Annali di Matematica (2017) 196:743–771
DOI 10.1007/s10231-016-0594-3

Double loop algebras and elliptic root systems

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Received: 12 December 2015 / Accepted: 9 July 2016 / Published online: 19 July 2016

Abstract In this note, we describe an elliptic root system and elliptic Weyl group, due to Saito (Publ RIMS Kyoto Univ 21:75–179, 1985), from view point of double loop algebra and its group. A natural action of the double loop group will be introduced on a trivial $\mathbb{C}^*$-bundle over the space of $\overline{\partial}$-connections on a C^∞-trivial principal bundle over an elliptic curve that would be constructed from 2-dimensional central extension of a double loop algebra. The invariant theory of the elliptic Weyl group will be also discussed.

Keywords Elliptic root system · 2-Toroidal Lie algebras · Invariant theory

Mathematics Subject Classification 14J17 · 14K25 · 17B65 · 22E65

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1 Introduction

Saito [14] arrived at the notions of elliptic root system and elliptic Weyl group in the course of his study on simply elliptic singularities [12]. An elliptic root system is a root system defined on a real vector space with positive semi-definite bilinear form whose radical is of dimension 2, extending finite and affine root system. The existence of imaginary roots which generate 2-dimensional radical indicates its relation to an elliptic curve.

It is natural to consider the construction of a Lie algebra with given elliptic root system as the next step, and there were several attempts to this problem by Wakimoto [19], Slodowy [18], Yamada (one of the authors of this article) [21] and Saito with Yoshii [16] etc. For recent developments, see e.g., [9]. Among these constructions, one has the 'maximal' one that is the universal central extension of a 2-toroidal Lie algebra $\mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ for some simple finite-dimensional Lie algebra $\mathfrak{g}$ (cf. [5]). Notice that the kernel of the universal central extension of such an algebra is of infinite dimension, whereas that of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ is 1-dimensional.

A relation between a simple singularity and the simple Lie algebra of the same type was clarified by the Grothendieck-Brieskorn-Slodowy theory [17]. This description in terms of Lie algebras was a key to construct the period map, in particular the primitive form [13], of the semi-universal deformation of a simple singularity [22]. Helmke and Slodowy [2] constructed the space of semi-universal deformation of an isolated hypersurface simply elliptic singularity in terms of a holomorphic affine Lie group, where the relation between holomorphic principal G-bundles, where G is the connected and simply connected simple Lie group over $\mathbb{C}$ whose Lie algebra is $\mathfrak{g}$, and the affine Lie group associated with G played an important role. Our project is to describe the semi-universal deformation of an isolated simply elliptic singularity in terms of Lie algebras that would be 2-toroidal Lie algebras, as one can see in [2]. This construction may allow us to describe the primitive form and hence the period map for an isolated simply elliptic singularity.

In this article, we explain the algebraic structure of 2-toroidal Lie algebras, i.e., in view of elliptic root systems etc. Next, we show that all of these concepts have some natural meaning in terms of the space $\mathcal{C}(\mathfrak{g})$ of $\overline{\partial}$-connections on a topologically trivial principal G-bundle over an elliptic curve. This note is organized as follows.

In Sect. 2, we explain the so-called elliptic root system, elliptic Weyl group W_{ell} and their hyperbolic extensions in view of 2-toroidal Lie algebras. In Sect. 3, we show that the elliptic Weyl group can be regarded as the quotient of the normalizer of its Cartan subalgebra of the C^{∞}-completion $\mathcal{E}(\mathfrak{g})$ of a 2-toroidal Lie algebras by its centralizer in the corresponding group $\mathcal{E}(G)$. An $\mathcal{E}(G)$-action on the space $\mathcal{C}(\mathfrak{g})$ will be studied and its relation to the invariant theory of W_{ell} will be explained in Sect. 4. In Sect. 5, an action of $SL(2, \mathbb{Z})$ will be studied.

2 Elliptic root systems and 2-toroidal Lie algebras

In this section, we will recall some facts about what is called an elliptic root system, its Weyl groups, and their hyperbolic extension, introduced by Saito [14]. We also describe them in terms of 2-toroidal Lie algebras and their central extensions.

2.1 Elliptic root systems and their Weyl groups

In this subsection, we recall the notion of an elliptic root system and an elliptic Weyl group. Let us generalize the classical notion of **root system** [14]:

Definition 2.1 Let F be a finite-dimensional vector space over $\mathbb{R}$, $(\cdot,\cdot)$ be a symmetric bilinear form of signature (l_+, l_0, l_-), i.e., l_+, l_0 and l_- signify the number of positive, zero and negative eigenvalues, respectively. We call a non-empty subset $R \subset F$, the **root system** associated with $(\cdot,\cdot)$ if it satisfies the next five axioms:

1. Let $Q(R)$ be the $\mathbb{Z}$-submodule of F generated by R. Then, $\dim(Q(R) \otimes \mathbb{R}) = \dim F$, i.e., $Q(R)$ is a full lattice in F.
2. For any $\alpha \in R$, $(\alpha, \alpha) \neq 0$.
3. For $\alpha \in R$, let $w_\alpha \in GL(F)$ be the reflection defined by
$$w_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad \lambda \in F.$$
Then, $w_\alpha(R) = R$ for any $\alpha \in R$.
4. For any $\alpha, \beta \in R$, $\dfrac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.
5. (Irreducibility) There is no non-empty subsets R_1 and R_2 such that $R = R_1 \cup R_2$ and $R_1 \perp R_2$.

As is well known, when the signature (l_+, l_0, l_-) of R is either of the form $(l, 0, 0)$ or $(l, 1, 0)$, R is a finite root system or an affine root system, respectively. In case when the signature of R is of the form $(l, 2, 0)$, R is called an **elliptic root system**. This is the root system introduced by K. Saito for his study on simply elliptic singularities.

Here and after, we will study the structure of an elliptic root system, say R. Set

$$\mathrm{Rad}\,(\cdot,\cdot) = \{\lambda \in F \mid (\lambda, \gamma) = 0 \quad \forall \gamma \in F\}.$$

It follows that the vector space $\mathrm{Rad}\,(\cdot,\cdot)$ is a 2-dimensional subspace of F defined over $\mathbb{Q}$, i.e., $\mathrm{Rad}\,(\cdot,\cdot) \cap Q(R)$ is a full sublattice of $Q(R)$. In this sense, fixing a 1-dimensional subspace E of $\mathrm{Rad}\,(\cdot,\cdot)$ defined over $\mathbb{Q}$ is called a **marking** for R, and the pair (R, E) is called a **marked elliptic root system**.

Remark 2.1 Two non-isomorphic marked elliptic root systems can be isomorphic as elliptic root systems. $G_2^{(1,3)}$ and $G_2^{(3,1)}$ are such examples. See [14] for detail.

For a marked elliptic root system (R, E), set

$$F_f = F/\mathrm{Rad}\,(\cdot,\cdot), \qquad F_a = F/E,$$
$$R_f = R/R \cap \mathrm{Rad}\,(\cdot,\cdot), \quad R_a = R/R \cap E.$$

Denote the symmetric bilinear form on F_f and F_a, induced from $(\cdot,\cdot)$, by $(\cdot,\cdot)_f$ and $(\cdot,\cdot)_a$. It is clear that R_f and R_a are, respectively, the finite and affine root systems associated with $(F_f, (\cdot,\cdot)_f)$ and $(F_a, (\cdot,\cdot)_a)$.

Remark 2.2 R_a is the real root system of the corresponding affine Lie algebra.

In the sequel, we regard F_f and F_a as vector subspaces of F and $R_f \subset R_a \subset R$, for simplicity.

Let $\{\alpha_1, \ldots, \alpha_l\}$ and $\{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ be root basis of R_f and R_a, respectively, i.e.,

$$F_f := \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i \quad F_a := \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i,$$

and each root is a $\mathbb{Z}$-linear combination of α_i's where all nonzero coefficients are either all positive or all negative. We express the fundamental imaginary root of the affine root system R_a as

$$\delta_1 = \sum_{i=0}^{l} a_i \alpha_i,$$

that is, a_i are coprime positive integers such that δ_1 is an imaginary root of the corresponding affine Lie algebra. We let

$$A = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{0 \le i, j \le l}$$

be the generalized Cartan matrix of R_a.

Now, following Wakimoto [19], we study the structure of the marked elliptic root system (R, E) in view of the generalized Cartan matrix A. An $(l+1)$-tuple of positive integers $(k_0, k_1, \ldots, k_l)$ is called **counting weight**, if the diagonal matrix $K = \mathrm{diag}(k_0, k_1, \ldots, k_l)$ satisfies the next two conditions:

1. KAK^{-1} is a generalized Cartan matrix (cf. [3]), and
2. G.C.D.$(k_0, k_1, \ldots, k_l) = 1$.

Example 2.1 1. If A is the generalized Cartan matrix of type $X_l^{(1)}$ with $X = A, D, E$, a counting weight is uniquely determined as

$$(k_0, k_1, \ldots, k_l) = (1, 1, \ldots, 1).$$

2. Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

be the generalized Cartan matrix of type $G_2^{(1)}$. There are two possible counting weights: $(1, 1, 1)$ and $(1, 3, 1)$. Namely, in general, a counting weight is not uniquely determined from a generalized Cartan matrix of affine type.

Let W_a be the affine Weyl group associated with the affine root system R_a. It is well known that R_a decomposes into a finite number of W_a-orbits (cf. Remark 2.2) and that each W_a-orbit of R_a contains a simple root.

Lemma 2.1 *If two simple roots α_i and α_j lie in the same W_a-orbit, then we have $k_i = k_j$.*

Proof We recall that α_i and α_j lie in the same W_a-orbit if and only if the vertices corresponding to these simple roots are connected by a subdiagram consisting of simple edges of the Dynkin diagram of R_a. Hence, since R_a is irreducible, it is sufficient to show this lemma in the case when the vertices corresponding to α_i and α_j are connected by one simple edge.

Set $A = (a_{i,j})$ and $B = (b_{i,j}) = KAK^{-1}$. As $b_{i,j} = a_{i,j}k_ik_j^{-1}$ and $a_{i,j} = a_{j,i} = -1$ by assumption, this implies that

$$b_{i,j}b_{j,i} = (a_{i,j}k_ik_j^{-1})(a_{j,i}k_jk_i^{-1}) = a_{i,j}a_{j,i} = 1.$$

from which it follows that $b_{i,j} = b_{j,i} = -1$, since B is a generalized Cartan matrix. Thus, we conclude that $k_i = k_j$. □

Now, for any $\alpha \in R_a$, there exists $w \in W_a$ and a simple root α_i such that $\alpha = w(\alpha_i)$. The above lemma assures that we can define the **counting weight of the root** α by

$$k(\alpha) = k_i.$$

Let δ_2 be a $\mathbb{Z}$-basis of the lattice $Q(R) \cap E$. The number $k(\alpha)$ is the smallest positive integer such that $\alpha + k(\alpha)\delta_2 \in R$. Indeed, we have

Proposition 2.2 (cf. [14,19]) $R = \{\alpha + mk(\alpha)\delta_2 | \alpha \in R_a,\ m \in \mathbb{Z}\}$.

Remark 2.3 Let us state the relation between a counting weight and the elliptic Dynkin diagram due to Saito [14]. Let $a^\vee = (a_0^\vee, a_1^\vee, \ldots, a_l^\vee) \in (\mathbb{Z}_{>0})^{l+1}$ be the vector satisfying

1. $a^\vee A = 0$, and
2. G.C.D.$(a_0^\vee, a_1^\vee, \ldots, a_l^\vee) = 1$.

Set

$$I = \left\{ j \in \{0, 1, \ldots, l\} \,\middle|\, \frac{a_j^\vee}{k_j} = \max\left\{\frac{a_0^\vee}{k_0}, \frac{a_1^\vee}{k_1}, \ldots, \frac{a_l^\vee}{k_l}\right\}\right\}.$$

For $j \in I$, we set $\alpha_j^* = \alpha_j + \delta_2$ and consider the set of vertices parametrized by

$$\{\alpha_0, \alpha_1, \ldots, \alpha_l\} \cup \{\alpha_j^* | j \in I\}.$$

The **elliptic Dynkin diagram** for the root system R is the graph with the vertices given by the above set where the vertices α and β are connected following the usual rules for Dynkin diagram depending on the values $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ and $\frac{2(\beta,\alpha)}{(\beta,\beta)}$. When these values are 2, we connect these vertices with two dashed edges.

In the rest of this article, we are only interested in the elliptic root systems of type $X_l^{(1,1)}$ $(X = A, B, C, D, E, F, G)$ in which case one always has $(k_0, k_1, \ldots, k_l) = (1, 1, \ldots, 1)$. Hence, the above proposition in this case implies

Corollary 2.3 *The root system of type $X_l^{(1,1)}$ is given by*

$$R = \{\alpha_f + m\delta_1 + n\delta_2 | \alpha_f \in R_f,\ m, n \in \mathbb{Z}\}.$$

Now, we discuss on the structure of the Weyl group associated with the elliptic root system of type $X_l^{(1,1)}$. Recall that, for any $\alpha \in R$, the reflection w_α with respect to α is, by definition, given by

$$w_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad \forall \lambda \in F.$$

The subgroup of $GL(F)$ generated by $\{w_\alpha\}_{\alpha\in R}$ is called **elliptic Weyl group** and will be denoted by W_{ell}.

