Annali di Matematica (2017) 196:983–1000
DOI 10.1007/s10231-016-0604-5

Pointwise regularity for a parabolic equation with log-term singularity

Angelo R. F. de Holanda[1] · **Olivaine S. de Queiroz**[2] · **Cesar K. S. dos Santos**[2]

Received: 13 March 2016 / Accepted: 9 August 2016 / Published online: 20 September 2016

Abstract We obtained existence and pointwise regularity results for the following parabolic free boundary problem:

$$u_t - \Delta u = \chi_{\{u>0\}} \log u \ \text{ in } \ \Omega \times (0, T],$$

with initial and boundary conditions in some appropriate spaces. The equation is singular along the set $\partial\{u > 0\}$, and the logarithmic nonlinearity does not have scaling properties. Thus, the machinery from regularity theory for free boundary problems, which strongly relies on the homogeneity of the problem, can not be applied directly. We prove that, near the free boundary, an approximate solution grows at most like $r^2 \log r$. This is the so-called supercharacteristic growth, and its study has intriguing open questions. Our estimates are crucial to understand further analytic and geometric properties of the free boundary.

Keywords Free boundary · Regularity theory · Existence · Logarithmic singularity

Mathematics Subject Classification 35K55 · 35R35

✉ Olivaine S. de Queiroz
olivaine@ime.unicamp.br

Angelo R. F. de Holanda
angelo@dme.ufcg.edu.br

Cesar K. S. dos Santos
cesarsantos@ime.unicamp.br

1 Universidade Federal de Campina Grande, Unidade Acadêmica de Matemática, Campina Grande, PB 58429-900, Brazil

2 Departamento de Matemática, IMECC, Universidade Estadual de Campinas, Rua Sérgio Buarque de Holanda, 651, Campinas, SP 13083-859, Brazil

1 Introduction

In this paper we study a model of parabolic equation with singular nonlinearity. Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and the associated local space–time cylinder $Q_T := \Omega \times (0, T]$, $T > 0$, we fix two prescribed nonnegative functions $u_0 \in C^1(\overline{\Omega})$, $g \in W_2^{1,1}(Q_T) \cap L^\infty(Q_T)$, and consider the singular parabolic problem

$$\begin{cases} u_t - \Delta u = \chi_{\{u>0\}} \log u & \text{in } Q_T, \\ u = g & \text{on } \partial\Omega \times (0, T], \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{1.1}$$

Our goal is to prove local existence of a solution and regularity properties for (1.1).

Parabolic equations with singular nonlinearities have been studied in a series of papers. One important model is

$$u_t - \Delta u = -\chi_{\{u>0\}} u^{\gamma-1} \quad \text{in } Q_T, \tag{1.2}$$

coupled with boundary and initial conditions, where $0 < \gamma < 2$. This problem arises as limit of equations modelling either quenching phenomena or chemical reactions (see [1] and [7]). Problem (1.2) when $\gamma \in [1, 2)$ (strong reaction) was studied in [4], where the authors proved optimal regularity and nondegeneracy estimates for the unique solution under some natural assumptions on the boundary and initial conditions. Moreover, they showed that the $(n + 1)$-dimensional Hausdorff measure (with respect to the parabolic metric) of the free boundary $\partial\{u > 0\}$ is locally bounded (see also [3], [14] and references therein).

The works [5] and [13] deal with (1.2) when $\gamma \in (0, 1)$. In both cases, the authors find a solution using a limit process. In fact, they studied the equation

$$u_t - \Delta u = f(\varepsilon, u) \quad \text{in } Q_T, \tag{1.3}$$

where, for $\varepsilon > 0$, the function $f(\varepsilon, u)$ is smooth and $f(\varepsilon, u) \to -u^{\gamma-1}$ pointwisely as $\varepsilon \to 0^+$ for $u > 0$.

The singular character of these equations gives rise to free boundary solutions. Thus, an important task is to prove optimal regularity of the solution close to the free boundary $\partial\{u > 0\}$. In the case of the power-type nonlinearity, the limit solution in [5] satisfies

$$u \in C^{1,\alpha_\gamma}_{\text{loc}}(Q_T), \quad 0 < \gamma < 1, \ \ \alpha_\gamma = \frac{\gamma}{2-\gamma}.$$

This means that, locally, u is $(1 + \alpha_\gamma)/2$-Hölder continuous in t and ∇u is α_γ-Hölder continuous in x (following the notation in [8]).

The power-type singular nonlinearity $-u^{\gamma-1}$ is stronger than $\log u$ for any $\gamma \in (0, 1)$. Thus, it is natural to think that the optimal regularity for solutions of (1.1) should be better than $C^{1,\alpha}$ for any $0 < \alpha < 1$. Of course it is not $C^{1,1}$ since the left-hand side in (1.1) blows up along the free boundary. We shall show that a limiting solution u of (1.1) satisfies

$$u \in C^{1,\log-\text{lip}}_{x,\text{loc}}(Q_T) \cap C^{0,\log-\text{lip}}_{t,\text{loc}}(Q_T).$$

Comparing with the results for the problem with power-type nonlinearity and observing that the right-hand side in (1.1) becomes unbounded, we see that this regularity is optimal.

For the existence result, we are going to use also an approximation procedure as explained in details below. For now, let us just mention that the logarithmic nonlinearity can change sign and this fact raises new challenges, specially concerning a priori estimates in L^∞ for u.

Parabolic and elliptic equations with logarithmic nonlinearity arise when we consider equations modelling the dynamic of thin films of viscous fluids (see [2] and references therein). From theoretical aspects of the free boundary theory, the lack of scaling of the log-type reaction is the main difference between (1.1) and (1.2). Recall that scaling arguments are essential in the study of regularity theory of free boundary problems. We are going to show that, for a fixed $t \in (0, T)$ and close to the free boundary, the approximated solution exhibits a *supercharacterictic growth* like $r^2 \log r$. Phenomenon of this type was studied first by Monneau & Weiss in [10], where the authors investigated an unstable obstacle problem:

$$-\Delta u = \chi_{\{u>0\}} \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.4}$$

Solutions of (1.4) have the supercharacteristic growth close to some free boundary points. Furthermore, the second variation of the energy associated to a solution of (1.4) takes the value $-\infty$ and the solutions are thus called *completely unstable*. Thus, Eq. (1.1) becomes an interesting example of a *highly unstable* parabolic free boundary problem, in the sense that every free boundary point grows like $r^2 logr$.

Finally, let us mention that the regularity theory for minimizers of the elliptic problem associated with (1.1) was recently considered by the second author and Shahgholian in [6].

We begin to describe in more details our main results and techniques.

1.1 Description of the results

A solution of (1.1) is a function $u \in W_2^{1,0}(Q_T)$ (see Sect. 1.2 for the definition of the spaces) satisfying the following: For any test function $\eta \in C^2(\overline{Q_T})$ vanishing on $\partial\Omega \times (0, T]$ and for every $\tau \in (0, T)$, one has the integral identity

$$\int_\Omega u(\tau)\eta(\tau)\mathrm{d}x - \int_\Omega u_0\eta(0)\mathrm{d}x + \int_0^\tau \int_\Omega (-u\eta_t + \langle \nabla u, \nabla\eta\rangle)\,\mathrm{d}x\mathrm{d}t \\ = \int_0^\tau \int_\Omega \eta\chi_{\{u>0\}} \log u \mathrm{d}x\mathrm{d}t, \tag{1.5}$$

with $u_0 \in C^1(\overline{\Omega})$.

