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Conformally flat Riemannian manifolds with finite L^p-norm curvature

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Abstract Let $(M^n, g)(n \geq 3)$ be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with constant scalar curvature S. Denote by T the trace-free Ricci curvature tensor of M. The main result of this paper states that T goes to zero uniformly at infinity if for $p \geq \frac{n}{2}$, the L^p-norm of T is finite. As applications, we prove that (M^n, g) is compact if the L^p-norm of T is finite and S is positive, and (M^n, g) is scalar flat if (M^n, g) is a noncompact manifold with nonnegative constant scalar curvature and the L^p-norm of T is finite. We prove that (M^n, g) is isometric to a sphere if S is positive and the L^p-norm of T is pinched in $[0, C)$, where C is an explicit positive constant depending only on n, p and S. Finally, we prove an $L^p(p \geq \frac{n}{2})$-norm of T pinching theorem for complete, simply connected, locally conformally flat Riemannian manifolds with negative constant scalar curvature.

Keywords Constant curvature space · Conformally flat manifold · Trace-free Ricci curvature tensor

Mathematics Subject Classification Primary 54C21 · 53C20

1 Introduction

Recall that a Riemannian manifold (M^n, g) of dimension n is said to be locally conformally flat if a neighborhood of each point of M can be conformally immersed into the standard sphere. When $n \geq 4$, it is well known that this is equivalent to the fact that the Weyl tensor identically vanishes (see [1,2,16], for example). According to the decomposition of the Riemannian curvature tensor, for $n \geq 3$, a locally conformally flat manifold has constant sectional curvature if and only it is Einstein, that is, the trace-free Ricci tensor, defined by

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$T = \text{Ric} - \frac{S}{n} g$, is identically equal to zero, where Ric is the Ricci curvature tensor and S is the scalar curvature. As a consequence, it follows from the H. Hopf classification theorem that the space forms are isometric to the only complete, simply connected, locally conformally flat, Einstein manifolds. Classification of locally conformally flat manifolds is one of the most important problem in the study of differential geometry, but is very difficult. Under various geometric conditions, many scholars have given some partial results to this classification [4–9, 11, 12, 14, 15, 17–20]. For example, some interesting rigidity results have been obtained under the pointwise pinching and L^p pinching assumptions for the traceless Ricci tensor, respectively [5–9, 11, 14, 17–20]. The curvature pinching phenomenon plays an important role in global differential geometry. We are interested in L^p pinching problems for complete locally conformally flat Riemannian manifold with constant scalar curvature. Throughout this paper, we always assume that M is an n-dimensional complete locally conformally flat Riemannian manifold with $n \geq 3$.

In 1996, Hebey and Vaugon [11] showed that a compact locally conformally flat Riemannian manifold (M, g) with Yamabe metric g and positive scalar curvature for a $L^{\frac{n}{2}}$ trace-free Ricci curvature pinching condition, is isometric to a quotient of sphere. In 2007, Pigola et al. [14] proved that a complete, simply connected, locally conformally flat Riemannian manifold M with zero scalar curvature for a $L^{\frac{n}{2}}$ trace-free Ricci curvature pinching condition is isometric to Euclidean space. In 2010, Xu and Zhao [19] proved that a complete, simply connected, locally conformally flat Riemannian manifold M with dimension $n \geq 6$ and constant nonzero scalar curvature for a $L^{\frac{n}{2}}$ trace-free Ricci curvature pinching condition is isometric to a constant curvature space form. In addition, they concluded that if the L^n-norm of trace-free Ricci curvature T is finite, then T goes to zero uniformly at infinity. As its application, they obtained that a complete, simply connected, locally conformally flat Riemannian manifold M with constant positive scalar curvature for a L^n trace-free Ricci curvature pinching condition is isometric to a sphere. In this note, we obtain the following rigidity theorems.

