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The Björling problem for timelike minimal surfaces in $\mathbb{R}^4_1$

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Abstract In this paper, we extend and solve the Björling problem for timelike surfaces in the ambient space $\mathbb{R}^4_1$. To do this, we define a Gauss map ideally suited to this setting using the split-complex variable and then we obtain a Weierstrass representation formula. We use this to construct new examples and give applications. In particular, we obtain one-parameter families of timelike surfaces in $\mathbb{R}^4_1$ which are solutions of the timelike Björling problem. In addition, we establish symmetry principles for the class of minimal timelike surfaces in $\mathbb{R}^4_1$.

Keywords Timelike surfaces · Björling problem · Lorentz–Minkowski space · Symmetry principles

Mathematics Subject Classfication 53A10 · 53B30 · 53C50

1 Introduction

The Björling problem and its solutions are an important, well-known problem. In the three-dimensional Euclidean space $\mathbb{R}^3$, given a real analytic strip the classical Björling problem was proposed by Björling in 1844 and consists in the construction of a minimal surface in $\mathbb{R}^3$ containing the strip in the interior. The solution was obtained by Schwarz in 1890 through an explicit formula in terms of initial data. After that, the Björling problem has been considered in other ambient spaces, including in larger codimension or with indefinite metrics. Some works in this direction are [3–8, 18], and [9]. For instance, in [3] the Björling problem was solved for spacelike surfaces in $\mathbb{L}^3 = \mathbb{R}^3_1$ using a complex representation formula, while in [11] we can find a version of the problem in the hyperbolic three-dimensional space $\mathbb{H}^3$. [18] focused on the problem in three-dimensional Lie groups, while in [7] they considered

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Lorentzian three-dimensional Lie groups. In addition, [5] considers spacelike surfaces in the Lorentz–Minkowski space $\mathbb{L}^4 = \mathbb{R}^4_1$.

For timelike surfaces the authors, more recently, proposed and solved in [6] and [9], the Björling problem for timelike surfaces in $\mathbb{R}^3_1$ and $\mathbb{R}^4_2$. In ([6,9]) the authors use the split-complex (or paracomplex) variable and theory of solutions to the homogeneous wave equation for constructing split-holomorphic extensions and then use the Weierstrass representation for minimal surfaces in $\mathbb{R}^4_1$. This representation was first derived by Konderak [14] for $\mathbb{R}^3_1$ and extended to other Lorentzian 3-manifolds in [16].

In this paper, we are mainly concerned with solving the Björling problem for timelike surfaces in the Lorentz space $\mathbb{L}^4 = \mathbb{R}^4_1$. We observe that this problem, besides its importance in geometry, it also is very interesting from the point of view of physics, since our ambient space is the simplest example of a relativistic spacetime. In order to solve this problem, we establish, using the split-complex variable, a convenient local frame for the immersion, which we use to describe the Gauss map. Then, we apply split-holomorphic extensions in a natural way to find the solution of the Björling problem and to show the uniqueness of the surface. We note that, since the initial curve c can be timelike or spacelike, we must consider two problems, the timelike Björling problem and the spacelike Björling problem.

After solving the Björling problem, we give explicit examples of minimal timelike surfaces which are solutions to the Björling problem and moreover, we recover the representation formula of the Björling problem for minimal timelike surface in the Lorentz–Minkowski space $\mathbb{L}^3 = \mathbb{R}^3_1$ ([6]). As another consequence, we rewrite versions of the timelike Björling problem which we also solve using the Weierstrass representation initially obtained [Formulas (13), (14)]). These solutions describe a one-parameter family of timelike surfaces in which each member is a solution of the timelike Björling problem.

As part of our study, we also prove the split-complex version of Schwarz reflection and we extend the notion of k-subspace of symmetry for timelike surface in $\mathbb{R}^4_1$ ([9]). Then we describe types of symmetries for minimal timelike surfaces in $\mathbb{R}^4_1$ with respect to non-degenerate k-subspaces.

2 Split-complex variable and preliminares

We begin our study of the Björling problem in $\mathbb{R}^4_1$, whose inner product is $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, by defining a version of the cross product, $\boxtimes$:

$$\langle \boxtimes(u, v, w), x\rangle = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} \tag{1}$$

for all $x \in \mathbb{R}^4_1$,

Then, it follows that $\langle \boxtimes(u, v, w), u\rangle = 0 = \langle \boxtimes(u, v, w), v\rangle = \langle \boxtimes(u, v, w), w\rangle$ and

$$\boxtimes(u, v, e_1) = -\tilde{u} \times \tilde{v},\quad \boxtimes(u, v, e_2) = \tilde{u} \times \tilde{v},\quad \boxtimes(u, v, e_3) = -\tilde{u} \times \tilde{v},\quad \boxtimes(u, v, e_4) = \tilde{u} \times \tilde{v},$$

where $\tilde{u}$ means dropping the first coordinate in the first case, the second coordinate in the second case, etc., and the cross product is taken in the three-dimensional space with the metric inherited from $\mathbb{R}^4_1$.

Now, we consider the split-complex variable $z = t + k's$ where $t, s \in \mathbb{R}$ and $k'^2 = 1$.

Definition 2.1 The split-complex numbers $\mathbb{C}' = \{t + k's | t, s \in \mathbb{R}, k'^2 = 1, 1k' = k'1\}$ are a commutative algebra over $\mathbb{R}$. If $z = t + k's$ then $\mathfrak{Re}(z) = t$, $\mathfrak{Im}(z) = s$, $\bar{z} = t - k's$. The indefinite metric on $\mathbb{C}'$ is given by $-z\bar{z} = -t^2 + s^2$.

We define $\mathbb{C}'^4_1$ to be $\mathbb{C}'^4$ to be the split-complex vector space with the indefinite split-hermitian structure:

$$h(z, w) = -z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3 + z_4\bar{w}_4.$$

This induces an indefinite inner product $g(,)$ on $\mathbb{R}^8_4$ by

$$\begin{aligned}
\mathfrak{Re}h(z, w) &= g((x_1 + k'y_1, \dots, x_4 + k'y_4), (u_1 + k'v_1, \dots, u_4 + k'v_4)) \\
&= g((x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4), (u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)) \\
&= -x_1u_1 + y_1v_1 + x_2u_2 - y_2v_2 + x_3u_3 - y_3v_3 + x_4u_4 - y_4v_4. \\
g(z, z) &= -{x_1}^2 + {y_1}^2 + {x_2}^2 - {y_2}^2 + {x_3}^2 - {y_3}^2 + {x_4}^2 - {y_4}^2.
\end{aligned}$$

The related symmetric bilinear product is:

$$(z, w) = -z_1w_1 + z_2w_2 + z_3w_3 + z_4w_4 = h(z, \bar{w}),$$

so that

$$\begin{aligned}
(z, z) = &-{x_1}^2 - {y_1}^2 + {x_2}^2 + {y_2}^2 + {x_3}^2 + {y_3}^2 + {x_4}^2 + {y_4}^2 \\
&+ 2k'(-x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)
\end{aligned}$$

We define an indefinite Riemannian metric $\tilde{g}_p(X, Y) = \frac{4}{c} g_z(X', Y')$ for $X, Y \in T_p(\mathbb{C}'\mathbb{P}^{3-}_1)$, $c > 0$ is fixed, z is any point in H^7_3 with $\pi(z) = p$ and $\pi_* X' = X$, $\pi_* Y' = Y$. Denote the Grassmannian of oriented timelike 2-planes in $\mathbb{R}^4_1$, $G\left(\mathbb{R}^2_1 \subset \mathbb{R}^4_1\right) := G^-_{2,4}$. Then, if $u, v \in \mathbb{R}^4_1$ are perpendicular vectors with $-\langle u, u\rangle = \eta^2 = \langle v, v\rangle$, we note that $u + k'v \in \mathbb{C}^4_1$ and

$$\begin{aligned}
g(u + k'v, u + k'v) &= -2\eta^2 \\
(u + k'v, u + k'v) &= 0,
\end{aligned}$$

thus this is an element in the quadric

$$Q^2_1 = \{[z] \in \mathbb{C}'\mathbb{P}^{3-}_1 \mid (z, z) = 0\}.$$

Let $G^+_{2,4} = G(\mathbb{R}^2 \subset \mathbb{R}^4_1)$ denote the Grassmannian of spacelike 2-planes of $\mathbb{R}^4_1$ with the induced orientation. Given $m, n \in \mathbb{R}^4_1$, with $\langle m, m\rangle = \langle n, n\rangle = \lambda^2 > 0$ and $\langle m, n\rangle = 0$, let $\mathrm{span}[m, n] \in G^+_{2,4}$. Then, we can identify $G^+_{2,4}$ with the quadric $QR = \{[z] \in \mathbb{C}'\mathbb{P}^3_1 \mid (z, z) \neq 0,\ g(z, z) = 0 = (z, \bar{z})\}$, through the mapping that sends each $\mathrm{span}[m, n] \in G^+_{2,4}$ into $[z] \in QR$ where $z = m + k'n$. To define the projective space $\mathbb{C}'\mathbb{P}^3$ here, we use lines in $\mathbb{C}'^4$ with the equivalence relation defined by multiplying by invertible $\lambda \in \mathbb{C}'$.