Our existence result relies on an approximation procedure that we now describe. For each $0 < \varepsilon < 1$ we define the perturbed term

$$\beta_\varepsilon(s) = \begin{cases} \log\left(\dfrac{s^2 + \varepsilon s + \varepsilon}{s + \varepsilon}\right) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0, \end{cases} \tag{1.6}$$

and the approximating problem

$$\begin{cases} u_t^\varepsilon - \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon) & \text{in } Q_T, \\ u = g & \text{on } \partial\Omega \times (0, T], \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{1.7}$$

The variational formulation of a solution of (1.7) is analogous to the one for the unperturbed problem (1.5): For $u_0 \in C^1(\overline{\Omega})$, we test the equation with any $\eta \in C^2(\overline{Q_T})$ vanishing on $\partial\Omega \times (0, T]$ and for every $\tau \in (0, T)$ we need to have

$$\int_\Omega u^\varepsilon(\tau)\eta(\tau)\mathrm{d}x - \int_\Omega u_0\eta(0)\mathrm{d}x + \int_0^\tau \int_\Omega \left(-u^\varepsilon\eta_t + \langle \nabla u^\varepsilon, \nabla\eta\rangle\right)\mathrm{d}x\mathrm{d}t \\ = \int_0^\tau \int_\Omega \eta\beta_\varepsilon(u^\varepsilon)\mathrm{d}x\mathrm{d}t. \tag{1.8}$$

Remark 1.1 Using density results, one could take as a test function $\eta \in W_2^{1,2}(Q_T)$ vanishing almost everywhere on $\partial\Omega \times (0, T]$ and satisfying $\eta = 0$ on $\Omega \times \{T\}$ in the sense of traces.

Proposition 1.2 *Let $T > 0$ be fixed and assume $u_0 \in C^1(\overline{\Omega})$, $g \in W_2^{1,1}(Q_T) \cap L^\infty(Q_T)$ with $u_0, g \geq 0$. Then, for each $\varepsilon \in (0, 1)$, problem (1.7) has a solution $u^\varepsilon \geq 0$ such that $u^\varepsilon \in C^3(Q_T)$. Furthermore, there exists a constant $M > 0$, $M = M(\Omega, g, u_0, T)$, such that*

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M, \quad \text{for every } \varepsilon \in (0, 1).$$

The existence part of Proposition 1.2 is proved by the method of sub- and supersolution. Since the logarithmic nonlinearity changes sign, we can not apply arguments based purely on the maximum principle to obtain the L^∞-estimate uniform in ε, differently from the power-type nonlinearity. We proceed as follows: First, we prove an L^2-estimate which enable us to control a truncated norm of u^ε; after that, the general machinery from [9] can be applied.

Once we have the approximated solution u^ε, pointwise uniform in $\varepsilon > 0$ estimates are proved in order to pass to the limit as $\varepsilon \to 0^+$ to obtain a candidate for a solution of (1.1).

Lemma 1.3 *Let $u_0 \in C^1(\overline{\Omega})$, $u_0, g \geq 0$, and suppose that $u^\varepsilon \geq 0$ satisfies (1.7). Then, for any $\Omega' \subset\subset \Omega$ and any $\tau \in (0, T]$, there are constants $C_1, C_2, \Lambda > 0$ such that*

$$|\nabla u^\varepsilon(x,t)|^2 \leq C_1\big(u^\varepsilon(x,t)|\log u^\varepsilon(x,t)| + \Lambda u^\varepsilon(x,t)\big), \quad \text{for all x} \in \Omega', \ \text{t} \in [0, \text{T}] \tag{1.9}$$

and

$$|u_t^\varepsilon(x,t)| \leq C_2|\log u^\varepsilon(x,t)|, \quad \text{for all x} \in \Omega', \text{t} \in (\tau, \text{T}]. \tag{1.10}$$

The constants C_1 and Λ depend on $\text{dist}(\Omega', \partial\Omega)$, N, g, u_0, *and* $\|u^\varepsilon\|_{L^\infty(Q_T)}$, *and, besides the dependence on these quantities, C_2 depends also on τ. In particular, these constants do not depend on $\varepsilon \in (0, 1)$.*

Let us mention that the proof of the estimate in time (1.10) is a delicate part of our work since it requires a new kind of intrinsic scaling. The proof of the gradient estimate (1.9) uses a Bernstein-type technique, and it was motivated by [5].

The purpose of Lemma 1.3 is twofold: It allows us to obtain uniform in ε estimates in the Hölder spaces $C_{x,\text{loc}}^{1,\alpha}$ and $C_{t,\text{loc}}^{0,\alpha}$ for the family $(u^\varepsilon)_{0<\varepsilon<1}$ and, by compactness, we find a candidate for a solution, which is the limit of u^ε when $\varepsilon \to 0$; it also gives the optimal regularity for the limit solution.

Our existence result reads as follows.

Theorem 1.4 *Let $u_0 \in C^1(\overline{\Omega})$, $u_0 \geq 0$, and suppose that $(u^\varepsilon)_{0<\varepsilon<1}$ is a family of solution of (1.7) for each $\varepsilon \in (0, 1)$. Then there exists a subsequence $\varepsilon_j \to 0^+$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Q_T for some $u \in C(\Omega \times [0, T])$. Furthermore, the function u is a weak solution of (1.1) in the sense (1.5).*

Remark 1.5 (Nonuniqueness) We emphasize that our regularity result (Theorem 1.6 below) is true for any limit solution from Theorem 1.4. Even for the power-type nonlinearity $-u^{\gamma-1}$, the uniqueness/nonuniqueness seems to be a challenging question. For instance, Winkler proved in [15] that there exist a number $0 < \gamma_c < 1$ such that, if $\gamma < \gamma_c$ and the dimension satisfies $N \leq 6$, then problem (1.2) has at least two solutions. A crucial role in the proof of this result is played by an explicit solution in the ball. Thus, in the case of the log-type nonlinearity, uniqueness/nonuniqueness is an interesting open question.

Finally, we present our regularity result.

Theorem 1.6 *Let* $u_0 \in C^1(\overline{\Omega})$, $u_0 \geq 0$, *and suppose that u is the limit solution from Theorem* 1.4. *Then we have the following:*

(i) *for any* $t \in (0, T)$, *we have that* $u(\cdot, t) \in C^{1,\log-\mathrm{lip}}_{\mathrm{loc}}(\Omega)$;
(ii) *for any* $x \in \Omega$, *we have that* $u(x, \cdot) \in C^{0,\alpha}_{\mathrm{loc}}(0, T)$ *for any* $\alpha \in (0, 1)$; *furthermore, if* $\Omega' \subset\subset \Omega$ *and* $\tau \in (0, T)$, *then*

$$|u_t(x, t)| \leq C|\log u(x, t)|, \quad \text{for all} \quad (x, t) \in \Omega' \times (\tau, T],$$

where $C = C(\tau, \mathrm{dist}(\Omega', \partial\Omega), \|u\|_{L^\infty(Q_T)})$.

1.2 Notation

We fix $\Omega \subset \mathbb{R}^N$ and $T > 0$. The parabolic cylinder is $Q_T = \Omega \times (0, T]$ and the local version is denoted by $Q_{t_1,t_2} = \Omega \times (t_1, t_2]$, where t_1, t_2 are fixed satisfying $(t_1, t_2] \subset (0, T]$.

For the parabolic functional spaces and norms, we use notations similar to those in [9]. For completeness we present here.

For $p \geq 1$, $L_{p,q}(Q_T)$ indicates the parabolic Lebesgue space of those functions $u : Q_T \to \mathbb{R}$ with finite norm given by

$$\|u\|_{L_{p,q}(Q_T)} = \left(\int_0^T \int_\Omega |u(x, t)|^p \mathrm{d}x\mathrm{d}t \right)^{1/q}.$$

The simplifications $L_p(Q_T) = L_{p,p}(Q_T)$ and $\|\cdot\|_p = \|\cdot\|_{L_{p,p}(Q_T)}$ are also used.

When we freeze the time variable, the Lebesgue space will be denoted by $L^p(\Omega)$, $p \geq 1$, and the norm in these spaces will be denoted by

$$\|u(\cdot, t)\|_{L^p(\Omega)} = \left(\int_\Omega |u(x, t)|^p \mathrm{d}x \right)^{1/p}.$$

We also denote by $W^{1,0}_q(Q_T)$ and $W^{1,1}_q(Q_T)$ the Banach (actually Hilbert) spaces generated by the norms

$$\|u\|_{W^{1,0}_q} = \|u\|_q + \|\nabla u\|_q,$$
$$\|u\|_{W^{1,1}_q} = \|u\|_q + \|\nabla u\|_q + \|u_t\|_q.$$

2 L^∞-bound uniform in ε

The L^∞-estimate uniform in ε is essential to obtain compactness. Since the logarithmic nonlinearity changes sign, this is not just an application of the maximum principle (as in the case of power-type nonlinearity). In our case, we use the slow growth of the function $s \mapsto \log s$ and the general results from [9].

We need first to prove an L^2-estimate.