Theorem 1 *Let M be a complete, simply connected, locally conformally flat Riemannian n-manifold with constant scalar curvature. For $p \geq \frac{n}{2}$, if $\int_M |T|^p < +\infty$, then, given any $\epsilon > 0$ and any $x_0 \in M$ there exists a geodesic ball $B_r(x_0)$ with center x_0 and radius r such that $|T|(x) < \epsilon$ for all $x \in M \setminus B_r(x_0)$.*

Theorem 2 *Let M be a complete, simply connected, locally conformally flat Riemannian n-manifold with positive constant scalar curvature. For $p \geq \frac{n}{2}$, if $\int_M |T|^p < +\infty$, then M must be compact.*

Corollary 1 *Let M be a complete, simply connected, noncompact locally conformally flat Riemannian n-manifold with nonnegative constant scalar curvature. For $p \geq \frac{n}{2}$, if $\int_M |T|^p < +\infty$, then M must be scalar flat.*

Remark 1 If $p = n$, Theorems 1 and 2 and Corollary 1 reduce to Lemmas 3.4 and 3.5 and Corollary 3.6 of [19], respectively.

Theorem 3 *Let M be a complete, simply connected, locally conformally flat Riemannian n-manifold with positive constant scalar curvature. Then for $p \geq \frac{n}{2}$, if*

$$\left(\int_M |T|^p\right)^{\frac{1}{p}} < C_1,$$

where

$$C_1 = \begin{cases} \frac{3\sqrt{6}}{4}\omega_3^{\frac{2}{3}}, & \text{if } n = 3 \quad \text{and} \quad p = \frac{3}{2}, \\ \left[\frac{6(2p-3)}{S}\right]^{\frac{3}{2p}} \frac{\sqrt{6}pS}{12(2p-3)}\omega_3^{\frac{1}{p}}, & \text{if } n = 3 \quad \text{and} \quad \frac{3}{2} < p < 2, \\ \left[\frac{n(n-1)}{S}\right]^{\frac{n}{2p}} \frac{S}{\sqrt{n(n-1)}}\omega_n^{\frac{1}{p}}, & \text{if } n = 3 \quad \text{and} \quad p \geq 2, \text{ or } n \geq 4, \end{cases}$$

then M is isometric to a sphere.

Remark 2 Some $L^{\frac{n}{2}}$ trace-free Ricci curvature pinching theorems have been shown in [11, 14, 19]. Theorem 3 extends the $L^{\frac{n}{2}}$ trace-free Ricci curvature pinching theorem given by [19] in dimension $n \geq 6$ and power $p = \frac{n}{2}$ to $n \geq 3$ and $p \geq \frac{n}{2}$. Theorem 3 improves the aforementioned result due to [11]. The pinching constant in Theorem 3 is better than the one in the L^n trace-free Ricci curvature pinching theorem given by [19].

Theorem 4 *Let M be a complete, simply connected, locally conformally flat Riemannian n-manifold with nonpositive constant scalar curvature. Assume that (i) or (ii) holds:*

(i) *scalar curvature $S = 0$ and $p = \frac{n}{2}$;*

(ii) *scalar curvature $S < 0$, $n \geq 6$ and $\frac{n}{2} \leq p < \frac{n-2}{2}(1 + \sqrt{1 - \frac{4}{n}})$.*

Then there exists a small number C_2 such that if

$$\left(\int_M |T|^p\right)^{\frac{1}{p}} < C_2,$$

then M is a constant curvature space form. In particular, when $p = \frac{n}{2}$, if $S = 0$, $C_2 = 2n^{-\frac{5}{2}}\sqrt{n-1}(n-2)(n^2-2n+4)\omega_n^{\frac{2}{n}}$; if $S < 0$, $C_2 = \sqrt{n(n-1)}\omega_n^{\frac{2}{n}}$.

Remark 3 If $p = \frac{n}{2}$, Theorem 4 reduces to some results of [14] and [19]. Theorems 3 and 4 can be considered as generalization of some main results in [14] and [19].

Remark 4 Let M be a complete Riemannian n-manifold with harmonic curvature and positive Yamabe constant. Using the same argument as in this note, we obtain some analog of Theorems in this note (see [7,8]). In 1996, Hebey and Vaugon [11] characterized M under $L^{\frac{n}{2}}$ pinching assumption for the traceless Ricci tensor and Weyl tensor.

