Continuity of attractors for a family of C^1 perturbations of the square

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Abstract We consider here the family of semilinear parabolic problems

$$\begin{cases} u_t(x, t) = \Delta u(x, t) - au(x, t) + f(u(x, t)), & x \in \Omega_\epsilon \text{ and } t > 0, \\ \frac{\partial u}{\partial N}(x, t) = g(u(x, t)), & x \in \partial\Omega_\epsilon \text{ and } t > 0, \end{cases}$$

where Ω is the unit square, $\Omega_\epsilon = h_\epsilon(\Omega)$, and h_ϵ is a family of diffeomorphisms converging to the identity in the C^1 -norm. We show that the problem is well posed for $\epsilon > 0$ sufficiently small in a suitable phase space, the associated semigroup has a global attractor $\mathcal{A}_\epsilon$, and the family $\{\mathcal{A}_\epsilon\}_{\epsilon \geq 0}$ is continuous at $\epsilon = 0$.

Keywords Parabolic problem · Perturbation of the domain · Lipschitz domain · Global attractor · Continuity of attractors

Mathematics Subject Classification Primary 35B41; Secondary 35K20 · 58D25

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be the unit square, a a positive constant, $f, g : \mathbb{R} \rightarrow \mathbb{R}$ real functions, and consider the semilinear parabolic problems with nonlinear Neumann boundary conditions:

$$\begin{cases} u_t(x, t) = \Delta u(x, t) - au(x, t) + f(u(x, t)), & x \in \Omega_\epsilon \text{ and } t > 0, \\ \frac{\partial u}{\partial N}(x, t) = g(u(x, t)), & x \in \partial\Omega_\epsilon \text{ and } t > 0, \end{cases} \quad (P_\epsilon)$$

where $\Omega_\epsilon = h_\epsilon(\Omega)$ and h_ϵ is the family of diffeomorphisms given by

$$h_\epsilon(x_1, x_2) = (x_1, x_2 + x_2 \epsilon \sin(x_1/\epsilon^\alpha)) \quad (1)$$

with $0 < \alpha < 1$ and $\epsilon > 0$ sufficiently small (see Fig. 1).

Our aim here is to prove well-posedness, establish the existence of a global attractor $\mathcal{A}_\epsilon$ for sufficiently small ϵ and prove the continuity of the family of attractors at $\epsilon = 0$, under appropriate conditions on the nonlinearities.

It well known that, under some smoothness hypotheses on $\Omega \subset \mathbb{R}^n$, f and g , the problem is well posed in appropriate phase spaces. The existence of a global compact attractor has been proved in [4, 10], under some additional growth and dissipative conditions on the nonlinearities f and g . In [12] the authors prove the continuity of the attractors with respect to C^2 -perturbations of a smooth domain of $\mathbb{R}^n$.

These results do not extend immediately to the case considered here, due to the lack of smoothness of the domains considered and the fact that the perturbations do not converge to the identity in the C^2 -norm.

Existence and continuity of global attractors for semilinear parabolic problems with respect to change of domains has also been considered in [3] with homogeneous boundary condition

$$\begin{cases} u_t = \Delta u + f(x, u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega_\epsilon \end{cases}$$

where Ω_ϵ , $0 \leq \epsilon \leq \epsilon_0$ are bounded domains with Lipschitz boundary in $\mathbb{R}^N$, $N \geq 2$. They prove that, if the perturbations are such that the convergence of the eigenvalues and eigenfunctions of the linear part of the problem can be shown, then the upper semicontinuity of attractors follows. With the additional assumption that the equilibria are all hyperbolic, the lower semicontinuity is also obtained.

The behavior of the equilibria of (P_ϵ) was studied in [1, 2]. In these papers, the authors consider a family of smooth domains $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geq 2$ and $0 \leq \epsilon \leq \epsilon_0$ whose boundary

Fig. 1 Map h_ϵ

oscillates rapidly when the parameter $\epsilon \rightarrow 0$ and prove that the solutions, as well as the spectra of the linearized problem around them, converge to the solution of a “limit problem.”

In this work, we follow the general approach of [12], which consists basically in “pull-backing” the perturbed problems to the fixed domain Ω and then considering the family of abstract semilinear problems thus generated.

We observe that the results obtained can be easily extended to *convex* domains, and more general families of C^1 -perturbations, but we have chosen to consider a specific setting, for the sake of clarity. The extension to more general Lipschitz domains is problematic for the lack of appropriate regularity results.

The paper is organized as follows: In Sect. 2 we show how the problem can be reduced to a family of problems in the initial domain and, in Sect. 3, we show that the perturbed linear operators are sectorial operators in suitable spaces. In Sect. 4 we show that the problem can be reformulated as an abstract problem in a scale of Banach spaces which are shown to be locally well posed in Sect. 5, under suitable growth assumptions on f and g . In Sect. 6, assuming a dissipative condition for the problem, using the properties of a Lyapunov functional, we prove that the solutions are globally defined. In Sect. 7 we prove the existence of global attractors. Finally, in Sect. 8, we show first that these attractors behave upper semicontinuously and, with some additional properties on the nonlinearities and on the set of equilibria, we show that they are also lower semicontinuous at $\epsilon = 0$ in the H^s -norm for $s < 1$.

2 Reduction to a fixed domain

One of the difficulties encountered in problems of perturbation of the domain is that the function spaces change with the change of the region. One way to overcome this difficulty is to perform a “change of variables” in order to bring the problem back to a fixed region. Here we use the approach developed by Henry in [7, Chapter 2].

If $\Omega \subset \mathbb{R}^n$ is a bounded region, we denote by $\text{Diff}^m(\Omega)$, $m \geq 0$, the set of C^m imbeddings. Thus, if $h \in \text{Diff}^m(\Omega)$, we may define the “pull-back” map $h^* : C^m(h(\Omega)) \rightarrow C^m(\Omega)$ by

$$h^*u(x) = (u \circ h)(x) = u(h(x)), x \in \Omega.$$

Now let Ω be the unit square in $\mathbb{R}^2$, and consider the family of maps $h_\epsilon : \Omega \rightarrow \mathbb{R}^2$, defined by (1). For simplicity we will denote by Ω_ϵ the corresponding family of “perturbed domains” $\Omega_{h_\epsilon} = h_\epsilon(\Omega)$.

Lemma 2.1 *If $\epsilon > 0$ is sufficiently small, the map h_ϵ defined by (1) belongs to $\text{Diff}^m(\Omega)$, for any $m \geq 1$ and $\|h_\epsilon - i_\Omega\|_{C^1(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0^+$.*

Proof It is not difficult to see that h_ϵ belongs to $\text{Diff}^m(\Omega)$, for any $m \geq 1$, and $\|h_\epsilon - i_\Omega\|_{C^1(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Indeed, if ϵ is small, $h_\epsilon \in C^m(\Omega)$ is injective and $|Jh_\epsilon(x)|^{-1} = |1 + \epsilon \sin(x_1/\epsilon^\alpha)|^{-1}$ is bounded in Ω , proving that $h_\epsilon \in \text{Diff}^m(\Omega)$. Now, a simple computation shows that

$$\begin{aligned} & \|h_\epsilon - i_\Omega\|_{C^1(\Omega)} \\ &= \max \left\{ \sup_{x \in \Omega} |\epsilon x_2 \sin(x_1/\epsilon^\alpha)|, \sup_{x \in \Omega} (\epsilon^{2-2\alpha} x_2^2 \cos^2(x_1/\epsilon^\alpha) + \epsilon^2 \sin^2(x_1/\epsilon^\alpha))^{\frac{1}{2}} \right\}. \end{aligned}$$

Therefore, if $0 < \alpha < 1$, it follows that $\|h_\epsilon - i_\Omega\|_{C^1(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0^+$. $\square$

Remark 2.2 The hypothesis $\alpha < 1$ is essential here. We do not have $\mathcal{C}^1$ -convergence of the family h_ϵ if $\alpha \geq 1$. Also, convergence in $\mathcal{C}^2$ -norm only holds if $\alpha < \frac{1}{2}$.

Lemma 2.3 If $s > 0$ and $\epsilon > 0$ is small enough, the map

$$h_\epsilon^* : H^s(\Omega_\epsilon) \rightarrow H^s(\Omega) : u \mapsto u \circ h_\epsilon$$

is an isomorphism with inverse $h_\epsilon^{*-1} = (h_\epsilon^{-1})^*$.

Proof We restrict ourselves to the case $0 \leq s < 1$, the proof for $s \geq 1$ is similar and is left to the reader. It is clear that h_ϵ^* is invertible with inverse $h_\epsilon^{*-1} = (h_\epsilon^{-1})^*$ as $\epsilon > 0$ is sufficiently small. Now, since the derivatives of h_ϵ^{-1} are bounded below by a constant M , we have $|h_\epsilon^{-1}(r) - h_\epsilon^{-1}(w)| \geq M|r - w|$ uniformly for $(r, w) \in \Omega_\epsilon \times \Omega_\epsilon$. Thus

$$\begin{aligned} & \|h_\epsilon^* u\|_{H^s(\Omega)} \\ &= \left\{ \int_{\Omega} |(u \circ h_\epsilon)(x)|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|(u \circ h_\epsilon)(x) - (u \circ h_\epsilon)(y)|^2}{|x - y|^{2+2s}} dx dy \right\}^{\frac{1}{2}} \\ &= \left\{ \int_{\Omega_\epsilon} |u(r)|^2 |Jh_\epsilon^{-1}(r)| dr \right. \\ &\quad \left. + \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} \frac{|u(r) - u(w)|^2}{|r - w|^{2+2s}} \frac{|r - w|^{2+2s}}{|h_\epsilon^{-1}(r) - h_\epsilon^{-1}(w)|^{2+2s}} |Jh_\epsilon^{-1}(r)| |Jh_\epsilon^{-1}(w)| dr dw \right\}^{\frac{1}{2}} \\ &\leq \left\{ K \int_{\Omega_\epsilon} |u(r)|^2 dr + \frac{K^2}{(2M)^{2+2s}} \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} \frac{|u(r) - u(w)|^2}{|r - w|^{2+2s}} dr dw \right\}^{\frac{1}{2}}, \end{aligned}$$

where K is a bound for $|Jh_\epsilon^{-1}(r)|$ in Ω_ϵ . A similar argument shows that $h_\epsilon^{*-1} = (h_\epsilon^{-1})^*$ is also bounded. $\square$

Let Δ_{Ω_ϵ} be the Laplacian operator in the region $\Omega_\epsilon = h_\epsilon(\Omega)$. We want to find an expression for the differential operator $h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}$ in the fixed region Ω , in terms of h_ϵ . Writing $h_\epsilon(x) = h_\epsilon(x_1, x_2) = ((h_\epsilon)_1(x), (h_\epsilon)_2(x)) = (y_1, y_2) = y$, we obtain, for $i = 1, 2$

$$\begin{aligned} \left(h_\epsilon^* \frac{\partial}{\partial y_i} h_\epsilon^{*-1}(u) \right)(x) &= \frac{\partial}{\partial y_i} (u \circ h_\epsilon^{-1})(h_\epsilon(x)) \\ &= \frac{\partial u}{\partial x_1}(h_\epsilon^{-1}(y)) \frac{\partial (h_\epsilon)_1^{-1}(y)}{\partial y_i}(y) + \frac{\partial u}{\partial x_2}(h_\epsilon^{-1}(y)) \frac{\partial (h_\epsilon)_2^{-1}(y)}{\partial y_i}(y) \\ &= \sum_{j=1}^2 \left[\left(\frac{\partial h_\epsilon}{\partial x_j} \right)^{-1} \right]_{j,i}(x) \frac{\partial u}{\partial x_j}(x) \\ &= \sum_{j=1}^2 b_{ij}(x) \frac{\partial u}{\partial x_j}(x), \end{aligned} \quad (2)$$

where $b_{ij}(x)$ is the i, j -entry of the inverse transpose of the Jacobian matrix of h_ϵ ,

$$[h_\epsilon^{-1}]_x^T = \begin{pmatrix} 1 & \frac{-x_2 \epsilon^{1-\alpha} \cos(x_1/\epsilon^\alpha)}{1 + \epsilon \sin(x_1/\epsilon^\alpha)} \\ 0 & \frac{1}{1 + \epsilon \sin(x_1/\epsilon^\alpha)} \end{pmatrix}. \quad (3)$$

Therefore,

$$\begin{aligned}
 h_{\epsilon}^* \Delta_{\Omega_{\epsilon}} h_{\epsilon}^{*-1}(u)(x) &= \sum_{i=1}^2 \left(h_{\epsilon}^* \frac{\partial^2}{\partial y_i^2} h_{\epsilon}^{*-1}(u) \right)(x) \\
 &= \sum_{i=1}^2 \left(\sum_{k=1}^2 b_{ik}(x) \frac{\partial}{\partial x_k} \left(\sum_{j=1}^2 b_{ij} \frac{\partial u}{\partial x_j} \right) \right)(x) \\
 &= \operatorname{div} \left(\frac{\partial u}{\partial x_1} + b_{12} \frac{\partial u}{\partial x_2}, (b_{12}^2 + b_{22}^2) \frac{\partial u}{\partial x_2} + b_{12} \frac{\partial u}{\partial x_1} \right)(x) \\
 &\quad - \frac{\partial b_{12}}{\partial x_2} \frac{\partial u}{\partial x_1}(x) - b_{12} \frac{\partial b_{12}}{\partial x_2} \frac{\partial u}{\partial x_2}(x). \tag{4}
 \end{aligned}$$

We also need to compute the boundary condition $h_{\epsilon}^* \frac{\partial}{\partial N_{\Omega_{\epsilon}}} h_{\epsilon}^{*-1} u = 0$ in the fixed region Ω in terms of h_{ϵ} . Let $N_{h_{\epsilon}(\Omega)}$ denote the outward unit normal to the boundary of $h_{\epsilon}(\Omega) = \Omega_{\epsilon}$. From (2), we obtain

$$\begin{aligned}
 \left(h_{\epsilon}^* \frac{\partial}{\partial N_{\Omega_{\epsilon}}} h_{\epsilon}^{*-1} u \right)(x) &= \sum_{i=1}^2 \left(h_{\epsilon}^* \frac{\partial}{\partial y_i} h_{\epsilon}^{*-1} u \right)(x) (N_{\Omega_{\epsilon}})_i(h_{\epsilon}(x)) \\
 &= \sum_{i,j=1}^2 b_{ij}(x) \frac{\partial u}{\partial x_j}(x) (N_{\Omega_{\epsilon}})_i(h_{\epsilon}(x)) \\
 &= N_{\Omega_{\epsilon}}(h_{\epsilon}(x)) \cdot (u_{x_1}(x) + b_{12}(x)u_{x_2}(x), b_{22}(x)u_{x_2}(x)).
 \end{aligned}$$

Since

$$\begin{aligned}
 h_{\epsilon}^* N_{\Omega_{\epsilon}}(x) &= N_{\Omega_{\epsilon}}(h_{\epsilon}(x)) = \frac{[h_{\epsilon}^{-1}]_x^T N_{\Omega}(x)}{\|[h_{\epsilon}^{-1}]_x^T N_{\Omega}(x)\|} \\
 &= \frac{((N_{\Omega}(x))_1 + b_{12}(x)(N_{\Omega}(x))_2, b_{22}(x)(N_{\Omega}(x))_2)}{\|[h_{\epsilon}^{-1}]_x^T N_{\Omega}(x)\|},
 \end{aligned}$$

the boundary condition $\left(h_{\epsilon}^* \frac{\partial}{\partial N_{\Omega_{\epsilon}}} h_{\epsilon}^{*-1} u \right)(x) = 0$ becomes

$$\begin{aligned}
 (N_{\Omega}(x))_1 [-u_{x_1}(x) - b_{12}(x)u_{x_2}(x)] \\
 + (N_{\Omega}(x))_2 [-b_{12}(x)u_{x_1}(x) - (b_{12}^2(x) + b_{22}^2(x))u_{x_2}(x)] = 0,
 \end{aligned}$$

which can be written as

$$\sum_{i,j=1}^2 (N_{\Omega}(x))_i (c_{ij} D_j u) = 0 \quad \text{on } \partial\Omega,$$

where

$$c_{11} = -1, \quad c_{12} = c_{21} = -b_{12}, \quad c_{22} = -(b_{12}^2 + b_{22}^2). \tag{5}$$

Consequently, we have that $v(\cdot, t)$ is a solution of (P_{ϵ}) in the perturbed region $\Omega_{\epsilon} = h_{\epsilon}(\Omega)$, if and only if $u(\cdot, t) = h_{\epsilon}^* v(\cdot, t)$ satisfies

$$\begin{cases} u_t(x, t) &= h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} u(x, t) - au(x, t) + f(u(x, t)), \quad x \in \Omega \text{ and } t > 0, \\ h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u(x, t) &= g(u(x, t)), \quad x \in \partial\Omega \text{ and } t > 0, \end{cases} \quad (6)$$

in the *fixed* region Ω .

3 Sectoriality of the perturbed operators

In this section we show that the family of differential operators $-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + aI$, appearing in (6), generate sectorial operators in various spaces.

3.1 Sectoriality in L^2

Consider the operator

$$A_\epsilon := (-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + aI) : L^2(\Omega) \rightarrow L^2(\Omega) \quad (7)$$

with domain

$$D(A_\epsilon) = \left\{ u \in H^2(\Omega) \left| h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u = 0, \text{ on } \partial\Omega \right. \right\}. \quad (8)$$

We will simply denote by A the unperturbed operator $(-\Delta_\Omega + aI)$.