Now, we focus on minimal timelike immersions.

Definition 2.2 A smooth immersion $X : M^2_1 \to \mathbb{R}^4_1$ of a two-dimensional oriented connected manifold is called a *timelike surface* if the induced metric has signature $(1, 1)$.

For our goals, we need to know that

Proposition 2.1 *If $S = X(M)$ is a timelike surface in $\mathbb{R}^4_1$ then $\Delta_M X = 2H$, where Δ and H denote the Laplacian and the mean curvature vector of the immersion, respectively.*

Definition 2.3 A timelike surface S in $\mathbb{R}^4_1$ is minimal if the mean curvature vector $H = 0$ for all points of S.

Let $z = t + k's$, where t and s are conformal coordinates in a neighborhood of a point p in M^2_1, so that $-\langle X_t, X_t\rangle = \eta^2 = \langle X_s, X_s\rangle$ and $\langle X_t, X_s\rangle = 0$. Let

$$\phi_j = \frac{\partial X_j}{\partial z} = \frac{1}{2}\left(\frac{\partial X_j}{\partial t} + k'\frac{\partial X_j}{\partial s}\right),$$

where X_j represents a component of timelike immersion. Observe that

$$-\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \langle X_s, X_s\rangle + \langle X_t, X_t\rangle + 2k\,\langle X_t, X_s\rangle = 0.$$

If we set $|a + k'b|^2 = b^2 - a^2$ then

$$-|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 = \frac{1}{4}(\langle X_s, X_s\rangle - \langle X_t, X_t\rangle) = \frac{\eta^2}{2} > 0.$$

Consider the split-complex 1-forms defined by $\Phi_j = \phi_j \mathrm{d}z$. By looking at a conformal change of coordinates and using Proposition 2.1 in [17], we can see that these forms are globally defined on M^2_1.

$$\begin{aligned} 2Re\int_\gamma \phi_j \mathrm{d}z &= Re\int_\gamma \left(\frac{\partial X_j}{\partial t} + k'\frac{\partial X_j}{\partial s}\right)(\mathrm{d}t + k'\mathrm{d}s) \\ &= \int_\gamma \left(\frac{\partial X_j}{\partial t}\mathrm{d}t + \frac{\partial X_j}{\partial s}\mathrm{d}s\right) = \int_\gamma \mathrm{d}X_j = X_j\,\big|_\gamma\ . \end{aligned}$$

Thus the integral over any closed curve has real part zero. The converse is also true.

Theorem 2.1 ([14]) *Let Σ be a Lorentzian surface and choose four split-holomorphic one-forms $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ globally defined on Σ satisfying:*

$$-\Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2 = 0. \tag{2}$$

$$-|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2 > 0. \tag{3}$$

$$\text{Each } \Phi_j \text{ has no real periods.} \tag{4}$$

Then the map $X : \Sigma \to \mathbb{R}^4_1$ given by

$$X(z) = 2Re\int_{\gamma_z}(\Phi_1, \Phi_2, \Phi_3, \Phi_4)\,\mathrm{d}z,$$

where γ_z is a path from the fixed basepoint z is a minimal immersion in $\mathbb{R}^4_1$.

Remark 2.1 We could also use the split-complex variable $w = k'z = s + k't$ in the above formulas, setting

$$\psi_j = \frac{\partial X_j}{\partial w} = \frac{1}{2}\left(\frac{\partial X_j}{\partial s} + k'\frac{\partial X_j}{\partial t}\right).$$

After replacing Φ_j by Ψ_j the formulas are the same, except that

$$-|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 = -\langle X_s, X_s\rangle + \langle X_t, X_t\rangle < 0.$$

We will use this alternative choice of variable when studying the Björling problem assuming that the initial curve is spacelike, as we will see in Sect. 4.

3 Weierstrass representation and the Gauss map

If $X : M_1^2 \to \mathbb{R}_1^4$ is a timelike immersion of $(M_1^2, z = t + k's)$, we can explicitly obtain three functions $x, y, \mu : M_1^2 \to \mathbb{C}'$ such that

$$\frac{\partial X}{\partial z} = \mu(1 + x^2 + y^2, 2x, 2y, -1 + x^2 + y^2).$$

In fact, setting

$$\mu W = (\phi_1, \phi_2, \phi_3, \phi_4)$$

where $(W, W) = 0$ and $(W, \bar{W}) \neq 0$, then $\phi_1\bar{\phi}_1 \neq 0$, and we obtain

$$\left(\frac{\phi_2}{\phi_1}\right)^2 + \left(\frac{\phi_3}{\phi_1}\right)^2 + \left(\frac{\phi_4}{\phi_1}\right)^2 = 1.$$

Inspired by classical stereographic projection, we define homogeneous coordinates for the Grassmannian of timelike planes, taking:

$$\mu(1 + x^2 + y^2) = \phi_1, \quad 2\mu x = \phi_2, \quad 2\mu y = \phi_3, \quad \mu(-1 + x^2 + y^2) = \phi_4.$$

Therefore,

$$W = (1 + x^2 + y^2, 2x, 2y, -1 + x^2 + y^2),$$

and we can solve for x, y and μ:

$$\mu = \frac{\phi_1 - \phi_4}{2}, \quad x = \frac{\phi_2}{\phi_1 - \phi_4}, \quad y = \frac{\phi_3}{\phi_1 - \phi_4}. \tag{5}$$

Note that $\phi_1 - \phi_4 \neq 0$, since $\phi_1 - \phi_4 = 0$ implies $(\phi_2)^2 + (\phi_3)^2 = 0$, and in $\mathbb{C}'$ the equation $a^2 + b^2 = 0$ has no non-trivial solutions. This would yield $\phi_2 = \phi_3 = 0$, but by formula (2), this is not possible.

We can also define two normal vectors N_1 and N_2 using x and y:

$$N_1 = (1 + x\bar{x} + y\bar{y}, x + \bar{x}, y + \bar{y}, -1 + x\bar{x} + y\bar{y}), \tag{6}$$
$$N_2 = k'(x\bar{y} - y\bar{x}, -y + \bar{y}, x - \bar{x}, x\bar{y} - y\bar{x}), \tag{7}$$

which are orthogonal to the vector $W = E + k'F \in \mathbb{R}_1^4 + k'\mathbb{R}_1^4$. So we have an orthonormal local frame defined by

$$\{F/\lambda, E/\lambda, N_1/\lambda, N_2/\lambda\} \tag{8}$$

where $\lambda^2 = 4(\mathfrak{Im}^2 x + \mathfrak{Im}^2 y) = (x - \bar{x})^2 + (y - \bar{y})^2$, with sign $+, -, +, +$, resp. Note that

$$E = \frac{W + \bar{W}}{2}, \quad F = k'\frac{W - \bar{W}}{2}, \quad N_2 = \boxtimes\left(\frac{F}{\lambda}, \frac{E}{\lambda}, N_1\right), \quad F = \boxtimes\left(E, \frac{N_2}{\lambda}, \frac{N_1}{\lambda}\right).$$