Lemma 2.1 *Let* $u_0 \in C^1(\overline{\Omega})$ *and* $g \in W^{1,1}_2(Q_T)$ *with* $u_0, g \geq 0$, *and suppose* $u^\varepsilon \geq 0$ *is weak solution of* (1.7) *with* $0 < \varepsilon \leq \varepsilon_0$, *for some fixed* $\varepsilon_0 > 0$. *Then*

$$\sup_{0\leq t\leq T} \|u^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq \sup_{0\leq t\leq T} \|u^\varepsilon(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u^\varepsilon\|_{L^2(Q_T)} \leq C,$$

for a constant $C = C(n, \Omega, T, \|g\|_{W^{1,1}_2(Q_T)}, \|u_0\|_{L^2(\Omega)}, \varepsilon_0)$ *(it does not depend on* $0 < \varepsilon < \varepsilon_0$*).*

Proof For simplicity we denote $u = u^\varepsilon$. Then we fix $\tau \in (0, T)$ and use $\eta = u - g$ as a test function to obtain, after integration by parts, the following identity:

$$\begin{aligned}\frac{3}{2}\int_\Omega u^2(\tau)\mathrm{d}x + \int_0^\tau\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}t &= \int_0^\tau\int_\Omega \beta_\varepsilon(u)(u-g)\mathrm{d}x\mathrm{d}t - \int_0^\tau\int_\Omega u g_t\mathrm{d}x\mathrm{d}t \\ &+ \int_\Omega u(\tau)g(\tau)\mathrm{d}x + \frac{1}{2}\int_\Omega u^2(0) - \int_\Omega u_0 g(0)\mathrm{d}x + \int_0^\tau\int_\Omega \langle\nabla u, \nabla g\rangle\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.1}$$

Let $a \in (0, 1)$ be fixed and $C_1 > 0$ be a constant depending only on a and ε_0 such that, for every $s \geq 0$, $|\beta_\varepsilon(s)| \leq C_1 s^a$. Thus, applying Hölder and Poincaré inequalities,

$$\int_0^\tau\int_\Omega \beta_\varepsilon(u)(u-g)\mathrm{d}x\mathrm{d}t \leq C_2\int_0^\tau \left(\|u\|^a_{L^2(\Omega)}\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)}\right)\right)\mathrm{d}t,$$

where the constant $C_2 > 0$ depends on a and Ω. By Young's inequality,

$$\begin{aligned}\int_0^\tau\int_\Omega \beta_\varepsilon(u)(u-g)\mathrm{d}x\mathrm{d}t &\leq C_2\delta_1\int_0^\tau \|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}t + C_3\int_0^\tau \|u\|^{2a}_{L^2(\Omega)}\mathrm{d}t \\ &+ \frac{C_2}{2}\int_0^\tau \|\nabla g\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned} \tag{2.2}$$

The constant C_3 depends on δ_1 and C_2.

We will use Young's inequality once again. First

$$\int_0^\tau\int_\Omega \langle\nabla u, \nabla g\rangle\mathrm{d}x\mathrm{d}t \leq \delta_2\int_0^\tau\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}t + \frac{1}{4\delta_2}\int_0^\tau\int_\Omega |\nabla g|^2\mathrm{d}x\mathrm{d}t, \tag{2.3}$$

and then:

$$\int_\Omega u(\tau)g(\tau)\mathrm{d}x \leq \delta_3\int_\Omega u^2(\tau)\mathrm{d}x + \frac{1}{4\delta_3}\int_\Omega g^2(\tau)\mathrm{d}x, \tag{2.4}$$

$$\int_0^\tau\int_\Omega u g_t\mathrm{d}x\mathrm{d}t \leq \int_0^\tau \left(\delta_4\int_\Omega u^2\mathrm{d}x + \frac{1}{4\delta_4}\int_\Omega g_t^2\mathrm{d}x\right)\mathrm{d}t. \tag{2.5}$$

Now we elect $\delta_1 = 1/(4C_2)$, $\delta_2 = 1/4$ and $\delta_3 = 1$ and then substitute (2.2)–(2.5) in (2.1) to obtain

$$\begin{aligned}\int_\Omega u^2(\tau)\mathrm{d}x + \int_0^\tau\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}t &\leq C_4 T\sup_{0\leq t\leq T}\|u(\cdot,t)\|^{2a}_{L^2(\Omega)} + \delta_4 T\sup_{0\leq t\leq T}\|u(\cdot,t)\|^2_{L^2(\Omega)} \\ &+ C_5\int_\Omega u^2(0)\mathrm{d}x + C_6\|g\|^2_{W_2^{1,1}(Q_T)}.\end{aligned}$$

All the constants appearing above depend only on the quantities specified in the lemma. We take the sup over the interval $(0, T]$ and $\delta_4 = 1/(2T)$ to obtain

$$\begin{aligned}\frac{1}{2}\sup_{0\leq t\leq T}\|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}t &\leq C_4 T\sup_{0\leq t\leq T}\|u(\cdot,t)\|^{2a}_{L^2(\Omega)} \\ &+ C_6\left(\|u_0\|^2_{L^2(\Omega)} + \|g\|^2_{W_2^{1,1}(Q_T)}\right).\end{aligned}$$

Since $2a < 2$ we obtain

$$\sup_{0\leq t\leq T}\|u(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q_T)} \leq C,$$

for a constant C as indicated in the lemma. □

Now we prove the L^∞-estimate.

Lemma 2.2 *Let us assume that the same hypothesis of Lemma 2.1 hold and adiationally that $g \in L^\infty(Q_T)$. There exists a constant $M = M(n, \Omega, T, \|g\|_{W_2^{1,1}(Q_T)}, \|u_0\|_{L^2(\Omega)}, \|g\|_{L^\infty(Q_T)}, \|u_0\|_{L^\infty(\Omega)}) > 0$ (independent of $\varepsilon > 0$) such that, for every weak solution $u^\varepsilon \geq 0$ of (1.7),*

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M.$$

Proof As before, we denote $u = u^\varepsilon$. Let us fix $k_0 \in \mathbb{N}$ such that

$$k_0 \geq \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(Q_T)}\}$$

and, for any $k \in \mathbb{N}$ with $k \geq k_0$, let us define

$$u^k(x,t) = \max\{u(x,t) - k, 0\}, \quad (x,t) \in Q_T,$$

and also

$$A_k(t) = \{x \in \Omega \mid u(x,t) > k\}, \quad 0 \leq t \leq T.$$

By the results in Chapter II, Section 4, of [9], we can use u^k as a test function in the definition of a solution for (1.7). This will give us the following identity:

$$\frac{1}{2}\int_{A_k(\tau)} (u^k)^2(\tau)\mathrm{d}x + \int_0^\tau \int_{A_k(t)} |\nabla u^k|^2 \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{A_k(t)} \beta_\varepsilon(u) u^k \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{A_k(0)} (u^k)^2(0)\mathrm{d}x, \tag{2.6}$$

which holds for every interval $(0,\tau) \subset (0,T]$.

For $0 < a < 1$ there exists $C > 0$ depending only on a, such that $\beta_\varepsilon(u) \leq Cu^a$. Applying this estimate and using Hölder inequality we have

$$\int_0^\tau \int_{A_k(t)} \beta_\varepsilon(u) u^k \mathrm{d}x\mathrm{d}t \leq C_1 \int_0^\tau \|u\|^a_{L^2(A_k(t))} |A_k(t)|^{(1-a)/2} \|u^k\|_{L^2(A_k(t))}\mathrm{d}t. \tag{2.7}$$

On the other hand, from the Sobolev inequality,

$$\|u^k\|_{L^2(A_k(t))} \leq C_2 \|\nabla u^k\|_{L^2(A_k(t))} |A_k(t)|^{\frac{1}{2}-\frac{1}{2^*}}, \tag{2.8}$$

for a constant $C_2 = C_2(\Omega) > 0$. Using (2.8) in (2.7) and Young's inequality we obtain

$$\int_0^\tau \int_{A_k(t)} \beta_\varepsilon(u) u^k \mathrm{d}x\mathrm{d}t \leq C_2\delta \int_0^\tau \|\nabla u^k\|^2_{L^2(A_k(t))}\mathrm{d}t + \frac{C_2}{4\delta}\int_0^\tau \|u\|^{2a}_{L^2(A_k(t))} |A_k(t)|^{2-a-\frac{2}{2^*}}\mathrm{d}t. \tag{2.9}$$