2 Proofs of Theorems

In what follows, we adopt, without further comment, the moving frame notation with respect to a chosen local orthonormal frame.

Let M be a locally conformally flat Riemannian n-manifold. Conformally flatness and decomposition of the Riemannian curvature tensor into irreducible components yield

$$R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) - \frac{S}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{1}$$

where R_{ij} denotes the components of the Ricci tensor Ric and S is the scalar curvature. Assuming that S is constant, from the second Bianchi identities and (1), we immediately obtain that the Ricci tensor is a Codazzi tensor, i.e.,

$$R_{ij,k} = R_{ik,j}.$$

Thus the traceless Ricci tensor

$$T = \mathrm{Ric} - \frac{S}{n} g$$

is again Codazzi, i.e., $T_{ij,k} = T_{ik,j}$. Then we get

$$|T|^2 = |\mathrm{Ric}|^2 - \frac{S^2}{n}.$$

We compute

$$\triangle |T|^2 = 2|\nabla T|^2 + 2\langle T, \triangle T\rangle = 2|\nabla T|^2 + 2T_{ij}T_{ij,kk}. \tag{2}$$

Since the traceless Ricci tensor is Codazzi, by the Ricci identities, we obtain

$$\begin{aligned} T_{ij,kk} &= T_{ik,jk} = T_{ki,kj} + T_{li}R_{lkjk} + T_{kl}R_{lijk} \\ &= T_{kk,ij} + T_{li}R_{lkjk} + T_{kl}R_{lijk} \\ &= T_{li}R_{lkjk} + T_{kl}R_{lijk}, \end{aligned} \tag{3}$$

which gives

$$T_{ij}T_{ij,kk} = T_{ij}T_{li}R_{lkjk} + T_{ij}T_{kl}R_{lijk}. \tag{4}$$

Now we calculate

$$T_{ij}T_{li}R_{lkjk} = T_{ij}T_{li}\left(T_{lj} + \frac{S}{n}\delta_{lj}\right) = T_{ij}T_{li}T_{lj} + \frac{S}{n}|T|^2 \tag{5}$$

and

$$T_{ij}T_{kl}R_{lijk} = \frac{2}{n-2}T_{ij}T_{il}T_{lj} + \frac{S}{n(n-1)}|T|^2. \tag{6}$$

By substituting (4), (5) and (6) into (2), we obtain

$$\triangle |T|^2 = 2|\nabla T|^2 + \frac{2n}{n-2}T_{ij}T_{il}T_{lj} + \frac{2S}{n-1}|T|^2. \tag{7}$$

Applying the simple algebraic lemma due to [13], we have

$$\triangle |T|^2 \geq 2|\nabla T|^2 - \frac{2n}{\sqrt{n(n-1)}}|T|^3 + \frac{2S}{n-1}|T|^2. \tag{8}$$

From (8), by the Kato inequality $|\nabla T|^2 \geq \frac{n+2}{n}|\nabla |T||^2$ (see Lemma in [11]), we obtain

$$|T|\triangle |T| \geq \frac{2}{n}|\nabla |T||^2 - \frac{n}{\sqrt{n(n-1)}}|T|^3 + \frac{S}{n-1}|T|^2. \tag{9}$$

Let $u = |T|$. By (9), we compute

$$\begin{aligned} u^\alpha \triangle u^\alpha &= u^\alpha\left(\alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 + \alpha u^{\alpha-1}\triangle u\right) \\ &= \frac{\alpha-1}{\alpha}|\nabla u^\alpha|^2 + \alpha u^{2\alpha-2}u\triangle u \\ &\geq \left(1 - \frac{n-2}{n\alpha}\right)|\nabla u^\alpha|^2 - \frac{n\alpha}{\sqrt{n(n-1)}}u^{2\alpha+1} + \frac{S\alpha}{n-1}u^{2\alpha}, \end{aligned} \tag{10}$$

where α is a positive constant.