In the case of smooth domains, it is not difficult to prove that A_ϵ is sectorial, for ϵ sufficiently small, but for general Lipschitz domains, we need to address some delicate questions of regularity. Fortunately, in our case, [5, Theorem 3.2.1.3] can be used, since we are dealing with a convex domain. However, this result is applicable only for operator in the divergence form. For this reason, using (4) we write our operator as

$$A_\epsilon = (C_\epsilon + aI + L_\epsilon) \quad (9)$$

where

$$C_\epsilon u = \sum_{i,j=1}^2 D_i (c_{ij} D_j u) \quad (10)$$

with the c_{ij} given in (5), and

$$L_\epsilon u = \frac{\partial b_{12}}{\partial x_2} \frac{\partial u}{\partial x_1} + b_{12} \frac{\partial b_{12}}{\partial x_2} \frac{\partial u}{\partial x_2}.$$

We now want to show that, if ϵ is small, the operator defined by (7) and (8) is sectorial. To this end, we need some auxiliary results for the first term in the decomposition (9).

Lemma 3.1 *If $\epsilon > 0$ is sufficiently small, the differential operator C_ϵ given by (10) is strongly elliptic.*

Proof If $\epsilon > 0$ is sufficiently small, we have $c_{11} = -1$, $-\frac{1}{4} < c_{12} = c_{21} = -b_{12} < \frac{1}{4}$ and $c_{22} = -(b_{12}^2 + b_{22}^2) < -\frac{1}{2}$. Therefore, if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
\sum_{i,j=1}^2 c_{ij} \xi_i \xi_j &\leq -\frac{1}{2} \xi_1^2 + \frac{1}{2} |\xi_1| |\xi_2| - \frac{1}{2} \xi_2^2 \\
&= -\frac{1}{4} (\xi_1^2 + \xi_2^2) - \left(\frac{1}{4} \xi_1^2 - \frac{1}{2} |\xi_1| |\xi_2| + \frac{1}{4} \xi_2^2 \right) \\
&\leq -\frac{1}{4} (\xi_1^2 + \xi_2^2).
\end{aligned} \tag{11}$$

□

Lemma 3.2 *If $\epsilon > 0$ is small enough, the differential operator C_ϵ defined by (10), with domain*

$$D(C_\epsilon) = \left\{ u \in H^2(\Omega) \mid \sum_{i,j=1}^2 (N_\Omega(x))_i (c_{ij} D_j u) = 0, x \in \partial\Omega \right\}, \tag{12}$$

is symmetric and bounded below in $L^2(\Omega)$.

Proof Using integration by parts, we have, for any $u, v \in D(C_\epsilon)$

$$\begin{aligned}
\langle C_\epsilon u, v \rangle_{L^2(\Omega)} &= \int_{\partial\Omega} v \left[\sum_{i,j=1}^2 (N_\Omega(x))_i (c_{ij} D_j u) \right] d\sigma(x) - \int_{\Omega} \sum_{i,j=1}^2 (c_{ij} D_j u D_i v) dx \\
&= - \int_{\Omega} \sum_{i,j=1}^2 (c_{ji} D_i v D_j u) dx = \langle u, C_\epsilon v \rangle_{L^2(\Omega)},
\end{aligned}$$

proving that C_ϵ is symmetric. Now, from (11) we get C_ϵ is bounded below since

$$\begin{aligned}
\langle C_\epsilon u, u \rangle_{L^2(\Omega)} &= \int_{\Omega} \sum_{i,j=1}^2 D_i (c_{ij} D_j u) \cdot u dx \\
&= - \int_{\Omega} \sum_{i,j=1}^2 (c_{ij} D_j u D_i u) dx \geq \frac{1}{4} \int_{\Omega} |\nabla u|^2 dx.
\end{aligned} \tag{13}$$

□

Remark 3.3 If u and v are complex valued with real and imaginary parts in $D(C_\epsilon)$, we still have

$$\begin{aligned}
\langle C_\epsilon u, u \rangle_{L^2(\Omega)} &= \int_{\Omega} \sum_{i,j=1}^2 D_i (c_{ij} D_j u) \cdot \bar{u} dx \\
&= - \int_{\Omega} \sum_{i,j=1}^2 (c_{ij} D_j u \overline{D_i u}) dx \geq \frac{1}{4} \int_{\Omega} |\nabla u|^2 dx.
\end{aligned} \tag{14}$$

Lemma 3.4 *If $\epsilon > 0$ is sufficiently small and $a > 0$, then the problem*

$$\begin{cases} C_\epsilon u + au &= f, \quad x \in \Omega \\ \sum_{i,j=1}^2 (N_\Omega(x))_i (c_{ij} D_j u) &= 0, \quad x \in \partial\Omega \end{cases}$$

has a unique solution $u \in H^2(\Omega)$, for any $f \in L^2(\Omega)$.

Proof Since Ω is a bounded and convex domain, it follows from [5, Theorem 3.2.1.3]. $\square$

Theorem 3.5 *If $\epsilon > 0$ is small enough, the operator C_ϵ defined by (10) with domain given by (12) is self-adjoint in $L^2(\Omega)$.*

Proof It is clear that C_ϵ is densely defined. It is also symmetric and lower bounded by Lemma 3.2. From Lemma 3.4 it follows that $C_\epsilon + aI$ is surjective for any $a > 0$ and therefore, an isomorphism from $D(C_\epsilon)$ to $L^2(\Omega)$. Thus $(C_\epsilon + aI)^{-1}$ is continuous as an operator in $L^2(\Omega)$ and has a closed graph. Hence, C_ϵ is closed. Then, due to [14, Proposition 3.11] we conclude that C_ϵ is self-adjoint. $\square$

From Theorem 3.5 it already follows that C_ϵ is a sectorial operator, but we give a direct proof to display the value of the constants and the sector.

Theorem 3.6 *If $\epsilon > 0$ is small enough, the operator C_ϵ in $L^2(\Omega)$ defined by (10) and (12) is sectorial and its sector can be chosen with vertex at any $b < 0$ and opening angle $0 < \theta < \frac{\pi}{2}$ with constant $M = \operatorname{cosec}(\theta)$.*

Proof We first observe that, as needed when treating spectral theory, we work in the complexification of the relevant spaces. For simplicity, we will not change notation, writing, for instance, $D(C_\epsilon)$ for the complexification of the domain of the operator C_ϵ .

Let $b < 0$, $0 < \phi < \frac{\pi}{2}$ and $\lambda = \alpha + i\beta$ a complex number in the sector $S_{b, \theta} = \{\lambda \in \mathbb{C} \mid \theta \leq |\arg(\lambda - b)| \leq \pi, \lambda \neq b\}$. If u is in (the complexification of) $D(C_\epsilon)$, we have

$$\begin{aligned} \|(C_\epsilon - \lambda)u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} &\geq |\langle (C_\epsilon - \lambda)u, u \rangle_{L^2(\Omega)}| \\ &= |\langle (C_\epsilon - \alpha)u - i\beta u, u \rangle_{L^2(\Omega)}| \\ &= \left[\langle (C_\epsilon - \alpha)u, u \rangle_{L^2(\Omega)}^2 + \beta^2 \langle u, u \rangle_{L^2(\Omega)}^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (15)$$

If $\alpha \leq b < 0$, it follows from (15) and (14) that

$$\begin{aligned} &\|(C_\epsilon - \lambda)u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\geq \left[\langle (C_\epsilon - b)u, u \rangle_{L^2(\Omega)}^2 + \langle (b - \alpha)u, u \rangle_{L^2(\Omega)}^2 + \beta^2 \langle u, u \rangle_{L^2(\Omega)}^2 \right]^{\frac{1}{2}} \\ &\geq \left[(b - \alpha)^2 + \beta^2 \right]^{\frac{1}{2}} \langle u, u \rangle_{L^2(\Omega)}. \end{aligned}$$

Hence, we obtain $\|(C_\epsilon - \lambda)u\|_{L^2(\Omega)} \geq |\lambda - b| \|u\|_{L^2(\Omega)}$, for any $u \in D(C_\epsilon)$, and then

$$\|(C_\epsilon - \lambda)^{-1}u\|_{L^2(\Omega)} \leq \frac{1}{|\lambda - b|} \|u\|_{L^2(\Omega)} \quad (16)$$

and the resolvent inequality holds with $M = 1$.

If $\alpha \geq b$, then $\beta \geq \tan(\theta)$, and it follows from (15) that

$$\|(C_\epsilon - \lambda)u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \geq \left[\beta^2 \langle u, u \rangle_{L^2(\Omega)}^2 \right]^{\frac{1}{2}} \geq |\beta| \langle u, u \rangle_{L^2(\Omega)}.$$

Then $\|(C_\epsilon - \lambda)u\|_{L^2(\Omega)} \geq |\lambda - b| \frac{|\beta|}{|\lambda - b|} \|u\|_{L^2(\Omega)} \geq |\lambda - b| \sin \theta \|u\|_{L^2(\Omega)}$ for any $u \in D(C_\epsilon)$, and thus

$$\|(C_\epsilon - \lambda)^{-1}u\|_{L^2(\Omega)} \leq \frac{\operatorname{cosec}(\theta)}{|\lambda - b|} \|u\|_{L^2(\Omega)} \quad (17)$$

and the resolvent inequality holds with $M = \operatorname{cosec}(\theta)$.

From (16) and (17), we conclude that C_ϵ is sectorial and the sector can be any one with vertex in $b < 0$ and opening angle $0 < \theta < \frac{\pi}{2}$, with constant $M = \operatorname{cosec}(\theta)$. $\square$

Theorem 3.7 *If $\epsilon > 0$ is small enough and $h_\epsilon \in \operatorname{Diff}^1(\Omega)$, then $A_\epsilon = (-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + aI)$ defined by (7) and (8) is sectorial.*

Proof We write the operator as in (9) $A_\epsilon = -h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + aI = C_\epsilon + aI + L_\epsilon$, and observe that, if ϵ is small enough, by Theorem 3.6 the operator $C_\epsilon + aI$ is sectorial with vertex in the origin and opening angle $0 < \theta < \frac{\pi}{2}$, with constant $M = \operatorname{cosec}(\theta)$. For definiteness, we may take $\theta = \frac{\pi}{6}$, $M = 2$. Furthermore, by (13), we obtain

$$\|(C_\epsilon + a)u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \geq K \|u\|_{H^1(\Omega)}^2,$$

where $K = \min\{\frac{1}{4}, a\}$ is a positive constant. Thus $\|(C_\epsilon + a)u\|_{L^2(\Omega)} \geq K \|u\|_{H^1(\Omega)}$.

Now, $D(C_\epsilon + aI + L_\epsilon) = D(C_\epsilon + aI)$ and, $\forall u \in D(C_\epsilon + aI)$, we have

$$\begin{aligned} & \|(C_\epsilon u + au + L_\epsilon u) - (C_\epsilon u + au)\|_{L^2(\Omega)} \\ &= \|L_\epsilon u\|_{L^2(\Omega)} \leq K_1 \left(\left\| \frac{\partial b_{12}}{\partial x_2} \right\|_\infty + \left\| b_{12} \frac{\partial b_{12}}{\partial x_2} \right\|_\infty \right) \|u\|_{H^1(\Omega)} \\ &\leq K_2 \left(\left\| \frac{\partial b_{12}}{\partial x_2} \right\|_\infty + \left\| b_{12} \frac{\partial b_{12}}{\partial x_2} \right\|_\infty \right) \|(C_\epsilon + aI)u\|_{L^2(\Omega)} \leq \eta(\epsilon) \|(C_\epsilon + aI)u\|_{L^2(\Omega)}, \end{aligned}$$

where $\eta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. So, from [12, Lemma 3.1], we get A_ϵ is sectorial with the sector

$$S_{0, \frac{\pi}{6}} = \left\{ \lambda \in \mathbb{C} \mid \frac{\pi}{6} \leq |\arg(\lambda)| \leq \pi, \lambda \neq 0 \right\} \quad (18)$$

satisfying

$$\|(\lambda - A_\epsilon)^{-1}\| \leq \frac{4\sqrt{5}}{|\lambda|} \quad \text{for all } \lambda \in S_{0, \frac{\pi}{6}}. \quad (19)$$

$\square$

Remark 3.8 In the above proof, we see that the sector and constant M in the resolvent inequality can be chosen independently of ϵ , for instance, as given in (18) and (19).

3.2 Sectoriality in $H^{-1}(\Omega)$

Henceforth we will denote by H^{-s} the dual space of H^s , either on Ω , Ω_ϵ , $\partial\Omega$ or $\partial\Omega_\epsilon$. Note that this symbol is usually reserved to denote the dual space of H_0^s . However, this notation should produce no confusion. The duality pairing between these spaces will be denoted by $\langle \cdot, \cdot \rangle_{-s, s}$. We now want to extend the operator A_ϵ to an operator $\tilde{A}_\epsilon$ in $H^{-1}(\Omega)$ with $D(\tilde{A}_\epsilon) = H^1(\Omega)$ and show that this extension is also sectorial for ϵ small.

If $u \in D(A_\epsilon) = \left\{ u \in H^2(\Omega) \mid h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u = 0 \right\}$, $\psi \in H^1(\Omega)$ and $v = u \circ h_\epsilon^{-1}$, we obtain

$$\begin{aligned}
 \langle A_\epsilon u, \psi \rangle_{-1,1} &= - \int_{\Omega} (h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x) \psi(x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx \\
 &= - \int_{\Omega} \Delta_{\Omega_\epsilon} (u \circ h_\epsilon^{-1})(h_\epsilon(x)) \psi(x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx \\
 &= - \int_{\Omega_\epsilon} \Delta_{\Omega_\epsilon} v(y) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, dy \\
 &\quad + a \int_{\Omega_\epsilon} u(h_\epsilon^{-1}(y)) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, dy \\
 &= - \int_{\partial \Omega_\epsilon} \frac{\partial v}{\partial N_{\Omega_\epsilon}}(y) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, d\sigma(y) \\
 &\quad + \int_{\Omega_\epsilon} \nabla_{\Omega_\epsilon} v(y) \cdot \nabla_{\Omega_\epsilon} \left(\psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \right) dy \\
 &\quad + a \int_{\Omega_\epsilon} u(h_\epsilon^{-1}(y)) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, dy \\
 &= \int_{\Omega_\epsilon} \nabla_{\Omega_\epsilon} v(y) \cdot \nabla_{\Omega_\epsilon} \left(\psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \right) dy \\
 &\quad + a \int_{\Omega_\epsilon} u(h_\epsilon^{-1}(y)) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, dy \\
 &= \int_{\Omega} (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x) \cdot \left(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} \frac{\psi}{|Jh_\epsilon|} \right)(x) |Jh_\epsilon(x)| \, dx \\
 &\quad + a \int_{\Omega} u(x) \psi(x) \, dx. \tag{20}
 \end{aligned}$$

Since (20) is well defined for $u \in H^1(\Omega)$, we may define an extension $\tilde{A}_\epsilon$ of A_ϵ , with values in $H^{-1}(\Omega)$ by this expression. For simplicity, we still denote this extension by A_ϵ , whenever there is no danger of confusion. We now show that this extension is a sectorial operator.

Theorem 3.9 *If $\epsilon > 0$ is sufficiently small, the operator A_ϵ defined by (20), with domain $H^1(\Omega)$ is sectorial.*

Proof We will apply [12, Lemma 3.1] again. If $\epsilon = 0$ we obtain, from (20)

$$\langle A_\epsilon u, u \rangle_{-1,1} = \langle Au, u \rangle_{-1,1} \geq \int_{\Omega} |\nabla_{\Omega} u|^2(x) \, dx + a \int_{\Omega} u^2(x) \, dx.$$

From the Lax-Milgram theorem, it follows that, for any $\psi \in H^{-1}(\Omega)$, there exists $u \in H^1(\Omega)$ with $Au = \psi$, so Au is well defined as an operator in $H^{-1}(\Omega)$ with $D(A) = H^1(\Omega)$. Arguing as in Theorems 3.5 and 3.6, we can prove A is self-adjoint and sectorial.