Using (2.9) in (2.6) with $\delta = 1/(2C_2)$ we get the following:

$$\frac{1}{2}\int_{A_k(\tau)} (u^k)^2(\tau)\mathrm{d}x + \frac{1}{2}\int_0^\tau \int_{A_k(t)} |\nabla u^k|^2 \mathrm{d}x\mathrm{d}t \leq C_3 \int_0^\tau \|u\|^{2a}_{L^2(A_k(t))} |A_k(t)|^{2-a-\frac{2}{2^*}}\mathrm{d}t. \tag{2.10}$$

Now we take the sup for $\tau \in [0,T]$ in (2.10) and apply Lemma 2.1 to obtain

$$\sup_{0\leq t\leq T} \|u^k(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u^k\|_2 \leq C_4 \left(\int_0^T |A_k(t)|^{2-a-\frac{2}{2^*}}\right)^{1/2}. \tag{2.11}$$

We observe that (2.11) can be written in the following way:

$$\sup_{0\le t\le T} \|u^k(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u^k\|_2 \le C_4 \left(\int_0^T |A_k(t)|^{r/q}\right)^{(1+\kappa)/r},$$

where $q = 2$ and

$$\begin{aligned} r &= 2, a = \frac{2}{n}, \kappa = 0 \quad \text{if } n > 2,\\ r &= 3, a = \frac{1}{2}, \kappa = \frac{1}{2} \quad \text{if } n = 2,\\ r &= 5, a = \frac{1}{2}, \kappa = \frac{3}{2} \quad \text{if } n = 1. \end{aligned}$$

With the last estimate and the values of q, r and κ we are in position to apply Theorem 6.1 on page 102 of [9], which implies the result. □

To finish this section let us prove that problem (1.7) has a solution for each $\varepsilon \in (0, 1)$.

Lemma 2.3 *Let $T > 0$ be fixed and suppose $u_0 \in C^1(\overline{\Omega})$, $g \in W_2^{1,1}(Q_T) \cap L^\infty(Q_T)$ with $u_0, g \ge 0$. Then, for each $\varepsilon \in (0, 1)$, problem (1.7) has a solution $u^\varepsilon \ge 0$ such that $u^\varepsilon \in C^3(Q_T)$.*

Proof We are going to use the sub- and supersolution method. Since $\underline{u} = 0$ is obviously a subsolution, we only need to find a supersolution. In order to do that, let Y be a solution of

$$\begin{cases} Y_t - \Delta Y = 1 & \text{in } Q_T,\\ Y = g & \text{on } \partial\Omega \times (0, T],\\ Y = u_0 & \text{in } \Omega \times \{0\}. \end{cases}$$

For $k > 0$ to be fixed we define $\overline{u} = kY$. Then,

$$\overline{u}_t - \Delta\overline{u} - \beta_\varepsilon(\overline{u}) \ge k - \log(\overline{u} + 1) \ge k - \log(k\|Y\|_{L^\infty} + 1).$$

By the growth of log we can choose $k > 0$ large enough such that $\overline{u}$ is a supersolution.

Now we can proceed as in the elliptic case (Lemma 2.3 in [11]) using the comparison principle for parabolic equations (Corollary 2.2.6 in [8]) to obtain a solution u satisfying $0 \le u \le \overline{u}$. The regularity of u follows from the general theory from [9]. □

Remark 2.4 Proposition 1.2 follows from Lemma 2.3 and Lemma 2.2.

3 Pointwise uniform estimates for ∇u^ε and u_t^ε

In this section we prove pointwise estimates for ∇u^ε and u_t^ε that are uniform in $\varepsilon > 0$. We start with the gradient of a solution of (1.7) following the lines of [5].

Lemma 3.1 *Let $u_0 \in C^1(\overline{\Omega})$ with $u_0 \ge 0$. Suppose that $u^\varepsilon \in C^3(Q_T) \cap C^1(\overline{Q}_T)$ satisfies (1.7) and let $\Omega' \subset\subset \Omega$. Then there are constants $C, \Lambda > 0$, depending only on* $\mathrm{dist}(\Omega', \Omega)$, N, $\|u^\varepsilon\|_{L^\infty(Q_T)}$, u_0, *such that*

$$|\nabla u^\varepsilon(x,t)|^2 \le Cu^\varepsilon(x,t)\left(\Lambda + \log\frac{1}{u^\varepsilon(x,t)}\right), \quad \text{for every } x \in \Omega',\ t \in [0, T]. \tag{3.1}$$

Proof The idea is to use a Bernstein-type technique. First, as in [5], we fix a function

$$\psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega \text{ and such that } \frac{|\nabla\psi|^2}{\psi} \text{ is bounded in } \Omega. \tag{3.2}$$

For instance, ψ could be a power of the first eigenfunction of the Dirichlet Laplacian. Notice that there is a constant $\delta' > 0$ such that $\psi \geq \delta'$ in Ω'.

To simplify notation, we denote $u = u^\varepsilon$ and define

$$Z(u) = -u \log u + \Lambda u \quad \text{with} \quad Z(0) = 0. \tag{3.3}$$

We fix $\Lambda \geq \max\{1, 2M\}$ where M is the constant from Lemma 2.2. Then $Z'(u) \geq 0$ and $Z''(u) \leq 0$, that is, Z is nondecreasing and concave. We also define

$$w = \frac{|\nabla u|^2}{Z(u)}, \quad v = w\psi. \tag{3.4}$$

The function v is continuous in $\overline{\Omega}$ and $v = 0$ in $\partial\Omega$. In fact, the only problem is when $u = 0$, but since $u \in C^3(\overline{\Omega})$ we have $|\nabla u|^2 \leq u \leq -u \log u + \Lambda u$ near the points where u vanishes and then w is bounded.

The proof is by contradiction. Assuming that estimate (3.1) fails, we have

$$\sup_{Q_T} v > C_1, \tag{3.5}$$

where $C_1 > 0$ will be fixed later (independent of ε) in order to obtain a contradiction.

The construction of v implies that, if $(x_0, t_0) \in \overline{Q}_T$ is a maximum point, then necessarily $x_0 \in \Omega$ and $t_0 > 0$. Furthermore,

$$\nabla v(x_0, t_0) = 0 \tag{3.6}$$

and

$$\Delta v(x_0, t_0) - v_t(x_0, t_0) \leq 0. \tag{3.7}$$

We manage to show that, if we choose C_1 very large, we get a contradiction with (3.7). Since similar computations have been done in details in [5], we just point out here the main differences. First, computing $\Delta v - v_t$ and evaluating in (x_0, t_0) we have

$$\begin{aligned}\Delta v - v_t \geq \frac{1}{Z(u)}\Bigg(\psi w^2\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right) + w(\psi Z'(u)\beta_\varepsilon(u) \\ - 2\psi Z(u)\beta_\varepsilon'(u) - KZ(u)) - K\psi^{1/2}w^{3/2}Z(u)^{1/2}Z'(u)\Bigg),\end{aligned} \tag{3.8}$$

where K is a positive constant depending on ψ. The following estimates hold uniform in $0 < \varepsilon < 1$:

$$Z(u)^{1/2}Z'(u) \leq C\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right), \tag{3.9}$$

$$Z(u)|\beta_\varepsilon'(u)| \leq C\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right), \tag{3.10}$$

$$-Z'(u)\beta_\varepsilon(u) \leq C\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right), \tag{3.11}$$

$$Z(u) \leq C\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right). \tag{3.12}$$

Here the constant $C > 0$ depends on M.

Using (3.9)–(3.12) in (3.8) we obtain

$$\Delta v - v_t \geq \frac{\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)}{Z(u)\psi}\Big(v^2 - C(v + v^{3/2})\Big).$$

Thus, if $v(x_0, t_0) > C_1$ for some large C_1 independent of ε, we obtain a contradiction to (3.7).

Now, in order to prove (3.9)–(3.12) we need to choose an adequate Λ and use the growth of the logarithmic nonlinearity.