In order to prove main results in this note, we need Lemma 1 as follow:

Lemma 1 *Let M be an $n(\geq 3)$-dimensional complete noncompact Riemannian manifold satisfying a Sobolev inequality of the following form:*

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq D_1 \int_M |\nabla f|^2 + F_1 \int_M |f|^2, \forall f \in C_0^\infty(M).$$

If a nonnegative function $u \in C^\infty(M)$ satisfies $\int_M u^{\frac{n}{2}} < +\infty$ and

$$\triangle u \geq au^2 + bu$$

for some constants a and b. Then, given any $\epsilon > 0$ and any $x_0 \in M$ there exists a geodesic ball $B_r(x_0)$ with center x_0 and radius r such that $u(x) < \epsilon$ for all $x \in M \setminus B_r(x_0)$.

Remark 5 Xiao and the first author [8] can carry out the proof of Lemma 1 by suitable modification to the proof of Theorem 1.1 in [3].

Proof of Theorem 1 Taking $\alpha = \frac{2p}{n} \geq 1$, from (10) we obtain

$$u^\alpha \triangle u^\alpha \geq -\frac{n\alpha}{\sqrt{n(n-1)}} u^{2\alpha+1} + \frac{S\alpha}{n-1} u^{2\alpha}. \tag{11}$$

Using the Young's inequality, from (11) we obtain

$$u^\alpha \triangle u^\alpha \geq au^{3\alpha} + bu^{2\alpha}, \tag{12}$$

where a and b are two constants depending only on n, α and S. Setting $w = u^\alpha$, we can rewrite (12) as

$$\triangle w \geq aw^2 + bw. \tag{13}$$

On the other hand, when M is a complete, simply connected, locally conformally flat Riemannian n-manifold, M satisfies the Sobolev inequality (see Corollary 3.2 in [10]):

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)\omega_n^{\frac{2}{n}}} \int_M \left(|\nabla f|^2 + \frac{(n-2)}{4(n-1)} Sf^2\right), f \in C_0^\infty(M). \tag{14}$$

Combining (13) with (14), we can prove Theorem 1 by using Lemma 1. This completes the proof of Theorem 1. □

Proof of Theorem 2 Take a local orthonormal frame $\{e_i\}$ such that Ric is diagonal. By (1), we have

$$R_{ijij} = \frac{T_{ii} + T_{jj}}{n-2} + \frac{S}{n(n-1)}. \tag{15}$$

Note that S is positive. From (15), we see from Theorem 1 that there is a positive constant δ such that $R_{ijij} > \delta$ in $M \setminus \Omega$ for some compact set Ω. The remainder of the argument is analogous to that in Lemma 3.5 of [19]. For completeness, we give the following proof.

Since M is complete, it suffices to show that M is bounded. Otherwise, there is a point $p_1 \in M$ such that $d(p_1, \Omega) = \inf_{q \in \Omega} d(p_1, q) = \frac{\pi}{\sqrt{\delta}} + 1$. Since Ω is compact, there is a point p_2 such that $d(p_1, p_2) = d(p_1, \Omega)$. Let γ be a minimal geodesic parametrized by arclength such that $\gamma(0) = p_1$ and $\gamma(\frac{\pi}{\sqrt{\delta}} + 1) = p_2$. Define $\gamma(t) = \gamma(\frac{\pi t}{\sqrt{\delta}})$. Taking $p_3 = \gamma(1)$. Then for $t \leq 1$, $\gamma(t) \in M \backslash \Omega$ is a minimal geodesic with $\gamma(0) = p_1$ and $\gamma(1) = p_3$. The formula for the second variation of arclength implies that for a parallel field