We now want to show that the family of operators $\{A_\epsilon\}$ satisfy the hypotheses of [12, Theorem 3.3] for ϵ small. Clearly $D(A_\epsilon) \supset D(A)$ for any $\epsilon \geq 0$. We now prove that there exists a positive function $\tau(\epsilon)$, with $\lim_{\epsilon \rightarrow 0^+} \tau(\epsilon) = 0$, such that

$$\| (A_\epsilon - A)u \|_{H^{-1}(\Omega)} \leq \tau(\epsilon) \| Au \|_{H^{-1}(\Omega)}, \quad \forall u \in D(A),$$

which is equivalent to

$$\left| \langle (A_\epsilon - A)u, \psi \rangle_{-1,1} \right| \leq \tau(\epsilon) \| u \|_{H^1(\Omega)} \| \psi \|_{H^1(\Omega)},$$

for all $u, \psi \in H^1(\Omega)$. In fact, for $\epsilon > 0$

$$\begin{aligned} \left| \langle (A_\epsilon - A)u, \psi \rangle_{-1,1} \right| &= \left| \int_{\Omega} \left\{ (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u - \nabla_{\Omega} u) \cdot \left(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} \frac{\psi}{|Jh_\epsilon|} \right) |Jh_\epsilon| \right. \right. \\ &\quad \left. \left. + \nabla_{\Omega} u \cdot \left(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} \frac{\psi}{|Jh_\epsilon|} \right) |Jh_\epsilon| - \nabla_{\Omega} u \nabla_{\Omega} \psi \right\} dx \right| \\ &\leq \int_{\Omega} \left| (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u - \nabla_{\Omega} u) \cdot \left(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} \frac{\psi}{|Jh_\epsilon|} \right) \right| |Jh_\epsilon| dx \end{aligned} \quad (21)$$

$$+ \int_{\Omega} \left| \nabla_{\Omega} u \cdot \left(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} - \nabla_{\Omega} \right) \left(\frac{\psi}{|Jh_\epsilon|} \right) \right| |Jh_\epsilon| dx \quad (22)$$

$$+ \int_{\Omega} \psi \nabla_{\Omega} u \cdot \nabla_{\Omega} \left(\frac{1}{|Jh_\epsilon|} \right) |Jh_\epsilon| dx. \quad (23)$$

To estimate (21), (22) and (23), we use the expression for $h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1}$ in terms of the coefficients b_{ij} given by (3). For (21), we have

$$\begin{aligned} &\int_{\Omega} \left| (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u - \nabla_{\Omega} u) \cdot \left(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} \frac{\psi}{|Jh_\epsilon|} \right) \right| |Jh_\epsilon| dx \\ &\leq \int_{\Omega} \left| \left(\sum_{j=1}^2 (b_{1j} - \delta_{1j}) \frac{\partial u}{\partial x_j} \right) \left(\sum_{j=1}^2 b_{1j} \frac{\partial}{\partial x_j} \left(\frac{\psi}{|Jh_\epsilon|} \right) \right) \right| \left| 1 + \epsilon \operatorname{sen}(x_1/\epsilon^\alpha) \right| dx \\ &\leq \max_{x \in \Omega} \{ |Jh_\epsilon| \} \underbrace{\left\{ \int_{\Omega} [b_{12}^2 + (b_{22} - 1)^2] \left(\frac{\partial u}{\partial x_2} \right)^2 dx \right\}^{\frac{1}{2}}}_{(I)} \\ &\quad \cdot \underbrace{\left\{ \int_{\Omega} \left[\frac{\partial}{\partial x_1} \left(\frac{\psi}{|Jh_\epsilon|} \right) + b_{12} \frac{\partial}{\partial x_2} \left(\frac{\psi}{|Jh_\epsilon|} \right) \right]^2 + b_{22}^2 \left[\frac{\partial}{\partial x_2} \left(\frac{\psi}{|Jh_\epsilon|} \right) \right]^2 dx \right\}^{\frac{1}{2}}}_{(II)}. \end{aligned}$$

For the integral (I), we have

$$\begin{aligned} \left\{ \int_{\Omega} [b_{12}^2 + (b_{22} - 1)^2] \left(\frac{\partial u}{\partial x_2} \right)^2 dx \right\}^{\frac{1}{2}} &\leq \left\{ \int_{\Omega} \frac{x_2^2 \epsilon^{2-2\alpha} + \epsilon^2}{[1 + \epsilon \operatorname{sen}(x_1/\epsilon^\alpha)]^2} \left(\frac{\partial u}{\partial x_2} \right)^2 dx \right\}^{\frac{1}{2}} \\ &\leq (K_1(\epsilon))^{\frac{1}{2}} \left\{ \int_{\Omega} \nabla_{\Omega} u \cdot \nabla_{\Omega} u dx \right\}^{\frac{1}{2}}, \end{aligned}$$

where $K_1(\epsilon) := \sup_{x \in \Omega} \frac{x_2^2 \epsilon^{2-2\alpha} + \epsilon^2}{[1 + \epsilon \operatorname{sen}(x_1/\epsilon^\alpha)]^2} \rightarrow 0$ as $h_\epsilon \rightarrow i_\Omega$ in $C^1(\Omega)$.

To estimate the integral (II), we first observe that

$$\frac{\partial}{\partial x_1} \left(\frac{1}{|Jh_\epsilon|} \right) = -\frac{\epsilon^{1-\alpha} \cos(x_1/\epsilon^\alpha)}{[1 + \epsilon \operatorname{sen}(x_1/\epsilon^\alpha)]^2} \quad \text{and} \quad \frac{\partial}{\partial x_2} \left(\frac{1}{|Jh_\epsilon|} \right) = 0.$$

Hence, we get the following estimate for (II)

$$\begin{aligned}
 & \left\{ \int_{\Omega} \left[\frac{\partial}{\partial x_1} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) + b_{12} \frac{\partial}{\partial x_2} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 + b_{22}^2 \left[\frac{\partial}{\partial x_2} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 dx \right\}^{\frac{1}{2}} \\
 & \leq \left\{ \int_{\Omega} 2 \left[\frac{\partial}{\partial x_1} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 + (2b_{12}^2 + b_{22}^2) \left[\frac{\partial}{\partial x_2} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 dx \right\}^{\frac{1}{2}} \\
 & = \left\{ \int_{\Omega} 2 \left[\psi \frac{\partial}{\partial x_1} \left(\frac{1}{|Jh_{\epsilon}|} \right) + \frac{1}{|Jh_{\epsilon}|} \frac{\partial \psi}{\partial x_1} \right]^2 + (2b_{12}^2 + b_{22}^2) \left[\frac{1}{|Jh_{\epsilon}|} \frac{\partial \psi}{\partial x_2} \right]^2 dx \right\}^{\frac{1}{2}} \\
 & \leq \underbrace{\left\{ \int_{\Omega} 4 \left(\frac{\partial}{\partial x_1} \left(\frac{1}{|Jh_{\epsilon}|} \right) \right)^2 \psi^2 dx \right\}^{\frac{1}{2}}}_{(III)} + \underbrace{\left\{ \int_{\Omega} \frac{4}{|Jh_{\epsilon}|^2} \left(\frac{\partial \psi}{\partial x_1} \right)^2 + \frac{2b_{12}^2 + b_{22}^2}{|Jh_{\epsilon}|^2} \left(\frac{\partial \psi}{\partial x_2} \right)^2 dx \right\}^{\frac{1}{2}}}_{(IV)}.
 \end{aligned}$$

To estimate (III), we first observe that if

$$K_2(\epsilon) := 4 \sup_{x \in \Omega} \left[\frac{\partial}{\partial x_1} \left(\frac{1}{|Jh_{\epsilon}|} \right) \right]^2 = \sup_{x \in \Omega} \frac{4\epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^{\alpha})}{[1 + \epsilon \sin(x_1/\epsilon^{\alpha})]^4}$$

then $K_2(\epsilon) \rightarrow 0$ as $h_{\epsilon} \rightarrow i_{\Omega}$ in $C^1(\Omega)$. Therefore

$$\begin{aligned}
 & \left\{ \int_{\Omega} \left[4 \left(\frac{\partial}{\partial x_1} \left(\frac{1}{|Jh_{\epsilon}|} \right) \right)^2 \right] \psi^2 dx \right\}^{\frac{1}{2}} \\
 & \leq (K_2(\epsilon))^{\frac{1}{2}} \left[\int_{\Omega} \psi^2 dx \right]^{\frac{1}{2}} \rightarrow 0 \text{ as } h_{\epsilon} \rightarrow i_{\Omega} \text{ in } C^1(\Omega).
 \end{aligned}$$

To estimate (IV), observe that

$$\frac{4}{|Jh_{\epsilon}|^2} = \frac{4}{[1 + \epsilon \sin(x_1/\epsilon^{\alpha})]^2} \quad \text{and} \quad \frac{2b_{12}^2 + b_{22}^2}{|Jh_{\epsilon}|^2} = \frac{2x_2^2 \epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^{\alpha}) + 1}{[1 + \epsilon \sin(x_1/\epsilon^{\alpha})]^4}$$

are bounded for $0 < \alpha < 1$ and $\epsilon > 0$ sufficiently small. Thus

$$\left\{ \int_{\Omega} \frac{4}{|Jh_{\epsilon}|^2} \left(\frac{\partial \psi}{\partial x_1} \right)^2 + \frac{2b_{12}^2 + b_{22}^2}{|Jh_{\epsilon}|^2} \left(\frac{\partial \psi}{\partial x_2} \right)^2 dx \right\}^{\frac{1}{2}} \leq K_3 \left\{ \int_{\Omega} \nabla_{\Omega} \psi \cdot \nabla_{\Omega} \psi dx \right\}^{\frac{1}{2}},$$

where K_3 is a positive constant. We then have the following estimate for (21):

$$\begin{aligned}
 & \int_{\Omega} \left| \left(h_{\epsilon}^* \nabla_{\Omega_{\epsilon}} h_{\epsilon}^{*-1} u - \nabla_{\Omega} u \right) \cdot \left(h_{\epsilon}^* \nabla_{\Omega_{\epsilon}} h_{\epsilon}^{*-1} \frac{\psi}{|Jh_{\epsilon}|} \right) \right| |Jh_{\epsilon}| dx \\
 & \leq \max_{x \in \Omega} \{ |Jh_{\epsilon}| \} [K_1(\epsilon) K_2(\epsilon)]^{\frac{1}{2}} \left\{ \int_{\Omega} \psi^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \nabla_{\Omega} u \cdot \nabla_{\Omega} u dx \right\}^{\frac{1}{2}} \\
 & \quad + \max_{x \in \Omega} \{ |Jh_{\epsilon}| \} [K_1(\epsilon)]^{\frac{1}{2}} K_3 \left\{ \int_{\Omega} \nabla_{\Omega} \psi \cdot \nabla_{\Omega} \psi dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \nabla_{\Omega} u \cdot \nabla_{\Omega} u dx \right\}^{\frac{1}{2}}.
 \end{aligned}$$

Taking

$$C_0(\epsilon) := \max_{x \in \Omega} \{ |Jh_{\epsilon}| \} [K_1(\epsilon) K_2(\epsilon)]^{\frac{1}{2}} \quad \text{and} \quad C'_0(\epsilon) := \max_{x \in \Omega} \{ |Jh_{\epsilon}| \} [K_1(\epsilon)]^{\frac{1}{2}} K_3,$$

we have that $C_0(\epsilon)$ and $C'_0(\epsilon) \rightarrow 0$ when $h_\epsilon \rightarrow i_\Omega$ in $C^1(\Omega)$. We estimate (22) in a similar way:

$$\begin{aligned} & \int_{\Omega} \left| \nabla_{\Omega} u \cdot (h_{\epsilon}^* \nabla_{\Omega_{\epsilon}} h_{\epsilon}^{*-1} - \nabla_{\Omega}) \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right| |Jh_{\epsilon}| \, dx \\ & \leq \max_{x \in \Omega} \{ |Jh_{\epsilon}| \} \left\{ \int_{\Omega} |\nabla_{\Omega} u|^2 \, dx \right\}^{\frac{1}{2}} \underbrace{\left\{ \int_{\Omega} \sum_{i=1}^2 \left[\sum_{j=1}^2 (b_{ij} - \delta_{ij}) \frac{\partial}{\partial x_j} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 \, dx \right\}^{\frac{1}{2}}}_{(V)}. \end{aligned}$$

For the integrand in (V), we have:

$$\begin{aligned} \sum_{i=1}^2 \left[\sum_{j=1}^2 (b_{ij} - \delta_{ij}) \frac{\partial}{\partial x_j} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 & \leq \sum_{i=1}^2 \left[2(b_{i1} - \delta_{i1})^2 \left[\frac{\partial}{\partial x_1} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 \right. \\ & \quad \left. + 2(b_{i2} - \delta_{i2})^2 \left[\frac{\partial}{\partial x_2} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 \right] \\ & = [2b_{12}^2 + 2(b_{22} - 1)^2] \frac{1}{|Jh_{\epsilon}|^2} \left(\frac{\partial \psi}{\partial x_2} \right)^2. \end{aligned}$$

We then have, for the integral (V)

$$\left\{ \int_{\Omega} \sum_{i=1}^2 \left[\sum_{j=1}^2 (b_{ij} - \delta_{ij}) \frac{\partial}{\partial x_j} \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right]^2 \, dx \right\}^{\frac{1}{2}} \leq \left\{ \int_{\Omega} \frac{2b_{12}^2 + 2(b_{22} - 1)^2}{|Jh_{\epsilon}|^2} \left(\frac{\partial \psi}{\partial x_2} \right)^2 \, dx \right\}^{\frac{1}{2}}$$

and if

$$K_4(\epsilon) := \sup_{x \in \Omega} \frac{2b_{12}^2 + 2(b_{22} - 1)^2}{|Jh_{\epsilon}|^2} = \sup_{x \in \Omega} \frac{2x_2^2 \epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^\alpha) + 2\epsilon^2 \sin^2(x_1/\epsilon^\alpha)}{|Jh_{\epsilon}|^2}$$

then $K_4(\epsilon) \rightarrow 0$ as $h_\epsilon \rightarrow i_\Omega$ in $C^1(\Omega)$. Thus, taking $C_1(\epsilon) := \max_{x \in \Omega} \{ |Jh_{\epsilon}(x)| \} [K_4(\epsilon)]^{\frac{1}{2}}$, we get from (22):

$$\int_{\Omega} \left| \nabla_{\Omega} u \cdot (h_{\epsilon}^* \nabla_{\Omega_{\epsilon}} h_{\epsilon}^{*-1} - \nabla_{\Omega}) \left(\frac{\psi}{|Jh_{\epsilon}|} \right) \right| |Jh_{\epsilon}| \, dx \leq C_1(\epsilon) \|\nabla_{\Omega} \psi\|_{L^2(\Omega)} \|\nabla_{\Omega} u\|_{L^2(\Omega)},$$

with $C_1(\epsilon) \rightarrow 0$ as $h_\epsilon \rightarrow i_\Omega$ in $C^1(\Omega)$.

Finally we have for the integral (23):

$$\begin{aligned} & \int_{\Omega} \left| \psi \nabla_{\Omega} u \cdot \nabla_{\Omega} \left(\frac{1}{|Jh_{\epsilon}|} \right) \right| |Jh_{\epsilon}| \, dx \\ & \leq \max_{x \in \Omega} \{ |Jh_{\epsilon}| \} \int_{\Omega} \psi \left[\sum_{j=1}^2 \left(\frac{\partial u}{\partial x_j} \right)^2 \right]^{\frac{1}{2}} \left[\sum_{j=1}^2 \left(\frac{\partial}{\partial x_j} \left(\frac{1}{|Jh_{\epsilon}|} \right) \right)^2 \right]^{\frac{1}{2}} \, dx \end{aligned}$$

$$\begin{aligned} &\leq \max_{x \in x} \{ |Jh_\epsilon| \} \left\{ \int_{\Omega} \psi^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \sum_{j=1}^2 \left(\frac{\partial u}{\partial x_j} \right)^2 \left[\sum_{j=1}^2 \left(\frac{\partial}{\partial x_j} \left(\frac{1}{|Jh_\epsilon|} \right) \right)^2 \right] dx \right\}^{\frac{1}{2}} \\ &= \max_{x \in \Omega} \{ |Jh_\epsilon| \} \left\{ \int_{\Omega} \psi^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \sum_{j=1}^2 \left(\frac{\partial u}{\partial x_j} \right)^2 \left[\frac{\epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^\alpha)}{|Jh_\epsilon|^2} \right] dx \right\}^{\frac{1}{2}}. \end{aligned}$$

Since $K_5(\epsilon) = \sup_{x \in \Omega} \frac{\epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^\alpha)}{|Jh_\epsilon|^2} \rightarrow 0$ as $h_\epsilon \rightarrow i_\Omega$ in $C^1(\Omega)$, we have for the integral (23)

$$\int_{\Omega} \left| \psi \nabla_{\Omega} u \cdot \nabla_{\Omega} \left(\frac{1}{|Jh_\epsilon|} \right) \right| |Jh_\epsilon| dx \leq C_2(\epsilon) \left\{ \int_{\Omega} \psi^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \nabla_{\Omega} u \cdot \nabla_{\Omega} u dx \right\}^{\frac{1}{2}},$$

with $C_2(\epsilon) := \max_{x \in \Omega} \{ |Jh_\epsilon| \} [K_5(\epsilon)]^{\frac{1}{2}} \rightarrow 0$ as $h_\epsilon \rightarrow i_\Omega$ in $C^1(\Omega)$.

Thus, we conclude that

$$|\langle (A_\epsilon - A)u, \psi \rangle_{-1,1}| \leq C(\epsilon) \|u\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}$$

with $\lim_{\epsilon \rightarrow 0^+} C(\epsilon) = 0$. Hence, the result follows from [12, Theorem 3.3] and

$$\|(A_\epsilon - A)u\|_{H^{-1}(\Omega)} \leq C(\epsilon) \|u\|_{H^1(\Omega)} \leq \tau(\epsilon) \|Au\|_{H^{-1}(\Omega)} \quad (24)$$

with $\lim_{\epsilon \rightarrow 0^+} \tau(\epsilon) = 0$. $\square$

Remark 3.10 From the above proof, it also follows that sector and the constant M in the resolvent inequality are the same as the ones for A_ϵ and can be chosen as in (18) and (19).

4 The abstract problem in a scale of Banach spaces

Our goal here is to pose the problem (P_ϵ) in a convenient abstract setting. We have proved in Theorem 3.7 that, if ϵ is small, the operator A_ϵ in $L^2(\Omega)$ defined by (7) with domain given in (8) is sectorial as well as its extension $\tilde{A}_\epsilon$ to $H^{-1}(\Omega)$. It is then well known that the domains X_ϵ^α (resp. $\tilde{X}_\epsilon^\alpha$), $\alpha \geq 0$, of the fractional powers of A_ϵ (resp. $\tilde{A}_\epsilon$), are Banach spaces, $X_\epsilon^0 = L^2(\Omega)$ (resp. $\tilde{X}_\epsilon^0 = H^{-1}(\Omega)$), $X_\epsilon^1 = D(A_\epsilon)$ (resp. $\tilde{X}_\epsilon^1 = D(\tilde{A}_\epsilon)$), X_ϵ^α ($\tilde{X}_\epsilon^\alpha$) is compactly imbedded in X_ϵ^β ($\tilde{X}_\epsilon^\beta$) when $\alpha > \beta \geq 0$, and $X_\epsilon^\alpha = H^{2\alpha}(\Omega)$ when 2α is an integer number.