Now let $(U, z = t + k's)$ be isothermal coordinates in a neighborhood of a point p in M_1^2, so that $-\langle X_t, X_t\rangle = \eta^2 = \langle X_s, X_s\rangle$ and $\langle X_t, X_s\rangle = 0$. If we write $\mu = \frac{1}{2}(\alpha + k'\beta)$ and $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, it follows that

$$\Phi = \frac{1}{2}(X_t + k'X_s) = \mu W = \mu(E + k'F) = \frac{1}{2}(\alpha E + \beta F) + \frac{k'}{2}(\beta E + \alpha F)$$

therefore

$$X_t = \alpha E + \beta F, \quad X_s = \beta E + \alpha F.$$

Now, take the local orthonormal ordered frame adapted to the immersion, with sign $+$, $-$, $+$, $+$, given by,

$$\left\{\frac{X_s}{\eta}, \frac{X_t}{\eta}, \frac{N_1}{\lambda}, \frac{N_2}{\lambda}\right\} \tag{9}$$

where $\eta^2 = \lambda^2(\alpha^2 - \beta^2)$, which we have assumed to be positive. We also have:

$$\boxtimes\left(X_t, \frac{N_2}{\lambda}, \frac{N_1}{\lambda}\right) = \boxtimes\left(\alpha E + \beta F, \frac{N_2}{\lambda}, \frac{N_1}{\lambda}\right) = \alpha F + \beta E = X_s. \tag{10}$$

Finally, taking $\left\{m := \frac{N_1}{\lambda}(z), n := \frac{N_2}{\lambda}(z)\right\}$ to be the orthonormal, normal frame defined by (9), we define the split-complex map $A : M_1^2 \to QR$, according the curve type which we will use as initial data for solving the Björling problem, as follows:

If the curve c is timelike, $A(z)$ is defined by

$$A(z) = [m(z) + k'n(z)]. \tag{11}$$

If the curve c is spacelike, $A(w),\ w = k'z$, is defined by

$$A(w) = [n(w) + k'm(w)]. \tag{12}$$

We see that in both cases, $[A(z)] \in QR$ and $[\Phi(z)] \in Q_1^2$.

4 Main results

We now consider the spacelike and timelike Björling problem for timelike minimal surfaces in $\mathbb{R}_1^4$. Let $c : I \subseteq \mathbb{R} \to \mathbb{R}_1^4$ be a C^ω timelike or spacelike curve in $\mathbb{R}_1^4$ and let $p : I \to \mathbb{C}'^4$ be a C^ω vector field along c, so that $\langle c', \mathfrak{Re}(p)\rangle = 0 = \langle c', \mathfrak{Im}(p)\rangle$, $\langle \mathfrak{Re}(p), \mathfrak{Re}(p)\rangle = 1 = \langle \mathfrak{Im}(p), \mathfrak{Im}(p)\rangle$ and $\langle \mathfrak{Re}(p), \mathfrak{Im}(p)\rangle = 0$. We ask that if we rotate $c'(0)$ to either e_1 or e_2 and add the missing vector, that $|e_1, e_2, \mathfrak{Im}(p), \mathfrak{Re}(p)| = 1$.

We call such a pair (c, p) an *analytic strip*. If the curve c is timelike, the *timelike Björling problem* is to find a timelike minimal surface S defined by $X : \Omega \subseteq \mathbb{C}' \to \mathbb{R}_1^4$ so that

1. $X(u, 0) = c(u)$
2. $A(u, 0) = [p(u)], \quad \forall u \in I.$

We can make the same definition for a C^∞ or a C^k strip.

If the curve c is spacelike, the *spacelike Björling problem* asks for a timelike minimal surface with

1. $X(0, u) = c(u)$
2. $A(0, u) = [p(u)], \quad \forall u \in I.$

Now, we solve these two Björling problems.

It's clear that if $X : \Omega \subseteq \mathbb{C}' \to \mathbb{R}_1^4$ is a timelike minimal immersion, then $c(u) := X(u, 0)$ or $X(0, u)$ and $[p(u)] := A(u, 0)$ or $A(0, u)$ satisfy the data and are either smooth or analytic, depending on the surface.

Now, we recall from [6] that, using basic properties of the homogeneous wave equation, a split-complex valued function $f(t, 0) = \gamma(t) + k'\delta(t), t \in [-R, R]$, can be extended split-holomorphicaly as

$$f(t, s) = u(t, s) + k'v(t, s) = (F(t + s) + G(t - s), F(t + s) - G(t - s)),$$

on the rhombus in $\mathbb{R}^2$ with vertices $(\pm R, 0)$ and $(0, \pm R)$, where F, G are C^2-real functions such that $F(t) + G(t) = \gamma(t)$ and $F(t) - G(t) = \delta(t)$. This extension is called the split-holomorphic deterministic extension. Using this extension, one proves the following proposition.

Proposition 4.1 *Let $\phi(t, s) = u + k'v$ and $\hat{\phi}(t, s) = a + k'b$ two split-holomorphic extensions of a split-complex valued function $\gamma(t) + k'\delta(t)$, so that $u(t, 0) = a(t, 0) = \gamma(t)$ and $v(t, 0) = b(t, 0) = \delta(t)$ in an open set I . Then, they agree everywhere on a subset of the intersection domain of the two extensions.*

Then, note from above that if γ and δ are at least of C^2-class, we have unique split-holomorphic extensions. Therefore, we are going to solve the Björling problem, only requiring that the curve c and the vector field p have components at least of C^2-class. We remind the reader that the classic Björling problem assumes the curve c and the vector field p, to be both analytic.

Theorem 4.1 *Let S be a timelike minimal surface in $\mathbb{R}^4_1$ given by $X : U \subseteq \mathbb{C}' \to \mathbb{R}^4_1$. Define the curve $c(u) = X(u, 0)$ and the split-complex vector field $[p(u)] := A(u, 0)$ along c on a real interval contained in U. Choose any simply connected open set $\Omega \subseteq U$ containing I in which we can define split-holomorphic extensions of c and p. Then, for all $z \in \Omega$ we have*

$$X(z) = \mathfrak{Re}\left(c(z) + k' \int_{z_o}^{z} \boxtimes(c'(\zeta), \mathfrak{Im}(p(\zeta)), \mathfrak{Re}(p(\zeta)))\mathrm{d}\zeta\right). \tag{13}$$

In the spacelike case we get equation:

$$X(w) = \mathfrak{Re}\left(c(w) - k' \int_{w_o}^{w} \boxtimes(c'(\zeta), \mathfrak{Im}(p(\zeta)), \mathfrak{Re}(p(\zeta)))\mathrm{d}\zeta\right). \tag{14}$$

Proof We set $\Phi = \frac{\partial X}{\partial z} = (\phi_1, \phi_2, \phi_3, \phi_4)$. Then we can see from the ordered basis (9), that

$$\Phi(z) = \frac{1}{2}(X_t(z) + k'X_s(z)) = \frac{1}{2}(X_t + k' \boxtimes (X_t(z), n(z), m(z))), \quad z \in \Omega. \tag{15}$$

Restricting to I we get:

$$\Phi(u, 0) = \frac{1}{2}[X_t(u, 0) + k' \boxtimes (X_t(u, 0), n(u, 0), m(u, 0))] \tag{16}$$

$$= \frac{1}{2}[c'(u) + k' \boxtimes (c'(u), \mathfrak{Im}(p(u)), \mathfrak{Re}(p(u)))], \tag{17}$$

where $A(u, 0) = [p(u)]$ is given by formula (11). Since these functions are at least of C^2-class, we can extend them uniquely to split-holomorphic functions on some open set containing I, more specifically, if I is the interval $(-R, R) \subset \Omega$, then shrinking Ω, if necessary, so that it is contained in the rhombus $\mathcal{R}$ with vertices $(\pm R, 0)$ and $(0, \pm R)$, we obtain from Proposition 3.1 that $\Phi(u, 0)$ has a unique extension to

$$\Phi(z) = \frac{1}{2}[c'(z) + k' \boxtimes (c'(z), \mathfrak{Im}(p(z)), \mathfrak{Re}(p(z)))], \text{ on } \Omega.$$

So when the curve is timelike, we are done.