Recall that

$$F = F_f \oplus \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2, \quad \mathrm{Rad}\,(\cdot,\cdot) = \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2.$$

We define the subspace F_f^* of F^* by

$$F_f^* = \{h \in F^* | h(\delta_1) = h(\delta_2) = 0\},$$

where F is identified with the dual of F^*. Let d_1, d_2 be the elements of F^* satisfying

$$d_i|_{F_f} = 0, \quad d_i(\delta_j) = \delta_{i,j} \quad (i, j \in \{1, 2\}).$$

It is clear that

$$F^* = F_f^* \oplus \mathbb{R}d_1 \oplus \mathbb{R}d_2.$$

Let $\mu : F \to F^*$ be the linear map satisfying

1. for any $\lambda \in F_f$, $\mu(\lambda) \in F_f^*$ such that

$$\mu(\lambda)(\kappa) = (\lambda, \kappa) \quad \forall\, \lambda, \kappa \in F_f,$$

2. $\mu|_{\mathbb{R}\delta_1\oplus\mathbb{R}\delta_2} = 0$.

It follows that the restriction of the linear map μ to F_f is injective. Hence, for $\alpha_f \in R_f$, we set

$$\alpha_f^\vee = \frac{2}{(\alpha_f, \alpha_f)}\mu(\alpha_f) \in F^*.$$

The elliptic group W_{ell} naturally acts on the dual F^*, and its action is explicitly given by

$$w_\alpha(h) = h - h(\alpha)\alpha_f^\vee \quad \forall\, h \in F^*,$$

for $\alpha = \alpha_f + m\delta_1 + n\delta_2 \in R$ with $\alpha_f \in R_f$. For $\alpha_f \in R_f$, we set

$$t^i_{\alpha_f^\vee} = w_{\delta_i - \alpha_f} w_{\alpha_f} \quad (i = 1, 2).$$

By direct computation, one can check that, for any $h + \omega_1 d_1 + \omega_2 d_2 \in F^*$ with $h \in F_f^*$, one has

$$t^i_{\alpha_f^\vee}(h + \omega_1 d_1 + \omega_2 d_2) = h + \omega_1 d_1 + \omega_2 d_2 + \omega_i \alpha_f^\vee \quad (i = 1, 2).$$

Let $Q_f^\vee$ be the coroot lattice of R_f, i.e.,

$$Q_f^\vee = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i^\vee.$$

The above computation implies the next proposition:

Proposition 2.4 (cf. [15]) *$W_{ell} \cong W_f \ltimes (Q_f^\vee \times Q_f^\vee)$, where W_f is the Weyl group associated with the finite root system R_f.*

Remark 2.4 The elliptic Weyl group W_{ell} is generated by $\{w_{\alpha_0}, w_{\alpha_1}, \ldots, w_{\alpha_l}, w_{\alpha_j^*}\ (j \in I)\}$.

2.2 Hyperbolic extension

As in the previous subsection, let F be an $(l+2)$-dimensional $\mathbb{R}$-vector space, $(\cdot,\cdot)$ be a symmetric bilinear form with signature $(l,2,0)$, $E \subset \mathrm{Rad}\,(\cdot,\cdot)$ be a marking and (R,E) be a marked elliptic root system in F. As we have seen before, the radical $\mathrm{Rad}\,(\cdot,\cdot)$ signifies the existence of the translation in 2 directions. But, the space $(F_f^*)_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} F_f^*$ is too small to consider the invariant functions with respect to the action of the elliptic Weyl group since the only invariant holomorphic function on $(F_f^*)_{\mathbb{C}}$ is a constant. Following Saito [14], the notion of the hyperbolic extension will be introduced depending upon the marking E.

Recall that we have chosen a basis $\{\delta_1,\delta_2\}$ of $\mathrm{Rad}\,(\cdot,\cdot)$ satisfying

$$\mathrm{Rad}\,(\cdot,\cdot) \cap Q(R) = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2, \quad E \cap Q(R) = \mathbb{Z}\delta_2.$$

We define the $(l+3)$-dimensional $\mathbb{R}$-vector space $\widehat{F}$ and a symmetric bilinear form $(\cdot,\cdot)_E$ as follows:

$$\begin{aligned}
&\widehat{F} = F \oplus \mathbb{R}\Lambda_1 = F_f \oplus \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2 \oplus \mathbb{R}\Lambda_1 = F_a \oplus \mathbb{R}\delta_2 \oplus \mathbb{R}\Lambda_1,\\
&(\cdot,\cdot)_E|_{F\times F} = (\cdot,\cdot), \quad (\delta_i,\Lambda_1)_E = \delta_{i,1}, \quad (F_f,\Lambda_1)_E = \{0\}, \quad (\Lambda_1,\Lambda_1)_E = 0.
\end{aligned}$$

We call $(\widehat{F},(\cdot,\cdot)_E)$ a **hyperbolic extension** of $(F,(\cdot,\cdot))$. By definition, we have

$$\mathrm{Rad}\,(\cdot,\cdot)_E = E = \mathbb{R}\delta_2,$$

and the restriction of $(\cdot,\cdot)_E$ to the $(l+2)$-dimensional subspace

$$\widehat{F}_a := F_a \oplus \mathbb{R}\Lambda_1$$

is non-degenerate.

For the elliptic root system R and $\alpha \in R$, define $\hat{w}_\alpha \in GL(\widehat{F})$ by

$$\hat{w}_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)_E}{(\alpha,\alpha)_E}\alpha \quad \forall\, \lambda \in \widehat{F}$$

and consider the subgroup $\widehat{W}_{ell}$ of $GL(\widehat{F})$ generated by $\{\hat{w}_\alpha\}_{\alpha\in R}$ called the **hyperbolic extension** of the elliptic Weyl group W_{ell}.

Let us study the structure of the group $\widehat{W}_{ell}$. Let d_1, d_2, c_1 be the elements of the dual $\widehat{F}^*$ satisfying

$$\begin{aligned}
&d_i(\Lambda_1) = 0, \quad d_i(\delta_j) = \delta_{i,j}, \quad d_i(F_f) = \{0\},\\
&c_1(\Lambda_1) = 1, \quad c_1(\delta_i) = 0, \quad c_1(F_f) = \{0\}.
\end{aligned}$$

By definition, one sees that the symmetric bilinear form $(\cdot,\cdot)_E$ is non-degenerate on $\widehat{F}_a$. Hence, introducing two subspaces of $\widehat{F}^*$ by

$$\widehat{F}_f^* := (\mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2 \oplus \mathbb{R}\Lambda_1)^\perp, \quad \widehat{F}_a^* := (\mathbb{R}\delta_2)^\perp,$$

one has

$$\begin{aligned}
\widehat{F}_a^* &= \widehat{F}_f^* \oplus \mathbb{R}d_1 \oplus \mathbb{R}c_1,\\
\widehat{F}^* &= \widehat{F}_f^* \oplus \mathbb{R}d_1 \oplus \mathbb{R}d_2 \oplus \mathbb{R}c_1 = \widehat{F}_a^* \oplus \mathbb{R}d_2.
\end{aligned}$$

Let $\hat{\mu} : \widehat{F} \to \widehat{F}^*$ be the linear map defined by

$$\hat{\mu}(\lambda)(\kappa) = (\lambda,\kappa)_E \quad \lambda,\kappa \in \widehat{F}.$$

Since $\mathrm{Rad}\,(\cdot,\cdot)_E = E = \mathbb{R}\delta_2$, it follows that $\hat{\mu}(\delta_2) = 0$. Nevertheless, the restriction

$$\hat{\mu}|_{\widehat{F}_a} : \widehat{F}_a \longrightarrow \widehat{F}_a^*,$$

is a linear isomorphism. Hence, we set $E^* = \mathbb{R}d_2$ and define the symmetric bilinear form $(\cdot,\cdot)_{E^*}$ on $\widehat{F}^*$ by

$$(\hat{\mu}(\lambda), \hat{\mu}(\kappa))_{E^*} := (\lambda, \kappa)_E \quad (\lambda, \kappa \in \widehat{F}_a), \quad (d_2, \widehat{F}^*)_{E^*} := \{0\}.$$

By definition, one has

$$\mathrm{Rad}\,(\cdot,\cdot)_{E^*} = E^* = \mathbb{R}d_2.$$

For $\alpha \in R \subset \widehat{F}$, set

$$\alpha^\vee = \frac{2}{(\alpha,\alpha)_E}\hat{\mu}(\alpha).$$

By definition, one has

$$\alpha^\vee(\lambda) = \frac{2(\alpha,\lambda)_E}{(\alpha,\alpha)_E}.$$

Th equalities $\hat{\mu}(\delta_1) = c_1$ and $\hat{\mu}(\delta_2) = 0$ imply that, for $\alpha = \alpha_f + m\delta_1 + n\delta_2 \in R$ with $\alpha_f \in R_f$, one has

$$\alpha^\vee = \alpha_f^\vee + \frac{2m}{(\alpha,\alpha)_E}c_1 = \alpha_f^\vee + m\frac{(\alpha_f^\vee, \alpha_f^\vee)_{E^*}}{2}c_1.$$

Hence, the natural action of the hyperbolic extension $\widehat{W}_{ell}$ on $\widehat{F}^*$ is given by

$$\hat{w}_\alpha(h) = h - h(\alpha)\alpha^\vee \quad h \in \widehat{F}^*$$

for any $\alpha \in R$. By direct computation, one obtains

Lemma 2.5 *For $\alpha = \alpha_f + m\delta_1 + n\delta_2 \in R$ with $\alpha_f \in R_f$ and $h = h_f + \omega_1 d_1 + \omega_2 d_2 + uc_1 \in \widehat{F}^*$ with $h_f \in F_f^*$, one has*

$$\begin{aligned}\hat{w}_\alpha(h) = h &- (h_f(\alpha_f) + m\omega_1 + n\omega_2)\alpha_f^\vee \\ &- m\left((h_f, \alpha_f^\vee)_{E^*} + (m\omega_1 + n\omega_2)\cdot\frac{(\alpha_f^\vee, \alpha_f^\vee)_{E^*}}{2}\right)c_1.\end{aligned}$$

For $\alpha_f \in R_f$, we set

$$\hat{t}^i_{\alpha_f^\vee} := \hat{w}_{\delta_i - \alpha_f}\hat{w}_{\alpha_f} \quad (i = 1, 2).$$

By Lemma 2.5, one has

Corollary 2.6 *For $\alpha_f, \beta_f \in R_f$ and $h = h_f + \omega_1 d_1 + \omega_2 d_2 + uc_1 \in \widehat{F}^*$ with $h_f \in F_f^*$, one has*

1. $\hat{t}^1_{\alpha_f^\vee}(h) = h + \omega_1\alpha_f^\vee - \left((h_f, \alpha_f^\vee)_{E^*} + \omega_1\cdot\frac{(\alpha_f^\vee, \alpha_f^\vee)_{E^*}}{2}\right)c_1,$
   $\hat{t}^2_{\alpha_f^\vee}(h) = h + \omega_2\alpha_f^\vee,$
2. $\hat{t}^1_{\alpha_f^\vee}\hat{t}^2_{\beta_f^\vee}(\hat{t}^2_{\beta_f^\vee}\hat{t}^1_{\alpha_f^\vee})^{-1}(h) = h - \omega_2(\alpha_f^\vee, \beta_f^\vee)_{E^*}c_1.$

Remark 2.5 For any $\alpha_f^\vee \in Q_f^\vee$, set

$$\hat{t}^1_{\alpha_f^\vee}(h) = h + \omega_1\alpha_f^\vee - \left((h_f, \alpha_f^\vee)_{E^*} + \omega_1 \cdot \frac{(\alpha_f^\vee, \alpha_f^\vee)_{E^*}}{2}\right)c_1,$$
$$\hat{t}^2_{\alpha_f^\vee}(h) = h + \omega_2\alpha_f^\vee.$$

It can be verified that for any $\beta_1^\vee, \beta_2^\vee \in Q_f^\vee$, one has

$$\hat{t}^i_{\beta_1^\vee}\hat{t}^i_{\beta_2^\vee} = \hat{t}^i_{\beta_1^\vee+\beta_2^\vee} \quad (i = 1, 2).$$

Thus, setting

$$H(Q_f^\vee) = \langle \hat{t}^1_{\alpha_f^\vee}, \hat{t}^2_{\beta_f^\vee} \,|\, \alpha_f^\vee, \beta_f^\vee \in Q_f^\vee \rangle,$$

Lemma 2.6 and Remark 2.5 imply that this group is a discrete Heisenberg group. Moreover, the next isomorphism is well known:

Proposition 2.7 ([14,21]) $\widehat{W}_{ell} \cong W_f \ltimes H(Q_f^\vee)$.

Notice that Moody and Shi [7] obtained similar results for n-toroidal Lie algebras.

2.3 2-Toroidal Lie algebras

In this subsection, we describe elliptic root systems and their Weyl groups in view of 2-toroidal Lie algebras $\mathfrak{g}_{\mathrm{tor}} = \mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra over $\mathbb{C}$. Recall that a 2-toroidal Lie algebra is the Lie algebra whose bracket structure is given as follows: for $X, Y \in \mathfrak{g}$,

$$[X \otimes s^m t^n, Y \otimes s^k t^l] := [X, Y] \otimes s^{m+k} t^{n+l},$$

where the bracket in the right-hand side is the Lie bracket in $\mathfrak{g}$. Here and after, we do not distinguish them.