E along γ, with $E(0) \perp \gamma'(0)$, the second derivative of the arclength of a variation induced by E is given by

$$I(E,E) = \int_0^1 \sin^2 \pi t[\pi^2 - l^2 Rm(\gamma', E, \gamma', E)],$$

where $l = \frac{\pi}{\sqrt{\delta}}$. Let $\{e_1, \ldots, e_{n-1}\}$ be orthonormal parallel vector fields along γ such that $e_i(t) \perp \gamma'(t)$. Then

$$\sum I(e_i, e_i) = \int_0^1 \sin^2 \pi t[(n-1)\pi^2 - l^2 \mathrm{Ric}(\gamma', \gamma')].$$

Since $R_{ijij} > \delta$ in $M \setminus \Omega$, we obtain

$$\sum I(e_i, e_i) < 0.$$

On the other hand, since $\gamma(t)$ is a minimizing geodesic, we have $\sum I(e_i, e_i) \geq 0$, which is a contradiction. Hence M is bounded and compact. This completes the proof of Theorem 2. □

Proof of Theorem 3 When $S > 0$, we see from Theorem 2 that M is compact. Taking $\alpha = \frac{2p}{n} \geq 1$. Using the Young's inequality, i.e., $fg \leq \frac{\epsilon f^p}{p} + \frac{\epsilon^{-\frac{q}{p}} g^q}{q}$, from (10) we obtain

$$u^\alpha \triangle u^\alpha \geq \left(1 - \frac{n-2}{n\alpha}\right) |\nabla u^\alpha|^2 - \frac{n\epsilon^{1-\alpha}}{\sqrt{n(n-1)}} u^{3\alpha} - \left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right] u^{2\alpha}. \quad (16)$$

Setting $w = u^\alpha$, we can rewrite (16) as

$$w \triangle w \geq \left(1 - \frac{n-2}{n\alpha}\right) |\nabla w|^2 - \frac{n\epsilon^{1-\alpha}}{\sqrt{n(n-1)}} w^3 - \left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right] w^2. \quad (17)$$

From (17), we obtain

$$w^\beta \triangle w^\beta \geq \left(1 - \frac{n-2}{n\alpha\beta}\right) |\nabla w^\beta|^2 - \frac{n\beta\epsilon^{1-\alpha}}{\sqrt{n(n-1)}} w^{2\beta+1} - \beta \left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right] w^{2\beta}, \quad (18)$$

where β is a positive constant. From (18), integrating by parts we get

$$\left(2 - \frac{n-2}{n\alpha\beta}\right) \int_M |\nabla w^\beta|^2 - \frac{n\beta\epsilon^{1-\alpha}}{\sqrt{n(n-1)}} \int_M w^{2\beta+1}$$
$$-\beta \left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right] \int_M w^{2\beta} \leq 0. \quad (19)$$

By the Hölder inequality and (19), we have

$$\left(2 - \frac{n-2}{n\alpha\beta}\right) \int_M |\nabla w^\beta|^2 - \frac{n\beta\epsilon^{1-\alpha}}{\sqrt{n(n-1)}} \left(\int_M w^{\frac{2n\beta}{n-2}}\right)^{\frac{n-2}{n}} \left(\int_M w^{\frac{n}{2}}\right)^{\frac{2}{n}}$$
$$-\beta \left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right] \int_M w^{2\beta} \leq 0. \quad (20)$$

Case 1. When $n = 3$ and $1 \leq \alpha < \frac{4}{3}$, if $\alpha > 1$, set $\epsilon = \frac{\sqrt{6}\alpha S}{24(\alpha-1)}$; if $\alpha = 1$, set $\epsilon = 1$. Take $\alpha\beta = \frac{1}{3}$. From (14) and (20), we get

$$\left[\frac{3\omega_3^{\frac{2}{3}}}{4} - \frac{\epsilon^{1-\alpha}}{\sqrt{6}\alpha} \left(\int_M |T|^p\right)^{\frac{2}{3}}\right] \left(\int_M w^{6\beta}\right)^{\frac{1}{3}} \leq 0. \quad (21)$$

We choose $\left(\int_M |T|^p\right)^{\frac{1}{p}} < C_1$ such that (21) implies $\left(\int_M w^{6\beta}\right)^{\frac{1}{3}} = 0$, that is, $T = 0$, i.e., M is Einstein manifold. Since M is a complete, simply connected, locally conformally flat manifold, M is isometric to a sphere.