In the case of the spacelike Björing problem, we use the variable $w = k'z$, and we get that

$$\Psi(w) = \frac{1}{2}(X_s(w) + k'X_t(w)) = \frac{1}{2}(X_s(w) + k' \boxtimes (X_s(w), n(w), m(w))). \tag{18}$$

Restricting to $(0, u)$, we find

$$\Psi(0,u) = \frac{1}{2}(X_s(0,u) + k' \boxtimes (X_s(0,u), n(0,u), m(0,u))) \quad (19)$$

$$= \frac{1}{2}(X_s(0,u) + k' \boxtimes (c'(u), \mathfrak{Re}(p(u)), \mathfrak{Im}(p(u)))). \quad (20)$$

Here, we have switched, of course to:

1. $X(0, u) = c(u)$
2. $A(0, u) = [p(u)], \forall u \in I$,

where $A(u, 0) = [p(u)]$ is given by formula (12). Similarly, since these functions are C^2, we can extend them uniquely to split-holomorphic functions on some open set containing I. Thus,

$$\Psi(s+kt) = \frac{1}{2}(c'(w) + k' \boxtimes (c'(w), \mathfrak{Re}(p(w)), \mathfrak{Im}(p(w)))), \text{ on } \Omega.$$

Thus, the theorem holds. □

Proposition 4.2 *The solutions given by formula* (13) *and* (14) *are independent of the choice of orthonormal basis of the normal plane.*

Proof For a real parameter θ, let $\cos(\theta)m(t) + \sin(\theta)n(t)$ and $-\sin(\theta)m(t) + \cos(\theta)n(t)$ be a new orthonormal basis of $N_t = \mathrm{Span}\{m(t), n(t)\}$. This defines

$$p_\theta(t) = (\cos(\theta)m(t) + \sin(\theta)n(t)) + k'\left(-\sin(\theta)m(t) + \cos(\theta)n(t)\right).$$

Then making the split-holomorphic extension, we have

$$p_\theta(z) = (\cos(\theta)m(z) + \sin(\theta)n(z)) + k'(-\sin(\theta)m(z) + \cos(\theta)n(z)),$$

so from the anti-commutativity of the exterior product it follows that

$$\begin{aligned}
X_\theta(z) &= \mathfrak{Re}\left\{c(z) + k'\int_{z_0}^{z} \boxtimes(c', \mathfrak{Im}(p_\theta), \mathfrak{Re}(p_\theta))\mathrm{d}\xi\right\}\\
&= \mathfrak{Re}\left\{c(z)+k'\int_{z_0}^{z} \boxtimes(c', -\sin(\theta)m(\xi)+\cos(\theta)n(\xi), \cos(\theta)m(\xi)+\sin(\theta)n(\xi))\mathrm{d}\xi\right\}\\
&= \mathfrak{Re}\left\{c(z) + k'\int_{z_0}^{z} [\cos^2(\theta) \boxtimes (c', n(\xi), m(\xi)) + \sin^2(\theta) \boxtimes (c', n(\xi), m(\xi))]\mathrm{d}\xi\right\}\\
&= \mathfrak{Re}\left\{c(z) + k'\int_{z_0}^{z} \boxtimes(c', \mathfrak{Im}(p), \mathfrak{Re}(p))\mathrm{d}\xi\right\} = X(z),
\end{aligned}$$

hence follows the claimed independence. □

For our examples, it is convenient to note that, using power series, we have split-holomorphic extensions of various functions:

$$\begin{aligned}
\cos(t + k's) &= \cos(t)\cos(s) - k'\sin(t)\sin(s)\\
\sin(t + k's) &= \sin(t)\cos(s) + k'\cos(t)\sin(s)\\
\cosh(t + k's) &= \cosh(t)\cosh(s) + k'\sinh(t)\sinh(s)\\
\sinh(t + k's) &= \sinh(t)\cosh(s) + k'\cosh(t)\sinh(s).
\end{aligned}$$

Example 4.1 This example begins with a timelike curve: $c(t) = (\sinh(t), 0, 0, \cosh(t))$, $p(t) = (\sinh(t), k' \sin(t), k' \cos(t), \cosh(t))$. Using formula (13), we obtain the minimal immersion

$$X(t,s) = (\sinh(t)\cosh(s), -\cos(t)\sin(s), \sin(t)\sin(s), \cosh(t)\cosh(s)).$$

Moreover, the split-holomorphic functions $\mu(z)$, $x(z)$ and $y(z)$, where $z = t + k's$, are

$$\mu(z) = \frac{1}{4}\mathrm{e}^{-z}, \quad x(z) = -k'\mathrm{e}^{z}\cos(z), \quad y(z) = k'\mathrm{e}^{z}\sin(z).$$

In fact,

$$\begin{aligned}
2\mu &= \frac{\partial}{\partial z}(X_1 - X_4) = \frac{\partial}{\partial z}(\cosh(s)(\sinh(t) - \cosh(t))) = \frac{1}{2}\mathrm{e}^{-z}.\\
2\mu x &= \frac{\partial}{\partial z}X_2 = \frac{1}{2}\mathrm{e}^{-z}x = \frac{\partial}{\partial z}(-\cos(t)\sin(s)) = -\frac{k'}{2}(\cos(t)\cos(s) - k'\sin(t)\sin(s))\\
2\mu y &= \frac{\partial}{\partial z}X_3 = \frac{1}{2}\mathrm{e}^{-z}y = \frac{\partial}{\partial z}(\sin(t)\sin(s)) = \frac{k'}{2}(\sin(t)\cos(s) + k'\cos(t)\sin(s)).
\end{aligned}$$

Note that $c(t) = X(t,0)$. Now, using the formulas (5)–(8), and $x(t,0)$, $y(t,0)$ we obtain $A(t,0) = [p(t)]$. In fact, since

$$x(t,0) = a = -k'\mathrm{e}^{t}\cos(t), \quad y(t,0) = b = k'\mathrm{e}^{t}\sin(t)$$

we have that

$$a\bar{a} = -\mathrm{e}^{2t}\cos^2(t), \quad a\bar{b} = \mathrm{e}^{2t}\sin(t)\cos(t),$$

therefore,

$$N_1(t,0) = (1 - \mathrm{e}^{2t}, 0, 0, -1 - \mathrm{e}^{2t}), \quad N_2(t,0) = (0, -2\mathrm{e}^{t}\sin(t), -2\mathrm{e}^{t}\cos(t), 0),$$

and

$$m(t) = \frac{N_1}{2\mathrm{e}^{t}} = (-\sinh(t), 0, 0, -\cosh(t)), \quad n(t) = \frac{N_2}{2\mathrm{e}^{t}} = (0, -\sin(t), -\cos(t), 0).$$

Thus $A(t,0) = [p(t)]$, as claimed. Note that in formula (13) we can change p to $-p$. We observe that the initial curve c is a geodesic.

Example 4.2 The Björling data are the spacelike curve $c(s) = (0, \cos s, \sin s, 0)$ and the vector field

$$p(s) = (0, \cos s, \sin s, 0) + k'(\sinh s, 0, 0, \cosh s).$$

Using formula (14), in the variable $w = s + k't$ we obtain the minimal immersion,

$$X(t,s) = (\sinh t \cosh s, \cos t \cos s, \cos t \sin s, \sinh t \sinh s).$$

Moreover, the split-holomorphic functions $\mu(w)$, $x(w)$ and $y(w)$ are

$$\mu(w) = \frac{1}{4}k'\mathrm{e}^{-w}, \quad x(w) = -k'\mathrm{e}^{w}\sin w, \quad y(w) = k'\mathrm{e}^{w}\cos w.$$

Note that for $t = 0$, $X(0,s) = c(s)$. Now, using the formulas (5)–(8), taking

$$x(0,s) = a = -k'\mathrm{e}^{s}\sin s \ \text{ and } \ y(0,s) = b = k'\mathrm{e}^{s}\cos s,$$

we obtain

$$a\bar{a} = -\mathrm{e}^{2s}\sin^2 s, \quad b\bar{b} = -\mathrm{e}^{2s}\cos^2 s, \quad a\bar{b} = \mathrm{e}^{2s}\sin s\cos s.$$
$$N_1(0,s) = (1-\mathrm{e}^{2s}, 0, 0, -1-\mathrm{e}^{2s}), \quad N_2(0,s) = (0, -2\mathrm{e}^{s}\cos s, -2\mathrm{e}^{s}\sin s, 0).$$

Therefore, we have

$$m(s) = (-\sinh s, 0, 0, -\cosh s) \quad \text{and} \quad n(s) = (0, -\cos s, -\sin s, 0).$$

Since, for each $s \in I$, the vector $c''(s)$ belongs to the normal plane along the curve $c(s) = X(0,s)$ this curve is a geodesic line of the surface solution $X(M)$.

Next, we establish the uniqueness for the solution of the Björling problem.

Theorem 4.2 *There exists a unique solution $X : \Omega \to \mathbb{R}^4_1$ to the Björling problem for timelike minimal surfaces which is given by the representations above, where Ω is a simply connected open subset of $\mathbb{C}'$ containing the real interval I and for which c and p have split-holomorphic extensions.*

Proof Assume the timelike curve case and define the split-holomorphic curve $\Phi : \Omega \subseteq \mathbb{C}' \to \mathbb{C}'^4$ by

$$\Phi(z) = c'(z) + k' \boxtimes (c'(z), \mathfrak{Im}(p(z)), \mathfrak{Re}(p(z))), \quad \forall z \in \Omega, \tag{21}$$

where Ω is a simply connected open subset of $\mathbb{C}'$ containing I on which the split-holomorphic extensions $c(z)$, $p(z)$ exist.