We first consider (3.9). Notice that

$$\begin{aligned} Z(u)^{1/2}Z'(u) &\leq (-u\log u + \Lambda u + 1)(-\log u + \Lambda - 1) \\ &\leq M\log^2 u - (C_2 u + C_3)\log u + C_4. \end{aligned} \tag{3.13}$$

Thus, if $\log u \leq 0$, since $C_2 u + C_3 \leq C_5\Lambda$ for some constant $C_5 > 0$, we have

$$\begin{aligned} Z(u)^{1/2}Z'(u) &\leq M\log^2 u - C_5\Lambda\log u + C_4 \\ &\leq C\left(\frac{1}{2}\log^2 u - \Lambda\log u + \frac{\Lambda^2+1}{2}\right) \\ &= C\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right). \end{aligned}$$

On the other hand, since

$$\Lambda \log M < \Lambda M$$

and $\Lambda \geq \max\{2M, 1\}$, we have

$$-\Lambda\log u + \frac{\Lambda^2+1}{2} \geq -\Lambda\log M + \frac{\Lambda^2+1}{2} > 0.$$

Assuming that $\log u > 0$ we see that

$$\begin{aligned} M\log^2 u - (C_2 u + C_3)\log u + C_4 &\leq M\log^2 u + C_4 \\ &\leq M\log^2 u - \Lambda\log M + C_5\frac{\Lambda^2+1}{2} \\ &\leq M\log^2 u - \Lambda\log u + C_5\frac{\Lambda^2+1}{2}. \\ &\leq C\left(\frac{1}{2}\log^2 u - \Lambda\log u + \frac{\Lambda^2+1}{2}\right). \end{aligned}$$

Now we prove (3.10). Once we have

$$\beta'_\varepsilon(u) = \frac{(u+\varepsilon)^2 - \varepsilon}{(u^2 + \varepsilon u + \varepsilon)(u+\varepsilon)},$$

if we assume that $u \leq \sqrt{\varepsilon} - \varepsilon$ we obtain

$$|\beta'_\varepsilon(u)| = \frac{\varepsilon - (u+\varepsilon)^2}{(u^2 + \varepsilon u + \varepsilon)(u+\varepsilon)} \leq \frac{\varepsilon}{(u^2 + \varepsilon u + \varepsilon)(u+\varepsilon)} \leq \frac{1}{u+\varepsilon} \leq \frac{1}{u}.$$

On the other hand, if $u > \sqrt{\varepsilon} - \varepsilon$,

$$|\beta'_\varepsilon(u)| = \frac{(u+\varepsilon)^2 - \varepsilon}{(u^2 + \varepsilon u + \varepsilon)(u+\varepsilon)} \leq \frac{u+\varepsilon}{u^2 + \varepsilon u + \varepsilon} \leq \frac{1}{u}.$$

Therefore,

$$Z(u)|\beta_\varepsilon'(u)| \le -\log u + \Lambda.$$

Now, if $\log u \le 0$, then $-\log u < -\Lambda \log u$ and we get

$$\begin{aligned} Z(u)|\beta_\varepsilon'(u)| &\le -\Lambda \log u + C\left(\frac{1}{2}\log^2 u + \frac{\Lambda^2+1}{2}\right) \\ &\le C\left(\frac{1}{2}\log^2 u - \Lambda \log u + \frac{\Lambda^2+1}{2}\right) \end{aligned}$$

for some constant $C > 0$. In the set where $\log u > 0$ we have

$$\begin{aligned} Z(u)|\beta_\varepsilon'(u)| &\le C\left(-\Lambda \log M + \frac{\Lambda^2+1}{2}\right) \\ &\le C\left(\frac{1}{2}\log^2 u - \Lambda \log u + \frac{\Lambda^2+1}{2}\right). \end{aligned}$$

Now we consider inequality (3.11). Once $\beta_\varepsilon(u) \le -\log u$, we have that

$$-Z'(u)\beta_\varepsilon(u) \le \log^2(u) - (\Lambda - 1)\log u.$$

But $-(\Lambda-1)\log u \le -\Lambda \log u$ if $\log u \le 0$. Therefore,

$$\begin{aligned} -Z'(u)\beta_\varepsilon(u) &\le \log^2(u) - \Lambda \log u \\ &\le C\left(\frac{1}{2}\log^2 u - \Lambda \log u + \frac{\Lambda^2+1}{2}\right). \end{aligned}$$

In the case $\log u > 0$ we obtain

$$\begin{aligned} -Z'(u)\beta_\varepsilon(u) &\le C\left(\frac{1}{2}\log^2 u - \Lambda \log M + \frac{\Lambda^2+1}{2}\right) \\ &\le C\left(\frac{1}{2}\log^2 u - \Lambda \log u + \frac{\Lambda^2+1}{2}\right). \end{aligned}$$

Finally, (3.12) follows from the definition of $Z(u)$ after some computations and dividing again in the cases $\log u \le 0$ and $\log u > 0$. This finishes the proof □

Now we prove the pointwise estimate in time. In the case of a power-type nonlinearity, next lemma can be seen as a maximum principle (see [4]). The lack of homogeneity complicates considerably the technique, and we need new ideas followed by the general parabolic theory.

Lemma 3.2 *Let $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, be the initial data and u^ε be the solution of* (1.7). *Then, for each $\Omega' \subset\subset \Omega$ and $\tau \in (0, T]$, there exists $C > 0$ such that*

$$|u_t^\varepsilon(x,t)| \le C \log \frac{1}{u^\varepsilon(x,t)}\ , \quad (x,t) \in \Omega' \times (\tau, T]$$

where $C = C(\tau, \mathrm{dist}(\Omega', \partial\Omega), \|u^\varepsilon\|_{L^\infty(Q_T)})$.

Proof Let us call again $u = u^\varepsilon$. Since $\|u^\varepsilon\|_{L^\infty(Q_T)}$ is bounded for a constant independent of ε, there is no loss of generality if we suppose $\|u^\varepsilon\|_{L^\infty(Q_T)} \le M < 1$.

Given $(x_0, t_0) \in \Omega' \times (\tau, T]$, we consider the rescaled and translated function

$$\tilde{u}(x,t) = e^{-\frac{\Lambda}{r}} u(rx + x_0, r^2 t + t_0),$$

where $\Lambda \geq \max\{1, 2M\}$ is the constant in Lemma 3.1. Now, fix $r_0 = \min(1, \mathrm{dist}(\Omega', \partial\Omega), \sqrt{\tau})$ and let $r > 0$ be given by

$$r^{-2}\mathrm{e}^{\frac{\Lambda}{r}} = Lr_0^{-2}\mathrm{e}^{\frac{2\Lambda}{r_0}} \log u(x_0, t_0),$$

where $L = 4(\log M)^{-1} < 0$. We have $0 < r < r_0/2$ and, by the choice of r_0, the function $\tilde{u}$ is defined in $\overline{B}_1 \times [-1, 0]$, it is of class C^1, and it satisfies

$$\tilde{u}_t - \Delta\tilde{u} = r^2\mathrm{e}^{-\frac{\Lambda}{r}}\beta_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}}\tilde{u}) \text{ in } B_1 \times (-1, 0]. \tag{3.14}$$

Furthermore, we can find a constant $C > 0$ satisfying

$$\begin{aligned} |u_t(x_0, t_0)| &= r^{-2}\mathrm{e}^{\frac{\Lambda}{r}}|\tilde{u}_t(0, 0)| \\ &= 4(\log M)^{-1}r_0^{-2}\mathrm{e}^{\frac{2\Lambda}{r_0}}|\tilde{u}_t(0, 0)| \log u(x_0, t_0) \\ &\leq C \log \frac{1}{u(x_0, t_0)} \end{aligned}$$

provided there is $C_1 > 0$ such that

$$|\tilde{u}_t(0, 0)| \leq C_1. \tag{3.15}$$

Thus, all we need to do is to prove (3.15). Recall that, from Lemma 3.1, we have

$$|\nabla u|^2 \leq C_2(-u \log u + \Lambda u) \text{ in } B_{r_0}(x_0) \times (t_0 - r^2, t_0).$$