Since $X_\epsilon^{1/2} = \tilde{X}_\epsilon^1$, it follows easily that $X_\epsilon^{\alpha-1/2} = \tilde{X}_\epsilon^\alpha$ for $1/2 \leq \alpha \leq 1$ and, by an abuse of notation, we will still write $X_\epsilon^{\alpha-1/2}$ instead of $\tilde{X}_\epsilon^\alpha$ for $0 \leq \alpha \leq 1/2$, so we may denote by $\{X_\epsilon^\alpha, -1/2 \leq \alpha \leq 1\} = \{X_\epsilon^\alpha, 0 \leq \alpha \leq 1\} \cup \{\tilde{X}_\epsilon^\alpha, 0 \leq \alpha \leq 1\}$ the whole family of fractional power spaces. We will denote simply by X^α the fractional power spaces associated to the unperturbed operator A .

For any $-1/2 \leq \beta \leq 0$, we may now define an operator in these spaces as the restriction of $\tilde{A}_\epsilon$. We then have the following result

Theorem 4.1 *For any $-1/2 \leq \beta \leq 0$ and ϵ sufficiently small, the operator $(A_\epsilon)_\beta$ in X_ϵ^β , obtained by restricting $\tilde{A}_\epsilon$ with domain $X_\epsilon^{\beta+1}$ is a sectorial operator.*

Proof Writing $\beta = -1/2 + \delta$, for some $0 \leq \delta \leq 1/2$, we have $(A_\epsilon)_\beta = \tilde{A}_\epsilon^{-\delta} \tilde{A}_\epsilon \tilde{A}_\epsilon^\delta$. Since $\tilde{A}_\epsilon^\delta$ is an isometry from X_ϵ^β to $X_\epsilon^{-1/2} = H^{-1}(\Omega)$, the result follows easily. $\square$

We now show that the scales $\{X_\epsilon^\alpha, -1/2 \leq \alpha \leq 1/2\} = \{\tilde{X}_\epsilon^\alpha, 0 \leq \alpha \leq 1\} = D(\tilde{A}_\epsilon^{\alpha+1/2})$ do not change when $\epsilon > 0$ varies. More precisely:

Theorem 4.2 *For any $0 \leq \alpha \leq 1$, let $\|\cdot\|_{\epsilon,\alpha}$ denote the norm in $X_\epsilon^{\alpha-1/2} = \tilde{X}_\epsilon^\alpha$ and $\|\cdot\|_\alpha$ the norm in $X^{\alpha-1/2} = \tilde{X}^\alpha$.*

Then, $\|u\|_{\epsilon,\alpha} \leq K_1(\epsilon)\|u\|_\alpha \leq K_2(\epsilon)\|u\|_{\epsilon,\alpha}$ with $K_1(\epsilon), K_2(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0^+$ uniformly in α . In particular, $X_\epsilon^\alpha = X^\alpha$ with equivalent norms, uniformly in ϵ .

Proof As observed above, using the same arguments of Theorems 3.5 and 3.6, we prove that $\tilde{A}_\epsilon$ is self-adjoint with respect to the usual inner product in $H^1(\Omega)$. Similarly, one can prove that $\tilde{A}_\epsilon$ is self-adjoint with respect to the inner product with “weight” $|Jh_\epsilon|$. It then follows, from well-known results, that the fractional powers of order α of these operators coincide isometrically with the interpolation spaces $[D(\tilde{A}), H^{-1}(\Omega)]_\alpha$ and $[D(\tilde{A}_\epsilon), H^{-1}(\Omega)]_\alpha$, respectively (see, for instance, Theorem 16.1 of [15]). Let $I : X \rightarrow Y$ denote the inclusion operator from $X \subset Y$ to Y and by $\|I\|_{\mathcal{L}(X,Y)}$ its norm as a linear operator. From [15, Theorem 1.15] we then have, for any $u \in D(\tilde{A}_\epsilon)$,

$$\begin{aligned} \|I\|_{\mathcal{L}([D(\tilde{A}), H^{-1}(\Omega)]_\alpha, [D(\tilde{A}_\epsilon), H^{-1}(\Omega)]_\alpha)} &\leq \|I\|_{\mathcal{L}(H^{-1}(\Omega), H^{-1}(\Omega))}^{(1-\alpha)} \|I\|_{\mathcal{L}(D(\tilde{A}), D(\tilde{A}_\epsilon))}^\alpha \\ &\leq \|I\|_{\mathcal{L}(D(\tilde{A}), D(\tilde{A}_\epsilon))}^\alpha \end{aligned} \quad (25)$$

Now, from (24) it follows that

$$\|Iu\|_{D(\tilde{A}_\epsilon)} = \|u\|_{D(\tilde{A}_\epsilon)} = \|\tilde{A}_\epsilon u\|_{H^{-1}(\Omega)} \leq (1 + \tau(\epsilon))\|\tilde{A}u\|_{H^{-1}(\Omega)} = (1 + \tau(\epsilon))\|u\|_{D(\tilde{A})},$$

where $\tau(\epsilon) \rightarrow 0$, uniformly for $u \in D(\tilde{A})$. Thus $\|I\|_{\mathcal{L}(D(\tilde{A}), D(\tilde{A}_\epsilon))} \leq (1 + \tau(\epsilon))$, and it follows from (25) that

$$\|u\|_{[D(\tilde{A}_\epsilon), H^{-1}(\Omega)]_\alpha} \leq (1 + \tau(\epsilon))^\alpha \|u\|_{[D(\tilde{A}), H^{-1}(\Omega)]_\alpha}$$

with $\tau(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. The reverse inequality follows similarly. $\square$

Using the results of Theorems 4.1 and 4.2, we can now pose the problem (6) as an abstract problem in the (fixed) scale of Banach spaces $\{X^\beta, -1/2 \leq \beta \leq 0\}$.

$$\begin{cases} u_t + (A_\epsilon)_\beta u = (H_\epsilon)_\beta u, & t > t_0; \\ u(t_0) = u_0 \in X^\eta, \end{cases} \quad (26)$$

where

$$(H_\epsilon)_\beta = H(\cdot, \epsilon) := (F_\epsilon)_\beta + (G_\epsilon)_\beta : X^\eta \rightarrow X^\beta, \quad \epsilon > 0 \text{ and } 0 \leq \eta \leq \beta + 1, \quad (27)$$

(i) $(F_\epsilon)_\beta = F(\cdot, \epsilon) : X^\eta \rightarrow X^\beta$ is given by

$$\langle F(u, \epsilon), \Phi \rangle_{\beta, -\beta} = \int_\Omega f(u) \Phi \, dx, \quad \text{for any } \Phi \in X^{-\beta}, \quad (28)$$

(ii) $(G_\epsilon)_\beta = G(\cdot, \epsilon) : X^\eta \rightarrow X^\beta$ is given by

$$\langle G(u, \epsilon), \Phi \rangle_{\beta, -\beta} = \int_{\partial\Omega} g(\gamma(u)) \gamma(\Phi) \left| \frac{J_{\partial\Omega} h_\epsilon}{Jh_\epsilon} \right| d\sigma(x), \quad \text{for any } \Phi \in X^{-\beta}, \quad (29)$$

where γ is the trace map and $J_{\partial\Omega} h_\epsilon$ is the determinant of the Jacobian matrix of the diffeomorphism $h_\epsilon : \partial\Omega \rightarrow \partial h_\epsilon(\Omega)$.

We show below that, under suitable conditions on η and β , the operators F_ϵ and G_ϵ are well defined. We note that F_ϵ does not depend on ϵ explicitly, but we have chosen to keep the ϵ in its notation for uniformity.

5 Local well-posedness

In order to prove local well-posedness for the abstract problem (26), we will assume the following growth conditions for the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$:

- (1) f is in $C^1(\mathbb{R}, \mathbb{R})$ and there exist real numbers $\lambda_1 > 0$ and $L_1 > 0$ such that

$$|f(u_1) - f(u_2)| \leq L_1 (1 + |u_1|^{\lambda_1} + |u_2|^{\lambda_1}) |u_1 - u_2| \text{ for all } u_1, u_2 \in \mathbb{R}. \quad (30)$$

- (2) g is in $C^2(\mathbb{R}, \mathbb{R})$ and there exist real numbers $\lambda_2 > 0$ and $L_2 > 0$ such that

$$|g(u_1) - g(u_2)| \leq L_2 (1 + |u_1|^{\lambda_2} + |u_2|^{\lambda_2}) |u_1 - u_2| \text{ for all } u_1, u_2 \in \mathbb{R}. \quad (31)$$

Lemma 5.1 *Suppose that f satisfies the growth condition (30) and $\eta > \frac{1}{2} - \frac{1}{2(\lambda_1+1)}$. Then, the map $(F_\epsilon)_\beta = F : X^\eta \rightarrow X^\beta$ given by (28) is well defined, and bounded in bounded sets.*

Proof If $u \in X^\eta$ and $\Phi \in X^{-\beta}$,

$$\begin{aligned} |\langle F(u, \epsilon), \Phi \rangle_{\beta, -\beta}| &\leq L_1 \int_{\Omega} |u| |\Phi| dx + L_1 \int_{\Omega} |u|^{\lambda_1+1} |\Phi| dx + \int_{\Omega} |f(0)| |\Phi| dx \\ &\leq L_1 \|u\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)} + L_1 \|u\|_{L^{2(\lambda_1+1)}(\Omega)}^{\lambda_1+1} \|\Phi\|_{L^2(\Omega)} + \|f(0)\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)}. \end{aligned}$$

By [8, Theorem 1.6.1] we have $X^{-\beta} \subset L^2(\Omega)$, $X^\eta \subset L^2(\Omega)$ and $X^\eta \subset L^{2(\lambda_1+1)}(\Omega)$, with embedding constants K_1 , K_2 and K_3 , respectively. Thus

$$\begin{aligned} |\langle F(u, \epsilon), \Phi \rangle_{\beta, -\beta}| &\leq L_1 K_1 K_2 \|u\|_{X^\eta} \|\Phi\|_{X^{-\beta}} + L_1 K_1 K_3^{\lambda_1+1} \|u\|_{X^\eta}^{\lambda_1+1} \|\Phi\|_{X^{-\beta}} \\ &\quad + K_1 \|f(0)\|_{L^2(\Omega)} \|\Phi\|_{X^{-\beta}}. \end{aligned}$$

Therefore, if $u \in X^\eta$ and $\Phi \in X^{-\beta}$, we have

$$\|F(u, \epsilon)\|_{X^\beta} \leq L_1 K_1 K_2 \|u\|_{X^\eta} + L_1 K_1 K_3^{\lambda_1+1} \|u\|_{X^\eta}^{\lambda_1+1} + K_1 \|f(0)\|_{L^2(\Omega)},$$

which concludes the proof. $\square$

Lemma 5.2 *Suppose that f satisfies the growth condition (30) and let p and q be conjugated exponents with $\frac{1}{\lambda_1} < p < \infty$. Then, if $\eta > \max \left\{ \frac{1}{2} - \frac{1}{2p\lambda_1}, \frac{1}{2} - \frac{1}{2q} \right\}$, the map $F(u, \epsilon)$ given by (28) is locally Lipschitz continuous in u .*

Proof If $u_1, u_2 \in X^\eta$, we have

$$\begin{aligned}
 & \left| \langle F(u_1, \epsilon) - F(u_2, \epsilon), \Phi \rangle_{\beta, -\beta} \right| \leq \int_{\Omega} L_1 (1 + |u_1|^{\lambda_1} + |u_2|^{\lambda_1}) |u_1 - u_2| |\Phi| dx \\
 & \leq L_1 \left\{ \int_{\Omega} (1 + |u_1|^{\lambda_1} + |u_2|^{\lambda_1})^2 |u_1 - u_2|^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} |\Phi|^2 dx \right\}^{\frac{1}{2}} \\
 & \leq L_1 \|1 + |u_1|^{\lambda_1} + |u_2|^{\lambda_1}\|_{L^{2p}(\Omega)} \|u_1 - u_2\|_{L^{2q}(\Omega)} \|\Phi\|_{L^2(\Omega)} \\
 & \leq L_1 \left(|\Omega|^{\frac{1}{2p}} + \|u_1^{\lambda_1}\|_{L^{2p}(\Omega)} + \|u_2^{\lambda_1}\|_{L^{2p}(\Omega)} \right) \|u_1 - u_2\|_{L^{2q}(\Omega)} \|\Phi\|_{L^2(\Omega)} \\
 & = L_1 (|\Omega|^{\frac{1}{2p}} + \|u_1\|_{L^{2p\lambda_1}(\Omega)}^{\lambda_1} + \|u_2\|_{L^{2p\lambda_1}(\Omega)}^{\lambda_1}) \|u_1 - u_2\|_{L^{2q}(\Omega)} \|\Phi\|_{L^2(\Omega)} \\
 & \leq L_1 (|\Omega|^{\frac{1}{2p}} + K_4^{\lambda_1} \|u_1\|_{X^\eta}^{\lambda_1} + K_4^{\lambda_1} \|u_2\|_{X^\eta}^{\lambda_1}) K_5 \|u_1 - u_2\|_{X^\eta} K_1 \|\Phi\|_{X^{-\beta}},
 \end{aligned}$$

where $|\Omega|$ is the measure of Ω .

Thus, by [8, Theorem 1.6.1], there exist constants K_1 , K_4 and K_5 such that

$$\begin{aligned}
 \left| \langle F(u_1, \epsilon) - F(u_2, \epsilon), \Phi \rangle_{\beta, -\beta} \right| & \leq L_1 (|\Omega|^{\frac{1}{2p}} + K_4^{\lambda_1} \|u_1\|_{X^\eta}^{\lambda_1} + K_4^{\lambda_1} \|u_2\|_{X^\eta}^{\lambda_1}) \\
 & \quad \cdot K_5 \|u_1 - u_2\|_{X^\eta} K_1 \|\Phi\|_{X^{-\beta}}.
 \end{aligned}$$

If U is a bounded subset of X^η and $u_1, u_2 \in U$, we have

$$\|F(u_1, \epsilon) - F(u_2, \epsilon)\|_{X^\beta} \leq K_{\lambda_1, U} \|u_1 - u_2\|_{X^\eta},$$

where $K_{\lambda_1, U}$ is a positive constant depending on λ_1 and U . This concludes the proof. $\square$

For the regularity properties of $(G_\epsilon)_\beta$ we will need to compute the function

$$\theta(x, \epsilon) := \left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon}(x) \right|, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

in each of the four segments of $\partial\Omega$. For $I_1 := \{(x_1, 1) \mid 0 \leq x_1 \leq 1\}$, we have $h_\epsilon(x_1, 1) = (x_1, 1 + \epsilon \sin(x_1/\epsilon^\alpha))$, so $\frac{d}{dx_1} h_\epsilon(x_1, 1) = Dh_\epsilon(x_1, 1) \cdot (1, 0) = (1, \epsilon^{1-\alpha} \cos(x_1/\epsilon^\alpha))$. Then, $h_\epsilon|_{I_1}$ takes the unit vector $(1, 0)$ tangent to I_1 to the vector $(1, \epsilon^{1-\alpha} \cos(x_1/\epsilon^\alpha))$. It follows that $\|Dh_\epsilon|_{I_1}(x_1, 1)\| = |J_{\partial\Omega} h_\epsilon(x_1, 1)| = \sqrt{1 + \epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^\alpha)}$. Thus

$$\theta(x, \epsilon) := \frac{\sqrt{1 + \epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^\alpha)}}{1 + \epsilon \sin(x_1/\epsilon^\alpha)}, \quad \text{for } x \in I_1. \quad (32)$$

Similar computations give

$$\begin{aligned}
 \theta(x, \epsilon) &:= \frac{1 + \epsilon \sin(1/\epsilon^\alpha)}{1 + \epsilon \sin(1/\epsilon^\alpha)} = 1 \text{ for } x \in I_2 := \{(1, x_2) \mid 0 \leq x_2 \leq 1\}, \\
 \theta(x, \epsilon) &:= \frac{1}{1 + \epsilon \sin(x_1/\epsilon^\alpha)} \text{ for } x \in I_3 := \{(x_1, 0) \mid 0 \leq x_1 \leq 1\}, \\
 \theta(x, \epsilon) &:= \frac{1}{1} = 1 \text{ for } x \in I_4 := \{(0, x_2) \mid 0 \leq x_2 \leq 1\}.
 \end{aligned} \quad (33)$$

Lemma 5.3 Suppose that g satisfies the growth condition (31), $\eta > \frac{1}{2} - \frac{1}{4(\lambda_2+1)}$, $\beta < -\frac{1}{4}$ and $\epsilon_0 < 1$. Then, the map $(G_\epsilon)_\beta = G : X^\eta \rightarrow X^\beta$ given by (29) is well defined, for $0 \leq \epsilon < \epsilon_0$ and bounded in bounded sets, uniformly in ϵ .