Two derivations $d_s = s\dfrac{\partial}{\partial s}$ and $d_t = t\dfrac{\partial}{\partial t}$ on $\mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ form a commutative Lie algebra $\mathfrak{d} = \mathbb{C}d_s \oplus \mathbb{C}d_t$ which also acts naturally on $\mathfrak{g}_{\mathrm{tor}}$; for $A \otimes s^m t^n \in \mathfrak{g}_{\mathrm{tor}}$,

$$[d_s, A \otimes s^m t^n] := mA \otimes s^m t^n, \quad [d_t, A \otimes s^m t^n] := nA \otimes s^m t^n.$$

Set $\mathfrak{g}_{\mathrm{tor}}^{\mathfrak{d}} = \mathfrak{d} \ltimes \mathfrak{g}_{\mathrm{tor}}$. With the aide of a non-degenerate symmetric invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, we define the symmetric bilinear form $(\cdot\,|\,\cdot)$ on $\mathfrak{g}_{\mathrm{tor}}^{\mathfrak{d}}$ as follows: for $X \otimes s^m t^n, Y \otimes s^k t^l \in \mathfrak{g}_{\mathrm{tor}}^{\mathfrak{d}}$,

$$(X \otimes s^m t^n | Y \otimes s^k t^l) = (X, Y)\delta_{m+k,0}\delta_{n+l,0},$$
$$(d_\sharp | X \otimes s^m t^n) = (d_s | d_t) = 0 \quad (\sharp \in \{s, t\}).$$

Unfortunately, this symmetric bilinear form is degenerate and is only $\mathfrak{g}_{\mathrm{tor}}$-invariant. Moreover, the restriction of $(\cdot\,|\,\cdot)$ to the commutative subalgebra $\mathfrak{h}^{\mathfrak{d}} := \mathfrak{d} \oplus \mathfrak{h}$ satisfies

$$\mathrm{Rad}\,(\cdot\,|\,\cdot)|_{\mathfrak{h}^{\mathfrak{d}} \times \mathfrak{h}^{\mathfrak{d}}} = \mathfrak{d},$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}^{\mathfrak{d}}_{\mathrm{tor}}$ admits the simultaneous eigenspace decomposition with respect to $\mathfrak{h}^{\mathfrak{d}}$:

$$\mathfrak{g}^{\mathfrak{d}}_{\mathrm{tor}} = \mathfrak{h}^{\mathfrak{d}} \oplus \left(\bigoplus_{\alpha \in (\mathfrak{h}^{\mathfrak{d}})^*} \mathfrak{g}^{\mathfrak{d}}_{\alpha} \right), \quad \mathfrak{g}^{\mathfrak{d}}_{\alpha} = \{A \in \mathfrak{g}_{\mathrm{tor}} | [h, A] = \alpha(h) A \quad (h \in \mathfrak{h}^{\mathfrak{d}})\}.$$

The set $\Delta_{ell} := \{\alpha \in (\mathfrak{h}^{\mathfrak{d}})^* | \mathfrak{g}^{\mathfrak{d}}_{\alpha} \neq \{0\}\}$ is called a **double affine root system**. Let us identify $\mathfrak{h}^*$ with a subspace of $(\mathfrak{h}^{\mathfrak{d}})^*$ as follows: for $\alpha \in \mathfrak{h}^*$, we set $\alpha(d_\sharp) = 0$ ($\sharp \in \{s, t\}$). Let δ_s, δ_t be the elements of $(\mathfrak{h}^{\mathfrak{d}})^*$ satisfying

$$\delta_\sharp|_{\mathfrak{h}} = 0, \quad \delta_\sharp(d_\flat) = \delta_{\sharp,\flat} \quad (\sharp, \flat \in \{s, t\}).$$

We have the next decomposition

$$(\mathfrak{h}^{\mathfrak{d}})^* = \mathfrak{h}^* \oplus \mathbb{C}\delta_s \oplus \mathbb{C}\delta_t.$$

We extend the symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ to $(\mathfrak{h}^{\mathfrak{d}})^*$ by

$$(\delta_\sharp | (\mathfrak{h}^{\mathfrak{d}})^*) = \{0\} \text{ for } \sharp \in \{s, t\}.$$

The next proposition is clear:

Proposition 2.8

$$\Delta_{ell} = \{\alpha_f + m\delta_s + n\delta_t | \alpha_f \in \Delta_f \cup \{0\},\ m, n \in \mathbb{Z}\} \setminus \{0\},$$

where Δ_f *is the root system of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$.

Set

$$\Delta^{re}_{ell} = \{\alpha \in \Delta_{ell} | (\alpha|\alpha) \neq 0\}, \quad \Delta^{im}_{ell} = \{\alpha \in \Delta_{ell} | (\alpha|\alpha) = 0\}.$$

The set Δ^{re}_{ell} (resp. Δ^{im}_{ell}) is called **real root system of** Δ_{ell} (resp. **imaginary root system of** Δ_{ell}). We have

$$\Delta^{re}_{ell} = \{\alpha_f + m\delta_s + n\delta_t | \alpha_f \in \Delta_f,\ m, n \in \mathbb{Z}\}, \quad \Delta^{im}_{ell} = \{m\delta_s + n\delta_t | (m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\},$$

and

$$\mathfrak{g}^{\mathfrak{d}}_{\alpha_f + m\delta_s + n\delta_t} = \mathfrak{g}_{\alpha_f} \otimes \mathbb{C} s^m t^n, \quad \mathfrak{g}^{\mathfrak{d}}_{m\delta_s + n\delta_t} = \mathfrak{h} \otimes \mathbb{C} s^m t^n.$$

Remark 2.6 For $\mathfrak{g}$ of type X_l, the set Δ^{re}_{ell} is the elliptic root system of type $X_l^{(1,1)}$ by Corollary 2.3.

Now, we consider the Weyl group of $\mathfrak{g}^{\mathfrak{d}}_{\mathrm{tor}}$. Define the linear isomorphism $\nu_f : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ by

$$\nu_f(h_f)(h'_f) = (h_f | h'_f) \quad h_f, h'_f \in \mathfrak{h},$$

and set

$$\alpha_f^\vee = \frac{2}{(\alpha_f|\alpha_f)} \nu_f^{-1}(\alpha_f) \quad (\alpha_f \in \Delta_f).$$

Let e_{α_f} ($\alpha_f \in \Delta_f$) be a root vector of root α_f in $\mathfrak{g}$ normalized by the relations $[e_{\alpha_f}, e_{-\alpha_f}] = \alpha_f^\vee$ for any α_f. For $\alpha = \alpha_f + m\delta_x + n\delta_y \in \Delta^{re}_{ell}$ with $\alpha_f \in \Delta_f$, we set $e_\alpha = e_{\alpha_f} \otimes s^m t^n$. The affine automorphism s_α of $\mathfrak{g}^{\mathfrak{d}}_{\mathrm{tor}}$ defined by

$$s_\alpha = \exp(\mathrm{ad}(e_\alpha)) \exp(-\mathrm{ad}(e_{-\alpha})) \exp(\mathrm{ad}(e_\alpha)).$$

stabilizes $\mathfrak{h}^{\eth}$, i.e.,

$$s_\alpha(h) = h - \alpha(h)\alpha_f^\vee \quad h \in \mathfrak{h}^{\eth}.$$

Notice that this is an isometry on $\mathfrak{h}$ but is only an affine transformation on $\mathfrak{h}^{\eth}$. We denote the restriction of s_α ($\alpha \in \Delta_{ell}^{re}$) to $\mathfrak{h}^{\eth}$ by w_α. The **double affine Weyl group** W_{daf} is, by definition, the subgroup of the group of affine transformations on $\mathfrak{h}^{\eth}$ generated by w_α ($\alpha \in \Delta_{ell}^{re}$).

For any $\alpha_f \in \Delta_f$, we set

$$t^{\sharp}_{\alpha_f^\vee} = w_{\delta_\sharp - \alpha_f} \cdot w_{\alpha_f} \quad (\sharp \in \{s, t\}).$$

It can be checked that, for any $h = h_f + \omega_s d_s + \omega_t d_t \in \mathfrak{h}^{\eth}$ with $h_f \in \mathfrak{h}$, we have

$$t^{\sharp}_{\alpha_f^\vee}(h) = h + \omega_\sharp \alpha_f^\vee \quad (\sharp \in \{s, t\}).$$

It follows that

Proposition 2.9 *One has the isomorphism*

$$W_{daf} \cong W_f \ltimes (Q_f^\vee \times Q_f^\vee).$$

This proposition shows that $W_{daf} \cong W_{ell}$, that is, we obtained a description of the elliptic Weyl group W_{ell} in terms of the 2-toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$.

Next, we consider the hyperbolic extension from view point of 2-toroidal Lie algebras. For this purpose, we consider a 2-dimensional central extension $\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}}$ of $\mathfrak{g}^{\eth}_{\mathrm{tor}}$ to obtain a *non-degenerate* symmetric invariant bilinear form on it. Namely, it is the vector space

$$\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}} := \mathfrak{g}^{\eth}_{\mathrm{tor}} \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t,$$

enjoying the next commutation relations

$$\begin{aligned}
&[\mathfrak{g}^{\eth}_{\mathrm{tor}}, c_s] = [\mathfrak{g}^{\eth}_{\mathrm{tor}}, c_t] = \{0\},\\
&[X \otimes s^m t^n, Y \otimes s^k t^l] = [X, Y] \otimes s^{m+k} t^{n+l} + (X, Y)\delta_{m+k,0}\delta_{n+l,0}(mc_s + nc_t),\\
&[d_s, X \otimes s^m t^n] = mX \otimes s^m t^n, \quad [d_t, X \otimes s^m t^n] = nX \otimes s^m t^n,\\
&[d_s, d_t] = 0,
\end{aligned}$$

where $X, Y \in \mathfrak{g}$ and $k, l, m, n \in \mathbb{Z}$.

The Lie algebra $\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}}$ possesses a non-degenerate symmetric $\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}}$-invariant bilinear form $\langle \cdot | \cdot \rangle$ whose non-trivial pairings are given by

$$\begin{aligned}
&\langle X \otimes s^m t^n | Y \otimes s^k t^l \rangle = (X, Y)\delta_{m+k,0}\delta_{n+l,0},\\
&\langle \partial_\sharp | d\flat \rangle = \delta_{\sharp,\flat} \quad (\sharp, \flat \in \{s, t\}).
\end{aligned}$$

We remark that the restriction of this symmetric $\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}}$-invariant bilinear form to the $(\mathrm{rk}\,\mathfrak{g} + 4)$-dimensional commutative subalgebra

$$\widetilde{\mathfrak{h}} := \mathfrak{h}^{\eth} \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t,$$

is again non-degenerate.

We identify $(\mathfrak{h}^{\eth})^*$ with a subspace of $\widetilde{\mathfrak{h}}^*$ as follows: for $\alpha \in (\mathfrak{h}^{\eth})^*$, we set $\alpha(c_\sharp) = 0$ ($\sharp \in \{s, t\}$). Since c_s and c_t are central in $\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}}$, the root space decomposition of $\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}}$ with respect to $\widetilde{\mathfrak{h}}$ looks as follows:

$$\widetilde{\mathfrak{g}}^{\eth}_{\mathrm{tor}} = \widetilde{\mathfrak{h}} \oplus \left(\bigoplus_{\alpha \in \Delta_{ell}} \widetilde{\mathfrak{g}}_\alpha \right), \quad \widetilde{\mathfrak{g}}_\alpha = \mathfrak{g}^{\eth}_\alpha.$$

Let Λ_s, Λ_t be the elements of $\widetilde{\mathfrak{h}}^*$ satisfying

$$\Lambda_\sharp(\mathfrak{h}^\partial) = 0, \quad \Lambda_\sharp(c_\flat) = \delta_{\sharp,\flat} \quad (\sharp, \flat \in \{s, t\}).$$

We have

$$\widetilde{\mathfrak{h}}^* = (\mathfrak{h}^\partial)^* \oplus \mathbb{C}\Lambda_s \oplus \mathbb{C}\Lambda_t = \mathfrak{h}^* \oplus \mathbb{C}\delta_s \oplus \mathbb{C}\delta_t \oplus \mathbb{C}\Lambda_s \oplus \mathbb{C}\Lambda_t.$$

Define the linear isomorphism $\nu : \widetilde{\mathfrak{h}} \longrightarrow \widetilde{\mathfrak{h}}^*$ by

$$\nu(\tilde{h})(\tilde{h}') = \langle \tilde{h} | \tilde{h}' \rangle \quad \tilde{h}, \tilde{h}' \in \widetilde{\mathfrak{h}}.$$

Via this isomorphism, we define a non-degenerate symmetric bilinear form on $\widetilde{\mathfrak{h}}^*$ by

$$\langle \tilde{\lambda} | \tilde{\kappa} \rangle = \langle \nu^{-1}(\tilde{\lambda}) | \nu^{-1}(\tilde{\kappa}) \rangle \quad \tilde{\lambda}, \tilde{\kappa} \in \widetilde{\mathfrak{h}}^*.$$

For $\alpha \in \Delta_{ell}^{re} \subset \widetilde{\mathfrak{h}}^*$, we set

$$\alpha^\vee = \frac{2}{\langle \alpha | \alpha \rangle} \nu^{-1}(\alpha) \in \widetilde{\mathfrak{h}}.$$

Notice that we have

$$\nu^{-1}(\delta_\sharp) = c_\sharp, \quad \nu^{-1}(\Lambda_\sharp) = d_\sharp \quad (\sharp \in \{s, t\}),$$

and for $\alpha = \alpha_f + m\delta_x + n\delta_y \in \Delta_{ell}^{re}$ with $\alpha_f \in \Delta_f$, we also have

$$\alpha^\vee = \alpha_f^\vee + \frac{2}{\langle \alpha_f | \alpha_f \rangle}(mc_s + nc_t).$$

In the same way as s_α ($\alpha \in \Delta_{ell}^{re}$), we define the automorphism $\tilde{s}_\alpha$ of $\widetilde{\mathfrak{g}}_{\text{tor}}^\partial$ by

$$\tilde{s}_\alpha = \exp(\text{ad}(e_\alpha)) exp(-\text{ad}(e_{-\alpha})) \exp(\text{ad}(e_\alpha)).$$

By direct computation, one can check that

$$\tilde{s}_\alpha(\tilde{h}) = \tilde{h} - \alpha(\tilde{h})\alpha^\vee,$$

for any $\tilde{h} \in \widetilde{\mathfrak{h}}$, namely, $\tilde{s}_\alpha$ stabilizes $\widetilde{\mathfrak{h}}$. We denote the restriction of $\tilde{s}_\alpha$ to $\widetilde{\mathfrak{h}}$ by $\underset{\sim}{\tilde{w}_\alpha}$ and define $\widetilde{W}_{daf}$ as the subgroup of $O(\widetilde{\mathfrak{h}}, \langle \cdot | \cdot \rangle)$ generated by $\underset{\sim}{\tilde{w}_\alpha}$ ($\alpha \in \Delta_{ell}^{re}$). The group $\widetilde{W}_{daf}$ is too big for our purpose, and we need to reduce the space $\widetilde{\mathfrak{h}}$ to a smaller space which we explain below.