Moreover,

$$|\nabla\tilde{u}|^2 = r^2e^{-\frac{2\Lambda}{r}}|\nabla u|^2.$$

It follows that

$$\begin{aligned} |\nabla\tilde{u}|^2 &\leq C_2r^2e^{-\frac{2\Lambda}{r}}(-u \log u + \Lambda u) \\ &\leq C_2r^2e^{-\frac{\Lambda}{r}}(-\tilde{u} \log \tilde{u} + \Lambda\tilde{u}(1 - 1/r)) \\ &\leq C\tilde{u} \log \frac{1}{\tilde{u}} \end{aligned} \tag{3.16}$$

whenever $0 < r < 1$. The inequality above holds in $B_1 \times (-1, 0]$ and $\log(\tilde{u})^{-1} \leq (\tilde{u})^{-1}$ since $\tilde{u} < 1$. So,

$$|\nabla\tilde{u}^{3/2}|^2 \leq \frac{9}{4}\frac{|\nabla\tilde{u}|^2}{\log\frac{1}{\tilde{u}}} \leq C, \text{ in } B_1 \times (-1, 0). \tag{3.17}$$

Now, let $\psi \in C_0^\infty(B_1 \times (0, 1])$. Multiplying Eq. (3.14) by $\tilde{u}_t\psi$ and integrating over B_1 we find

$$\int_{B_1} \tilde{u}_t^2\psi\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_1} |\nabla\tilde{u}|^2\psi\mathrm{d}x - \int_{B_1} \tilde{u}_t\nabla\tilde{u}\nabla\psi\mathrm{d}x + r^2\mathrm{e}^{-\frac{2\Lambda}{r}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_1} B_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}}\tilde{u})\psi\mathrm{d}x,$$

where

$$B_\varepsilon(u) = \int_0^u \beta_\varepsilon(s)\mathrm{d}s.$$

Hence, by Young's inequality

$$\frac{1}{2}\int_{B_1} \tilde{u}_t^2\psi\mathrm{d}x \leq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_1} |\nabla\tilde{u}|^2\psi\mathrm{d}x + C\int_{B_1} |\nabla\tilde{u}|^2\mathrm{d}x + r^2\mathrm{e}^{-\frac{2\Lambda}{r}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_1} B_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}}\tilde{u})\psi\mathrm{d}x.$$

Integrating from $t_1 \in (-1, 0)$ to 0 we obtain

$$\begin{aligned}\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t \leq & \int_{B_1} |\nabla \tilde{u}(\cdot, t_1)|^2 \mathrm{d}x + 2C\int_{t_1}^{0}\int_{B_1} |\nabla \tilde{u}|^2 \mathrm{d}x\mathrm{d}t \\ & + 2r^2 \mathrm{e}^{-\frac{2\Lambda}{r}} \int_{B_1} B_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}} \tilde{u}(\cdot, 0))\psi \,\mathrm{d}x \\ & - 2r^2 \mathrm{e}^{-\frac{2\Lambda}{r}} \int_{B_1} B_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}} \tilde{u}(\cdot, t_1))\psi \,\mathrm{d}x. \end{aligned} \tag{3.18}$$

Assuming $0 < \varepsilon < 1 - M$ we see that

$$\beta_\varepsilon(u) = \log \frac{u^2 + \varepsilon u + \varepsilon}{u + \varepsilon} \leq 0 < u,$$

which implies

$$B_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}} \tilde{u}) \leq \frac{1}{2}\mathrm{e}^{\frac{2\Lambda}{r}} \tilde{u}^2.$$

On the other hand, $-\beta_\varepsilon(u) \leq u^{-1/2}$ and

$$-B_\varepsilon(\mathrm{e}^{\frac{\Lambda}{r}} \tilde{u}) \leq 2\mathrm{e}^{\frac{\Lambda}{2r}} \tilde{u}^{1/2}.$$

Using the previous inequalities in (3.18) we get

$$\begin{aligned}\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t \leq & -C\int_{B_1} \tilde{u}(\cdot, t_1)\log \tilde{u}(\cdot, t_1)\mathrm{d}x - C\int_{t_1}^{0}\int_{B_1} \tilde{u}\log \tilde{u}\,\mathrm{d}x\mathrm{d}t \\ & + r^2\int_{B_1} \tilde{u}^2(\cdot, 0)\mathrm{d}x + 4r^2\mathrm{e}^{-\frac{3\Lambda}{r}}\int_{B_1} \tilde{u}^{1/2}(\cdot, t_1)\mathrm{d}x. \end{aligned} \tag{3.19}$$

Since $0 < r < r_0/2$ and $\tilde{u}\log \tilde{u} \leq \tilde{u}^{1/2}$, we see that there is a constant $C > 0$ such that

$$\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t \leq C\left[1 + \int_{t_1}^{0}\int_{B_1} \tilde{u}^{1/2}\mathrm{d}x\mathrm{d}t + \int_{B_1} \tilde{u}^{1/2}(\cdot, t_1)\mathrm{d}x\right] \tag{3.20}$$

Now, (3.17) implies that, for any $y \in B_1$ and $t \in (-1/2, 0)$,

$$\tilde{u}(y, t) \leq C(1 + \tilde{u}(\overline{x}, t))$$

where $\overline{x} \in B_{r_1}$ with $r_1 \leq (\frac{1}{2})^{1/3N} < 1$. It follows that (see [5])

$$\begin{aligned}\tilde{u}^{1/2}(y, t) &\leq \left(C\left(1 + |t|^{1/3}\left(\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t\right)^{1/2}\right)\right)^{1/2} \\ &\leq C\left(1 + |t|^{1/6}\left(\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t\right)^{1/4}\right). \end{aligned}$$

For $|t| \leq |t_1|$ the Young's inequality implies

$$\tilde{u}^{1/2}(y, t) \leq C\left(1 + |t_1|^{1/6}\left(\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t\right)\right).$$

It follows from (3.20) that

$$\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t \leq C\left(1 + |t_1|^{1/6}\left(\int_{t_1}^{0}\int_{B_1} \tilde{u}_t^2 \psi \,\mathrm{d}x\mathrm{d}t\right)\right).$$

Then, choosing $\delta > 0$ small we see that, for $t_1 \in (-\delta, 0]$, there is a constant C (which does not depend on t_1) such that

$$\int_{t_1}^{0} \int_{B_1} \tilde{u}_t^2 \psi \, \mathrm{d}x \mathrm{d}t \leq C.$$

From (3.17) there exist $\rho > 0$ small and $C_1 > 0$ with $C_1 = C_1(\rho) \to 0$ as $\rho \to 0$ such that

$$\tilde{u}(x,t)^{3/2} \geq \tilde{u}(x_0,t)^{3/2} - C_1, \text{ in } B_\rho(0) \times (-\rho, 0].$$

The continuity of $\tilde{u}$ implies the existence of $C_2 > 0$, $C_2 = C_2(\rho) \to 0$ as $\rho \to 0$, such that

$$\tilde{u}(x_0,t) \geq \tilde{u}(0,0) - C_2 \text{ in } B_\rho(0) \times (-\rho, 0].$$

Since $\tilde{u}(0,0) > 0$ does not depend on ρ,

$$\tilde{u}(x,t) \geq C > 0 \text{ in } B_\rho(0) \times (-\rho, 0].$$

Thus,

$$|\tilde{u}_t - \Delta \tilde{u}| \leq C \;\; in \;\; B_\rho(0) \times (-\rho, 0].$$

Now, parabolic regularity theory implies inequality (3.15). □

Remark 3.3 Lemma 1.3 follows from Lemma 3.1 and Lemma 3.2.

4 Existence of a solution

In this section we justify the passage to the limit as $\varepsilon \to 0^+$ proving Theorem 1.4. Let us start with the existence of a candidate.

Proposition 4.1 *Let u^ε be a solution of (1.7), $0 < \varepsilon \leq \varepsilon_0$ with initial data $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. Then, if $\Omega' \subset\subset \Omega$ and $\tau \in (0, T)$ are fixed, for every $\alpha \in (0, 1)$, there exists a constant $C > 0$, depending only on $\Omega', n, \tau, T, \alpha$ and $\|u_0\|_{L^\infty(\Omega)}$ such that*

$$|u^\varepsilon(x,t) - u^\varepsilon(x,s)| \leq C|t-s|^\alpha, \tag{4.1}$$

and

$$|\nabla u^\varepsilon(x,t) - \nabla u^\varepsilon(y,t)| \leq C|x-y|^\alpha, \tag{4.2}$$

for every $x, y \in \Omega'$ and $t, s \in (\tau, T)$. In particular, there exists $u \colon \Omega \times (0, T) \to \mathbb{R}$ such that, at least for a subsequence,

$$u^\varepsilon \to u \quad as \; \varepsilon \to 0^+, \quad \text{locally uniformly in } Q_T.$$

Furthermore, u satisfies

$$|u_t(x,t)| \leq C_1 |\log u(x,t)|, \quad \text{for every } x \in \Omega', \; t \in [0, T] \tag{4.3}$$

and

$$|\nabla u(x,t)|^2 \leq C_1 u(x,t) \left(\Lambda + |\log u(x,t)|\right), \quad \text{for every } x \in \Omega', \; t \in [0, T], \tag{4.4}$$

where the constant C_1 depends only on N, Ω', τ, T and $\|u_0\|_{L^\infty(\Omega)}$ and Λ is the one from Lemma 3.1.