Case 2. When $n = 3$ and $\alpha \geq \frac{4}{3}$, or $n \geq 4$, set $\epsilon = \frac{S}{\sqrt{(n-1)n}}$ and $\frac{1}{\alpha\beta} = \frac{n}{n-2}(1+\sqrt{1-\frac{4}{n\alpha}})$. We also get

$$\left[(2-\frac{n-2}{n\alpha\beta})\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4} - \frac{n\beta\epsilon^{1-\alpha}}{\sqrt{n(n-1)}}\left(\int_M |T|^p\right)^{\frac{2}{n}}\right]\left(\int_M w^{\frac{2n\beta}{n-2}}\right)^{\frac{n-2}{n}} \leq 0. \tag{22}$$

We choose $\left(\int_M |T|^p\right)^{\frac{1}{p}} < C_1$ such that (22) implies $\left(\int_M w^{\frac{2n\beta}{n-2}}\right)^{\frac{n-2}{n}} = 0$, that is, $T = 0$, i.e., M is Einstein manifold. Since M is a complete, simply connected, locally conformally flat manifold, M is isometric to a sphere. This completes the proof of Theorem 3. □

Proof of Theorem 4 Let ϕ be a smooth compactly supported function on M and $\alpha = \frac{2p}{n} \geq 1$. First choosing $\beta = \frac{n}{4}$ in (18), multiplying (18) by ϕ^2 and integrating over M, we obtain

$$\begin{aligned}\left[1-\frac{4(n-2)}{n^2\alpha}\right]\int_M |\nabla w^{\frac{n}{4}}|^2\phi^2 &\leq \frac{n^2\epsilon^{1-\alpha}}{4\sqrt{n(n-1)}}\int_M w^{\frac{n}{2}+1}\phi^2 + \int_M w^{\frac{n}{4}}\phi^2 \triangle w^{\frac{n}{4}} \\ &\quad + \frac{n}{4}\left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right]\int_M w^{\frac{n}{2}}\phi^2 \\ &= \frac{n^2\epsilon^{1-\alpha}}{4\sqrt{n(n-1)}}\int_M w^{\frac{n}{2}+1}\phi^2 - 2\int_M w^{\frac{n}{4}}\phi\langle\nabla\phi, \nabla w^{\frac{n}{4}}\rangle \\ &\quad - \int_M |\nabla w^{\frac{n}{4}}|^2\phi^2 + \frac{n}{4}\left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right]\int_M w^{\frac{n}{2}}\phi^2,\end{aligned}$$

which gives

$$\begin{aligned}\left[2-\frac{4(n-2)}{n^2\alpha}\right]\int_M |\nabla w^{\frac{n}{4}}|^2\phi^2 &\leq \frac{n^2\epsilon^{1-\alpha}}{4\sqrt{n(n-1)}}\int_M w^{\frac{n}{2}+1}\phi^2 - 2\int_M w^{\frac{n}{4}}\phi\langle\nabla\phi, \nabla w^{\frac{n}{4}}\rangle \\ &\quad + \frac{n}{4}\left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right]\int_M w^{\frac{n}{2}}\phi^2.\end{aligned} \tag{23}$$

Using the Cauchy–Schwarz inequality, we can rewrite (23) as

$$\begin{aligned}\left[2-\frac{4(n-2)}{n^2\alpha}-\varepsilon\right]\int_M |\nabla w^{\frac{n}{4}}|^2\phi^2 &\leq \frac{n^2\epsilon^{1-\alpha}}{4\sqrt{n(n-1)}}\int_M w^{\frac{n}{2}+1}\phi^2 + \frac{1}{\varepsilon}\int_M w^{\frac{n}{2}}|\nabla\phi|^2 \\ &\quad + \frac{n}{4}\left[\frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{S\alpha}{n-1}\right]\int_M w^{\frac{n}{2}}\phi^2,\end{aligned} \tag{24}$$

