag{3.7}$$

Finally, using Eqs. (3.6) and (3.7), $\lambda_1 = n\lambda$ and Lemma 2.3, we have

$$\int_M \left(\mathrm{Ric}\,(\nabla f + \lambda\xi, \nabla f + \lambda\xi) - \lambda\,\|\nabla f + \lambda\xi\|^2\right) = \int_M \left((\Delta f)^2 - \left\|A_f\right\|^2 - n(n-1)\lambda^2 f^2\right)$$
$$= \int_M \left(\frac{1}{n}(\Delta f)^2 - \left\|A_f\right\|^2\right).$$

Using Ric $\geq \lambda$ and the Schwartz inequality $\left\|A_f\right\|^2 \geq \frac{1}{n}(\Delta f)^2$, in the above equation, we conclude the equality $\left\|A_f\right\|^2 = \frac{1}{n}(\Delta f)^2$ and it holds if and only if

$$A_f = \frac{\Delta f}{n} I = -\lambda f I,$$

where the potential function f is non-constant owing to the fact that ξ is nontrivial conformal vector field and Eq. (2.2); and the constant $\lambda > 0$. Thus, the above equation gives Obata's differential equation (cf. [15]) and hence M is isometric to $S^n(\lambda)$.

The converse is trivial as the unit sphere $S^n(\lambda)$ admits a nontrivial conformal vector field ξ induced by a nonzero constant vector field on the Euclidean space R^n that satisfies $\Delta\xi = -\lambda\xi$ (see paragraph before Lemma 2.1) and all the requirements of the statement.

In the rest of this section, we study the impact of the presence of a nontrivial null conformal vector field ξ on the geometry of a complete connected Riemannian manifold (M, g). Let ξ be a null conformal vector field on a Riemannian manifold (M, g). Then taking $X = Y = \xi$ in Eq. (2.5) and using Eq. (2.1), we get

$$\xi(f)\xi = \|\xi\|^2 \nabla f,$$

that is, vector fields ∇f and ξ are parallel. Hence, there exists a smooth function ρ on M such that

$$\nabla f = \rho\xi. \tag{3.8}$$

We call this smooth function ρ associated with a null conformal vector field ξ the *connecting* function of the null conformal vector field ξ. The Eq. (3.8) is used in [8] to characterize φ-analytic conformal vector fields. Thus, we see that a null conformal vector field ξ is a φ-analytic vector field.

Finally, we use a null conformal vector field ξ with connecting function ρ to obtain the following characterization of the Euclidean space $(R^n, \langle , \rangle)$ and the sphere $S^n(c)$.

Theorem 3.2 *An n-dimensional ($n \geq 3$) complete and connected Riemannian manifold (M, g) with nonnegative scalar curvature S admits a nontrivial null conformal vector field ξ with connecting function ρ such that the function $S + 2(n-1)\rho$ is a constant along the integral curves of ξ, if and only if it is either isometric the Euclidean space $(R^n, \langle , \rangle)$ or isometric to the sphere $S^n(\lambda)$.*

Proof Let ξ be a nontrivial null conformal vector field on an n-dimensional Riemannian manifold (M, g) with potential function f and the connecting function ρ. Then taking covariant derivative with respect to $X \in \mathfrak{X}(M)$ in Eq. (3.8) and using Eq. (2.1), we get

$$A_f X = X(\rho)\xi + f\rho X + \rho\varphi X, \quad X \in \mathfrak{X}(M). \tag{3.9}$$

Using symmetry and skew symmetry of the operators A_f and φ, respectively, in the above equation, we conclude

$$2\rho\varphi X = \eta(X)\nabla\rho - X(\rho)\xi, \quad X \in \mathfrak{X}(M), \tag{3.10}$$

and

$$2A_f X = \eta(X)\nabla\rho + X(\rho)\xi + f\rho X, \quad X \in \mathfrak{X}(M), \tag{3.11}$$

where η is smooth 1-form dual to the vector field ξ. Taking squared norm in Eq. (3.10), gives

$$2\rho^2 \|\varphi\|^2 = \|\xi\|^2 \|\nabla\rho\|^2 - \xi(\rho)^2. \tag{3.12}$$

Also, taking divergence in Eq. (3.8), we get

$$\Delta f = \xi(\rho) + nf\rho. \tag{3.13}$$

Now, taking squared norm in Eq. (3.11) and using above equation, we arrive at

$$\|A_f\|^2 - \frac{1}{n}(\Delta f)^2 = \frac{1}{2}\left(\|\xi\|^2 \|\nabla\rho\|^2\right) + \frac{n-2}{2n}\xi(\rho)^2 \tag{3.14}$$