Proof The proof of estimates (4.1) and (4.2) follows exactly as in [5], Corollary 3.2. Just notice that Lemma 3.1 and Lemma 3.2 imply that u^ε satisfies the estimates (16) and the one from Lemma 3.1, both in [5]. Now, the existence of the limit and the estimates for u follow from the compactness results available between Hölder spaces (see, for instance, [8]).

We also need an integrability result.

Lemma 4.2 *Let u be the limit function from Proposition 4.1. Then,*

$$\chi_{\{u>0\}} \log u \in L^1_{\mathrm{loc}}(Q_T).$$

Proof For $0 < \varepsilon \leq \varepsilon_0$ we fix $0 < a_0 < 1$ satisfying $\beta_{\varepsilon_0}(a_0) = 0$. We let $\Omega' \subset\subset \Omega$ and $\eta \in C_0^\infty(\Omega)$ be such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in Ω'. Then, for some constants $\tilde{M}, \overline{M} > 0$ depending on ε_0 we have:

$$\begin{aligned}
\int_0^T \int_{\Omega' \cap \{u^\varepsilon \leq a_0\}} |\beta_\varepsilon(u^\varepsilon)| \mathrm{d}x\mathrm{d}t &\leq -\int_0^T \int_{\Omega'} \beta_\varepsilon(u^\varepsilon) \mathrm{d}x\mathrm{d}t + \tilde{M}T \\
&\leq -\int_0^T \int_{\Omega'} \eta\beta_\varepsilon(u^\varepsilon) \mathrm{d}x\mathrm{d}t + \tilde{M}T \\
&\leq -\int_0^T \int_{\Omega} \eta\beta_\varepsilon(u^\varepsilon) \mathrm{d}x\mathrm{d}t + \overline{M}T \\
&= -\int_0^T \int_{\Omega} \left(\eta u_t^\varepsilon + \langle \nabla u^\varepsilon, \nabla \eta \rangle\right) \mathrm{d}x\mathrm{d}t + \overline{M}T \\
&= -\int_0^T \int_{\Omega} \eta \left(u^\varepsilon(T) - u_0\right) \mathrm{d}x - \int_0^T \int_{\Omega} \langle \nabla u^\varepsilon, \nabla \eta \rangle \mathrm{d}x\mathrm{d}t \\
&\quad + \overline{M}T.
\end{aligned}$$

We conclude the proof using Lemma 2.1, Lemma 2.2 and Fatou's Lemma. □

Finally, we give the proof of existence.

Proof of Theorem 1.4 Let us show that the function u from Proposition 4.1 is a solution in the sense of (1.5). To this end, let $\xi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $0 \leq \xi(s) \leq 1$ and

$$\xi(s) = \begin{cases} 1, & s \geq 1, \\ 0, & s \leq 1/2. \end{cases}$$

For $m > 0$ we use the test function $\eta\xi(u^\varepsilon/m)$, $\eta \in C_c^\infty(\Omega \times [0, T))$ in the definition (1.8) (for a general test function we use an approximation procedure). We obtain:

$$\begin{aligned}
&\int_\Omega u^\varepsilon(\tau)\eta(\tau)\xi(u^\varepsilon(\tau)/m)\mathrm{d}x - \int_\Omega u_0\eta(0)\xi(u_0/m)\mathrm{d}x - \int_0^\tau \int_\Omega u^\varepsilon \left(\eta\xi(u^\varepsilon/m)\right)_t \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \langle \nabla u^\varepsilon, \nabla(\eta\xi(u^\varepsilon/m))\rangle \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_\Omega \beta_\varepsilon(u^\varepsilon)\eta\xi(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.5}$$

From the dominated convergence theorem we can pass to the limit in the first and second integral in the first line of (4.5) as $\varepsilon \to 0$ and $m \to 0$. As for the third integral we have

$$\int_0^\tau \int_\Omega u^\varepsilon \left(\eta\xi(u^\varepsilon/m)\right)_t \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_\Omega u^\varepsilon \eta_t \xi(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t + \frac{1}{m}\int_0^\tau \int_\Omega u^\varepsilon u_t^\varepsilon \eta\xi'(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t, \tag{4.6}$$

and as above

$$\lim_{m\to 0}\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega -u^\varepsilon\eta_t\xi(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t = -\int_{t_1}^{t_2}\int_\Omega u\eta_t\mathrm{d}x\mathrm{d}t.$$

On the other hand, let $\Omega' \subset\subset \Omega$ be such that $\eta = 0$ in $(\Omega\setminus\Omega')\times[0,\tau]$. Then Lemma 3.2 implies that

$$\begin{aligned}\frac{1}{m}\left|\int_0^\tau\int_\Omega u^\varepsilon u_t^\varepsilon\eta\xi'(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t\right| &\le \frac{1}{m}\sup|\eta|\sup|\xi'|\int_0^\tau\int_{\Omega'}\chi_{\{1/2\le u^\varepsilon/m\le 1\}}u^\varepsilon|u_t^\varepsilon|\mathrm{d}x\mathrm{d}t\\ &\le C\int_0^\tau\int_{\Omega'}\chi_{\{1/2\le u^\varepsilon/m\le 1\}}\frac{u^\varepsilon}{m}|\log u^\varepsilon|\mathrm{d}x\mathrm{d}t\\ &\le C\int_0^\tau\int_{\Omega'}\chi_{\{1/2\le u^\varepsilon/m\le 1\}}|\log u^\varepsilon|\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0}\frac{1}{m}\left|\int_0^\tau\int_\Omega u^\varepsilon u_t^\varepsilon\eta\xi'(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t\right| \le C\int_{t_1}^{t_2}\int_{\Omega'}\chi_{\{1/2\le u/m\le 1\}}|\log u|. \tag{4.7}$$

Lemma 4.2 and dominated convergence theorem implies that the right-hand side of (4.7) tends to zero as $m\to 0$.

Back to (4.5) we compute

$$\begin{aligned}\int_0^\tau\int_\Omega\langle\nabla u^\varepsilon,\nabla(\eta\xi(u^\varepsilon/m))\rangle\mathrm{d}x\mathrm{d}t &= \int_0^\tau\int_\Omega\xi(u^\varepsilon/m)\langle\nabla u^\varepsilon,\nabla\eta\rangle\mathrm{d}x\mathrm{d}t\\ &\quad+\frac{1}{m}\int_0^\tau\int_\Omega\eta|\nabla u^\varepsilon|^2\xi'(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Again, it is not difficult to show that

$$\lim_{m\to 0}\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega\xi(u^\varepsilon/m)\langle\nabla u^\varepsilon,\nabla\eta\rangle\mathrm{d}x\mathrm{d}t = \int_0^\tau\int_\Omega\langle\nabla u,\nabla\eta\rangle\mathrm{d}x\mathrm{d}t.$$

On the other hand, applying Lemma 3.2 we have

$$\begin{aligned}\frac{1}{m}\left|\int_0^\tau\int_\Omega\eta|\nabla u^\varepsilon|^2\xi'(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t\right| &\le C\int_0^\tau\int_{\Omega'}\chi_{\{1/2\le u^\varepsilon/m\le 1\}}\frac{u^\varepsilon}{m}(|\log u^\varepsilon|+\Lambda)\mathrm{d}x\mathrm{d}t\\ &\le C\int_0^\tau\int_{\Omega'}\chi_{\{1/2\le u^\varepsilon/m\le 1\}}(|\log u^\varepsilon|+\Lambda)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

It follows from Lemma 4.2 that

$$\lim_{m\to\infty}\lim_{\varepsilon\to 0}\frac{1}{m}\left|\int_0^\tau\int_\Omega\eta|\nabla u^\varepsilon|^2\xi'(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t\right| = 0.$$

Finally, using the pointwise convergence of the function $\beta_\varepsilon(u^\varepsilon)\eta\xi(u^\varepsilon/m)$ as $\varepsilon\to 0$ and Lemma 4.2 we see that

$$\lim_{m\to 0}\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega\beta_\varepsilon(u^\varepsilon)\eta\xi(u^\varepsilon/m)\mathrm{d}x\mathrm{d}t = \int_0^\tau\int_\Omega\chi_{\{0<u\}}\eta\log u\mathrm{d}x\mathrm{d}t. \tag{4.8}$$

This finishes the proof of existence. □

5 Optimal regularity

In this last section we prove the log −lip regularity for the approximated solution and finish the proof of Theorem 1.6.