Set

$$\mathcal{H}^\pm = \left\{ (\omega_s, \omega_t) \in \mathbb{C} \times \mathbb{C}^* \;\middle|\; \pm\text{Im}\left(\frac{\omega_s}{\omega_t}\right) > 0 \right\},$$

and

$$\widetilde{\mathfrak{h}}_{\mathcal{H}^\pm} = \{ \tilde{h} = h_f + \omega_s d_s + \omega_t d_t + u_s c_s + u_t c_t \in \widetilde{\mathfrak{h}} \mid h_f \in \mathfrak{h}, \ (\omega_s, \omega_t) \in \mathcal{H}^\pm \}.$$

Notice that we treat d_s and d_t unequally. For an elliptic root system R, the choice of a marking E causes unequal treatment, and it is natural to choose $\mathbb{C}d_t$ as E^*. Let

$$X(\widetilde{\mathfrak{h}}_{\mathcal{H}^\pm}) = \{ \tilde{h} \in \widetilde{\mathfrak{h}}_{\mathcal{H}^\pm} \mid \langle \tilde{h} | \tilde{h} \rangle = 0 \}$$

be a complex submanifold of $\widetilde{\mathfrak{h}}_{\mathcal{H}^\pm}$. Since the symmetric bilinear form $\langle \cdot | \cdot \rangle$ is $\widetilde{W}_{daf}$-invariant, $\widetilde{W}_{daf}$ acts on $X(\widetilde{\mathfrak{h}}_{\mathcal{H}^\pm})$. By definition, this action commutes with the natural $\mathbb{C}^*$-action. Hence, $\widetilde{W}_{daf}$ acts on the projectified space

$$\mathbb{P}(X(\widetilde{\mathfrak{h}}_{\mathcal{H}^\pm})) := X(\widetilde{\mathfrak{h}}_{\mathcal{H}^\pm})/\mathbb{C}^*.$$

Before studying the $\widetilde{W}_{daf}$-action on $\mathbb{P}(X(\widetilde{\mathfrak{h}}_{\mathcal{H}^{\pm}}))$, let us describe this latter space explicitly.

We remark that $\widetilde{h} = h_f + \omega_s d_s + \omega_t d_t + u_s c_s + u_t c_t \in X(\widetilde{\mathfrak{h}}_{\mathcal{H}^{\pm}})$ with $h_f \in \mathfrak{h}$ implies the next equalities:

$$\langle h_f | h_f \rangle + 2(\omega_s u_s + \omega_t u_t) = 0 \quad \text{i.e.,} \quad u_t = -\frac{1}{\omega_t}\left(\omega_s u_s + \frac{1}{2}\langle h_f | h_f \rangle\right).$$

Hence, setting $\widehat{\mathfrak{h}}_{\mathbb{H}} = \mathbb{H} \times \mathfrak{h} \times \mathbb{C}$, where $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}\, \tau > 0\}$, we have

Proposition 2.10 *The holomorphic mapping*

$$\varphi : \mathbb{P}(X(\widetilde{\mathfrak{h}}_{\mathcal{H}^-})) \longrightarrow \widehat{\mathfrak{h}}_{\mathbb{H}}$$

defined by

$$[h_f + \omega_s d_s + \omega_t d_t + u_s c_s + u_t c_t] \longmapsto \left(-\frac{\omega_s}{\omega_t}, -\frac{h_f}{\omega_t}, -\frac{u_s}{\omega_t}\right)$$

is an isomorphism of complex manifolds.

Set $\tau = -\dfrac{\omega_s}{\omega_t} \in \mathbb{H}$. Let us compute the action of $\widetilde{W}_{daf}$ on $\widehat{\mathfrak{h}}_{\mathbb{H}} \cong \mathbb{P}(X(\widetilde{\mathfrak{h}}_{\mathcal{H}^-}))$.

For $\alpha \in \Delta_{ell}^{re}$, we denote by $\hat{w}_\alpha$ the element of $\mathrm{Aut}(\widehat{\mathfrak{h}}_{\mathbb{H}})$ induced from $\tilde{w}_\alpha \in \widetilde{W}_{daf}$.

Lemma 2.11 *For $(\tau, h_f, u) \in \widehat{\mathfrak{h}}_{\mathbb{H}}$ and $\alpha = \alpha_f + m\delta_s + n\delta_t \in \Delta_{ell}^{re}$ with $\alpha_f \in \Delta_f$, one has*

$$\begin{aligned}&\hat{w}_\alpha(\tau, h_f, u)\\ &= \left(\tau, h_f - (\alpha_f(h_f) + m\tau - n)\alpha_f^\vee, u - m\left(\langle \alpha_f^\vee | h_f \rangle + (m\tau - n)\frac{\langle \alpha_f^\vee | \alpha_f^\vee \rangle}{2}\right)\right).\end{aligned}$$

For $\alpha_f, \beta_f \in \Delta_f$, we set

$$\hat{t}^s_{\alpha_f^\vee} := \hat{w}_{\delta_s - \alpha_f} \cdot \hat{w}_{\alpha_f}, \quad \hat{t}^t_{\beta_f^\vee} := \hat{w}_{\delta_t + \beta_f} \cdot \hat{w}_{\beta_f}.$$

Notice that the difference of the sign in two formulas occurs because of the definition $\tau = -\dfrac{\omega_s}{\omega_t}$. By Lemma 2.11, we have

Corollary 2.12 *For any $(\tau, h_f, u) \in \widehat{\mathfrak{h}}_{\mathbb{H}}$ and $\alpha_f, \beta_f \in \Delta_f$, we have*

1. $\hat{t}^s_{\alpha_f^\vee}(\tau, h_f, u) = \left(\tau, h_f + \tau\alpha_f^\vee, u - \left(\langle \alpha_f^\vee | h_f \rangle + \tau \cdot \dfrac{\langle \alpha_f^\vee | \alpha_f^\vee \rangle}{2}\right)\right).$
2. $\hat{t}^t_{\beta_f^\vee}(\tau, h_f, u) = (\tau, h_f + \beta_f^\vee, u).$
3. $\hat{t}^s_{\alpha_f^\vee}\hat{t}^t_{\beta_f^\vee}(\hat{t}^t_{\beta_f^\vee}\hat{t}^s_{\alpha_f^\vee})^{-1}(\tau, h_f, u) = (\tau, h_f, u - \langle \alpha_f^\vee | \beta_f^\vee \rangle).$

Remark 2.7 Motivated by Corollary 2.12 1. and 2., we introduce the automorphisms $\hat{t}^s_{\alpha_f^\vee}$ and $\hat{t}^t_{\alpha_f^\vee}$ of $\widehat{\mathfrak{h}}_{\mathbb{H}}$ for $\alpha_f^\vee \in Q_f^\vee$ by

$$\begin{aligned}\hat{t}^s_{\alpha_f^\vee}(\tau, h_f, u) &= \left(\tau, h_f + \tau\alpha_f^\vee, u - \left(\langle \alpha_f^\vee | h_f \rangle + \tau \cdot \frac{\langle \alpha_f^\vee | \alpha_f^\vee \rangle}{2}\right)\right),\\ \hat{t}^t_{\alpha_f^\vee}(\tau, h_f, u) &= (\tau, h_f + \alpha_f^\vee, u).\end{aligned}$$

It can be checked that, for any $\beta_1^\vee, \beta_2^\vee \in Q_f^\vee$, one has

$$\hat{t}^s_{\beta_1^\vee}\hat{t}^s_{\beta_2^\vee} = \hat{t}^s_{\beta_1^\vee + \beta_2^\vee}, \quad \hat{t}^t_{\beta_1^\vee}\hat{t}^t_{\beta_2^\vee} = \hat{t}^t_{\beta_1^\vee + \beta_2^\vee}.$$

Set

$$\widehat{W}_{daf} = \langle \hat{w}_\alpha | \alpha \in \Delta_{ell}^{re} \rangle.$$

By Corollary 2.12 and Remark 2.7, the group generated by $\hat{t}^s_{\alpha_f^\vee}$ and $\hat{t}^t_{\beta_f^\vee}$ ($\alpha_f^\vee, \beta_f^\vee \in Q_f^\vee$) is a discrete Heisenberg group (cf. Corollary 2.12 (3)) isomorphic to $H(Q_f^\vee)$. Indeed, we have

Proposition 2.13 $\widehat{W}_{daf} \cong W_f \ltimes H(Q_f^\vee)$. *In particular, we have* $\widehat{W}_{daf} \cong \widehat{W}_{ell}$.

Thus, we conclude that the group $\widehat{W}_{daf}$ realizes the **hyperbolic extension of** W_{ell}. We call the space $\widehat{\mathfrak{h}}_{\mathbb{H}} \cong \mathbb{P}(X(\widetilde{\mathfrak{h}}_{\mathcal{H}}))$ the **hyperbolic extension of** $\mathbb{H} \times \mathfrak{h}$. Let us end up this subsection with a remark which would be related to invariant theory of the Weyl group W_{ell} that will be discussed in Sect. 4.3:

Remark 2.8 By definition, we have

$$\langle \alpha_f^\vee | \beta_f^\vee \rangle = (\alpha_f^\vee, \beta_f^\vee) \in \mathbb{Z}$$

for any $\alpha_f^\vee, \beta_f^\vee \in Q_f^\vee$. This means that the $\widehat{W}_{ell}$-action on $\widehat{\mathfrak{h}}_{\mathbb{H}}$ induces a W_{ell}-action on $\mathbb{H} \times \mathfrak{h} \times \mathbb{C}^*$ via the exponential map $\mathbb{C} \longrightarrow \mathbb{C}^*; z \mapsto \exp(2\pi\sqrt{-1}z)$.

3 Double loop groups and elliptic Weyl groups

In this section, we will show that the elliptic Weyl group, recalled in Sect. 2.1 and related to 2-toroidal Lie algebras in Sect. 2.3, can be obtained naturally from double loop groups associated with a connected and simply connected simple Lie group G over $\mathbb{C}$. Here, a double loop group signifies the Fréchet Lie group

$$\mathcal{E}(G) := C^\infty(S^1 \times S^1, G)$$

with its Lie algebra

$$\mathcal{E}(\mathfrak{g}) := C^\infty(S^1 \times S^1, \mathfrak{g})$$

called a double loop algebra, where $\mathfrak{g}$ signifies the Lie algebra of G. As usual, their structures are defined by pointwise operations. In Sect. 2.3, we used the terminology '**toroidal**', but in this section, as we work on C^∞-class, we use '**double loop**' to distinguish with the former. We realize S^1 as $\sqrt{-1}\mathbb{R}/2\pi\sqrt{-1}\mathbb{R}$ and set

$$\partial_x = \frac{\partial}{\partial x} = s\frac{\partial}{\partial s} = d_s, \quad \partial_y = \frac{\partial}{\partial y} = t\frac{\partial}{\partial t} = d_t \quad (s = e^x,\ t = e^y).$$

As in Sect. 2.3, $\mathfrak{d} := \mathbb{C}\partial_x \oplus \mathbb{C}\partial_y$ acts naturally on $\mathcal{E}(\mathfrak{g})$:

$$[\partial_\sharp, X \otimes f] := X \otimes \partial_\sharp f \quad X \in \mathfrak{g},\ f \in C^\infty(\mathbb{T}, \mathbb{C}),\ \sharp \in \{x, y\}.$$

Via this action, we introduce a Lie algebra structure on $\mathcal{E}(\mathfrak{g})^{\mathfrak{d}} := \mathfrak{d} \ltimes \mathcal{E}(\mathfrak{g})$.

Remark 3.1 Notice that the group $\mathcal{E}(G)$ is not only a **regular F Lie group** in the sense of [10] but also **locally exponential** in the sense of [8], where the exponential map $\exp : \mathcal{E}(\mathfrak{g}) \longrightarrow \mathcal{E}(G)$ is a local diffeomorphism between neighborhoods of $0 \in \mathcal{E}(\mathfrak{g})$ and of $e \in \mathcal{E}(G)$.