Taking covariant derivative in Eq. (3.10) and using Eq. (2.1), we get

$$\begin{aligned} &2X(\rho)\varphi Y + 2\rho(\nabla\varphi)(X, Y) + \rho\eta(\nabla_X Y)\nabla\rho - (\nabla_X Y)(\rho)\xi \\ &\quad = X(\eta(Y))\nabla\rho + \eta(Y)A_\rho X - XY(\rho)\xi - Y(\rho)(fX + \varphi X), \end{aligned}$$

that is,

$$\begin{aligned} 2X(\rho)\varphi Y + 2\rho(\nabla\varphi)(X, Y) &= fg(X, Y)\nabla\rho + g(\varphi X, Y)\nabla\rho \\ &\quad + \eta(Y)A_\rho X - H_\rho(X, Y)\xi - fY(\rho)X - Y(\rho)\varphi X. \end{aligned}$$

For a local orthonormal frame $\{e_1, \ldots, e_n\}$, take $X = Y = e_i$ in the above equation and sum, which in view of Eq. (2.1) gives

$$3\varphi(\nabla\rho) - 2\rho Q\xi - 2\rho(n-1)\nabla f = \quad (n-1)f\nabla\rho + A_\rho\xi - (\Delta\rho)\xi.$$

Taking inner product in above equation with ξ, we get

$$\begin{aligned} &-2\rho\mathrm{Ric}(\xi, \xi) - 2\rho(n-1)\xi(f) \\ &\quad = H_\rho(\xi, \xi) + (n-1)f\xi(\rho) - \|\xi\|^2 \Delta\rho. \end{aligned} \tag{3.15}$$

Equation (3.10) with $X = \xi$ gives

$$\|\xi\|^2 \nabla\rho = \xi(\rho)\xi,$$

which, on taking divergence on both sides and using Eq. (2.1), leads to

$$2g\left(f\nabla\rho + \varphi\nabla\rho, \xi\right) + \|\xi\|^2 \Delta\rho = \xi\xi(\rho) + nf\xi(\rho).$$

Since, the Eq. (2.1) with $\varphi(\xi) = 0$ gives $\nabla_\xi \xi(\rho) = f\xi(\rho)$; consequently, the above equation takes the form

$$2f\xi(\rho) + \|\xi\|^2 \Delta\rho = H_\rho(\xi, \xi) + f\xi(\rho) + nf\xi(\rho),$$

that is,

$$(n-1)f\xi(\rho)+H_\rho(\xi,\xi)-\|\xi\|^2\,\Delta\rho=0.$$

Using this equation in Eq. (3.15), we arrive at

$$\rho\mathrm{Ric}(\xi,\xi)=-\rho(n-1)\xi(f),$$

which on using Eq. (3.8), that is, $\rho\xi=\nabla f$ gives

$$\mathrm{Ric}(\nabla f,\xi)=-(n-1)\,\|\nabla f\|^2\,. \tag{3.16}$$

Finally, taking divergence on both sides of equation $\varphi\xi=0$ and using Eqs. (2.1) and (2.6), we arrive at

$$\mathrm{Ric}(\xi,\xi)+(n-1)\xi(f)=\|\varphi\|^2\,,$$

which on multiplying by ρ and using Eq. (3.6) gives

$$\rho\,\|\varphi\|^2=0,$$

which on connected M implies either $\rho=0$ or $\varphi=0$ and we discuss these two cases separately.