for the positive constant ε. By (14) and (24), we have

$$\begin{aligned}\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4}\left(\int_M(\phi^2 w^{\frac{n}{2}})^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} &\le \int_M\left(|\nabla(\phi w^{\frac{n}{4}})|^2+\frac{(n-2)Sw^{\frac{n}{2}}\phi^2}{4(n-1)}\right)\\ &=\int_M\Big(w^{\frac{n}{2}}|\nabla\phi|^2+\phi^2|\nabla w^{\frac{n}{4}}|^2\\ &\quad+2\phi w^{\frac{n}{4}}\langle\nabla\phi,\nabla w^{\frac{n}{4}}\rangle+\frac{(n-2)Sw^{\frac{n}{2}}\phi^2}{4(n-1)}\Big)\\ &\le\left(1+\frac{1}{\eta}\right)\int_M w^{\frac{n}{2}}|\nabla\phi|^2+(1+\eta)\int_M\phi^2|\nabla w^{\frac{n}{4}}|^2\\ &\quad+\int_M\frac{(n-2)Sw^{\frac{n}{2}}\phi^2}{4(n-1)}\\ &\le B\int_M w^{\frac{n}{2}}|\nabla\phi|^2+E\int_M w^{\frac{n}{2}+1}\phi^2+D\int_M w^{\frac{n}{2}}\phi^2,\end{aligned} \tag{25}$$

where

$$B=1+\frac{1}{\eta}+\frac{1+\eta}{\varepsilon\left(2-\frac{4(n-2)}{n^2\alpha}-\varepsilon\right)},$$

$$E=\frac{(1+\eta)n^2\epsilon^{1-\alpha}}{4\sqrt{n(n-1)}\left(2-\frac{4(n-2)}{n^2\alpha}-\varepsilon\right)},$$

$$D=\frac{(n-2)S}{4(n-1)}-\frac{(1+\eta)n\alpha S}{4(n-1)\left(2-\frac{4(n-2)}{n^2\alpha}-\varepsilon\right)}+\frac{(1+\eta)n^2(\alpha-1)\epsilon}{4\sqrt{n(n-1)}\left(2-\frac{4(n-2)}{n^2\alpha}-\varepsilon\right)}.$$

Case 1. $S=0$.

Taking $\alpha=1$, we have $D=0$. Thus from (25) we have

$$\begin{aligned}\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4}\left(\int_M(\phi w^{\frac{n}{4}})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\le B\int_M w^{\frac{n}{2}}|\nabla\phi|^2+E\int_M w^{\frac{n}{2}+1}\phi^2\\ &\le B\int_M w^{\frac{n}{2}}|\nabla\phi|^2+E\left(\int_M(\phi w^{\frac{n}{4}})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\left(\int_M w^{\frac{n}{2}}\right)^{\frac{2}{n}}.\end{aligned}$$

Since $(\int_M w^{\frac{n}{2}})^{\frac{2}{n}}<C_2$, there exists a constant $F=\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4}-E(\int_M w^{\frac{n}{2}})^{\frac{2}{n}}>0$, such that

$$F\left(\int_M(\phi u^{\frac{n}{4}})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\le B\int_M u^{\frac{n}{2}}|\nabla\phi|^2. \tag{26}$$

Let us choose a cutoff function ϕ satisfying the properties that

$$\phi(x)=\begin{cases}1, & \text{on } B(r),\\ 0, & \text{on } M\setminus B(2r),\end{cases}$$

and $|\nabla\phi|\le\frac{2}{r}$. In particular, if M is compact, and if $r>d$, where d is the diameter of M, then $\phi=1$ on M. From (26), we get

$$F\left(\int_{B_r}u^{\frac{n^2}{2(n-2)}}\right)^{\frac{n-2}{n}}\le\frac{4}{r^2}B\int_M u^{\frac{n}{2}}. \tag{27}$$

Let $r \to +\infty$, by assumption that $\int_M u^{\frac{n}{2}} < \infty$, from (27), we have $T = 0$, i.e., M is Einstein manifold. Since M is a complete, simply connected, locally conformally flat manifold, M is a constant curvature space form.