We define a left action of $\mathcal{E}(G)$ on $\mathcal{E}(\mathfrak{g})^{\mathfrak{d}}$ as follows:

$$\begin{aligned} L : \mathcal{E}(G) \times \mathcal{E}(\mathfrak{g})^{\mathfrak{d}} &\longrightarrow \mathcal{E}(\mathfrak{g})^{\mathfrak{d}}; \\ (g, (\xi, A)) &\longmapsto (\xi, \mathrm{Ad}(g)A - dg \cdot g^{-1}(\xi)). \end{aligned} \tag{1}$$

We sometimes denote $L(g, (\xi, A))$ by $L_g(\xi, A)$.

We choose a commutative ad-diagonalizable subalgebra of $\mathcal{E}(\mathfrak{g})^{\mathfrak{d}}$ by

$$\mathfrak{h}^{\mathfrak{d}} = \mathfrak{d} \oplus \mathfrak{h},$$

and set

$$\begin{aligned} N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}}) &= \{g \in \mathcal{E}(G) | L_g(\mathfrak{h}^{\mathfrak{d}}) = \mathfrak{h}^{\mathfrak{d}}\}, \\ Z_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}}) &= \{g \in \mathcal{E}(G) | L_g(\xi, h) = (\xi, h) \text{ for } (\xi, h) \in \mathfrak{h}^{\mathfrak{d}}\}. \end{aligned}$$

Our purpose of this subsection is to show that the group

$$W'_{ell} := N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})/Z_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}}),$$

is isomorphic to W_{ell}, i.e., we have the next theorem:

Theorem 3.1 $W'_{ell} \cong W_f \ltimes (Q_f^{\vee} \times Q_f^{\vee})$.

Proof For $g \in N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})$, $(\xi, h) \in \mathfrak{h}^{\mathfrak{d}}$, we have

$$L_g(\xi, h) = (\xi, \mathrm{Ad}(g)(h) - dg \cdot g^{-1}(\xi)) \in \mathfrak{h}^{\mathfrak{d}}$$

by definition. In particular, letting $h = 0$, we obtain $t := dg \cdot g^{-1}(\xi) \in \mathfrak{h}$. □

Now, suppose that $\xi = \omega_x \partial_x + \omega_y \partial_y \in \mathfrak{d}$ satisfies $-\dfrac{\omega_x}{\omega_y} \in \mathbb{H}$. For $(a, b) \in \mathbb{R}^2$, setting

$$K_{a,b}(x, y) = g(x, y)^{-1} g(x + a, y + b) \in \mathcal{E}(G),$$

we see that

$$\begin{aligned} \xi K_{a,b}(x, y) &= \xi(g(x, y)^{-1}) g(x + a, y + b) + g(x, y)^{-1} \xi(g(x + a, y + b)) \\ &= -g(x, y)^{-1}(\xi g(x, y) g(x, y)^{-1}) g(x + a, y + b) \\ &\quad + g(x, y)^{-1}(\xi g(x + a, y + b) g(x + a, y + b)^{-1}) g(x + a, y + b) \\ &= -g(x, y)^{-1} t g(x + a, y + b) + g(x, y)^{-1} t g(x + a, y + b) = 0, \end{aligned}$$

that is, $K_{a,b}(x, y)$ is ξ-holomorphic. Since $\mathbb{T} = S^1 \times S^1$ is compact, $K_{a,b}$ has to be a constant, say $A(a, b) \in G$. We regard A as a C^{∞}-function $\mathbb{T} \longrightarrow G$. By definition, one has $A(a, b) = g(0, 0)^{-1} g(a, b)$. Moreover, we have

Lemma 3.2

$$A(a, b) \in Z_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}}) = H,$$

where H signifies the Cartan subgroup of G whose Lie algebra is $\mathfrak{h}$.

Proof Since $g \in N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})$ implies $dg \cdot g^{-1}(\xi) \in \mathfrak{h}$ as we have already shown, we have $\mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h}$ which implies $A(a, b) \in N_G(\mathfrak{h})$ for each (a, b). By the connectivity of $\mathbb{T}$, it follows that $\mathrm{Im}A \subset N_G(\mathfrak{h})$ is connected since A is continuous. Now, $A(0, 0) = e \in G$ implies $\mathrm{Im}\, A \subset H$ since G is simply-connected by assumption. □

Next, we show

Lemma 3.3

$$A \in \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{T}, H).$$

Proof By direct computation, one has

$$\begin{aligned} A(a+a', b+b') &= g(0,0)^{-1} g(a+a', b+b') \\ &= g(0,0)^{-1} g(a,b) \cdot g(a,b)^{-1} g(a+a', b+b') \\ &= A(a,b) K_{a',b'}(a,b) = A(a,b) A(a',b'). \end{aligned}$$

□

We remark that $\mathrm{Hom}_{\mathrm{Grp}}(\mathbb{T}, H) \cong Q_f^\vee \times Q_f^\vee$. By definition and Lemma 3.3, we have

$$g(a,b) = g(0,0)A(a,b) \in N_G(\mathfrak{h}) \ltimes (Q_f^\vee \times Q_f^\vee),$$

which implies the existence of the surjection $N_G(\mathfrak{h}) \ltimes (Q_f^\vee \times Q_f^\vee) \twoheadrightarrow N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})$. In particular, we obtain $Z_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}}) = Z_G(\mathfrak{h}) = H$. As $W_f = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$, we see that there is a surjection $W_f \ltimes (Q_f^\vee \times Q_f^\vee) \twoheadrightarrow N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})/Z_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})$.

Now, for $\alpha \in \Delta_{ell}^{re}$, we define the element s'_α of $\mathcal{E}(G)$ by

$$s'_\alpha = \exp(e_\alpha)\exp(-e_{-\alpha})\exp(e_\alpha).$$

We denote the subgroup of $\mathcal{E}(G)$ generated by $\{s'_\alpha | \alpha \in \Delta_{ell}^{re}\}$ by $\mathcal{W}_{ell}$. By direct calculation, we have

$$L_{s'_\alpha}(\xi, h) = (\xi, h - \alpha(\xi, h)\alpha_f^\vee) \qquad (\xi, h) \in \mathfrak{h}^{\mathfrak{d}},$$

i.e., $s'_\alpha \in N_{\mathcal{E}(G)}(\mathfrak{h}^{\mathfrak{d}})$. Set

$$w'_\alpha = L_{s'_\alpha}\big|_{\mathfrak{h}^{\mathfrak{d}}}.$$

It follows that

Lemma 3.4 *The group generated by* $\{w'_\alpha | \alpha \in \Delta_{ell}^{re}\}$ *is isomorphic to* $W_f \ltimes (Q_f^\vee \times Q_f^\vee)$.

Proof First of all, it can be checked easily that $\langle w'_\alpha | \alpha \in \Delta_f \rangle = W_f$. For $\alpha \in \Delta_f$, we set

$$(t^\sharp_{\alpha^\vee})' = w'_{\delta_\sharp - \alpha} \circ w'_\alpha \quad \sharp \in \{x, y\}.$$

It can be verified by direct calculation that these elements act on $\mathfrak{h}^{\mathfrak{d}}$ and satisfy

$$(t^\sharp_{\alpha^\vee})'(\xi, h) = (\xi, h + \omega_\sharp \alpha^\vee),$$

where we set $\xi = \omega_x \partial_x + \omega_y \partial_y \in \mathfrak{d}$. Hence the result follows. □

By this lemma, Theorem 3.1 follows.

Let $P_f^\vee$ be the co-weight lattice of Δ_f, i.e., the dual lattice of the root lattice Q_f, and for $\Lambda^\vee \in P_f^\vee$, we define two elements of $C^\infty(\mathbb{R}^2, G)$ by

$$\phi^x_{\Lambda^\vee} := \exp(-x\Lambda^\vee), \quad \phi^y_{\Lambda^\vee} := \exp(-y\Lambda^\vee).$$

(The group $C^\infty(\mathbb{R}^2, G)$ acts on the Lie algebra $C^\infty(\mathbb{R}^2, \mathfrak{g})$ via the gauge transformation as in (1)). Notice that these elements preserve $\mathcal{E}(\mathfrak{g}) \subset C^\infty(\mathbb{R}^2, \mathfrak{g})$ as can be seen from the next formulae: for $\sharp \in \{x, y\}$,

$$\begin{aligned} L_{\phi^\sharp_{\Lambda^\vee}}(\xi, h) &= (\xi, h + \omega_\sharp \Lambda^\vee), \\ L_{\phi^\sharp_{\Lambda^\vee}}(\xi, e_\alpha) &= (\xi, e^{-\alpha(\Lambda^\vee)\sharp} e_\alpha + \omega_\sharp \Lambda^\vee). \end{aligned} \tag{2}$$

It turns out that $\phi^x_{\Lambda^\vee}, \phi^y_{\Lambda^\vee} \in \mathcal{E}(G)$ if and only if $\Lambda^\vee \in Q^\vee_f$. Indeed, it can be checked that $\phi^\sharp_{\alpha^\vee} = s'_{\delta_\sharp - \alpha} s'_\alpha$ for $\alpha \in \Delta_f$ and $\sharp \in \{x, y\}$ (cf. see, e.g., Sect. 1). In particular, we have $L_{\phi^\sharp_{\alpha^\vee}}\Big|_{\mathfrak{h}^\mathfrak{d}} = (t^\sharp_{\alpha^\vee})'$ for $\alpha^\vee \in Q^\vee_f$. Here and after, for $\Lambda^\vee \in P^\vee_f$ and $\sharp \in \{x, y\}$, we set

$$(t^\sharp_{\Lambda^\vee})' = L_{\phi^\sharp_{\Lambda^\vee}}\Big|_{\mathfrak{h}^\mathfrak{d}}.$$

We call the subgroup of $GL(\mathfrak{h}^\mathfrak{d})$ generated by W_f and $\langle (t^x_{\Lambda^\vee})', (t^y_{\Lambda^\vee})' | \Lambda^\vee \in P^\vee_f \rangle$ the **extended elliptic Weyl group** and denote it by W^e_{ell}.

Remark 3.2 (cf. [14]) Similarly, one can show that the group $W^e_{ell} \cong W_f \ltimes (P^\vee_f \times P^\vee_f)$ is isomorphic to $N_{\mathcal{E}(G^{ad})}(\mathfrak{h}^\mathfrak{d})/Z_{\mathcal{E}(G^{ad})}(\mathfrak{h}^\mathfrak{d})$, i.e., the Weyl group of $\mathcal{E}(G^{ad})$, where G^{ad} is the Chevalley group of adjoint type associated with the Lie algebra $\mathfrak{g}$.

4 Central extensions of $\mathcal{E}(\mathfrak{g})$ and $\mathcal{E}(G)$

Here, we reconstruct the hyperbolic extension of the elliptic Weyl group recalled in Sect. 2.3, from viewpoint of a central extension of the double loop group $\mathcal{E}(G)$.

4.1 $\widetilde{\mathcal{E}}(G)$-Action on $\widetilde{\mathcal{E}}(\mathfrak{g})^\mathfrak{d}$

Let $\widetilde{\mathcal{E}}(\mathfrak{g})^\mathfrak{d}$ be the Fréchet space

$$\widetilde{\mathcal{E}}(\mathfrak{g})^\mathfrak{d} := \mathcal{E}(\mathfrak{g})^\mathfrak{d} \oplus (\mathbb{C}dx \oplus \mathbb{C}dy)$$

with the smooth Lie bracket satisfying

$$\begin{aligned} &[\widetilde{\mathcal{E}}(\mathfrak{g})^\mathfrak{d}, \mathbb{C}dx \oplus \mathbb{C}dy] = \{0\}, \\ &[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)\left[\left(\int_\mathbb{T} (\partial_x f) g\omega\right) dx + \left(\int_\mathbb{T} (\partial_y f) g\omega\right) dy\right], \\ &[\partial_x, X \otimes f] = X \otimes \partial_x f, \quad [\partial_y, X \otimes f] = X \otimes \partial_y f, \\ &[\partial_x, \partial_y] = 0. \end{aligned}$$

Here, $X, Y \in \mathfrak{g}$, $f, g \in C^\infty(\mathbb{T}, \mathbb{C})$ with $\mathbb{T} = \left(\sqrt{-1}\mathbb{R}/2\pi\sqrt{-1}\mathbb{Z}\right)^2$ and $\omega = -\frac{1}{4\pi^2} dx \wedge dy$ is a volume form on $\mathbb{T}$.

The Lie algebra $\widetilde{\mathcal{E}}(\mathfrak{g})^\mathfrak{d}$ possesses the non-degenerate symmetric invariant bilinear form $\langle \cdot | \cdot \rangle$ whose non-trivial pairings are given by

$$\begin{aligned} \langle A | B \rangle &= \int_\mathbb{T} (A, B)\omega \quad (A, B \in \mathcal{E}(\mathfrak{g})), \\ \langle \partial_\sharp | d\flat \rangle &= \delta_{\sharp,\flat} \quad (\sharp, \flat \in \{x, y\}). \end{aligned} \tag{3}$$

Notice that this bilinear form is the smooth extension of the bilinear form defined on $\widetilde{\mathfrak{g}}^{\mathfrak{d}}_{\mathrm{tor}}$.

Next, we introduce a central extension of $\mathcal{E}(G)$ by $\mathbb{C}$, denoted by $\widehat{\mathcal{E}}(G)$, as follows.