Case (i) If $\rho=0$, then Eq. (3.8) confirms that f is a constant. Note that this constant f is nonzero, for otherwise ξ will be a Killing vector field, which contradicts the assumption that ξ is a nontrivial conformal vector field. Now define a smooth function h on M by

$$h=\frac{1}{2}\,\|\xi\|^2\,,$$

which by Eq. (2.1) gives $\nabla h=f\xi-\varphi(\xi)=f\xi$ (as ξ is null conformal vector field) and, consequently,

$$A_h(X)=f^2X+f\varphi(X),\quad X\in\mathfrak{X}(M).$$

The above equation gives

$$H_h(X,X)=cg(X,X),\quad c=f^2>0,\quad X\in\mathfrak{X}(M).$$

Note that the function h is not a constant, for then in view of $\nabla h=f\xi$ and constant $f\neq 0$, will give $\xi=0$ contradicting the fact that ξ is a nontrivial conformal vector field. Thus on polarization, the above equation gives

$$H_h(X,Y)=cg(X,Y),\quad \text{constant } c>0,\quad X,Y\in\mathfrak{X}(M).$$

Hence, M is isometric to the Euclidean space $(R^n,\langle,\rangle)$ (cf. [19,21]).
Case (ii) Assume that $\varphi=0$. Then Eqs. (2.6) and (3.8) give

$$Q(\xi)=-(n-1)\rho\xi.$$

Taking divergence on both sides of this equation and using Eq. (2.7) and $\operatorname{div}\xi=nf$, we get

$$fS+\frac{1}{2}\xi(S)=-(n-1)\xi(\rho)-n(n-1)\rho f,$$

that is,

$$f(S+n(n-1)\rho)+\frac{1}{2}\xi(S+2(n-1)\rho)=0.$$

As $S+2(n-1)\rho$ is a constant along the integral curves of the vector field ξ, we have $f(S+n(n-1)\rho)=0$, and as seen above $f\neq 0$, thus on connected M, we have

$$S=-n(n-1)\rho. \tag{3.17}$$

Thus in view of above equation, we have $S+2(n-1)\rho=(n-1)(2-n)\rho$ is a constant, and as $n\geq 3$, we conclude ρ is a constant. Using this information in Eqs. (3.13) and (3.14), we arrive at

$$\Delta f=n\rho f,\quad \left\|A_f\right\|^2=\frac{1}{n}(\Delta f)^2. \tag{3.18}$$

The constant $\rho=0$ is already dealt with in Case (i); therefore, we assume $\rho\neq 0$, and this in view of the hypothesis and Eq. (3.17) gives $\rho<0$ and therefore $\rho=-c$ for positive constant c. The second equation in Eq. (3.18) is the equality in the Schwartz inequality $\left\|A_f\right\|^2\geq\frac{1}{n}(\Delta f)^2$. Hence, $A_f=\frac{\Delta f}{n}I$ and consequently in view of the first equation in Eq. (3.18) gives

$$H_f(X,Y)=-cfg(X,Y),\quad X,Y\in\mathfrak{X}(M). \tag{3.19}$$

Note that if the potential function f is a constant, then Eq. (3.8) gives $-c\xi=0$ with constant $c\neq 0$ that leads to a contradiction $\xi=0$ as ξ is a nontrivial conformal vector field. Hence Eq. (3.19) is Obata's differential equation and therefore M is isometric to $S^n(c)$.

The converse is trivial as the Euclidean space $(R^n,\langle,\rangle)$ admits a nontrivial gradient conformal vector field $\xi=\psi$ (ψ position vector field) with potential function $f=\frac{1}{2}\left\|\psi\right\|^2$, $\varphi=0$ and is a null conformal vector field with connecting function $\rho=1$, which satisfies all the requirements of the hypothesis ($S+2(n-1)\rho=2(n-1)$). Also, the sphere $S^n(c)$ admits a nontrivial gradient conformal vector field ξ induced by a nonzero constant vector field $Z=\xi+hN$ on the Euclidean space R^{n+1} with potential function $f=-\sqrt{c}h$, N being unit normal vector field to $S^n(c)$. As this vector field ξ is gradient conformal vector field, we have $\varphi=0$, it is nontrivial null conformal vector field with connecting function $\rho=-c$. Hence the conformal vector field ξ satisfies all the requirements in the hypothesis ($S+2(n-1)\rho=(n-1)(n+2)c$).

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