Proof If $u \in X^\eta$ and $\Phi \in X^{-\beta}$, we have

$$\begin{aligned} |\langle G(u, \epsilon), \Phi \rangle_{\beta, -\beta}| &\leq \int_{\partial\Omega} |g(\gamma(u))| |\gamma(\Phi)| \left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x) \\ &\leq \|\theta\|_\infty L_2 \left(\int_{\partial\Omega} |\gamma(u)|^2 d\sigma(x) \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |\gamma(\Phi)|^2 d\sigma(x) \right)^{\frac{1}{2}} \\ &\quad + \|\theta\|_\infty L_2 \left(\int_{\partial\Omega} |\gamma(u)|^{2(\lambda_2+1)} d\sigma(x) \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |\gamma(\Phi)|^2 d\sigma(x) \right)^{\frac{1}{2}} \\ &\quad + \|\theta\|_\infty \|g(\gamma(0))\|_{L^2(\partial\Omega)} \|\gamma(\Phi)\|_{L^2(\partial\Omega)} \\ &\leq \|\theta\|_\infty L_2 \left[\|\gamma(u)\|_{L^2(\partial\Omega)} + \|\gamma(u)\|_{L^{2(\lambda_2+1)}(\partial\Omega)}^{\lambda_2+1} \right] \|\gamma(\Phi)\|_{L^2(\partial\Omega)} \\ &\quad + \|\theta\|_\infty \|g(\gamma(0))\|_{L^2(\partial\Omega)} \|\gamma(\Phi)\|_{L^2(\partial\Omega)}, \end{aligned}$$

where $\|\theta\|_\infty = \sup \{|\theta(x, \epsilon)| : x \in \partial\Omega, 0 \leq \epsilon \leq \epsilon_0\}$ is finite by (32) and (33). If $s = 2\eta$, then $X^\eta \subset H^s(\Omega)$, by [8, Theorem 1.6.1], and $\gamma : H^s(\Omega) \mapsto L^{2(\lambda_2+1)}(\partial\Omega)$, by [9, Theorem 4.7]. Thus, we obtain $\|\gamma(\Phi)\|_{L^2(\partial\Omega)} \leq \bar{K}_1 \|\Phi\|_{X^{-\beta}}$, $\|\gamma(u)\|_{L^{2(\lambda_2+1)}(\partial\Omega)} \leq \bar{K}_2 \|u\|_{X^\eta}$, and $\|\gamma(u)\|_{L^2(\partial\Omega)} \leq \bar{K}_3 \|u\|_{X^\eta}$ where $\bar{K}_1$, $\bar{K}_2$ and $\bar{K}_3$ are embedding constants. It follows that

$$\begin{aligned} \|G(u, \epsilon)\|_{X^\beta} &\leq L_2 \bar{K}_3 \bar{K}_1 \|\theta\|_\infty \|u\|_{X^\eta} + L_2 \bar{K}_2^{\lambda_2+1} \bar{K}_1 \|\theta\|_\infty \|u\|_{X^\eta} \\ &\quad + \bar{K}_1 \|\theta\|_\infty \|g(\gamma(0))\|_{L^2(\partial\Omega)}, \end{aligned}$$

proving that $(G_\epsilon)_\beta$ is well defined. $\square$

Lemma 5.4 Suppose that g satisfies the growth condition (31) and let p and q be conjugated exponents, with $\frac{1}{2\lambda_2} < p < \infty$ and $\epsilon_0 < 1$. Then, if $\eta > \max \left\{ \frac{1}{2} - \frac{1}{4p\lambda_2}, \frac{1}{2} - \frac{1}{4q} \right\}$ and $\beta < -\frac{1}{4}$, the map $G(u, \epsilon) = G(u) : X^\eta \times [0, \epsilon_0] \rightarrow X^\beta$ given by (29) is uniformly continuous in ϵ , for u in bounded sets of X^η , and locally Lipschitz continuous in u , uniformly in ϵ .

Proof We first show that $(G_\epsilon)_\beta$ is locally Lipschitz continuous in $u \in X^\eta$. Let $u_1, u_2 \in X^\eta$, $\Phi \in X^{-\beta}$ and $\epsilon \in [0, \epsilon_0]$. Then

$$\begin{aligned} |\langle G(u_1, \epsilon) - G(u_2, \epsilon), \Phi \rangle_{\beta, -\beta}| &\leq \int_{\partial\Omega} |g(\gamma(u_1)) - g(\gamma(u_2))| |\gamma(\Phi)| \left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x) \\ &\leq L_2 \|\theta\|_\infty \left\{ \int_{\partial\Omega} (1 + |\gamma(u_1)|^{\lambda_2} + |\gamma(u_2)|^{\lambda_2})^2 |\gamma(u_1) - \gamma(u_2)|^2 d\sigma(x) \right\}^{\frac{1}{2}} \\ &\quad \cdot \left\{ \int_{\partial\Omega} |\gamma(\Phi)|^2 d\sigma(x) \right\}^{\frac{1}{2}} \\ &\leq L_2 \|\theta\|_\infty \left\{ \int_{\partial\Omega} (1 + |\gamma(u_1)|^{\lambda_2} + |\gamma(u_2)|^{\lambda_2})^{2p} d\sigma(x) \right\}^{\frac{1}{2p}} \\ &\quad \cdot \left\{ \int_{\partial\Omega} |\gamma(u_1) - \gamma(u_2)|^{2q} d\sigma(x) \right\}^{\frac{1}{2q}} \|\gamma(\Phi)\|_{L^2(\partial\Omega)} \\ &\leq L_2 \|\theta\|_\infty \left(|\partial\Omega|^{\frac{1}{2p}} + \|\gamma(u_1)\|_{L^{2p\lambda_2}(\partial\Omega)}^{\lambda_2} + \|\gamma(u_2)\|_{L^{2p\lambda_2}(\partial\Omega)}^{\lambda_2} \right) \\ &\quad \cdot \|\gamma(u_1) - \gamma(u_2)\|_{L^{2q}(\partial\Omega)} \|\gamma(\Phi)\|_{L^2(\partial\Omega)}. \end{aligned}$$

Reasoning as in Lemma 5.3, we obtain the following estimates:

$$\begin{aligned} \|\gamma(\Phi)\|_{L^2(\partial\Omega)} &\leq \bar{K}_1 \|\Phi\|_{X^{-\beta}}, \quad \|\gamma(u)\|_{L^{2p\lambda_2}(\partial\Omega)} \\ &\leq \bar{K}_4 \|u\|_{X^\eta}, \quad \|\gamma(u)\|_{L^{2q}(\partial\Omega)} \leq \bar{K}_5 \|u\|_{X^\eta}, \end{aligned}$$

where $\bar{K}_1$, $\bar{K}_4$ and $\bar{K}_5$ are imbedding constants. Thus, we obtain:

$$\begin{aligned} |\langle G(u_1, \epsilon) - G(u_2, \epsilon), \Phi \rangle_{\beta, -\beta}| &\leq L_2 \|\theta\|_\infty \left(|\partial\Omega|^{\frac{1}{2p}} + \bar{K}_4^{\lambda_2} \|u_1\|_{X^\eta}^{\lambda_2} + \bar{K}_4^{\lambda_2} \|u_2\|_{X^\eta}^{\lambda_2} \right) \\ &\quad \cdot \bar{K}_5 \|u_1 - u_2\|_{X^\eta} \bar{K}_1 \|\Phi\|_{X^{-\beta}}. \end{aligned}$$

If U is a bounded subset of X^η , $u_1, u_2 \in U$, we have

$$\|G(u_1, \epsilon) - G(u_2, \epsilon)\|_{X^\beta} \leq \bar{K}_{\lambda_2, U, h_\epsilon} \|u_1 - u_2\|_{X^\eta},$$

where $\bar{K}_{\lambda_2, U} > 0$ depends on λ_2 and U . Therefore, $(G_\epsilon)_\beta$ is locally Lipschitz in u .

Now, if $u \in X^\eta$, $\Phi \in X^{-\beta}$ and $\epsilon_1, \epsilon_2 \in [0, \epsilon_0]$, we finally get

$$|\langle G(u, \epsilon_1) - G(u, \epsilon_2), \Phi \rangle_{\beta, -\beta}| \leq \|\theta_{\epsilon_1} - \theta_{\epsilon_2}\|_\infty \int_{\partial\Omega} |g(\gamma(u))| |\gamma(\Phi)| d\sigma(x),$$

with $\|\theta_{\epsilon_1} - \theta_{\epsilon_2}\|_\infty = \sup_x |\theta(x, \epsilon_1) - \theta(x, \epsilon_2)| \rightarrow 0$ as $|\epsilon_1 - \epsilon_2| \rightarrow 0$ by (32) and (33). $\square$

Theorem 5.5 Suppose f and g satisfy the growth conditions (30) and (31), respectively. Suppose also, that β and η satisfy the hypotheses of Lemmas 5.1, 5.2, 5.3, 5.4, $\eta < 1 + \beta$ and $\epsilon > 0$ is sufficiently small. Then, for any $(t_0, u_0) \in \mathbb{R} \times X^\eta$, the problem (26) has a unique solution $u(t, t_0, u_0, \epsilon)$ with initial value $u(t_0) = u_0$. The map $\epsilon \mapsto u(t, t_0, u_0, \epsilon) \in X^\eta$ is continuous at $\epsilon = \epsilon_0$, uniformly for u_0 in bounded sets of X^η and $t_0 \leq t \leq T < \infty$.

Proof From Theorem 4.1 it follows that $(A_\epsilon)_\beta$ is a sectorial operator in X_ϵ^β , with domain $X_\epsilon^{1+\beta}$, if ϵ is small enough. From Lemmas 5.1, 5.2, 5.3 and 5.4 it follows that $(H_\epsilon)_\beta$ is well defined and locally Lipschitz continuous in X^η , and bounded in bounded sets of X^η . Then, the result follows from [8, Theorems 3.3.3 and 3.4.1] and [12, Lemma 3.7]. $\square$

6 Lyapunov functionals and global existence

We now want to show that the solutions given by Theorem 5.5 are globally defined, if an additional (dissipative) hypotheses on f and g is assumed. Here are these hypotheses:

There exist constants c_0 , d_0 and d'_0 such that

$$\limsup_{|u| \rightarrow \infty} \frac{f(u)}{u} \leq c_0, \quad \limsup_{|u| \rightarrow \infty} \frac{g(u)}{u} \leq d'_0 \quad (34)$$

and, if $d_0 > d'_0$, the first eigenvalue μ_1 of the problem

$$\begin{cases} -\Delta u + (a - c_0)u = \mu u & \text{em } \Omega \\ \frac{\partial u}{\partial N_\Omega} = d_0 u & \text{em } \partial\Omega \end{cases} \quad (35)$$

is positive.

Remark 6.1 Observe that the hypotheses (34) and (35) still hold for the perturbed operator $h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}$ with perturbed boundary conditions $h_\epsilon^* \frac{\partial u}{\partial N_\Omega} h_\epsilon^{*-1}$, if $\epsilon > 0$ is small enough, since the eigenvalues change continuously with ϵ by (24).

In order to prove global existence, we work first in the natural “energy space” $H^1(\Omega)$, that is, we choose $\eta = 1/2$ (and $\beta < -1/4$). It is also convenient to work first in the *perturbed domain* Ω_ϵ . More precisely, we consider initially the following abstract version of problem (P_ϵ) in the Banach space Y^β , where Y^α with $-1/2 \leq \alpha \leq 1$, now denote the fractional power spaces of the operator $-\Delta_{\Omega_\epsilon} + aI$ in the perturbed domain Ω_ϵ ,

$$\begin{cases} v_t + A_\beta v = H_\beta v, & t > t_0 \\ v(t_0) = v_0 \in H^1(\Omega_\epsilon), \end{cases} \quad (36)$$

where $H_\beta := F_\beta + G_\beta : H^1(\Omega_\epsilon) \rightarrow Y^\beta$, $\epsilon > 0$,

(i) $F_\beta : H^1(\Omega_\epsilon) \rightarrow Y^\beta$ is given by

$$\langle F(v), \Psi \rangle_{\beta, -\beta} = \int_{\Omega_\epsilon} f(v) \Psi \, dy, \quad \text{for any } \Psi \in Y^{-\beta},$$

(ii) $G_\beta : H^1(\Omega_\epsilon) \rightarrow Y^\beta$ is given by

$$\langle G(v), \Psi \rangle_{\beta, -\beta} = \int_{\partial\Omega_\epsilon} g(\gamma(v)) \gamma(\Psi) \left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(y), \quad \text{for any } \Psi \in Y^{-\beta},$$

where γ is the trace map.

It is not difficult to see that v is a solution of (36) if and only if $u = v \circ h_\epsilon$ satisfies (26) with $\eta = 1/2$. Thus, the local well-posedness of (36) follows immediately from Theorem 5.5.

We now prove the existence of a Lyapunov functional for the dynamical system set by (36).

Lemma 6.2 *Suppose that the hypotheses of Theorem 5.5 are satisfied with $\eta = 1/2$ and, additionally, that f and g satisfy the dissipative conditions (34) and (35). Consider the map*

$$\begin{aligned} W_\epsilon : H^1(\Omega_\epsilon) &\longrightarrow \mathbb{R} \\ v &\longmapsto W_\epsilon(v) = \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{a}{2} \int_{\Omega_\epsilon} |v|^2 dx - \int_{\Omega_\epsilon} F(v) dx \\ &\quad - \int_{\partial\Omega_\epsilon} G(\gamma(v)) dS \end{aligned}$$

where a is a positive number, F and $G : \mathbb{R} \rightarrow \mathbb{R}$ are primitives of f and g , respectively. Then, if $\epsilon > 0$ is sufficiently small, W_ϵ is a Lyapunov functional for the problem (36) and there exist constants K_1 and K_2 such that $W_\epsilon(v) \leq K_1 \|v\|_{H^1(\Omega_\epsilon)}^2 + K_2$ for any $v \in H^1(\Omega_\epsilon)$.

Proof If v is a solution of (36) in $H^2(\Omega)$, we have

$$\begin{aligned} \frac{d}{dt} W_\epsilon(v(t)) &= - \left(\int_{\Omega_\epsilon} v_t \Delta v \, dx - a \int_{\Omega_\epsilon} v_t v \, dx + \int_{\Omega_\epsilon} v_t f(v) \, dx \right) \\ &= - \int_{\Omega_\epsilon} |v_t|^2 dx = - \|v_t\|_{L^2(\Omega_\epsilon)}^2. \end{aligned}$$

The equality $\frac{d}{dt} W_\epsilon(v(t)) = - \|v_t\|_{L^2(\Omega_\epsilon)}^2$ is established, supposing that v is a solution in $H^2(\Omega)$. But, since both sides are well-defined and continuous functions of $v \in H^1(\Omega_\epsilon)$, it remains true for any solution. Therefore W_ϵ is decreasing along the solutions of (36). It is clear from its formula, that W_ϵ is continuous. We now want to obtain an estimate for $W_\epsilon(v)$

in terms of the norm of $v \in H^1(\Omega_\epsilon)$. From (34), there exist $\epsilon_f > 0$ and $M(\epsilon_f) > 0$ such that $\frac{f(s)}{s} - c_0 \leq \epsilon_f$ for $|s| > M(\epsilon_f)$. Therefore, if $s > 0$ we have

$$\int_{\Omega_\epsilon} F(v) \, dx = \int_{\Omega_\epsilon} \left(\int_0^v f(s) \, ds \right) dx \leq \frac{c_0}{2} \int_{\Omega_\epsilon} |v|^2 dx + k_0,$$

where k_0 depends on ϵ_f and $M(\epsilon_f)$. A similar argument gives the same estimate for $s < 0$.

Now, from (34) there exist $\epsilon_g > 0$ and $N(\epsilon_g) > 0$ such that $\frac{g(s)}{s} - d'_0 \leq \epsilon_g$ for $|s| > N(\epsilon_g)$. Therefore, arguing as before, we get

$$\int_{\partial\Omega_\epsilon} G(\gamma(v)) \, dS \leq \frac{d'_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS + k'_0,$$

with k'_0 depending on ϵ_g and $N(\epsilon_g)$.

Hence, we can use these estimates to obtain

$$\begin{aligned} W_\epsilon(v) &\leq \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{a}{2} \int_{\Omega_\epsilon} |v|^2 dx + \frac{c_0}{2} \int_{\Omega_\epsilon} |v|^2 dx \\ &\quad + k_0 + \frac{d'_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS + k'_0 \\ &\leq K_1 \|v\|_{H^1(\Omega_\epsilon)}^2 + K_2. \end{aligned}$$

On the other hand, from the same estimates, we also have

$$\begin{aligned} W_\epsilon(v) &\geq \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{a}{2} \int_{\Omega_\epsilon} |v|^2 dx \\ &\quad - \frac{c_0}{2} \int_{\Omega_\epsilon} |v|^2 dx - k_0 - \frac{d'_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS - k'_0 \\ &= \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{(a-c_0)}{2} \int_{\Omega_\epsilon} |v|^2 dx - \frac{d'_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS - (k_0 + k'_0), \end{aligned}$$

and, since $d_0 > d'_0$, we may choose $d''_0 \neq 0$ such that

$$d_0 > d''_0 > d'_0 \quad \text{and} \quad \left(\frac{d_0}{d''_0} - 1 \right) \frac{a-c_0}{2} + \frac{\lambda_0}{2} > 0,$$

where λ_0 is the first eigenvalue of (35) (for the perturbed domain—see Remark 6.1). Thus

$$\begin{aligned} W_\epsilon(v) &\geq \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{(a-c_0)}{2} \int_{\Omega_\epsilon} |v|^2 dx - \frac{d''_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS - (k_0 + k'_0) \\ &= \frac{d''_0}{d_0} \left[\frac{d_0}{d''_0} \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{d_0}{d''_0} \frac{(a-c_0)}{2} \int_{\Omega_\epsilon} |v|^2 dx - \frac{d_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS - \frac{d_0}{d''_0} (k_0 + k'_0) \right] \\ &= \frac{d''_0}{d_0} \left[\left(\frac{d_0}{d''_0} - 1 \right) \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \left(\frac{d_0}{d''_0} - 1 \right) \frac{(a-c_0)}{2} \int_{\Omega_\epsilon} |v|^2 dx \right. \\ &\quad \left. + \frac{(a-c_0)}{2} \int_{\Omega_\epsilon} |v|^2 dx - \frac{d_0}{2} \int_{\partial\Omega_\epsilon} |\gamma(v)|^2 dS - \frac{d_0}{d''_0} (k_0 + k'_0) \right] \\ &\geq \frac{d''_0}{d_0} \left\{ \left(\frac{d_0}{d''_0} - 1 \right) \frac{1}{2} \int_{\Omega_\epsilon} |\nabla v|^2 dx + \left[\left(\frac{d_0}{d''_0} - 1 \right) \frac{(a-c_0)}{2} + \frac{\lambda_0}{2} \right] \int_{\Omega_\epsilon} |v|^2 dx - \frac{d_0}{d''_0} (k_0 + k'_0) \right\}. \end{aligned}$$

From this evaluation, we see that $W_\epsilon(v)$ is bounded below by a positive constant times $\|v\|_{H^1(\Omega_\epsilon)}^2$, which depends on the first eigenvalue of (35), plus a constant depending on the nonlinearities. Thus $|W_\epsilon(v)| \rightarrow \infty$ as $\|v\|_{H^1(\Omega_\epsilon)} \rightarrow \infty$.