Let Θ be the two-cocycle defined by

$$\Theta(g_1, g_2) = \frac{1}{8\pi^2}\int_{\mathbb{T}}(g_1^{-1}dg_1 \wedge dg_2 \cdot g_2^{-1}), \tag{4}$$

for $g_1, g_2 \in \mathcal{E}(G)$. Here, the two-cocyle condition means, for $g_i \in \mathcal{E}(G)$ $(1 \leq i \leq 3)$,

$$\Theta(g_1, g_2) + \Theta(g_1g_2, g_3) = \Theta(g_1, g_2g_3) + \Theta(g_2, g_3). \tag{5}$$

The Fréchet Lie group $\widehat{\mathcal{E}}(G)$ is the central extension of $\mathcal{E}(G)$ by $\mathbb{C}$ with the two-cocyle Θ, i.e., for $(g_i, c_i) \in \widehat{\mathcal{E}}(G)$ $(i = 1, 2)$, we define their product by

$$(g_1, c_1) \cdot (g_2, c_2) := (g_1g_2, c_1 + c_2 + \Theta(g_1, g_2)). \tag{6}$$

The next lemma describes a central extension of (1):

Lemma 4.1 *Let $\widehat{L} : \widehat{\mathcal{E}}(G) \times \widetilde{\mathcal{E}}(\mathfrak{g})^{\mathfrak{d}} \longrightarrow \widetilde{\mathcal{E}}(\mathfrak{g})^{\mathfrak{d}}$ be the map defined by*

$$\begin{aligned}&\widehat{L}((g,c),(\xi,A,\alpha)) = \widehat{L}_{(g,c)}(\xi,A,\alpha)\\ &:= \left(\xi, \mathrm{Ad}(g)A - dg\cdot g^{-1}(\xi), \alpha - c\cdot\xi \lrcorner (-4\pi^2\omega) + \langle A|g^{-1}dg\rangle - \frac{1}{2}\langle dg\cdot g^{-1}(\xi)|dg\cdot g^{-1}\rangle\right).\end{aligned}$$

Here, $\xi \in \mathfrak{d}$, $A \in \mathcal{E}(\mathfrak{g})$ and $\alpha \in \mathbb{C}dx \oplus \mathbb{C}dy$.

1. *$\widehat{L}$ defines a left $\widehat{\mathcal{E}}(G)$-action on $\widetilde{\mathcal{E}}(\mathfrak{g})^{\mathfrak{d}}$.*
2. *This left $\widehat{\mathcal{E}}(G)$-action keeps the bilinear form $\langle\cdot|\cdot\rangle$ invariant.*

Remark 4.1 1. Since $\mathbb{T}$ has no boundary, we have

$$\Theta(g_1, g_2) = 0 \quad g_i = exp(A_i \otimes f_i) \in \mathcal{E}(G)\ (i = 1, 2).$$

2. If $\mathrm{Im}\,\dfrac{\omega_x}{\omega_y} < 0$, $\xi = \omega_x\partial_x + \omega_y\partial_y \in \mathfrak{d}$ defines a holomorphic structure on $\mathbb{T}$.

We set

$$\begin{aligned}X &= \{\widetilde{A} = (\xi, A, \alpha) \in \widetilde{\mathcal{E}}(\mathfrak{g})^{\mathfrak{d}} \mid \langle\widetilde{A}|\widetilde{A}\rangle = 0\},\\ X_{\mathcal{H}^\pm} &= \{(\omega_x\partial_x + \omega_y\partial_y, A, \alpha) \in X \mid (\omega_x, \omega_y) \in \mathcal{H}^\pm\},\end{aligned}$$

where $\mathcal{H}^\pm$ is defined in Sect. 2.1. By Lemma 4.1, we have

Corollary 4.2 *The group $\widehat{\mathcal{E}}(G)$ acts on $X_{\mathcal{H}^\pm}$.*

By definition, this action commutes with the natural $\mathbb{C}^*$-action. Hence, $\widehat{\mathcal{E}}(G)$ acts on the projectified space

$$\mathbb{P}(X_{\mathcal{H}^\pm}) := X_{\mathcal{H}^\pm}/\mathbb{C}^*.$$

We remark that $\widetilde{A} = (\omega_x\partial_x + \omega_y\partial_y, A, u_xdx + u_ydy) \in X_{\mathcal{H}^\pm}$ implies the next equalities:

$$\langle A|A\rangle + 2(\omega_xu_x + \omega_yu_y) = 0 \quad \text{i.e.,} \quad u_y = -\frac{1}{\omega_y}\left(\omega_xu_x + \frac{1}{2}\langle A|A\rangle\right).$$

Proposition 4.3 *The map*

$$\psi^{\pm} : \mathbb{P}(X_{\mathcal{H}^{\pm}}) \longrightarrow \mathbb{H} \times \mathcal{E}(\mathfrak{g}) \times \mathbb{C}$$

defined by

$$[\omega_x \partial_x + \omega_y \partial_y, A, u_x dx + u_y dy] \longmapsto \left(\pm\frac{\omega_x}{\omega_y}, \pm\frac{A}{\omega_y}, -\frac{u_x}{\omega_y}\right)$$

is an isomorphism of complex Fréchet manifolds.

We set $\tau = -\dfrac{\omega_x}{\omega_y} \in \mathbb{H}$ and $\overline{\partial} = \tau \partial_x - \partial_y$. We denote the elliptic curve $(\mathbb{T}, \overline{\partial})$ by E_τ. Set

$$\mathcal{C}(\mathfrak{g}) = \mathbb{H} \times \mathcal{E}(\mathfrak{g}), \quad \mathcal{C}(\mathfrak{g})_\tau = \{\tau\} \times \mathcal{E}(\mathfrak{g}).$$

The latter space can be identified with the space of $\overline{\partial}$-connections on C^∞-trivial principal G-bundles over E_τ. By Remark 4.1.2, we consider the map ψ^- in the rest of this section.

Denoting the induced action of $\widehat{\mathcal{E}}(G)$ on $\mathcal{C}(\mathfrak{g}) \times \mathbb{C}$ via ψ^- by the same letter $\widehat{L}$, we obtain the next proposition:

Proposition 4.4 *For $(g, c) \in \widehat{\mathcal{E}}(G)$ and $(\tau, A, u) \in \mathcal{C}(\mathfrak{g}) \times \mathbb{C}$, we have*

$$\begin{aligned}&\widehat{L}_{(g,c)}(\tau, A, u)\\ &\quad = \left(\tau, \mathrm{Ad}(g)(A) - (\overline{\partial} g)g^{-1}, u - c + \langle A | g^{-1}\partial_x g\rangle - \frac{1}{2}\langle(\overline{\partial} g)g^{-1} | (\partial_x g)g^{-1}\rangle\right).\end{aligned}$$

Remark 4.2 The canonical projection $\mathcal{C}(\mathfrak{g}) \times \mathbb{C} \twoheadrightarrow \mathcal{C}(\mathfrak{g})$ induces the left $\mathcal{E}(G)$-action on $\mathcal{C}(\mathfrak{g})$ which we denote by L.

Now, we reconstruct the hyperbolic extension of W_{ell} in terms of the left $\widehat{\mathcal{E}}(\mathfrak{g})$-action on $\mathcal{C}(\mathfrak{g}) \times \mathbb{C}$. For $\alpha \in \Delta_{ell}^{re}$, we define the element $\hat{s}'_\alpha$ of $\widehat{\mathcal{E}}(G)$ by

$$\hat{s}'_\alpha = (\exp(e_\alpha), 0) \cdot (\exp(-e_{-\alpha}), 0) \cdot (\exp(e_\alpha), 0).$$

Lemma 4.5 *For $\alpha \in \Delta_{ell}^{re}$, we have*

1. $\Theta(\exp(e_\alpha), \exp(-e_{-\alpha})) = \Theta(\exp(e_\alpha) \cdot \exp(-e_{-\alpha}), \exp(e_\alpha)) = 0.$
2. $\hat{s}'_\alpha = (s'_\alpha, 0).$

Proof (1) For $\alpha = \alpha_f + m\delta_x + n\delta_y$ $(\alpha_f \in \Delta_f)$,

$$e_\alpha = e_{\alpha_f} \otimes s^m t^n, \quad e_{-\alpha} = e_{-\alpha_f} \otimes s^{-m} t^{-n}.$$

Hence, by Remark 4.1, we obtain the result.

(2) This follows from (1) and the definition of $\hat{s}'_\alpha$.

□

Set

$$\widehat{\mathfrak{h}}'_{\mathbb{H}} = \mathbb{H} \times \mathfrak{h} \times \mathbb{C} \subset \mathcal{C}(\mathfrak{g}) \times \mathbb{C}.$$

By Proposition 4.4 and Lemma 4.5, we obtain

Corollary 4.6 *For* $(\tau, h, u) \in \widehat{\mathfrak{h}}'_{\mathbb{H}}$ *and* $\alpha = \alpha_f + m\delta_x + n\delta_y \in \Delta_{ell}^{re}$ $(\alpha_f \in \Delta_f)$*, we have*

$$\widehat{L}_{\hat{s}'_\alpha}(\tau, h, u) = \left(\tau, h - (\alpha_f(h) + (m\tau - n))\alpha_f^\vee, u - m\left(\langle \alpha_f^\vee | h\rangle + (m\tau - n)\frac{\langle \alpha_f^\vee | \alpha_f^\vee\rangle}{2}\right)\right).$$

In particular, $\hat{s}'_\alpha \in \widehat{\mathcal{E}}(G)$ stabilizes $\widehat{\mathfrak{h}}'_{\mathbb{H}}$. Hence, we set

$$\hat{w}'_\alpha = \hat{L}_{\hat{s}'_\alpha}\Big|_{\widehat{\mathfrak{h}}'_{\mathbb{H}}}.$$

For $\alpha_f, \beta_f \in \Delta_f$, we set

$$(\hat{t}^x_{\alpha_f^\vee})' = \hat{w}'_{\delta_x - \alpha_f} \circ \hat{w}'_{\alpha_f}, \quad (\hat{t}^y_{\beta_f^\vee})' = \hat{w}'_{\delta_y + \beta_f} \circ \hat{w}'_{\beta_f}.$$

Notice that we have

$$(\hat{t}^x_{\alpha_f^\vee})' = \hat{L}_{(s'_{\delta_x - \alpha_f} \cdot s'_{\alpha_f}, 0)}\Big|_{\widehat{\mathfrak{h}}'_{\mathbb{H}}}, \quad (\hat{t}^y_{\beta_f^\vee})' = \hat{L}_{(s'_{\delta_y + \beta_f} \cdot s'_{\beta_f}, 0)}\Big|_{\widehat{\mathfrak{h}}'_{\mathbb{H}}}.$$

From Proposition 4.4 and Corollary 4.6, the next lemma follows:

Lemma 4.7 *For any* $(\tau, h, u) \in \widehat{\mathfrak{h}}'_{\mathbb{H}}$ *and any* $\alpha_f, \beta_f \in \Delta_f$*, one has*

1. $(\hat{t}^x_{\alpha_f^\vee})'(\tau, h, u) = \left(\tau, h + \tau\alpha_f^\vee, u - \left(\langle \alpha_f^\vee | h\rangle + \tau\frac{\langle \alpha_f^\vee | \alpha_f^\vee\rangle}{2}\right)\right)$.
2. $(\hat{t}^y_{\beta_f^\vee})'(\tau, h, u) = (\tau, h + \beta_f^\vee, u)$.
3. $(\hat{t}^x_{\alpha_f^\vee})'(\hat{t}^y_{\beta_f^\vee})'(\tau, h, u) = (\hat{t}^y_{\beta_f^\vee})'(\hat{t}^x_{\alpha_f^\vee})'(\tau, h, u - \langle \alpha_f^\vee | \beta_f^\vee\rangle)$.

Let $\widehat{W}'_{ell}$ be the subgroup of $\mathrm{Aut}(\widehat{\mathfrak{h}}'_{\mathbb{H}})$ generated by $\hat{w}'_\alpha$ $(\alpha \in \Delta_{ell}^{re})$.

By Lemma 2.11, Corollary 2.12 and Lemma 4.7, $\widehat{W}'_{ell}$ is isomorphic to the hyperbolic extension $\widehat{W}_{ell}$ of W_{ell}. Namely, we obtain

Theorem 4.8 $\widehat{W}'_{ell} \cong W_f \ltimes H(Q_f^\vee) \cong \widehat{W}_{ell}$.

4.2 $\mathcal{E}(G)$-Action on $\widetilde{\mathcal{C}}(\mathfrak{g})$

In the previous subsection, we have studied the $\widehat{\mathcal{E}}(G)$-action on $\mathcal{C}(\mathfrak{g}) \times \mathbb{C}$. Here, via the exponential map $\mathbb{C} \to \mathbb{C}^*; z \mapsto e^{2\pi\sqrt{-1}z}$, we will show that the group $\mathcal{E}(G)$ acts on $\widetilde{\mathcal{C}}(\mathfrak{g}) := \mathcal{C}(\mathfrak{g}) \times \mathbb{C}^*$. In addition, we will study the W_{ell}-action on $\widetilde{\mathfrak{h}}_{\mathbb{H}} := \mathbb{H} \times \mathfrak{h} \times \mathbb{C}^* \subset \widetilde{\mathcal{C}}(\mathfrak{g})$ which will play an important role in the invariant theory.

Let θ be the Maurer–Cartan form of G and define the 3-form σ on G as follows:

$$\sigma = \frac{1}{24\pi^2}(\theta \wedge d\theta). \tag{7}$$

Here and after, the invariant form $(\cdot, \cdot)$ is normalized so that the square length of the long root is 2. We normalize the exterior product as in Sect. 5.2.