We know that

$$\langle c'(u), \boxtimes(c'(u), \mathfrak{Im}(p(u)), \mathfrak{Re}(p(u)))\rangle = 0$$
$$\langle \boxtimes (c'(u), \mathfrak{Im}(p(u)), \mathfrak{Re}(p(u))), \boxtimes(c'(u), \mathfrak{Im}(p(u)), \mathfrak{Re}(p(u)))\rangle = -\langle c'(u), c'(u)\rangle.$$

This means that $\phi(u, 0) = v + k'w$ with v and w orthogonal and of opposite length, so that

$$-\phi_1(u,0)^2 + \phi_2(u,0)^2 + \phi_3(u,0)^2 + \phi_4(u,0)^2 = 0.$$

Similarly,

$$-|\phi_1(u,0)|^2 + |\phi_2(u,0)|^2 + |\phi_3(u,0)|^2 + |\phi_4(u,0)|^2 = -2|c'(u)|^2 \neq 0.$$

Using the split-holomorphic deterministic extension, we see that

$$-\phi_1(z)^2 + \phi_2(z)^2 + \phi_3(z)^2 + \phi_4(z)^2 = 0$$

where the extensions exist. By shrinking the simply connected set, we can also assume that

$$-|\phi_1(z)|^2 + |\phi_2(z)|^2 + |\phi_3(z)|^2 + |\phi_4(z)|^2 \neq 0.$$

Using the fact that if C is a curve in the $\mathbb{C}'$ plane and $f(z)$ is a split-holomorphic function on C with a continuous derivative $f'(z)$, then

$$\int_C f'(z)\mathrm{d}z = f(z)|_C \tag{22}$$

and that the integral is clearly path independent, we can see that the split-holomorphic curve Φ has no real periods in a simply connected domain.

Thus, using Theorem 2.1, $X(z) = \mathfrak{Re}\int_{x_o}^{z} \Phi(\zeta)\mathrm{d}\zeta$ defines a timelike minimal surface $X(\Omega)$ in $\mathbb{R}^4_1$.

Next, we check that $X(u, 0) = c(u)$ and $A(u, 0) = [p(u)]$. The first condition is clear, since the second part of the integrand is purely imaginary. For the second condition, we note that

$$X_t(u, 0) = c'(u)$$
$$X_s(u, 0) = \boxtimes(c'(u), \mathfrak{Im}(p(u)), \mathfrak{Re}(p(u))).$$

Meanwhile, from (15) we see that $X_s(u, 0) = \boxtimes(c'(u), n(u), m(u))$. So it follows that $A(u, 0) = [m(u) + k'n(u)] = [\mathfrak{Re}(p(u)) + k'\mathfrak{Im}(p(u))] = [p(u)]$.

The uniqueness follows easily from Proposition 4.1.

For spacelike curves the proof follows the same line as the timelike case. □

As noted in [6], one cannot solve the Björling problem uniquely for a null curve. One cannot even guarantee existence. The null Björling problem would state that for any null curve $x(u)$ with a spacelike frame $[m(u), n(u)]$ along the curve we could find a minimal surface $X(u, v)$, where u and v are null coordinates, so that $X(u, 0) = x(u)$ and the normal space to the surface along the curve is spanned by $m(u)$ and $n(u)$. [1]

Proposition 4.3 *Let $x(u)$ be a null curve in $\mathbb{R}^4_1$, with a given orthonormal spacelike frame $[m(u), n(u)]$ along the curve. If there is a solution $X(u, v)$ to the Björling problem along the curve then, locally $X(u, v) = x(u) + y(v)$ for another null curve $y(v)$ so that $y(0) = \mathbf{0}$ and $y'(0) \perp m(u), n(u)$ in a neighborhood of 0.*

Proof We recall that every minimal surface in $\mathbb{R}^4_1$ can be written as a sum of null curves. It's easy to see we can find such a $y(v)$ and, since $X(u, 0) = x(u)$ we must have $y(0) = \mathbf{0}$. Furthermore, the tangent space along the curve is given by

$$X_u(u, v) = x'(u)$$
$$X_v(u, v) = y'(v).$$

Thus $X_u(u, 0) = x'(u)$ and $X_v(u, 0) = y'(0)$. □

Next, we give an example of a null curve with a normal frame that cannot be contained in any minimal surface.

Example 4.3 Take the null helix $x(u) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), \cos(u), \sin(u))$. The tangent vector to the curve is $\alpha(u) = \frac{1}{\sqrt{2}}(\cosh(u), \sinh(u), -\sin(u), \cos(u))$.

Define the normal frame by:

$$m(u) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), -\cos(u), -\sin(u)) \text{ and}$$
$$n(u) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), \cos(u), \sin(u)).$$

Here, $y'(0)$ would need be orthogonal to $m(u)$ and $n(u)$ for all u around 0, but this is impossible because $m(u)$ and $n(u)$ span all of $\mathbb{R}^4_1$ in any small neighborhood of 0.

Example 4.4 In this example, we construct a one-parameter family of solutions of the Björling problem. Let $c(t) = (\sinh t, 0, 0, \cosh t)$ be the curve in Example 4.1, and consider the vector fields

$$m(t) = (\sinh t, 0, 0, \cosh t), \quad n_1(t) = (0, \sin t, \cos t, 0), \quad n_2(t) = (0, \cos t, -\sin t, 0).$$

[1] We thank the referee for suggesting this problem.

We use the normal plane given by span$[m(t), n_\theta(t)]$ where

$$n_\theta(t) = \cos\theta n_1(t) + \sin\theta n_2(t) = (0, \sin(\theta+t), \cos(\theta+t), 0),$$

for $\theta \in [0, 2\pi]$. Then given the initial data $c(t)$ and

$$p_\theta(t) = m(t) + k' n_\theta(t) = (\sinh t, 0, 0, \cosh t) + k'(0, \sin(\theta+t), \cos(\theta+t), 0),$$

we obtain a one-parameter family of solutions of the Björling problem given by

$$X_\theta(t,s) = (\sinh t \cosh s, -\cos(\theta+t)\sin s, \sin(\theta+t)\sin s, \cosh t \cosh s).$$

Note that $X_\theta(t, 0) = c(t)$ for all θ. For $\theta = 0$, we have the Example 4.1. Moreover

$$c''(t) = m(t) = (\sinh t, 0, 0, \cosh t)$$

implies that the curve is a geodesic line of each surface $X_\theta(t, s)$.

Using the split-complex representation formulas (13), (14), we recover the timelike (spacelike) Björling problem in $\mathbb{L}^3 = \mathbb{R}^3_1$, namely Theorem 3.1 and 3.2 in ([6]), as follows.

Corollary 4.1 *Let $c : I \to \mathbb{R}^3_1 = \{x_4 = 0\} \subset \mathbb{R}^4_1$, be a regular real timelike (spacelike) curve at least of C^2-class, and let $p : I \to \mathbb{C}'^4$ be a real vector field along c at least of C^2-class, such that $p(u) = w + k'e_4$ if the curve is timelike ($p(u) = e_4 + k'w$ if the curve is spacelike), where $w(u) \in \mathbb{R}^3_1$ is a unit spacelike vector field with $\langle c'(u), w(u)\rangle = 0$ for all $u \in I$. Then, there exists a unique solution to the timelike (spacelike) Björling problem for minimal timelike surfaces in $\mathbb{R}^3_1$, which is given by*

$$\mathfrak{Re}\left(c(z) + k'\int_{t_o}^{z} (w(\xi) \times c'(\xi))\mathrm{d}\xi\right), \tag{23}$$

where $\times$ is the cross product of $\mathbb{R}^3_1$.