Finally, if $v(t)$ is a solution for all $t \in \mathbb{R}$ and $W_\epsilon(v(t)) = W_\epsilon(v_0)$, with $v_0 \in H^1(\Omega_\epsilon)$,

$$\frac{d}{dt} W_\epsilon(v(t)) = \frac{d}{dt} W_\epsilon(v_0) = 0 \implies \|v_t\|_{L^2(\Omega_\epsilon)}^2 = 0,$$

and so, v must be an equilibrium of (36).

Therefore W_ϵ is a Lyapunov function for the flow generated by (36), as claimed. $\square$

Next let us see that the Lyapunov functionals W_ϵ in the perturbed regions Ω_ϵ approach the functional $W = W_0$ when $\epsilon \rightarrow 0^+$ in the following sense:

Lemma 6.3 *If $W_\epsilon : H^1(\Omega_\epsilon) \rightarrow \mathbb{R}$ is as in Lemma 6.2, then*

$$K_1(h_\epsilon) |W(u)| \leq |W_\epsilon(u \circ h_\epsilon^{-1})| \leq K_2(h_\epsilon) |W(u)|, \quad \forall u \in H^1(\Omega),$$

with $K_1(h_\epsilon), K_2(h_\epsilon) \rightarrow 1$ when $\epsilon \rightarrow 0^+$.

Proof The result was proved in [11] in the case where $h_\epsilon \rightarrow i_\Omega$ in $\mathcal{C}^2$. However, since the inequalities involve only first order derivatives the extension $\mathcal{C}^1$ is immediate. $\square$

We now define a functional $V_\epsilon : H^1(\Omega) \rightarrow \mathbb{R}$ by

$$V_\epsilon(u) = W_\epsilon(u \circ h_\epsilon^{-1}). \quad (37)$$

Lemma 6.4 *The functional V_ϵ defined by (37) is a Lyapunov functional for (26) with $\eta = 1/2$ and $-1/2 < \beta < -1/4$, and the following estimates hold:*

- (1) $K_1 \|u\|_{H^1(\Omega)}^2 - K_2 \leq V_\epsilon(u) \leq K_1 \|u\|_{H^1(\Omega)}^2 + K_2$, for some constants K_1 and K_2 ,
- (2) $K_1(h_\epsilon) |V(u)| \leq |V_\epsilon(u)| \leq K_2(h_\epsilon) |V(u)|$, for all $u \in H^1(\Omega)$ with $K_1(h_\epsilon), K_2(h_\epsilon) \rightarrow 1$ when $\epsilon \rightarrow 0^+$.

Proof The required properties for V_ϵ follow easily from the properties of W_ϵ and the fact that $h_\epsilon^{*-1} : H^1(\Omega) \rightarrow H^1(\Omega_\epsilon)$ is an isomorphism which takes solutions of (26) into solutions of (36). $\square$

Using the properties of the Lyapunov functional V_ϵ , we now prove the following result of global existence for (26):

Theorem 6.5 *Suppose that β, η, f and g satisfy the conditions of Theorem 5.5 and, additionally, that f and g satisfy the dissipative conditions (34) and (35). Then if $\epsilon > 0$ is sufficiently small the solutions of (26) are globally defined.*

Proof We just consider the case $\eta = 1/2$. The other ones follow from existence and uniqueness. From Theorem 5.5, for each $(t_0, u_0) \in \mathbb{R} \times H^1(\Omega)$, there exists $T = T(t_0, u_0) > 0$ such that the problem (26) has a unique solution u in $(t_0, t_0 + T)$, with $u(t_0) = u_0$. From Lemmas 5.2 and 5.4, it follows that $(H_\epsilon)_\beta$ is locally Lipschitz. So, it takes bounded sets in $X^\eta = H^1(\Omega)$, into bounded sets of X^β . Suppose that $T < \infty$. Then, by [8, Theorem 3.3.4], there exists a sequence $t_n \rightarrow T^-$ such that $\|u(t_n)\|_{H^1(\Omega)} \rightarrow \infty$, and thus $|V_\epsilon(u(t_n))| \rightarrow \infty$, which is a contradiction with the fact that V_ϵ is decreasing along orbits. $\square$

From now on, we denote the flow generated by (26) by $T_{\eta,\beta}(t, t_0, u)$ or $T_{\eta,\beta}(t, u)$. Sometimes we also do not include the parameters η and β to simplify the notation.

7 Existence of global attractors

In this section, we prove that the flow $T_{\epsilon, \eta, \beta}(t, t_0, u)$ admits a global attractor. As in the previous section, it is convenient to start with the especial case $\eta = 1/2$.

Lemma 7.1 *If $\epsilon > 0$ is sufficiently small, the nonlinear semigroup $T_\epsilon(t)$ generated by (26), with $\eta = 1/2$ and $-1/2 < \beta < -1/4$ in $X^{1/2} = H^1(\Omega)$ is a gradient flow.*

Proof We know, from Lemma 6.4 that the map V_ϵ defined by (37) is a Lyapunov functional for the flow. Since $(A_\epsilon)_\beta$ has compact resolvent and $(H_\epsilon)_\beta$ takes bounded subsets of $H^1(\Omega)$ into bounded subsets of X^β , it follows from [8, Theorem 3.3.6] that bounded positive orbits are precompact, so $T_\epsilon(t)$ is gradient. $\square$

We now want to show that the set of equilibria of (26) is bounded. As in the previous sections, it is convenient to prove this first for the problem in the perturbed domain.

Lemma 7.2 *Suppose that f and g satisfy the conditions of Theorem 5.5 and the dissipative conditions (34) and (35). Then if $\epsilon > 0$ is sufficiently small, the set E_ϵ of equilibria of the system generated by (36) is uniformly bounded in $H^1(\Omega_\epsilon)$.*

Proof The equilibria of (36) are the solutions of the problem:

$$\begin{cases} \Delta u(x) - au(x) + f(u(x)) = 0, & x \in \Omega_\epsilon \\ \frac{\partial u}{\partial N}(x) = g(u(x)), & x \in \partial\Omega_\epsilon. \end{cases}$$

Multiplying by u and integrating, we obtain

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} u \Delta u \, dx - a \int_{\Omega_\epsilon} |u|^2 \, dx + \int_{\Omega_\epsilon} u f(u) \, dx \\ &= - \int_{\Omega_\epsilon} |\nabla u|^2 \, dx + \int_{\partial\Omega_\epsilon} u g(u) \, dS - a \int_{\Omega_\epsilon} |u|^2 \, dx + \int_{\Omega_\epsilon} u f(u) \, dx. \end{aligned} \quad (38)$$

From (34), for any $\delta > 0$ there exist constants K_δ such that for all $u \in \mathbb{R}$

$$f(u)u \leq (c_0 + \delta)u^2 + K_\delta \quad \text{and} \quad g(u)u \leq (d'_0 + \delta)u^2 + K_\delta.$$

Hence, we obtain from (38) that

$$\int_{\Omega_\epsilon} |\nabla u|^2 \, dx \leq -a \int_{\Omega_\epsilon} |u|^2 \, dx + (c_0 + \delta) \int_{\Omega_\epsilon} u^2 \, dx + (d'_0 + \delta) \int_{\partial\Omega_\epsilon} u^2 \, dS + K_\delta(|\Omega| + |\partial\Omega|). \quad (39)$$

On the other hand, since the first eigenvalue $\lambda_0(\epsilon)$ of (35) is positive, we have

$$\lambda_0(\epsilon) \int_{\Omega_\epsilon} |u|^2 \, dx \leq \int_{\Omega_\epsilon} |\nabla u|^2 \, dx - d_0 \int_{\partial\Omega_\epsilon} |u|^2 \, dS + (a - c_0) \int_{\Omega_\epsilon} |u|^2 \, dx.$$

Consequently, it follows from (39) that

$$\begin{aligned} \lambda_0(\epsilon) \int_{\Omega_\epsilon} |u|^2 \, dx &\leq -a \int_{\Omega_\epsilon} |u|^2 \, dx + (c_0 + \delta) \int_{\Omega_\epsilon} u^2 \, dx + (d'_0 + \delta) \int_{\partial\Omega_\epsilon} u^2 \, dS \\ &\quad + K_\delta(|\Omega| + |\partial\Omega|) - d_0 \int_{\partial\Omega_\epsilon} |u|^2 \, dS + (a - c_0) \int_{\Omega_\epsilon} |u|^2 \, dx \\ &\leq \delta \int_{\Omega_\epsilon} u^2 \, dx + (d'_0 - d_0 + \delta) \int_{\partial\Omega_\epsilon} u^2 \, dS + K_\delta(|\Omega| + |\partial\Omega|). \end{aligned}$$

Choosing $\delta < \min\{\lambda_0(\epsilon), (d_o - d'_0)\}$ and setting $l_\delta = \min\{\lambda_0(\epsilon) - \delta, (d_o - d'_0) - \delta\}$, we obtain

$$\int_{\Omega_\epsilon} u^2 dx + \int_{\partial\Omega_\epsilon} u^2 dS \leq \frac{K_\delta(|\Omega| + |\partial\Omega|)}{l_\delta}. \quad (40)$$

Finally, using (39) again, we get

$$\int_{\Omega_\epsilon} |\nabla u|^2 dx \leq (c_0 + d'_0 + 2\delta) \frac{K_\delta(|\Omega| + |\partial\Omega|)}{l_\delta} + K_\delta(|\Omega| + |\partial\Omega|). \quad (41)$$

The claim now follows immediately from (40) and (41), observing that the constants in those estimates can be chosen uniformly in ϵ , for ϵ close to 0. $\square$

Corollary 7.3 *Suppose that f and g satisfy the conditions of Lemma 7.2. Then if $\epsilon > 0$ is sufficiently small, the set E_ϵ of equilibria of the system generated by (26) is uniformly bounded in $H^1(\Omega)$.*

Proof Since u is an equilibria of (6) if and only if $v = h_\epsilon^{*-1}u$ is an equilibria of (36), the result follows from Lemmas 2.3 and 7.2. $\square$

We are now in a position to prove the existence of attractors to the case $X^{1/2} = H^1(\Omega)$.

Theorem 7.4 *Suppose that f and g satisfy the conditions of Theorem 5.5 with $\eta = 1/2$, and the dissipative conditions (34) and (35). Then, if $\epsilon > 0$ is sufficiently small, the flow $T_\epsilon(t, u)$ generated by (26) has a global attractor $\mathcal{A}_\epsilon$ in $X^{1/2} = H^1(\Omega)$.*

Proof We apply [6, Theorem 3.8.5]. We know from Lemma 7.1 that the flow is gradient and, from Corollary 7.3, its equilibria set is bounded. So, it remains to be proved that it is asymptotically smooth. But this follows from regularizing properties of the flow obtained by [8, Theorem 3.3.6]. $\square$

We now extend the previous result to other phase spaces.

Theorem 7.5 *Suppose f and g satisfy the conditions of Theorem 5.5, and the dissipative conditions (34) and (35). Then, if $\epsilon > 0$ is sufficiently small, the flow $T_{\epsilon, \eta, \beta}(u)$ generated by (26) has a global attractor $\mathcal{A}_\epsilon$ in X^η . Furthermore $\mathcal{A}_\epsilon$ does not depend on either η or β .*

Proof Suppose first that $\eta = \eta_0 < 1/2$.

Let B be a bounded set in X^η . We may suppose that $B = B_R$, the open ball of radius R in X^η centered at the origin. By Lemmas 5.1 and 5.3, there exists N such that $\|(H_\epsilon)_\beta(u)\|_{X^\beta} < N$ for any u with $\|u\|_{X^\eta} < 2R$. Now, let $T(t)u_0 = u(t; t_0, u_0)$ be the solution of (26) with $\eta = \eta_0$ and $u_0 \in B_R$, and take $\operatorname{Re} \sigma((A_\epsilon)_\beta) > w > 0$. While $\|u(t; t_0, u_0)\|_{X^\eta} \leq 2R$, we have by variation of constants formula:

$$\begin{aligned} \|u(t; t_0, u_0)\|_{X^\eta} &\leq \|(A_\epsilon)_\beta^{\eta-\beta} e^{-(A_\epsilon)_\beta(t-t_0)} u_0\|_{X^\beta} \\ &\quad + \int_{t_0}^t \|(A_\epsilon)_\beta^{\eta-\beta} e^{-(A_\epsilon)_\beta(t-s)} (H_\epsilon)_\beta u(s)\|_{X^\beta} ds \\ &\leq \overline{C}_{\beta, \eta} (t - t_0)^{\beta-\eta} e^{-w(t-t_0)} \|u_0\|_{X^\beta} \\ &\quad + N C_{\beta, \eta} \int_{t_0}^t (t - s)^{-(\eta-\beta)} e^{-w(t-s)} ds. \end{aligned} \quad (42)$$

Let $T = \sup \{t \geq t_0 \mid u(s; t_0, u_0) \in B_{2R}, \text{ for all } s \leq t\}$ and $\delta > 0$ such that

$$\overline{C}_{\beta, \eta}(t - t_0)^{\beta - \eta} e^{-w(t - t_0)} \|u_0\|_{X^\beta} + NC_{\beta, \eta} \int_{t_0}^t (t - s)^{-(\eta - \beta)} e^{-w(t - s)} ds < 2R, \quad \forall t \in (t_0, \delta).$$

From (42), it follows that $T \geq \delta$, so the solutions with initial conditions in the ball of radius R in X^η remain in the ball of radius $2R$ for $0 \leq t \leq T$. Now, using the variation of constants formula again, we obtain, for any $0 \leq t \leq T$

$$\begin{aligned} \|u(t; t_0, u_0)\|_{X^{\frac{1}{2}}} &\leq \|e^{-(A_\epsilon)_\beta(t - t_0)} u_0\|_{X^{\frac{1}{2}}} + \int_{t_0}^t \|e^{-(A_\epsilon)_\beta(t - s)} (H_\epsilon)_\beta u(s)\|_{X^{\frac{1}{2}}} ds \\ &\leq \|(A_\epsilon)_\beta^{\frac{1}{2} - \eta} e^{-(A_\epsilon)_\beta(t - t_0)} u_0\|_{X^\eta} \\ &\quad + \int_{t_0}^t \|(A_\epsilon)_\beta^{\frac{1}{2} - \beta} e^{-(A_\epsilon)_\beta(t - s)} (H_\epsilon)_\beta u(s)\|_{X^\beta} ds \\ &\leq \overline{C}_\beta(t - t_0)^{-(\frac{1}{2} - \eta)} e^{-w(t - t_0)} \|u_0\|_{X^\beta} \\ &\quad + NC_\beta \int_{t_0}^t (t - s)^{-(\frac{1}{2} - \beta)} e^{-w(t - s)} ds. \end{aligned}$$

Then $T(t)B_R$ is in a bounded set of $X^{1/2}$ as $0 < t \leq T$. By Theorem 7.4 the attractor $\mathcal{A}_\epsilon$ of (26) with $\eta = 1/2$ attracts $T(t)B$ in the norm of $X^{1/2}$. Thus $\mathcal{A}_\epsilon$ also attracts B in the norm of X^η . Since $\mathcal{A}_\epsilon$ is invariant for the flow $T(t)$, it must be the attractor of (26) with $\eta = \eta_0$.