The next lemma, called the **Polyakov–Wiegmann identity**, follows by direct computation:

Lemma 4.9 ([11]) *Let $f, g : D \times [0,1] \to G$ be a C^∞-map, where $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is a closed disk. One has the next identity:*

$$(f \cdot g)^*\sigma = f^*\sigma + g^*\sigma + \frac{1}{8\pi^2} d(f^{-1} \cdot df \wedge dg \cdot g^{-1}).$$

Let $S\mathbb{T}$ be the solid torus bounded by $\mathbb{T}$ and let $g \in \mathcal{E}(G)$. Regarding g as a loop on the loop group $L(G) := C^\infty(S^1, G)$, it follows that there exists $\overline{g} \in C^\infty(S\mathbb{T}, G)$ whose restriction to $\mathbb{T}$ is g since $\pi_1(L(G)) \cong \pi_1(G) \times \pi_1(\Omega G) \cong 0$. Hence, we fix such element $\overline{g}$ and set

$$\lambda(g) = \int_{S\mathbb{T}} \overline{g}^*\sigma. \tag{8}$$

It follows that $\lambda(g)$ mod $\mathbb{Z}$ does not depend on the choice of $\overline{g}$. In particular, the number $e^{2\pi\sqrt{-1}\lambda(g)}$ is well defined (see, e.g., [6] for detail). By Lemma 4.9, we obtain the next lemma:

Lemma 4.10 *For any $g, g' \in \mathcal{E}(G)$, the next identity holds:*

$$\lambda(g \cdot g') \equiv \lambda(g) + \lambda(g') + \Theta(g, g') \qquad \mathrm{mod}\ \mathbb{Z}.$$

We set

$$\widetilde{\mathcal{E}}(G) = \mathcal{E}(G) \times \mathbb{C}^*,$$

and define the structure of the group as follows:

$$(g, u) \cdot (g', v) = (gg', uve^{-2\pi\sqrt{-1}\Theta(g,g')}).$$

By Lemma 4.10, we have

Corollary 4.11 *The central extension*

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{\mathcal{E}}(G) \longrightarrow \mathcal{E}(G) \longrightarrow 1,$$

splits. Indeed, $\mathcal{E}(G) \hookrightarrow \widetilde{\mathcal{E}}(G)$; $g \mapsto (g, e^{-2\pi\sqrt{-1}\lambda(g)})$ is a section of this short exact sequence.

Thus, by setting

$$\widetilde{\mathcal{C}}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g}) \times \mathbb{C}^*, \quad \widetilde{\mathcal{C}}(\mathfrak{g})_\tau = \mathcal{C}(\mathfrak{g})_\tau \times \mathbb{C}^*,$$

the $\widehat{\mathcal{E}}(G)$-action given in Proposition 4.4 lifts to an $\widetilde{\mathcal{E}}(G)$-action on $\widetilde{\mathcal{C}}(\mathfrak{g})$. In particular, by the splitting $\mathcal{E}(G) \hookrightarrow \widetilde{\mathcal{E}}(G)$ as above, it induces naturally the left action $\widetilde{L}$ of $\mathcal{E}(G)$ on $\widetilde{\mathcal{C}}(\mathfrak{g})$. This can be explicitly given as follows:

Proposition 4.12 *For $g \in \mathcal{E}(G)$ and $(\tau, A, u) \in \widetilde{\mathcal{C}}(\mathfrak{g})$, we have*

$$\widetilde{L}_g(\tau, A, u) = \left(\tau, \mathrm{Ad}(g)(A) - \overline{\partial} g \cdot g^{-1}, u \cdot e^{2\pi\sqrt{-1}\left\{\langle A | g^{-1}\partial_x g\rangle - \frac{1}{2}\langle \overline{\partial} g \cdot g^{-1} | \partial_x g \cdot g^{-1}\rangle\right\}} \cdot e^{-2\pi\sqrt{-1}\lambda(g)}\right).$$

Remark 4.3 1. By Remark 4.2, the canonical projection $\widetilde{\mathcal{C}}(\mathfrak{g}) \twoheadrightarrow \mathcal{C}(\mathfrak{g})$ is $\mathcal{E}(G)$-equivariant.
2. For $\tau \in \mathbb{H}$, the embedding $\mathcal{C}(\mathfrak{g})_\tau \hookrightarrow \mathcal{E}(\mathfrak{g})^{\mathfrak{d}}$; $(\tau, A) \mapsto \overline{\partial} + A$ is $\mathcal{E}(G)$-equivariant.

Now, we study the action of W_{ell} on

$$\widetilde{\mathfrak{h}}_{\mathbb{H}} := \mathbb{H} \times \mathfrak{h} \times \mathbb{C}^* \subset \widetilde{\mathcal{C}}(\mathfrak{g}).$$

For $\alpha \in \Delta_{ell}^{re}$, we define the element $\tilde{s}_\alpha$ of $\widetilde{\mathcal{E}}(G)$ by

$$\begin{aligned}\tilde{s}_\alpha = &(\exp(e_\alpha), e^{-2\pi\sqrt{-1}\lambda(\exp(e_\alpha))}) \cdot (\exp(-e_{-\alpha}), e^{-2\pi\sqrt{-1}\lambda(\exp(-e_{-\alpha}))})\\ &\cdot (\exp(e_\alpha), e^{-2\pi\sqrt{-1}\lambda(\exp(e_\alpha))}).\end{aligned}$$

By Corollary 4.11, we have

$$\tilde{s}_\alpha = (s'_\alpha, e^{-2\pi\sqrt{-1}\lambda(s'_\alpha)}).$$

Lemma 4.13 *For any $\alpha \in \Delta_{ell}^{re}$, one has*

$$\lambda(s'_\alpha) = 0.$$

Proof By Lemmas 4.5 and 4.10, we have

$$\lambda(s'_\alpha) = \lambda(\exp(e_\alpha)) + \lambda(\exp(-e_{-\alpha})) + \lambda(\exp(e_\alpha)).$$

Suppose that $\alpha = \alpha_f + m\delta_x + n\delta_y$ with $\alpha_f \in \Delta_f$. We set $g(x, y) = \exp(e_\alpha)$ and $\overline{g}(x, y, r) = \exp(r^{|m+n|+2} e_\alpha)$ $(0 \le r \le 1)$. Then $\overline{g}$ is a extension of g to a C^∞-function on $S\mathbb{T}$. By the Maurer-Cartan equation

$$d\theta = -\frac{1}{2}[\theta, \theta],$$

we have $\overline{g}^*\sigma = 0$. □

By this lemma, we have

$$\tilde{s}_\alpha = (s'_\alpha, 1) \quad (\alpha \in \Delta_{ell}^{re}).$$

Hence, by Proposition 4.12, the next lemma follows:

Lemma 4.14 *For $\alpha = \alpha_f + m\delta_x + n\delta_y$ $(\alpha_f \in \Delta_f)$ and $(\tau, h, u) \in \widetilde{\mathfrak{h}}_{\mathbb{H}}$, one has*

$$\begin{aligned}&\widetilde{L}_{s'_\alpha}(\tau, h, u)\\ &= \left(\tau, h - (\alpha_f(h) + (m\tau - n))\alpha_f^\vee, u \cdot e^{-2\pi\sqrt{-1}m\left(\langle\alpha_f^\vee|h\rangle + (m\tau - n)\frac{\langle\alpha_f^\vee|\alpha_f^\vee\rangle}{2}\right)}\right).\end{aligned}$$

This lemma implies that s'_α stabilizes $\widetilde{\mathfrak{h}}_{\mathbb{H}}$. We set

$$\tilde{w}_\alpha = \widetilde{L}_{s'_\alpha}\big|_{\widetilde{\mathfrak{h}}_{\mathbb{H}}},$$

and

$$\tilde{t}^x_{\alpha_f^\vee} = \tilde{w}_{\delta_x - \alpha_f} \cdot \tilde{w}_{\alpha_f}, \quad \tilde{t}^y_{\beta_f^\vee} = \tilde{w}_{\delta_y + \beta_f} \cdot \tilde{w}_{\beta_f},$$

for $\alpha \in \Delta_{ell}^{re}$ and $\alpha_f, \beta_f \in \Delta_f$. We denote by $\widetilde{W}_{ell}$ the subgroup of $\mathrm{Aut}(\widetilde{\mathfrak{h}}_{\mathbb{H}})$ generated by $\tilde{w}_\alpha$ $(\alpha \in \Delta_{ell}^{re})$. Since $\langle\alpha_f^\vee|\beta_f^\vee\rangle = (\alpha_f^\vee, \beta_f^\vee)$ is an integer (cf. Remark 2.8), by Lemma 4.7, it follows that $\widetilde{W}_{ell}$ is isomorphic to W_{ell}. Namely, we obtain the next result:

Theorem 4.15 $\widetilde{W}_{ell} \cong W_f \ltimes (Q_f^\vee \times Q_f^\vee) \cong W_{ell}$.

Remark 4.4 It can be shown that the $\mathcal{E}(G)$-orbit of $\{\tau\} \times \mathfrak{h} \times \mathbb{C}^* \subset \widetilde{\mathcal{C}}(\mathfrak{g})_\tau$ is dense and

$$\mathcal{O}_{\mathcal{E}(G)}(\tau, h) \cap (\{\tau\} \times \mathfrak{h}) = \mathcal{O}_{W_{ell}}(\tau, h) \text{ for any } h \in \mathfrak{h},$$

where $\mathcal{O}_{\mathcal{G}}(X)$ $(\mathcal{G} = \mathcal{E}(G), W_{ell})$ signifies the $\mathcal{G}$-orbit of X. Thus, the latter can be seen as an analogue of the **Chevalley restriction theorem**. The proof of these statements require some arguments on principal G-bundles over E_τ and will be discussed in our future publication.

4.3 Invariant theory of W_{ell}

In this subsection, we discuss on the structure of W_{ell}-invariants holomorphic functions on $\widetilde{\mathfrak{h}}_{\mathbb{H}} = \mathbb{H} \times \mathfrak{h} \times \mathbb{C}^*$. For simplicity, we denote the ring of holomorphic functions on $\mathbb{H}$ and $\widetilde{\mathfrak{h}}_{\mathbb{H}}$ by $\mathcal{O}_{\mathbb{H}}$ and $\mathcal{O}_{\widetilde{\mathfrak{h}}_{\mathbb{H}}}$, respectively.

For $\alpha_f^\vee \in Q_f^\vee$, we define $\tilde{t}^x_{\alpha_f^\vee}, \tilde{t}^y_{\alpha_f^\vee}$, as in Remark 2.7, as follows:

$$\tilde{t}^x_{\alpha_f^\vee}(\tau, h, u) = \left(\tau, h + \tau\alpha_f^\vee, u e^{-2\pi\sqrt{-1}\left(\langle \alpha_f^\vee | h\rangle + \tau \frac{\langle \alpha_f^\vee | \alpha_f^\vee\rangle}{2}\right)}\right),$$

$$\tilde{t}^y_{\alpha_f^\vee}(\tau, h, u) = (\tau, h + \alpha_f^\vee, u).$$

It turns out that, as in Corollary 2.12, this is compatible with the action given by Lemma 4.14 for $\alpha_f \in \Delta_f$. Moreover, for $\alpha_f^\vee, \beta_f^\vee \in Q_f^\vee$, one has

$$\tilde{t}^x_{\alpha_f^\vee}\tilde{t}^x_{\beta_f^\vee} = \tilde{t}^x_{\alpha_f^\vee+\beta_f^\vee}, \quad \tilde{t}^y_{\alpha_f^\vee}\tilde{t}^y_{\beta_f^\vee} = \tilde{t}^y_{\alpha_f^\vee+\beta_f^\vee}.$$

Hence, $\tilde{t}^x_{\alpha_f^\vee}, \tilde{t}^y_{\alpha_f^\vee}$ are elements of W_{ell}, viewed as the subgroup of $\mathrm{Aut}(\widetilde{\mathfrak{h}}_{\mathbb{H}})$ generated by $\{\tilde{w}_\alpha\}_{\alpha \in \Delta^{re}_{ell}}$. We denote the subgroup of W_{ell} generated by $\{\tilde{t}^x_{\alpha_f^\vee}\}_{\alpha_f^\vee \in Q_f^\vee}$, $\{\tilde{t}^y_{\alpha_f^\vee}\}_{\alpha_f^\vee \in Q_f^\vee}$ and $\{\tilde{t}^x_{\alpha_f^\vee}, \tilde{t}^y_{\alpha_f^\vee}\}_{\alpha_f^\vee \in Q_f^\vee}$ by T_x, T_y and T, respectively.