Proof If curve c is timelike and $p(u) = w + k'e_4$ we have
$\boxtimes(c'(u), \mathfrak{Im}\, p(u), \mathfrak{Re}\, p(u)) = \tilde{w}(u) \times \tilde{c}'(u)$ and from Theorem 4.2's formula, (23) follows.
For the spacelike curve c with $p(u) = e_4 + k'w$, we have
$\boxtimes(c'(u), \mathfrak{Im}\, p(u), \mathfrak{Re}\, p(u)) = -\tilde{w}(u) \times \tilde{c}'(u)$ and from Theorem 4.2 we have the claim. □

Example 4.5 The Björling data are the spacelike curve $c(s) = (0, \cosh(s), s, 0)$ and the vector field

$$p(s) = \left(0, \frac{1}{\cosh(s)}, -\frac{\sinh(s)}{\cosh(s)}, k'\right).$$

Using formula (14), in the variable $w = s + k't$, we obtain the minimal immersion,

$$X(w) = (\sinh(t)\cosh(s), \cosh(t)\cosh(s), s, 0).$$

Note that for $t = 0$, $X(0, s) = c(s)$.

5 Applications and more examples

We now use formula (13) and Theorem 4.2 in order to construct families of examples by considering the Frenet frame on the curve c, as we see next.

Example 5.1 Assume that $\gamma(t)$ is a non-degenerate timelike curve with $\langle\gamma'(t), \gamma'(t)\rangle = -1$, and let $T(t)$, $N(t)$, $B(t)$, $R(t)$ be the Frenet frame adapted to it. We consider two cases for Björling problem, namely, given the strip (c, p) with

$$c(t) = \gamma(t) \quad \text{and} \quad p(t) = N(t) + k'B(t) \tag{24}$$

$$c(t) = \gamma(t) \quad \text{and} \quad p(t) = N(t) + k'R(t), \tag{25}$$

for each case, we find a timelike minimal surface S defined by $X : \Omega \subset \mathbb{C}' \to \mathbb{R}^4_1$ so that $X(t, 0) = c(t)$ and $A(t, 0) = [p(t)]$.

Since in both cases the curve c and the Frenet frame satisfy the conditions of the timelike Björling problem, it follows from formula (13) that the surfaces which are local solutions for the Björling problem are, respectively, given by

$$X(z) = \mathfrak{Re}\left(c(z) + k'\int_{z_0}^{z} \boxtimes(c'(\xi), B(\xi), N(\xi))\mathrm{d}\xi\right), \quad X_s(t, 0) = R(t),$$

$$Y(z) = \mathfrak{Re}\left(c(z) + k'\int_{z_0}^{z} \boxtimes(c'(\xi), R(\xi), N(\xi))\mathrm{d}\xi\right), \quad Y_s(t, 0) = B(t).$$

Note that, for $(t, 0)$, $X_t(t, 0) = T(t) = Y_t(t, 0)$, both surfaces stay orthogonal along the curve $\gamma(t)$ and $N(t)$ is a common normal field to both surfaces.

For a real parameter $\theta \in [0, 2\pi]$, we define $Z(\theta, z) = \cos(\theta)X(z) + \sin(\theta)Y(z)$. This is the solution given by

$$Z(\theta, z) = \mathfrak{Re}\left(c(z) + k'\int_{z_0}^{z} \boxtimes(c'(\xi), \cos(\theta)B(\xi) + \sin(\theta)R(\xi), N(\xi))\mathrm{d}\xi\right),$$

which satisfies $Z_s(\theta, t, 0) = -\sin(\theta)B(t) + \cos(\theta)R(t)$.

We can take the domain of the split-holomorphic extensions to be compact. The resulting surfaces are then transversal to each other.

Example 5.2 Let $c(t)$, $t \in \mathbb{R}$, a timelike curve given by

$$c(t) = \left(\sqrt{2}\sinh t, \frac{\sqrt{3}}{3}\cos(t\sqrt{3}), \frac{\sqrt{3}}{3}\sin(t\sqrt{3}), \sqrt{2}\cosh t\right).$$

The first, the second and the third curvatures will be denoted by $k(t)$, $\tau(t)$, $\rho(t)$. The Frenet frame $\{T, N, B, R\}$ satisfy the Frenet formulas

$$\begin{cases} T' = kN \\ N' = kT + \tau B \\ B' = -\tau N + \rho R \\ R' = -\rho B, \end{cases}$$

where $B(t)$ belongs to the subspace $\mathrm{Span}[T(t), N(t), c'''(t)]$, it preserves this orientation and $R(t) = -\boxtimes(T(t), N(t), B(t))$. Note that

$$R' = \boxtimes(T', N, B) + \boxtimes(T, N', B) + \boxtimes(T, N, B') = \rho \boxtimes (T, N, R) = -\rho B.$$

Now,

$$T(t) = \left(\sqrt{2}\cosh t, -\sin(t\sqrt{3}), \cos(t\sqrt{3}), \sqrt{2}\sinh t\right)$$
$$N(t) = \frac{1}{\sqrt{5}}\left(\sqrt{2}\sinh t, -\sqrt{3}\cos(t\sqrt{3}), -\sqrt{3}\sin(t\sqrt{3}), \sqrt{2}\cosh t\right) \text{ with } k(t) = \sqrt{5}$$
$$B(t) = \frac{1}{\sqrt{2}}\left(-\sqrt{2}\cosh t, 2\sin(t\sqrt{3}), -2\cos(t\sqrt{3}), -\sqrt{2}\sinh t\right) \text{ with } \tau(t) = \frac{4\sqrt{10}}{5}$$
$$R(t) = \frac{1}{\sqrt{5}}\left(\sqrt{3}\sinh t, \sqrt{2}\cos(t\sqrt{3}), \sqrt{2}\sin(t\sqrt{3}), \sqrt{3}\cosh t\right) \text{ with } \rho(t) = \frac{\sqrt{15}}{5}.$$

Then, we take the initial data $c(t)$ and the normal plane along the curve $c(t)$ given by

$$p_\theta(t) = N(t) + k'\left(\cos(\theta)B(t) + \sin(\theta)R(t)\right), \quad \theta \in [0, 2\pi].$$

Then from Theorem 4.2 it follows that, for each $\theta \in [0, 2\pi]$, Z_θ is a timelike minimal surface. In addition the curve $Z_\theta(t, 0) = c(t)$ is a common geodesic and $N(t)$ is the normal to the all these surfaces Z_θ along the curve $c(t)$. The surfaces are given parametrically by

$$Z_\theta(z) = \mathfrak{Re}(c(z)) + \cos(\theta)\mathfrak{Im}\left(\int_{z_0}^{z} R(\xi)\mathrm{d}\xi\right) + \sin(\theta)\mathfrak{Im}\left(\int_{z_0}^{z} B(\xi)\mathrm{d}\xi\right).$$

For the next two results, we take as reference the Schwarz Principle of Reflection established in Theorem 3.3 of [9], and we prove a version of Propositions 3.2 and 3.3 of [9].

Proposition 5.1 *(Timelike version) Let $X : \Omega \subseteq \mathbb{C}' \in \mathbb{R}^4_1$ be the solution of the timelike Björling problem for a given strip (c, p) in $\mathbb{R}^4_1$, where Ω is simply connected open set containing the real interval I which is symmetric with respect to the real axis and for which $c(z)$ and $p(z)$ have split-holomorphic extensions. For all $z \in \Omega$, we have*

$$X(\bar{z}) = \mathfrak{Re}\left(c(z) - k'\int_{z_o}^{z} \boxtimes(c'(\zeta), \mathfrak{Im}(p(\zeta), \mathfrak{Re}(p(\zeta))\mathrm{d}\zeta\right). \tag{26}$$

Proof Let $\tilde{S} = \tilde{X}(\Omega)$ be defined by $\tilde{X}(t, s) = X(t, -s)$. The surface is still timelike and minimal. Using our definitions of $\tilde{m}$ and $\tilde{n}$, we see that

$$\tilde{m}(t, s) = m(t, -s)$$
$$\tilde{n}(t, s) = -n(t, -s)$$
$$\tilde{A}(t, s) = \bar{A}(t, -s).$$

Then $\tilde{A}(t, 0) = \bar{A}(t, 0) = [\bar{p}(t)]$ and $\tilde{X}(t, 0) = X(t, 0) = c(t)$. Thus $\tilde{X}$ is a solution of the Björling problem for $\tilde{c} = c$, $\tilde{p} = \bar{p}$. This implies

$$\tilde{X}(z) = 2\mathfrak{Re}\int_{z_o}^{z} \tilde{\Phi}(w)\mathrm{d}w,$$

where $\tilde{\Phi}(\xi) = \frac{1}{2}\left(\tilde{X}_t + k' \boxtimes (\tilde{X}_t(\xi), \tilde{n}(\xi), \tilde{m}(\xi))\right)$. Restricting to $(t, 0)$ we get:
$\tilde{\Phi}(t, 0) = \frac{1}{2}\left(X_t(t, 0) + k' \boxtimes (X_t(t, 0), -n(t, 0), m(t, 0))\right)$, and we are done. □

For the spacelike case, we have the following result for which we omit the proof since it follows the same line as the proof of Proposition 5.1. Now, we just recall that $w = s + k't$ and we have to use the spacelike Björling representation formula.