Theorem 5.1 *Let* $u_0 \in C^1(\overline{\Omega})$, $u_0 \geq 0$, *and suppose that* u *is the limit solution of* (1.1) *from Theorem 1.4. Then there are positive constants* C *and* R_0 *such that*

$$|\nabla u(y, t_0) - \nabla u(x, t_0)| \leq C|y - x||\log |y - x||$$

for all $x, y \in \Omega' \cap B_{R_0}(x_0)$, *where* $\Omega' \subset\subset \Omega$ *and* $0 < t_0 < T$.

Proof Let us fix $x_0, y_0 \in \Omega' \cap \{u(\cdot, t_0) > 0\}$ and consider the inequality

$$|x - x_0|^2 |\log |x - x_0|| < \max \{u(x_0, t_0), u(y_0, t_0)\}. \tag{5.1}$$

Notice also that we can assume that $u < 1$. Besides, without loss of generality, we also assume $u(x_0, t_0) > u(y_0, t_0)$.

Let us define

$$\tilde{u}(x, t) = \mathrm{e}^{-\frac{\Lambda}{r}} u(rx + x_0, r^2 t + t_0), \quad (x, t) \in B_\rho \times (-\rho, 0],$$

where Λ, r and r_0 were chosen in the proof of Lemma 3.2 and $\rho > 0$ will be fixed later. Proceeding as in that lemma, there exist $C, \rho > 0$ such that

$$\begin{aligned}
&|\tilde{u}_t - \Delta \tilde{u}| \leq C \text{ in } B_\rho \times (-\rho, 0],\\
&|\tilde{u}_t| \leq C \text{ in } B_\rho \times (-\rho, 0],\\
&|\tilde{u}| \leq C \text{ on } B_\rho \times \{-\rho\} \text{ and } \partial B_\rho \times (-\rho, 0].
\end{aligned}$$

Besides, $\tilde{u}(\cdot, t_0)$ satisfies $\Delta \tilde{u}(\cdot, t_0) = h$ in B_ρ, with h having all the requirements of the regularity theory for the obstacle problem developed in details in [12] (see Theorem 2.14). Thus, we deduce that $\nabla \tilde{u}$ is $C^{0,1}$ in the variable $x \in B_{\rho/2} \times (-\rho/2, 0]$. Let $r_0 > 0$ be fixed small enough and y_0 be adjusted such that $y_0 \in \Omega' \subset\subset \Omega$ and $|y_0 - x_0| < r_0$. Then, choose ρ in such way that

$$\frac{y_0 - x_0}{r} \in B_{\rho/2}(0).$$

Then,

$$\left| \nabla \tilde{u} \left(\frac{y_0 - x_0}{r}, 0 \right) - \nabla \tilde{u}(0, 0) \right| \leq C r^{-1} |y_0 - x_0|.$$

Back to u we have

$$\begin{aligned}
|\nabla u(y_0, t_0) - \nabla u(x_0, t_0)| &\leq C r^{-2} \mathrm{e}^{\frac{\Lambda}{r}} |y_0 - x_0|\\
&= C r_0^{-2} \mathrm{e}^{\frac{2\Lambda}{r_0}} |L| |\log u(x_0, t_0)| |x - x_0|.
\end{aligned} \tag{5.2}$$

If (5.1) is true, then

$$|\log u(x_0, t_0)| \leq \left|\log \left(|x - x_0|^2 |\log |x - x_0||\right)\right| \leq C |\log |x - x_0||,$$

and from (5.2),

$$|\nabla u(x, t_0) - \nabla u(x_0, t_0)| \leq C |x - x_0| |\log |x - x_0||,$$

where C is a constant depending on r_0 and ρ.

Now we assume that (5.1) does not happen. From the gradient estimate (4.4) and using that the function $s \mapsto -s \log s$ is increasing for $0 < s < s_0$, s_0 small, we get

$$\begin{aligned}
&|\nabla u(y_0, t_0) - \nabla u(x_0, t_0)| \\
&\quad \le |\nabla u(y_0, t_0)| + |\nabla u(x_0, t_0)| \\
&\quad \le C \left((-u(y_0, t_0) \log u(y_0, t_0))^{1/2} + (-u(x_0, t_0) \log u(x_0, t_0))^{1/2}\right) \\
&\quad \le C \left(|y_0 - x_0|^2 \log \frac{1}{|y_0 - x_0|}\right)^{1/2} \left(- \log \left(|y_0 - x_0|^2 \log \frac{1}{|y_0 - x_0|}\right)\right)^{1/2} \\
&\quad \le C|y_0 - x_0| \log \frac{1}{|y_0 - x_0|} \left(2 + \frac{2 \log (- \log |y_0 - x_0|)}{\log |y_0 - x_0|}\right)^{1/2} \\
&\quad \le C|y_0 - x_0| \log \frac{1}{|y_0 - x_0|}.
\end{aligned}$$

This finishes the proof of the regularity. □

Acknowledgements OSQ thanks the support from CNPq/Brasil and from FAPESP/SP. The authors would like to thank the referee for his/her useful comments and suggestions.

References

1. Aris, R.: The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts. Clarendon Press, Oxford (1975)
2. Bertozzi, A.L., Pugh, M.C.: Finite-time blow-up of solutions of some long-wave unstable thin film equations. Indiana Univ. Math. J. **49**(4), 1323–1366 (2000)
3. Chen, X.-Y., Matano, H., Mimura, M.: Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption. J. Rein. Angew. Math. **459**, 1–36 (1995)
4. Choe, H.J., Weiss, G.S.: A semilinear parabolic equation with free boundary. Indiana Univ. Math. J. **52**(1), 19–50 (2003)
5. Dávila, J., Montenegro, M.: Existence and asymptotic behavior for a singular parabolic equation. Trans. Am. Math. Soc. **357**(5), 1801–1828 (2005)
6. de Queiroz, O.S., Shahgolian, H.: A free boundary problem with log–term singularity. Preprint, (2016)
7. Díaz, J.I.: Nonlinear Partial Differential Equations and Free Boundaries: Elliptic Equations, vol. I. Pitman (Advanced Publishing Program), Boston (1985)
8. Imbert, C., Silvestre, L.: An introduction to fully nonlinear parabolic equations. An introduction to the Kähler-Ricci flow. Lecture Notes in Mathematics, vol. 2086, pp. 7–88. Springer, Cham (2013)
9. Ladyzenskaja, O.A., Solonnikov, V.A., Uraltseva, N.N.: Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence (1967) (Russian)
10. Monneau, R., Weiss, G.S.: An unstable elliptic free boundary problem arising in solid combustion. Duke Math. J. **136**(2), 321–341 (2007)
11. Montenegro, M., de Queiroz, O.S.: Existence and regularity to an elliptic equation with logarithmic nonlinearity. J. Differ. Equ. **246**(2), 482–511 (2009)
12. Petrosyan, A., Shahgholian, H., Uraltseva, N.: Regularity of free boundaries in obstacle-type problems. Graduate Studies in Mathematics, vol. 136. American Mathematical Society, Providence (2012)
13. Phillips, D.: Existence of solutions of quenching problems. Appl. Anal. **24**(4), 253–264 (1987)
14. Weiss, G.S.: Self-similar blow-up and Hausdorff dimension estimates for a class of parabolic free boundary problems. SIAM J. Math. Anal. **30**(3), 623–644 (1999)
15. Winkler, M.: Nonuniqueness in the quenching problem. Math. Ann. **339**(3), 559–597 (2007)