Case 2. $S < 0$.

When $n \geq 6$, noting that ε, ϵ and η are sufficiently small, we choose $1 \leq \alpha < \frac{n-2}{n}(1 + \sqrt{1 - \frac{4}{n}})$ such that $D \leq 0$. The rest of the proof runs as before.

In particular, when $p = \frac{n}{2}$, i.e., $\alpha = 1$. We choose η such that $D = 0$, i.e., $(2-4\frac{n-2}{n^2}-\varepsilon) = \frac{(1+\eta)n}{(n-2)}$. Thus we have

$$\frac{\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4}}{E} = \sqrt{n(n-1)}\omega_n^{\frac{2}{n}}.$$

So we choose $\left(\int_M |T|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \sqrt{n(n-1)}\omega_n^{\frac{2}{n}}$ such that $F > 0$. The rest of the proof runs as before. This completes the proof of Theorem 4. □

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References

1. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer, Berlin (1998)
2. Besse, A.L.: Einstein Manifolds. Springer, Berlin (1987)
3. Brard, P., do Carmo, M., Santos, W.: Complete hypersurfaces with constant mean curvature and finite total curvature. Ann. Glob. Anal. Geom. **16**, 273–290 (1998)
4. Carron, G.: Some old and new results about rigidity of critical metric. arXiv:1012.0685
5. Carron, G., Herzlich, M.: Conformally flat manifolds with nonnegative Ricci curvature. Compos. Math. **142**, 798–810 (2006)
6. Cheng, Q.M.: Compact locally conformally flat Riemannian manifolds. Bull. Lond. Math. Soc. **33**, 459–465 (2001)
7. Fu, H.P., Dan, P.P., Song, S.L.: Complete manifolds with harmonic curvature and finite L^p-norm curvature. J. Math. Res. Appl.
8. Fu, H.P., Xiao, L.Q.: Some L^p rigidity results for complete manifolds with harmonic curvature. arXiv:1510.07121
9. Goldberg, S.I.: An application of Yaus maximum principle to conformally flat spaces. Proc. Am. Math. Soc. **79**, 268–270 (1980)
10. Hebey, E.: Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities. Courant Lecture Notes in Mathematics **5**, New York University Courant Institute of Mathematical Sciences, New York. American Mathematical Society, Providence, RI (1999)
11. Hebey, E., Vaugon, M.: Effective L^p pinching for the concircular curvature. J. Geom. Anal. **6**, 531–553 (1996)
12. Kim, S.: Rigidity of noncompact complete manifolds with harmonic curvature. Manuscripta Math. **135**, 107–116 (2011)
13. Okumura, M.: Hypersurfaces and a pinching problem on the second fundamental tensor. Am. J. Math. **96**, 207–213 (1974)
14. Pigola, S., Rigoli, M., Setti, A.G.: Some characterizations of space-forms. Trans. Am. Math. Soc. **359**, 1817–1828 (2007)
15. Schoen, R., Yau, S.T.: Conformally flat manifolds, Kleinian groups and scalar curvature. Invent. Math. **92**, 47–71 (1988)
16. Schoen, R., Yau, S.T.: Lectures on Differential Geometry. Conference Proceedings and Lecture Notes in Geometry and Topology, vol. I. International Press, Cambridge (1994)
17. Tani, M.: On a conformally flat Riemannian space with positive Ricci curvature. Tohoku Math. J. **19**, 227–231 (1967)

18. Tian, G., Viaclovsky, J.: Bach-flat asymptotically locally Euclidean metrics. Invent. Math. **160**, 357–415 (2005)
19. Xu, H.W., Zhao, E.T.: L^p Ricci curvature pinching theorems for conformally flat Riemannian manifolds. Pac. J. Math. **245**, 381–396 (2010)
20. Zhu, S.H.: The classification of complete locally conformally flat manifolds of nonnegative Ricci curvature. Pac. J. Math. **163**, 189–199 (1994)