Proposition 5.2 *(Spacelike version) Let $X : \Omega \subseteq \mathbb{C}' \in \mathbb{R}^4_1$ be the solution of the spacelike Björling problem for a given strip (c, p) in $\mathbb{R}^4_1$, where Ω is a w-symmetric simply connected open set containing the real interval I and for which $c(w)$ and $p(w)$ have split-holomorphic extensions. For all $w \in \Omega$ we have*

$$X(\bar{w}) = \mathfrak{Re}\left(c(w) + k' \int_{w_o}^{w} \boxtimes(c'(\zeta), \mathfrak{Im}(p(\zeta), \mathfrak{Re}(p(\zeta))\mathrm{d}\zeta\right). \tag{27}$$

5.1 k-subspaces of symmetry

In this section, we study k-subspaces of symmetry for timelike minimal surfaces in $\mathbb{R}^4_1$ and we establish some results describing various symmetries for this kind of surface.

We start defining orthogonal intersection and degenerate or non-degenerate subspace of symmetry.

Definition 5.1 Let $X : M^2_1 \to \mathbb{R}^4_1$ be a regular surface in $\mathbb{R}^4_1$, and consider a k-dimensional plane $\Pi^k \subset \mathbb{R}^4_1$. We say that Π^k intersects the surface orthogonally provided at any point $X(p)$ of $X(M^2_1) \cap \Pi^k \neq \emptyset$ both $T_pM^2 \cap T_{X(p)}\Pi^k$ and $T_pM^2 \cap (T_{X(p)}\Pi^k)^{\perp}$ are 1-dimensional.

Definition 5.2 A k-plane Π is a plane of symmetry for the surface $X(M) = S$ if there exists a linear transformation A of $\mathbb{R}^4_1$ with $A^2 = \mathrm{Id}$ which fixes Π pointwise and a diffeomorphism T which satisfies $A \circ X = X \circ T$.

If Π is non-degenerate, we ask that A be orthogonal. Specifically, $\mathbb{R}^4_1 = \Pi \oplus W$, with $A(p) = p$ if $p \in \Pi$ and $A(w) = -w$ if $w \in W$.

If Π is degenerate, we have $\mathbb{R}^4_1 = V \oplus W$ with $A_{|W}$ being an orthogonal transformation and $A_{|V}$ being anti-orthogonal, meaning $\langle Av, Av' \rangle = -\langle v, v' \rangle$. We still ask that $A(p) = p$ if $p \in \Pi$ and only Π is fixed pointwise. Finally, we require V and W to be invariant by A.

Even in the degenerate case, V and W are non-degenerate, since $\oplus$ means orthogonal direct sum.

In $\mathbb{R}^4_1$ there are three one-dimensional types of lines: timelike, spacelike and null lines. We have three types of planes, with signature:

$$(+, +)\ (-, +)\ (0, +).$$

Here $+$ means a spacelike vector, $-$ a timelike vector and 0 a null vector. Finally, there are three kinds of three-dimensional subspaces: $(-, +, +)$ $(+, +, +)$ $(0, +, +)$. Next, we give every type of obtainable example below. Each one of them represents a timelike minimal surfaces with k-plane of symmetry.

Example 5.3

$$X(t, s) = \left(\frac{2}{3}\left(t^3 + 3ts^2\right) + t,\ t^2 + s^2,\ 2ts,\ \frac{2}{3}\left(t^3 + 3ts^2\right) - t\right)$$

has $[e_2]$ as line of symmetry with $T(t, s) = (-t, s)$. It has $[e_2, e_3]$ as a plane of symmetry with $T(t, s) = (-t, -s)$, and $[e_1, e_2, e_4]$ as three-space of symmetry with $T(t, s) = (t, -s)$. This example covers the spacelike line and $(+, +)$, $(-, +, +)$ cases, respectively.

Example 5.4

$$X(t, s) = (\cosh(s)\sinh(t), -\cos(t)\sin(s), \sin(s)\sin(t), \cosh(s)\cosh(t)).$$

This has $[e_2, e_4]$ as a plane of symmetry with $T(t, s) = (-t, s)$. It has $[e_3, e_4]$ as a plane of symmetry with $T(t, s) = (-t, -s)$, and $[e_1, e_4]$ as a plane of symmetry with $T(t, s) = (t, -s)$. So this example covers $(+, +)$ and $(-, +)$ cases.

Example 5.5

$$X(t, s) = (t + s, \cos t + \cos s, \sin t, \sin s).$$

is symmetric with respect to $[e_1]$ using $T(t, s) = (t + \pi, s - \pi)$. So it covers the timelike line case.

When the fixed subspace is degenerate, there is an orthonormal basis with respect to which the symmetry A has the following forms, with subspaces $V = [e_1, e_4]$ and $W = [e_2, e_3]$. In fact,

If $\Pi = [e_1 + e_4]$ then there is an orthonormal basis so that

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

If $\Pi = [e_1 + e_4, e_3]$, we have the following A

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

If $\Pi = [e_1 + e_4, e_2, e_3]$, we have the following A

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

So, using this information, we have

Example 5.6 The parametric timelike minimal surface

$$X(t, s) = (t, 0, 0, s)$$

is symmetric with respect to the degenerate line $\Pi = [e_1 + e_4]$.

It is also symmetric with respect to $\Pi = [e_1 + e_4, e_3]$ and symmetric with respect to $\Pi = [e_1 + e_4, e_3, e_2]$. In all of them, we use $T(t, s) = (s, t)$, and $V = [e_1, e_4]$, $W = [e_2, e_3]$.

Finally, the surface is symmetric with respect to the three-dimensional subspace $[e_2, e_3, e_4]$ using $T(t, s) = (-t, s)$. So this examples covers the degenerate cases (0), $(0, +)$, $(0, +, +)$, and the non-degenerate $(+, +, +)$ case.

Thus, we have examples of all possible cases.

We observe that, according to our definition of orthogonal intersection, in Example 5.3 the only subspace of symmetry which intersects X orthogonally is $[e_2]$. In Example 5.4, the only orthogonal intersection is with respect to $[e_1, e_4]$ and in Example 5.6, the null line $[e_1 + e_4]$ orthogonally intersects the surface

The main theorem of this section concerns minimal surfaces which are symmetric with respect to non-degenerate subspaces.

Theorem 5.1 *Let S be a timelike minimal surface in $\mathbb{R}^4_1$ given by $X : U \subset \mathbb{C}' \to \mathbb{R}^4_1$. Then*

1. *every non-degenerate straight line contained in S is an axis of symmetry of S;*
2. *if S intersects any timelike or spacelike 2-plane Π^2 orthogonally along a regular curve of S then Π^2 is a plane of symmetry of S;*
3. *if S intersects any timelike or spacelike 3-space Π^3 orthogonally along a regular curve of S, then Π^3 is a 3-plane of symmetry of S.*

To prove this theorem, we need Lemma 5.1 below which will imply Theorem 5.1 after using a special coordinate system. For constructing that system, we follow the technique of Hoffman and Karcher [12], which employs the conjugate minimal surface. We learned of this in [5].

Definition 5.3 The map $X^* : \Omega \subset \mathbb{C}' \to \mathbb{R}^4_1$ given by

$$X^*(z) = \mathfrak{Im} \int_{\gamma_z} (\Phi_1, \Phi_2, \Phi_3, \Phi_4)\, \mathrm{d}z,$$

where γ_z is a path from the fixed basepoint z is a minimal immersion in $\mathbb{R}^4_1$ called the conjugate immersion.

Equivalently X^* is the harmonic conjugate of X, in that $X^*{}_{jt} = X_{js}$ and $X^*{}_{js} = X_{jt}$.

It is easy to see that (X_j, X_j^*) gives a new coordinate system on an open set in Ω, unless $X_{jt} = \pm X_{js}$ in a subset of Ω, in which case X_j is a function of $t \pm s$. Of course, one coordinate could be identically zero, since our immersion might be contained in a three-dimensional subspace.