Suppose now $1/2 < \eta = \eta_0 < \beta + 1$. If B is a bounded set in X^η , it is also a bounded set in $X^{1/2}$. So, it is attracted by the global attractor $\mathcal{A}_\epsilon$ of (26) with $\eta = 1/2$ in the space $X^{1/2}$ and under the flow $T_{1/2}(t)$, which coincides with the flow in X^η . Now let us prove $T_{1/2}(t)$ is continuous from $X^{1/2}$ into X^η for $t > 0$. If $u_1, u_2 \in X^{1/2}$ and $u_i(t; t_0, u_i) = T_{1/2}(t)u_i$, $i = 1, 2$

$$\begin{aligned} &\|u(t; t_0, u_1) - u(t; t_0, u_2)\|_{X^\eta} \\ &\leq \|e^{-(A_\epsilon)_\beta(t - t_0)}(u_1 - u_2)\|_{X^\eta} + \int_{t_0}^t \|e^{-(A_\epsilon)_\beta(t - s)}((H_\epsilon)_\beta(u_1(s)) \\ &\quad - (H_\epsilon)_\beta(u_2(s)))\|_{X^\eta} ds \\ &\leq \|(A_\epsilon)_\beta^{\eta - \frac{1}{2}} e^{-(A_\epsilon)_\beta(t - t_0)}(u_1 - u_2)\|_{X^{\frac{1}{2}}} \\ &\quad + \int_{t_0}^t \|(A_\epsilon)_\beta^{\eta - \frac{1}{2}} e^{-(A_\epsilon)_\beta(t - s)}((H_\epsilon)_\beta u_1(s) - (H_\epsilon)_\beta u_2(s))\|_{X^{\frac{1}{2}}} ds \\ &\leq C_\eta(t - t_0)^{\frac{1}{2} - \eta} e^{-w(t - t_0)} \|u_1 - u_2\|_{X^{\frac{1}{2}}} \\ &\quad + L_{\beta, \eta} C_{\beta, \eta} \int_{t_0}^t (t - s)^{\frac{1}{2} - \eta} e^{-w(t - s)} \|u_1(s) - u_2(s)\|_{X^{\frac{1}{2}}} ds \end{aligned}$$

where $L_{\beta, \eta}$ is a (local) Lipschitz constant of $(H_\epsilon)_\beta$, which exists by Lemmas 5.2 and 5.4. The claimed continuity follows then from the Gronwall's inequality [8, Theorem 7.1.1].

Therefore, if V_δ is a neighborhood of $\mathcal{A}_\epsilon$ in $X^{1/2}$ which contains $T_{1/2}(t)B = T_\eta(t)B$, then $T_{1/2}(1)B$ is a small neighborhood of $\mathcal{A}_\epsilon$ in X^η which contains $T_{1/2}(t + 1)B = T_\eta(t + 1)B$. Since $\mathcal{A}_\epsilon = T_{1/2}(1)\mathcal{A}_\epsilon \in X^\eta$ is invariant, it must be the attractor of T_η . $\square$

8 Continuity of the attractors

Here we prove that the family of attractors of (26) is continuous in X^η at $\epsilon = 0$ for $\eta < 1/2$.

8.1 Upper semicontinuity

We start proving that the family of attractors $\mathcal{A}_\epsilon$ is uniformly bounded in $H^1(\Omega)$. More precisely,

Theorem 8.1 *Suppose the hypotheses of Theorem 7.5 hold. Then, the family of attractors $\{\mathcal{A}_\epsilon, \epsilon < \epsilon_0\}$ of (26) is uniformly bounded in $H^1(\Omega)$, for some $\epsilon_0 > 0$, and $\mathcal{A}_\epsilon$ is bounded in $L^\infty(\Omega)$ for each $\epsilon \geq 0$.*

Proof Denote by $T_\epsilon(t)$, the flow generated by (26), and $T = T_0$. By Corollary 7.3, there is an $\epsilon_0 > 0$, such that the set E_ϵ of equilibria of (26) is in the open ball B_r of radius r in $H^1(\Omega)$ for $\epsilon < \epsilon_0$. If $u \in \mathcal{A}_\epsilon$, by [6, Theorem 3.8.5], there is a t_u such that $u = T(t_u)u_0$ for some $u_0 \in B_r$. Let V_ϵ be the Lyapunov functional of (26) given by Lemma 6.4.

We have

$$V_\epsilon(u_0) \leq K_1 \|u_0\|_{H^1(\Omega)}^2 + K_2 \leq K_1 r^2 + K_2.$$

It follows that $V_\epsilon(u_0) \leq R$, for some constant R depending only on r , and then

$$V_\epsilon(u) \leq V_\epsilon(T(t_u)u_0) \leq V_\epsilon(u_0) \leq R.$$

From Lemma 6.4, there exists $\epsilon_0 > 0$, such that

$$\|u\|_{H^1(\Omega)}^2 \leq \frac{1}{K_1} (V_\epsilon(u) + K_2), \quad \forall u \in \bigcup_{\epsilon \leq \epsilon_0} \mathcal{A}_\epsilon.$$

The $L^\infty(\Omega)$ estimate follows immediately from the fact that the attractors do not depend on η , and X^η is continuously imbedded in $L^\infty(\Omega)$ for $\eta > 1/2$ (see [9, Theorem 3.8]). $\square$

We are now ready to prove the upper semicontinuity property of the family.

Theorem 8.2 *Suppose the hypotheses of Theorem 7.5 hold with $\beta = -1/2$. Then the family of attractors $\{\mathcal{A}_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$ of the flow $T_{\epsilon,\eta}(t, u)$, generated by (26), whose existence is guaranteed by Theorem 7.5 is upper semicontinuous in X^η . (We observe that the conditions on η hold if $1/2 - \delta < \eta < 1/2$, with δ sufficiently small).*

Proof We have to check that the conditions of [12, Theorem 3.9] hold. In fact:

- (1) We have showed, during the proof of Theorem 3.9 [see estimate (24)], that the family of operators $A_\epsilon = (A_\epsilon)_{-1/2}$ defined in $X^{-1/2} = H^{-1}(\Omega)$ with domain $X^{1/2} = H^1(\Omega)$ satisfy the conditions of [12, Theorem 3.3].
- (2) From Lemmas 5.1 and 5.3, it follows that $H_\epsilon = F_\epsilon + G_\epsilon$ takes bounded sets in X^η into bounded sets of $X^{-1/2} = H^{-1}(\Omega)$.
- (3) From Lemmas 5.2 and 5.4, H_ϵ is continuous in ϵ , uniformly for u in bounded sets of X^η and locally Lipschitz in u , uniformly in ϵ .
- (4) By Theorem 8.1, the set $\bigcup_{\epsilon \leq \epsilon_0} \mathcal{A}_\epsilon$ is bounded in $X^{1/2}$, and thus, by (2) H_ϵ , takes bounded sets from $X^{1/2}$ into bounded sets of $X^{-1/2}$. $\square$

From the semicontinuity of attractors, we can easily prove the corresponding property for the equilibria set.

Corollary 8.3 *Suppose the hypotheses of Theorem 7.5 hold with $\beta = -1/2$. Then, the family of equilibria sets $\{E_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$ of (26), is upper semicontinuous in X^η .*

Proof The result is well known, but we sketch a proof here for completeness. Suppose u_n with $\lim_{n \rightarrow \infty} \epsilon_n = 0$. We choose an arbitrary subsequence that we still call it (u_n) . It is enough to show that, there exists a subsequence (u_{n_k}) which converges to a point $u_0 \in E_0$. Since $(u_n) \in \mathcal{A}_\epsilon$, and $\mathcal{A}_\epsilon$ is compact, there exists a subsequence (u_{n_k}) which converges to a point $u_0 \in \mathcal{A}_\epsilon$. Now, since the flow $T_\epsilon(t)$ is continuous in ϵ we have, for any $t > 0$

$$u_{n_k} \rightarrow u_0 \Leftrightarrow T_{\epsilon_{n_k}}(t)u_{n_k} \rightarrow T_0(t)u_0 \Leftrightarrow u_{n_k} \rightarrow T_0(t)u_0.$$

Hence, by uniqueness of the limit $T_0(t)u_0 = u_0$, for any $t > 0$, we get $u_0 \in E_0$. $\square$

8.2 Lower semicontinuity

For the lower semicontinuity, we need to assume additional properties for the nonlinearities. We assume:

$$f \text{ and } g \in C^1(\mathbb{R}, \mathbb{R}) \text{ with bounded derivatives.} \quad (43)$$

Lemma 8.4 *If f satisfies (43) and $\eta > 0$, then the map $F : X^\eta \times \mathbb{R} \rightarrow X^\beta$ given by (28) is Gateaux differentiable with respect to u , with Gateaux differential $\frac{\partial F}{\partial u}(u, \epsilon)w$ given by*

$$\left\langle \frac{\partial F}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{\beta, -\beta} = \int_{\Omega} f'(u)w \Phi \, dx, \quad (44)$$

for all $w \in X^\eta$ and $\Phi \in X^{-\beta}$.

Proof Observe first that $F(u, \epsilon)$ is well defined, since the conditions of Lemma 5.1 are satisfied with $\lambda_1 = 0$. It is clear that $\frac{\partial F}{\partial u}(u, \epsilon)$ is linear. We now show that it is bounded. In fact, for all $u, w \in X^\eta$ and $\Phi \in X^{-\beta}$, we have

$$\left| \left\langle \frac{\partial F}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{\beta, -\beta} \right| \leq \|f'\|_{\infty} \|w\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)} \, dx \leq K \|f'\|_{\infty} \|w\|_{X^\eta} \|\Phi\|_{X^{-\beta}}$$

where K comes from embedding constants. This proves boundedness.

Now, we have, for all $u, w \in X^\eta$ and $\Phi \in X^{-\beta}$

$$\begin{aligned} & \left| \frac{1}{t} \left\langle F(u + tw, \epsilon) - F(u, \epsilon) - t \frac{\partial F}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{\beta, -\beta} \right| \\ & \leq \frac{1}{|t|} \int_{\Omega} | [f(u + tw) - f(u) - tf'(u)w] \Phi | \, dx \\ & \leq K_1 \frac{1}{|t|} \left(\int_{\Omega} | f(u + tw) - f(u) - tf'(u)w |^2 \, dx \right)^{\frac{1}{2}} \|\Phi\|_{X^{-\beta}} \\ & \leq K_1 \left(\underbrace{\int_{\Omega} | (f'(u + \bar{t}w) - f'(u)) w |^2 \, dx}_{(I)} \right)^{\frac{1}{2}} \|\Phi\|_{X^{-\beta}}, \end{aligned}$$

where K_1 is the embedding constant of $X^{-\beta}$ into $L^2(\Omega)$, and $0 \leq \bar{t} \leq t$. Since f' is continuous and bounded, the integrand of (I) is bounded by an integrable function and goes to 0 as $t \rightarrow 0$. Thus, the integral (I) goes to 0 as $t \rightarrow 0$ from Lebesgue's Dominated Convergence Theorem, and $\lim_{t \rightarrow 0} \frac{F(u+tw, \epsilon) - F(u, \epsilon)}{t} = \frac{\partial F}{\partial u}(u, \epsilon)w$ in X^β for all $u, w \in X^\eta$. Therefore, F is Gateaux differentiable with Gateaux differential given by (44). $\square$

We now want to prove that the Gateaux differential of $F(u, \epsilon)$ is continuous in u . Let us denote by $\mathcal{B}(X, Y)$ the space of linear bounded operators from X to Y . We will need the following result, whose simple proof is omitted.

Lemma 8.5 *Suppose X, Y are Banach spaces and $T_n : X \rightarrow Y$ is a sequence of linear operators converging strongly to the linear operator $T : X \rightarrow Y$. Suppose also that $X_1 \subset X$ is a Banach space, the inclusion $i : X_1 \hookrightarrow X$ is compact and let $\tilde{T}_n = T_n \circ i$ and $\tilde{T} = T \circ i$. Then $\tilde{T}_n \rightarrow \tilde{T}$ uniformly for x in a bounded subset of X_1 (that is, in the norm of $\mathcal{B}(X_1, Y)$).*

Lemma 8.6 *If f satisfies (43) and $\eta > 0$, the Gateaux differential of $F(u, \epsilon)$ with respect to u is continuous in u , and then, the map $u \mapsto \frac{\partial F}{\partial u}(u, \epsilon) \in \mathcal{B}(X^{-\beta}, X^\beta)$ is continuous.*

Proof Let u_n be a sequence converging to u in X^η , and choose $0 < \tilde{\eta} < \eta$. Then, we have for any $\Phi \in X^{-\beta}$ and $w \in X^{\tilde{\eta}}$:

$$\begin{aligned} & \left| \left\langle \left(\frac{\partial F}{\partial u}(u_n, \epsilon) - \frac{\partial F}{\partial u}(u, \epsilon) \right) w, \Phi \right\rangle_{\beta, -\beta} \right| \\ & \leq \int_{\Omega} \left| (f'(u) - f'(u_n))w \Phi \right| dx \\ & \leq \left(\int_{\Omega} |(f'(u) - f'(u_n))w|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\Phi|^2 dx \right)^{\frac{1}{2}} \\ & \leq K_1 \underbrace{\left(\int_{\Omega} |(f'(u) - f'(u_n))w|^2 dx \right)^{\frac{1}{2}}}_{(I)} \|\Phi\|_{X^{-\beta}}, \end{aligned}$$

where K_1 is the embedding constant of $X^{-\beta}$ in $L^2(\Omega)$.

Now, the integrand in (I) is bounded by the integrable function $\|f'\|_{\infty}^2 w^2$ and goes to zero a.e. as $u_n \rightarrow u$ in X^η . Therefore the sequence of operators $\frac{\partial F}{\partial u}(u_n, \epsilon)$ converges strongly in the space $\mathcal{B}(X^{\tilde{\eta}}, X^\beta)$ to the operator $\frac{\partial F}{\partial u}(u, \epsilon)$. From Lemma 8.5 the convergence holds in the norm of $\mathcal{B}(X^\eta, X^\beta)$, since X^η is compactly embedded in $X^{\tilde{\eta}}$ by [8, Theorem 1.4.8]. $\square$

Lemma 8.7 *If g satisfies (43) and $\eta > 1/4$ then the map $G : X^\eta \times \mathbb{R} \rightarrow X^\beta$ given by (29) is Gateaux differentiable with respect to u , with Gateaux differential*

$$\left\langle \frac{\partial G}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{\beta, -\beta} = \int_{\partial\Omega} g'(\gamma(u))\gamma(w)\gamma(\Phi) \left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x), \quad (45)$$

for all $w \in X^\eta$ and $\Phi \in X^{-\beta}$.

Proof First we note that $G(u, \epsilon)$ is well defined by Lemma 5.3. $\frac{\partial G}{\partial u}(u, \epsilon)$ is linear and bounded. Indeed, for all $u, w \in X^\eta$ and $\Phi \in X^{-\beta}$

$$\begin{aligned} \left| \left\langle \frac{\partial G}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{\beta, -\beta} \right| &= \left| \int_{\partial\Omega} g'(\gamma(u))\gamma(w)\gamma(\Phi) \left| \frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon} \right| d\sigma(x) \right| \\ &\leq \|\theta\|_\infty \|g'\|_\infty \int_{\partial\Omega} |\gamma(w)| |\gamma(\Phi)| d\sigma(x) \\ &\leq K_1 K_2 \|\theta\|_\infty \|g'\|_\infty \|w\|_{X^\eta} \|\Phi\|_{X^{-\beta}} \end{aligned}$$

where K_1 and K_2 are embedding constants.

Now, we have, for all $u, w \in X^\eta$ and $\Phi \in X^{-\beta}$

$$\begin{aligned} &\left| \frac{1}{t} \left\langle G(u + tw, \epsilon) - G(u, \epsilon) - t \frac{\partial G}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{\beta, -\beta} \right| \\ &\leq \frac{1}{|t|} \int_{\partial\Omega} | [g(\gamma(u + tw)) - g(\gamma(u)) - tg'(\gamma(u))]\gamma(w) | |\gamma(\Phi)| \left| \frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon} \right| d\sigma(x) \\ &\leq K_1 \|\theta\|_\infty \frac{1}{|t|} \left\{ \int_{\partial\Omega} | [g(\gamma(u + tw)) - g(\gamma(u)) - tg'(\gamma(u))]\gamma(w) |^2 d\sigma(x) \right\}^{\frac{1}{2}} \|\Phi\|_{X^{-\beta}} \\ &\leq K_1 \|\theta\|_\infty \left\{ \underbrace{\int_{\partial\Omega} | [g'(\gamma(u + \bar{t}w)) - g'(\gamma(u))]\gamma(w) |^2 d\sigma(x)}_{(I)} \right\}^{\frac{1}{2}} \|\Phi\|_{X^{-\beta}} \end{aligned}$$

where K_1 is an embedding constant, and $0 \leq \bar{t} \leq t$. Since g' is bounded, the integrand of (I) is bounded by an integrable function and goes to 0 as $t \rightarrow 0$. Thus, $(I) \rightarrow 0$ as $t \rightarrow 0$, and $\lim_{t \rightarrow 0} \frac{G(u+tw, \epsilon) - G(u, \epsilon)}{t} = \frac{\partial G}{\partial u}(u, \epsilon)w$ in X^β , for all $u, w \in X^\eta$. Thus G is Gateaux differentiable with Gateaux differential given by (45). $\square$

Lemma 8.8 *If g satisfies (43) and $\eta > 1/4$, the Gateaux differential of $G(u, \epsilon)$ with respect to u is continuous and uniformly continuous in ϵ for u in bounded sets of X^η and $0 \leq \epsilon \leq \epsilon_0 < 1$.*

Proof Let $0 \leq \epsilon \leq \epsilon_0$, u_n be a sequence converging to u in X^η , and choose $1/4 < \tilde{\eta} < \eta$. Then, for any $\Phi \in X^{-\beta}$ and $w \in X^{\tilde{\eta}}$:

$$\begin{aligned} &\left| \left\langle \left(\frac{\partial G}{\partial u}(u_n, \epsilon) - \frac{\partial G}{\partial u}(u, \epsilon) \right) w, \Phi \right\rangle_{\beta, -\beta} \right| \\ &\leq \int_{\partial\Omega} |(g'(\gamma(u)) - g'(\gamma(u_n)))\gamma(w)\gamma(\Phi)| \left| \frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon} \right| d\sigma(x) \\ &\leq \|\theta_\epsilon\|_\infty \left\{ \int_{\partial\Omega} |(g'(\gamma(u)) - g'(\gamma(u_n)))\gamma(w)|^2 d\sigma(x) \right\}^{\frac{1}{2}} \left\{ \int_{\partial\Omega} |\gamma(\Phi)|^2 d\sigma(x) \right\}^{\frac{1}{2}} \\ &\leq K \|\theta_\epsilon\|_\infty \left\{ \underbrace{\int_{\partial\Omega} |(g'(\gamma(u)) - g'(\gamma(u_n)))\gamma(w)|^2 d\sigma(x)}_{(I)} \right\}^{\frac{1}{2}} \|\Phi\|_{X^{-\beta}} \end{aligned}$$

where K is the constant due to continuity of the trace map from $X^{-\beta}$ into $L^2(\partial\Omega)$.