In every case below, we just have to have one of the complementary coordinates not being a function of $t \pm s$. If any of the nonzero complementary functions is a function of $t \pm s$ then the definition of orthogonal intersection says that the curve $\gamma(w) = (t(w), s(w))$ defined by $X_j(\gamma(w)) = 0$ would satisfy $t(w) \pm s(w) = c$, where c is a constant. But this is a null curve, which is a contradiction. We see easily that in every case there is a complementary function which can be used as a coordinate.

In general, we use (X_j, X_j^*) as a new coordinate system on Ω. For instance, to look at case (3) in the Theorem, let's assume that S intersects the x_1, x_2, x_3 three-space orthogonally. Then $c(z) = (c_1(z), c_2(z), c_3(z), 0)$ and in the tangent space, $c'^{\perp} = c^{*\prime}$ is in $[e_4]$, because of the definition of orthogonal intersection.

We can set $u = X_3$ and $v = X_3^*$. To use the Lemma, we need $u + kv$ so that $X(u, 0)$ or $X(0, v) \subset [e_1, e_2, e_3]$. $v = 0$ means that we are in $X^* \cap [e_1, e_2, e_4]$, which is the curve c^*. We have $c(u) = (c_1(u), c_2(u), u, 0)$, so that $x_4(u, 0) = 0$.

Now, we have the following version of the Lemma 4.1 of [9]

Lemma 5.1 *Let S be a timelike minimal surface in $\mathbb{R}^4_1$, given by $X : \Omega \subseteq \mathbb{C}' \to \mathbb{R}^4_1$, where Ω is symmetric and simply connected.*

1. *If, for all $t \in I$, the curve $c(t) = X(t, 0)$ is contained in the x_1-axis, then*
$$X(t, -s) = (x_1(t, s), -x_2(t, s), -x_3(t, s), -x_4(t, s)). \tag{28}$$
2. *If, for all $s \in I$, the curve $c(s) = X(s, 0)$ is contained in the x_4-axis, then*
$$X(s, -t) = (-x_1(s, t), -x_2(s, t), -x_3(s, t), x_4(s, t)). \tag{29}$$
3. *If, for all $t \in I$, the curve $X(t, 0)$ is contained in the timelike plane $\Pi = [e_1, e_4]$, then*
$$X(t, -s) = (x_1(t, s), -x_2(t, s), -x_3(t, s), x_4(t, s)). \tag{30}$$

4. *If, for all $s \in I$ the curve $X(s,0)$ is contained in the spacelike plane $\Pi = [e_3, e_4]$ then*

$$X(s,-t) = (-x_1(s,t), -x_2(s,t), x_3(s,t), x_4(s,t)).$$

5. *If, for all $s \in I$, the curve $X(s,0)$ is contained in the timelike 3-space $\Pi = [e_1, e_2, e_3]$ and if the surface S intersects Π orthogonally along c, then*

$$X(s,-t) = (x_1(s,t), x_2(s,t), x_3(s,t), -x_4(s,t)). \tag{31}$$

6. *If, for all $t \in I$, the curve $X(t,0)$ is contained in the 3-space $\Pi = [e_1, e_2, e_3]$ and if the surface S intersects Π orthogonally along c, then*

$$X(t,-s) = (x_1(t,s), x_2(t,s), x_3(t,s), -x_4(t,s)). \tag{32}$$

7. *If, for all $s \in I$, the curve $X(s,0)$ is contained in the positive definite 3-space $\Pi = [e_2, e_3, e_4]$ and if the surface S intersects Π orthogonally along c, then*

$$X(s,-t) = (-x_1(s,t), x_2(s,t), x_3(s,t), x_4(s,t)). \tag{33}$$

Proof We include some of the proofs; the missing ones are similar, of course.

1. Set $c(t) := X(t,0)$ and $p(t) = A(t,0)$. By definition, we have $c(t) = (c_1(t), 0, 0, 0)$, $\mathfrak{Im}\, p(t) = (0, m_2(t,0), m_3(t,0), m_4(t,0))$ and $\mathfrak{Re}\, p(t) = (0, n_2(t,0), n_3(t,0), n_4(t,0))$. We see that $\boxtimes(c'(t), \mathfrak{I}(p(t,0))), \mathfrak{R}(p(t,0))$ has zero for its first coordinate and we write the vector as $(0, \boxtimes_2(t), \boxtimes_3(t), \boxtimes_4(t))$. Using the formulas for $X(z)$, $X(\bar{z})$ we find:

$$X(z) = \left(\mathfrak{Re}(c_1(z)), \mathfrak{Im}\int^z \boxtimes_2(\zeta)\mathrm{d}\zeta, \mathfrak{Im}\int^z \boxtimes_3(\zeta)\mathrm{d}\zeta, \mathfrak{Im}\int^z \boxtimes_4(\zeta)\mathrm{d}\zeta\right)$$
$$X(\bar{z}) = \left(\mathfrak{Re}(c_1(z)), -\mathfrak{Im}\int^z \boxtimes_2(\zeta)\mathrm{d}\zeta, -\mathfrak{Im}\int^z \boxtimes_3(\zeta)\mathrm{d}\zeta, -\mathfrak{Im}\int^z \boxtimes_4(\zeta)\mathrm{d}\zeta\right).$$

For (2), $c(s) := X(s,0)$ and $p(s)=A(s,0)$. By definition, we have $c(s) = (0,0,0,c_4(s))$, $\mathfrak{Im}\, p(s) = (m_1(s,0), m_2(s,0), m_3(s,0), 0)$ and $\mathfrak{Re}\, p(s) = (n_1(s,0), n_2(s,0), n_3(s,0), 0)$. We see that $\boxtimes(c'(s), \mathfrak{I}(p(s,0)), \mathfrak{R}(p(s,0)))$ has zero for its last coordinate and we write the vector as $(\boxtimes_1(s), \boxtimes_2(s), \boxtimes_3(s), 0)$. Using the formulas for $X(w)$, $X(\bar{w})$ we find:

$$X(w) = \left(-\mathfrak{Im}\int^w \boxtimes_1(\zeta)\mathrm{d}\zeta, -\mathfrak{Im}\int^w \boxtimes_2(\zeta)\mathrm{d}\zeta, -\mathfrak{Im}\int^w \boxtimes_3(\zeta)\mathrm{d}\zeta, \mathfrak{Re}(c_4(w))\right)$$
$$X(\bar{w}) = \left(\mathfrak{Im}\int^w \boxtimes_1(\zeta)\mathrm{d}\zeta, \mathfrak{Im}\int^w \boxtimes_2(\zeta)\mathrm{d}\zeta, \mathfrak{Im}\int^w \boxtimes_3(\zeta)\mathrm{d}\zeta, \mathfrak{Re}(c_4(w))\right).$$

3. The hypothesis means that $c(t) = (c_1(t), 0, 0, c_4(t))$. The plane given by $[\mathfrak{Re}\, p(t), \mathfrak{Im}\, p(t)]$ is orthogonal to $T_{c(t)}S$ along $c(t)$. In addition, the hypothesis means $(c'(t)^{\perp} \cap T_{c(t)}S) \perp sp[e_1, e_4]$. Thus, $\boxtimes(\mathfrak{Re}\, p(t), \mathfrak{Im}\, p(t), c'(t))$ is in $[e_2, e_3]$. Now, the result follows as before:

$$X(\bar{z}) = \mathfrak{Re}(c(z)) - k'\int_{z_o}^{z} (0, \alpha(\zeta), \beta(\zeta), 0) d\zeta.$$

5. By the hypothesis $c(s) = (c_1(s), c_2(s), c_3(s), 0)$. Since S intersects Π orthogonally, it follows that $X_t(s,0) \in \Pi^{\perp}$, then $X_t(s,0)$ is parallel to e_4. Then $\mathfrak{Im}\, p(s)$ and $\mathfrak{Re}\, p(s)$ lie in Π, which implies that $\boxtimes(\mathfrak{Re}\, p(s), \mathfrak{Im}\, p(s), c'(s))$ is of the form $(0,0,0,\boxtimes_4(s))$. Then,

$$X(w) = (\mathfrak{Re}(c_1(w)), \mathfrak{Re}(c_3(w)), \mathfrak{Re}(c_3(w)), \mathfrak{Im}\int^w \boxtimes_4(s)),$$
$$X(\bar{w}) = (\mathfrak{Re}(c_1(w)), \mathfrak{Re}(c_2(w)), \mathfrak{Re}(c_3(w)), -\mathfrak{Im}\int^w \boxtimes_4(s)).$$

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