Now, the integrand in (I) is bounded by the integrable function $4\|g'\|_\infty^2 |\gamma(w)|^2$ and goes to 0 a.e. as $u_n \rightarrow u$ in X^η . Therefore the sequence of operators $\frac{\partial G}{\partial u}(u_n, \epsilon)$ converges strongly in the space $\mathcal{B}(X^\eta, X^\beta)$ to the operator $\frac{\partial G}{\partial u}(u, \epsilon)$. From Lemma 8.5 the convergence holds in the norm of $\mathcal{B}(X^\eta, X^\beta)$ since X^η is compactly embedded in X^η .

On the other hand, if $0 \leq \epsilon_1 \leq \epsilon_2 < \epsilon_0$, we have for any $\Phi \in X^{-\beta}$ and $w \in X^\eta$:

$$\begin{aligned} & \left| \left\langle \left(\frac{\partial G}{\partial u}(u, \epsilon_1) - \frac{\partial G}{\partial u}(u, \epsilon_2) \right) w, \Phi \right\rangle_{\beta, -\beta} \right| \\ & \leq \int_{\partial\Omega} |g'(\gamma(u))\gamma(w)\gamma(\Phi)| |\theta_{\epsilon_1} - \theta_{\epsilon_2}| \, d\sigma(x) \\ & \leq \|\theta_{\epsilon_1} - \theta_{\epsilon_2}\|_\infty \left\{ \int_{\partial\Omega} |g'(\gamma(u))\gamma(w)|^2 \, d\sigma(x) \right\}^{\frac{1}{2}} \left\{ \int_{\partial\Omega} |\gamma(\Phi)|^2 \, d\sigma(x) \right\}^{\frac{1}{2}} \\ & \leq K K' \|g'\|_\infty \|w\|_{X^\eta} \|\Phi\|_{X^{-\beta}} \|\theta_{\epsilon_1} - \theta_{\epsilon_2}\|_\infty, \end{aligned}$$

where K' is the constant due to continuity of the trace map from X^η into $L^2(\partial\Omega)$. This proves uniform continuity in ϵ . $\square$

Lemma 8.9 *If f and g satisfy the condition (43) and $\eta > \frac{1}{4}$, then the map $(H_\epsilon)_\beta = (F_\epsilon)_\beta + (G_\epsilon)_\beta : X^\eta \times \mathbb{R} \rightarrow X^\beta$ given by (27) is continuously Fréchet differentiable with respect to u and the derivative $\frac{\partial G}{\partial u}$ is uniformly continuous with respect to ϵ , for u in bounded sets of X^η and $0 \leq \epsilon \leq \epsilon_0 < 1$.*

Proof The proof follows from Lemmas 8.6, 8.8 and Proposition 2.8 in [13]. $\square$

We now prove lower semicontinuity for the equilibria.

Theorem 8.10 *If f and g satisfy the conditions of Theorem 7.5 and also (43), and the equilibria of (26) with $\epsilon = 0$ are all hyperbolic with $1/4 < \eta < 1/2$, then the family of equilibria sets $\{E_\epsilon, 0 \leq \epsilon < \epsilon_0\}$ of (26) is lower semicontinuous in X^η at $\epsilon = 0$.*

Proof A point $e \in X^\eta$ is an equilibrium of (26) if and only if it is a root of the map

$$\begin{aligned} Z : H^1(\Omega) \times \mathbb{R} &\longrightarrow X^{-\frac{1}{2}} \\ (u, \epsilon) &\longmapsto (A_{h_\epsilon})_{-\frac{1}{2}}(u) + (H_\epsilon)_{-\frac{1}{2}}(u). \end{aligned}$$

By Lemma 8.9 the map $(H_\epsilon)_{-\frac{1}{2}} : X^\eta \rightarrow X^{-\frac{1}{2}}$ is continuously Fréchet differentiable with respect to u , and by Lemmas 5.2 and 5.4, it is also continuous in ϵ if $\eta = 1/2 - \delta$ with $\delta > 0$ sufficiently small. Therefore, the same holds if $\eta = 1/2$.

The map $A_\epsilon = -h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^* + aI$ is a bounded linear operator from $H^1(\Omega)$ to $X^{-1/2}$. It is also continuous in ϵ since it is analytic as a function of h_ϵ , and h_ϵ is continuous in ϵ .

Thus, the map Z is continuously differentiable in u and continuous in ϵ . The derivative of $\frac{\partial Z}{\partial u}(e, 0)$ is an isomorphism by hypotheses. Therefore, the Implicit Function Theorem apply, implying that the zeroes of $Z(\cdot, \epsilon)$ are given by a continuous function $e(\epsilon)$. This proves the claim. $\square$

In order to get the lower semicontinuity, we also need to prove the continuity of local unstable manifolds at equilibria, more precisely

Theorem 8.11 *Suppose f and g satisfy the conditions of Theorem 8.10, u_0 is an equilibrium of (26) with $\epsilon = 0$, and for each $\epsilon > 0$ sufficiently small, let u_ϵ be the unique equilibrium*

of (26), whose existence is asserted by Corollary 8.3 and Theorem 8.10. Then, for ϵ and δ sufficiently small, there exists a local unstable manifold $W_{\text{loc}}^u(u_\epsilon)$ of u_ϵ , and if we denote $W_\delta^u(u_\epsilon) = \{w \in W_{\text{loc}}^u(u_\epsilon) \mid \|w - u_\epsilon\|_{X^\eta} < \delta\}$, then

$$\beta(W_\delta^u(u_\epsilon), W_\delta^u(u_0)) \quad \text{and} \quad \beta(W_\delta^u(u_0), W_\delta^u(u_\epsilon))$$

approach zero as $\epsilon \rightarrow 0^+$, where $\beta(O, Q) = \sup_{o \in O} \inf_{q \in Q} \|q - o\|_{X^\eta}$ for $O, Q \subset X^\eta$.

Proof Let $H_\epsilon(u) = H(u, \epsilon)$ be the map defined by (27) with $\beta = -1/2$ and u_ϵ a hyperbolic equilibrium of (26). Since $H(u, \epsilon)$ is differentiable by Lemma 8.9, it follows that $H_\epsilon(u_\epsilon + w, \epsilon) = H_\epsilon(u_\epsilon, \epsilon) + H_u(u_\epsilon, \epsilon)w + r(w, \epsilon) = A_\epsilon u_\epsilon + H_u(u_\epsilon, \epsilon)w + r(w, \epsilon)$, with $r(w, \epsilon) = o(\|w\|_{X^\eta})$, as $\|w\|_{X^\eta} \rightarrow 0$. The claimed result was proved in [12], assuming that

- (a) $\|r(w, 0) - r(w, \epsilon)\|_{X^\beta} \leq C(\epsilon)$, with $C(\epsilon) \rightarrow 0$ when $\epsilon \rightarrow 0^+$, uniformly for w in a neighborhood of 0 in X^η .
- (b) $\|r(w_1, \epsilon) - r(w_2, \epsilon)\|_{X^\beta} \leq k(\rho)\|w_1 - w_2\|_{X^\eta}$, for $\|w_1\|_{X^\eta} \leq \rho$, $\|w_2\|_{X^\eta} \leq \rho$, with $k(\rho) \rightarrow 0$ when $\rho \rightarrow 0^+$ and k non-decreasing.

Property a) follows easily from the fact that both H_ϵ and $\partial_u H_\epsilon$ are uniformly continuous in ϵ for u in bounded sets of X^η by Lemmas 5.2, 5.4 and 8.9. It remains to prove b).

If $w_1, w_2 \in X^\eta$ and $\epsilon \in [0, \epsilon_0]$, with $0 < \epsilon_0 < 1$ small enough, we have

$$\begin{aligned} \|r(w_1, \epsilon) - r(w_2, \epsilon)\|_{X^\beta} &= \|H(u_\epsilon + w_1, \epsilon) - H(u_\epsilon, \epsilon) - H_u(u_\epsilon, \epsilon)w_1 \\ &\quad - H(u_\epsilon + w_2, \epsilon) + H(u_\epsilon, \epsilon) + H_u(u_\epsilon, \epsilon)w_2\|_{X^\beta} \\ &\leq \|F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F_u(u_\epsilon, \epsilon)w_1 \end{aligned} \quad (46)$$

$$\begin{aligned} &\quad - F(u_\epsilon + w_2, \epsilon) + F(u_\epsilon, \epsilon) + F_u(u_\epsilon, \epsilon)w_2\|_{X^\beta} \\ &\quad + \|G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G_u(u_\epsilon, \epsilon)w_1 \\ &\quad - G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) + G_u(u_\epsilon, \epsilon)w_2\|_{X^\beta}. \end{aligned} \quad (47)$$

We first estimate (46). Since f' is bounded by (43), we have

$$\begin{aligned} &\left| \langle F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F_u(u_\epsilon, \epsilon)w_1 \right. \\ &\quad \left. - F(u_\epsilon + w_2, \epsilon) + F(u_\epsilon, \epsilon) + F_u(u_\epsilon, \epsilon)w_2, \Phi \rangle_{\beta, -\beta} \right| \\ &\leq \int_\Omega | [f(u_\epsilon + w_1) - f(u_\epsilon) - f'(u_\epsilon)w_1 - f(u_\epsilon + w_2) + f(u_\epsilon) + f'(u_\epsilon)w_2] \Phi | \, dx \\ &= \int_\Omega | [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)](w_1(x) - w_2(x)) \Phi | \, dx \\ &\leq K_2 \left\{ \int_\Omega | [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^2 (w_1(x) - w_2(x))^2 | \, dx \right\}^{\frac{1}{2}} \|\Phi\|_{H^1(\Omega)}, \end{aligned}$$

where K_2 is the embedding constant of $X^{\frac{1}{2}} = H^1(\Omega)$ into $L^2(\Omega)$, and $w_1(x) \leq \xi_x \leq w_2(x)$ or $w_2(x) \leq \xi_x \leq w_1(x)$.

Now choosing $q = \frac{1}{1-2\eta}$, we have by [8, 9] that X^η is continuously imbedded in $L^{2q}(\Omega)$. Therefore, if p is the conjugate exponent of q we have, by Hölder's inequality

$$\begin{aligned} & \left\{ \int_{\Omega} [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^2 (w_1(x) - w_2(x))^2 dx \right\}^{\frac{1}{2}} \\ & \leq \left\{ \int_{\Omega} [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^{2p} dx \right\}^{\frac{1}{2p}} \left\{ \int_{\Omega} (w_1(x) - w_2(x))^{2q} dx \right\}^{\frac{1}{2q}} \\ & \leq \left\{ \int_{\Omega} [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^{2p} dx \right\}^{\frac{1}{2p}} \|w_1 - w_2\|_{L^{2q}(\Omega)} \\ & \leq K_3 \left\{ \int_{\Omega} [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^{2p} dx \right\}^{\frac{1}{2p}} \|w_1 - w_2\|_{X^\eta}, \end{aligned}$$

where K_3 is the imbedding constant of the imbedding X^η into $L^{2q}(\Omega)$. Therefore, we have

$$\begin{aligned} & \|F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F_u(u_\epsilon, \epsilon)w_1 - F(u_\epsilon + w_2, \epsilon) + F(u_\epsilon, \epsilon) + F_u(u_\epsilon, \epsilon)w_2\|_{X^\beta} \\ & \leq K_2 K_3 \left\{ \int_{\Omega} [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^{2p} dx \right\}^{\frac{1}{2p}} \|w_1 - w_2\|_{X^\eta}. \end{aligned}$$

Now the integrand above is bounded by $4^p \|f'\|_\infty^{2p}$ and goes to 0 as $\rho \rightarrow 0^+$, since $\|w_1\|_{X^\eta} \leq \rho$, $\|w_2\|_{X^\eta} \leq \rho$ and $w_1(x) \leq \xi_x \leq w_2(x)$. Thus, the integral goes to 0 by Lebesgue's Theorem.

We now estimate (47):

$$\begin{aligned} & \left| \langle G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G_u(u_\epsilon, \epsilon)w_1 \right. \\ & \quad \left. - G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) + G_u(u_\epsilon, \epsilon)w_2, \Phi \rangle_{\beta, -\beta} \right| \\ & \leq \int_{\partial\Omega} \left| [g(\gamma(u_\epsilon + w_1)) - g(\gamma(u_\epsilon)) - g'(\gamma(u_\epsilon))w_1 \right. \\ & \quad \left. - g(\gamma(u_\epsilon + w_2)) + g(\gamma(u_\epsilon)) + g'(\gamma(u_\epsilon))w_2] \gamma(\Phi) \gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right| d\sigma(x) \\ & = \int_{\partial\Omega} \left| [g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))] \gamma(w_1(x) - w_2(x)) \gamma(\Phi) \gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right| d\sigma(x) \\ & \leq \overline{K}_2 \left\{ \int_{\partial\Omega} [(g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon)))]^2 [\gamma(w_1(x) \right. \\ & \quad \left. - w_2(x))]^2 \left[\gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right]^2 d\sigma(x) \right\}^{\frac{1}{2}} \|\Phi\|_{X^{\frac{1}{2}}}, \end{aligned}$$

where $\overline{K}_2$ is the constant of the embedding $H^1(\Omega)$ into $L^2(\partial\Omega)$, and $w_1(x) \leq \xi_x \leq w_2(x)$ or $w_2(x) \leq \xi_x \leq w_1(x)$. Now choosing $\overline{q} = \frac{1}{1-2\eta}$, we have that X^η is continuously imbedded in $L^{2\overline{q}}(\partial\Omega)$. Hence, if $\overline{p}$ is the conjugate exponent of $\overline{q}$, we have by Hölder's inequality

$$\begin{aligned}
& \left\{ \int_{\partial\Omega} [g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))]^2 [\gamma(w_1(x) - w_2(x))]^2 \left[\gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right]^2 d\sigma(x) \right\}^{\frac{1}{2}} \\
& \leq \left\{ \int_{\partial\Omega} [g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))]^{2\bar{p}} \left[\gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right]^{2\bar{p}} d\sigma(x) \right\}^{\frac{1}{2\bar{p}}} \\
& \quad \cdot \left\{ \int_{\partial\Omega} [\gamma(w_1(x) - w_2(x))]^{2\bar{q}} d\sigma(x) \right\}^{\frac{1}{2\bar{q}}} \\
& \leq \left\{ \int_{\partial\Omega} [g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))]^{2\bar{p}} \left[\gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right]^{2\bar{p}} d\sigma(x) \right\}^{\frac{1}{2\bar{p}}} \|w_1 - w_2\|_{L^{2\bar{q}}(\partial\Omega)} \\
& \leq \bar{K}_3 \left\{ \int_{\partial\Omega} [g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))]^{2\bar{p}} \left[\gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right]^{2\bar{p}} d\sigma(x) \right\}^{\frac{1}{2\bar{p}}} \|w_1 - w_2\|_{X^\eta},
\end{aligned}$$

where $\bar{K}_3$ is the embedding constant of X^η into $L^{2\bar{q}}(\partial\Omega)$. Therefore, we have

$$\begin{aligned}
& \|G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G_u(u_\epsilon, \epsilon)w_1 - G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) + G_u(u_\epsilon, \epsilon)w_2\|_{X^\beta} \\
& \leq \bar{K}_2 \bar{K}_3 \left\{ \int_{\partial\Omega} [g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))]^{2\bar{p}} \left[\gamma \left(\left| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \right| \right) \right]^{2\bar{p}} d\sigma(x) \right\}^{\frac{1}{2\bar{p}}} \|w_1 - w_2\|_{X^\eta}.
\end{aligned}$$

Now the integrand above is bounded by $4^{\bar{p}} \|g'\|_\infty^{2\bar{p}} \|\theta\|_\infty^{2\bar{p}}$, where θ is given by (32) and (33), thus it goes to 0 as $\rho \rightarrow 0^+$, since $\|w_1\|_{X^\eta} \leq \rho$, $\|w_2\|_{X^\eta} \leq \rho$ and $w_1(x) \leq \xi_x \leq w_2(x)$. Thus, the integral goes to 0 by Lebesgue's Theorem. $\square$

We are now in a position to prove the main result of this section

Theorem 8.12 *Assume the hypotheses of Theorem 8.10 hold. Then, the family of attractors $\{\mathcal{A}_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$, of the problem (26), whose existence is guaranteed by Theorem 7.5, is lower semicontinuous in X^η . (We observe that the conditions on η hold if $1/2 - \delta < \eta < 1/2$, with δ sufficiently small.)*

Proof The system generated by (26) is gradient for any ϵ and its equilibria are all hyperbolic for ϵ in a neighborhood of zero. Also, the equilibria are continuous in ϵ by Theorem 8.10. The linearization is continuous in ϵ , as shown during the proof of Theorem 8.10, and the local unstable manifolds of the equilibria are continuous in ϵ , by Theorem 8.11. Then, the result follows from [12, Theorem 3.10]. $\square$

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