Let $P_f \subset \mathfrak{h}^*$ be the weight lattice of $\mathfrak{g}$, i.e., the dual lattice of $Q_f^\vee$. A T_y-invariant function on $\widetilde{\mathfrak{h}}_{\mathbb{H}}$ is nothing but the function on τ, u and $e^{2\pi\sqrt{-1}\lambda}$ $(\lambda \in P_f)$, where $e^{2\pi\sqrt{-1}\lambda}$ is the function on $\mathfrak{h}$ defined by $e^{2\pi\sqrt{-1}\lambda} : h \mapsto e^{2\pi\sqrt{-1}\lambda(h)}$. For $K \in \mathbb{Z}$, we set

$$f_{\lambda,K}(\tau, h, u) = u^{-K} e^{2\pi\sqrt{-1}\lambda(h)}.$$

By computing $\sum_{\alpha_f^\vee \in Q_f^\vee} f_{\lambda,K}(\tilde{t}^x_{\alpha_f^\vee}(\tau, h, u))$, we see that the function

$$\theta_{\lambda,K}(\tau, h, u) := u^{-K} \sum_{\gamma \in \nu_f(Q_f^\vee)} q^{\frac{1}{2K}||\lambda + K\gamma||^2} e^{2\pi\sqrt{-1}(\lambda+K\gamma)(h)}$$

on $\widetilde{\mathfrak{h}}_{\mathbb{H}}$ where $q = e^{2\pi\sqrt{-1}\tau} \in \mathbb{C}^*$ and the linear map $\nu_f : \mathfrak{h} \to \mathfrak{h}^*$ is defined in Sect. 2.3, is T-invariant and is holomorphic on $\widetilde{\mathfrak{h}}_{\mathbb{H}}$ for $K \in \mathbb{Z}_{>0}$.

Fix a set of simple roots Π_f and let Δ_f^+ be the set of positive roots of Δ_f with respect to Π_f. We denote the highest root of Δ_f^+ by θ_f and the half sum of roots in Δ_f^+ by ρ_f. Let P_f^+ be the set of dominant weights with respect to Π_f, i.e., the set $\{\lambda \in P_f | \lambda(\alpha_f^\vee) \geq 0\ (\forall \alpha_f \in \Pi_f)\}$. We remark that $\rho_f \in P_f^+$. The number $h^\vee := 1 + \rho_f(\theta_f^\vee)$, called the **dual Coxeter number**, plays an important role.

For $K \in \mathbb{Z}_{\geq 0}$, we set

$$P_K^+ = \{(\lambda, K) | \lambda \in P_f^+, \ \lambda(\theta_f^\vee) \leq K\}, \quad P_{af}^+ := \bigcup_{K \in \mathbb{Z}_{\geq 0}} P_K^+.$$

It is clear that $P_0^+ = \{(0, 0)\}$. Now for $(\lambda, K) \in P_{af}^+$, we set

$$\chi_{\lambda,K}(\tau, h, u) = \frac{\sum_{w \in W_f} \varepsilon(w)\theta_{w(\lambda+\rho_f),K+h^\vee}(\tau, h, u)}{\sum_{w \in W_f} \varepsilon(w)\theta_{w(\rho_f),h^\vee}(\tau, h, u)},$$

where $\varepsilon(w) = \det_{\mathfrak{h}}(w)$ stands for the signature of $w \in W_f$. It is well known that $\chi_\lambda \in \mathcal{O}_{\widetilde{\mathfrak{h}}_{\mathbb{H}}}^{W_{ell}}$ is the character of the integrable highest weight modules over the affine Lie algebra associated with $\mathfrak{g}$ (cf. [3,4]). Indeed, the next stronger statement is known:

Proposition 4.16 (cf. [1]) *$\mathcal{O}_{\widetilde{\mathfrak{h}}_{\mathbb{H}}}^{W_{ell}} = \bigoplus_{(\lambda,K) \in P_{af}^+} \mathcal{O}_{\mathbb{H}}\chi_{\lambda,K}$ as $\mathcal{O}_{\mathbb{H}}$-module.*

As an $\mathcal{O}_{\mathbb{H}}$-algebra, the following description of $\mathcal{O}_{\widetilde{\mathfrak{h}}_{\mathbb{H}}}^{W_{ell}}$ is known.

Let $\{\Lambda_{\alpha_f}\}_{\alpha_f \in \Pi_f} \subset \mathfrak{h}^*$ be the dual basis of $\Pi_f^\vee := \{\alpha_f^\vee\}_{\alpha_f \in \Pi_f} \subset \mathfrak{h}$, i.e., $\Lambda_{\alpha_f}(\beta_f^\vee) = \delta_{\alpha_f,\beta_f}$. Set $a_0^\vee = 1$ and $a_{\alpha_f}^\vee = \Lambda_{\alpha_f}(\theta_f^\vee)$. One has

Theorem 4.17 (cf. [1,20]) *For each $\alpha_f \in \Pi_f$, there exists $\chi_{\alpha_f} \in \bigoplus_{(\lambda,a_{\alpha_f}^\vee) \in P_{af}^+} \mathcal{O}_{\mathbb{H}}\chi_{\lambda,a_{\alpha_f}^\vee}$ such that*

1. *$\chi_{0,1}$ and χ_{α_f} $(\alpha_f \in \Pi_f)$ are algebraically independent, and*
2. *$\mathcal{O}_{\widetilde{\mathfrak{h}}_{\mathbb{H}}}^{W_{ell}}$ is generated by $\chi_{0,1}$ and χ_{α_f} $(\alpha_f \in \Pi_f)$ as $\mathcal{O}_{\mathbb{H}}$-algebra.*

Remark 4.5 The above theorem together with Remark 4.4 implies that there might exist $\mathcal{E}(G)$-invariant holomorphic functions on $\widetilde{\mathcal{C}}(\mathfrak{g})_\tau$ that provide us an analogue of the adjoint quotient map $\widetilde{\mathcal{C}}(\mathfrak{g})_\tau \twoheadrightarrow \mathfrak{h} \oplus \mathbb{C}$. We will discuss on the existence of such a map with its application in our future publication.

In 1984, Kac and Peterson [4] gave formulas of the Jacobian of the fundamental characters $\chi_{0,1}, \chi_{\Lambda_{\alpha_f},a_{\alpha_f}^\vee}$ $(\alpha_f \in \Pi_f)$ for type $A_l^{(1)}, B_l^{(1)}, C_l^{(1)}, D_l^{(1)}, G_2^{(1)}$ and for some twisted cases, *without proof*. Their formulas suggest

Conjecture 4.18 *$\mathcal{O}_{\widetilde{\mathfrak{h}}_{\mathbb{H}}}^{W_{ell}} = \mathcal{O}_{\mathbb{H}}[\chi_{0,1}, \chi_{\Lambda_{\alpha_f},a_{\alpha_f}^\vee} \ (\alpha_f \in \Pi_f)]$ as $\mathcal{O}_{\mathbb{H}}$-algebra. In particular, $\chi_{0,1}$ and $\chi_{\Lambda_{\alpha_f},a_{\alpha_f}^\vee}$ $(\alpha_f \in \Pi_f)$ are algebraically independent.*

5 $\mathrm{SL}_2(\mathbb{Z})$-Action on $\mathbb{H} \times \mathcal{E}(\mathfrak{g}) \times \mathbb{C}$

In the previous subsection, we have studied an $\mathcal{E}(G)$-action on $\widetilde{\mathcal{C}}(\mathfrak{g})$ with the aid of Proposition 4.3 via the isomorphism ψ^-. Here, via the isomorphism ψ^+ in the same proposition, we will study an $SL(2, \mathbb{Z})$-action on $\widetilde{\mathcal{C}}(\mathfrak{g}) = \mathbb{H} \times \mathcal{E}(\mathfrak{g}) \times \mathbb{C}$.

A natural left $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{R}^2$ is denoted by φ, i.e., for $\gamma \in SL(2, \mathbb{Z})$, we set

$$\varphi_\gamma : \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \gamma \begin{pmatrix} x \\ y \end{pmatrix}.$$

The next lemma is easy to check:

Lemma 5.1 *For $\gamma \in SL(2,\mathbb{Z})$ and $(\xi, A, \alpha) \in \widetilde{\mathcal{E}}(\mathfrak{g})^{\mathfrak{d}}$, we set*

$$\gamma.(\xi, A, \alpha) := (\varphi_{\gamma *}(\xi), (\varphi_{\gamma^{-1}})^*(A), (\varphi_{\gamma^{-1}})^*(\alpha)).$$

1. *This is a left action and keeps the bilinear form $\langle\cdot|\cdot\rangle$ invariant.*
2. *For $\gamma \in SL(2,\mathbb{Z})$ and $(g, c) \in \widehat{\mathcal{E}}(G)$, the next identity holds:*

$$\gamma \circ \hat{L}_{(g,c)} \circ \gamma^{-1} = \hat{L}_{((\varphi_{\gamma^{-1}})^* g, c)}.$$

The induced action of $SL(2,\mathbb{Z})$ on Im ψ^+:

Proposition 5.2 *For $(\tau, A, u) \in \widetilde{\mathcal{C}}(\mathfrak{g})$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, we have*

$$\gamma.(\tau, A, u) = \left(\frac{a\tau + b}{c\tau + d}, \frac{(\varphi_{\gamma^{-1}})^* A}{c\tau + d}, u - \frac{c\langle A|A\rangle}{2(c\tau + d)} \right).$$

This action coincides exactly with the $SL(2,\mathbb{Z})$-action of the $\mathbb{C}$-span of the characters of integrable modules over an affine Lie algebra given in [4].

Appendix 1: Lifting of a Weyl group

In Sect. 3, we have calculated a lifting of a reflection

$$s'_\alpha = \exp(e_\alpha)\exp(-e_{-\alpha})\exp(e_\alpha) \in \mathcal{E}(G)$$

and its action on $\mathfrak{h}^\delta$. Here, for the sake of leader's convenience, we will show how one can calculate its action on $\mathcal{E}(\mathfrak{g})^{\mathfrak{d}}$.

5.1 On $\mathfrak{h}^{\mathfrak{d}}$

For $\alpha \in \Delta_{ell}^{re}$ and $t \in \mathbb{C}^*$, set

$$\sigma_\alpha(t) = \exp(te_\alpha)\exp(-t^{-1}e_{-\alpha})\exp(te_\alpha).$$

Let us compute rapidly $L_{\sigma_\alpha(t)}(\xi, h)$ for $(\xi, h) \in \mathfrak{h}^{\mathfrak{d}}$. By definition, we have

$$\begin{aligned} L_{\sigma_\alpha(t)}(\xi, h) &= L_{\exp(te_\alpha)} \circ L_{\exp(-t^{-1}e_{-\alpha})} \circ L_{\exp(te_\alpha)}(\xi, h) \\ &= L_{\exp(te_\alpha)} \circ L_{\exp(-t^{-1}e_{-\alpha})}(\xi, h - t\alpha(\xi, h)e_\alpha) \\ &= L_{\exp(te_\alpha)}(\xi, h - \alpha(\xi, h)\alpha^\vee - t\alpha(\xi, h)e_\alpha) \\ &= (\xi, h - \alpha(\xi, h)\alpha^\vee) = w'_\alpha(\xi, h). \end{aligned}$$

In particular, this action does not depend upon the choice of $t \in \mathbb{C}^*$.

5.2 On $\mathcal{E}(\mathfrak{g})^{\mathfrak{d}}$

Let $\alpha \in \Delta_{ell}^{re}$ and $\beta \in \Delta_{ell}$ such that $\beta \neq \pm\alpha$. Let $p, q \in \mathbb{Z}_{\geq 0}$ be integers satisfying

$$(\beta + \mathbb{Z}\alpha) \cap \Delta = \{\beta + i\alpha | -q \leq i \leq p\} : \ \alpha\text{-string through}\beta.$$

Since $\{e_\alpha, e_{-\alpha}, \alpha^\vee\}$ forms an $\mathfrak{sl}_2$-triplet, i.e., $[\alpha^\vee, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$ and $[e_\alpha, e_{-\alpha}] = \alpha^\vee$, setting $e_{\beta+i\alpha} = \dfrac{1}{(p-i)!}(\mathrm{ad}e_{-\alpha})^{p-i}(e_{\beta+p\alpha})$ for $-q \le i \le p$, one can check

$$\begin{cases} \left[\alpha^\vee, e_{\beta+i\alpha}\right] = (2i - p + q)e_{\beta+i\alpha}, \\ \left[e_\alpha, e_{\beta+i\alpha}\right] = (i + q + 1)e_{\beta+(i+1)\alpha}, \\ \left[e_{-\alpha}, e_{\beta+i\alpha}\right] = (p + 1 - i)e_{\beta+(i-1)\alpha}, \end{cases}$$

where we set $e_{\beta-(q+1)\alpha} = 0 = e_{\beta+(p+1)\alpha}$ for simplicity. Let us compute $L_{\sigma_\alpha(t)}(\xi, e_\beta)$. Since we have

$$L_{\sigma_\alpha(t)}(\xi, e_\beta) = L_{\sigma_\alpha(t)}(\xi, 0) + L_{\sigma_\alpha(t)}(0, e_\beta)$$

by definition, and the first term can be computed using the result of the last subsection, we may assume that $\xi = 0$. Let us calculate step by step. First, we have

$$\begin{aligned} &L_{\exp(te_\alpha)}(0, e_\beta) \\ &\quad= \left(0, \sum_{i=0}^{p} \frac{1}{i!} t^i (\mathrm{ad}e_\alpha)^i (e_\beta)\right) = \left(0, \sum_{i=0}^{p} \frac{(q+1)(q+2)\cdots(q+i+1)}{i!} t^i e_{\beta+i\alpha}\right) \\ &\quad= \left(0, \sum_{i=0}^{p} (-1)^i \binom{-q-1}{i} t^i e_{\beta+i\alpha}\right). \end{aligned}$$

Second, we have

$$\begin{aligned} &L_{\exp(-t^{-1}e_{-\alpha})}\left(0, \sum_{i=0}^{p} \binom{-q-1}{i} (-t)^i e_{\beta+i\alpha}\right) \\ &\quad= \left(0, \sum_{i=0}^{p} \binom{-q-1}{i} (-t)^i \sum_{j\ge 0} \frac{1}{j!} (-t)^{-j} (\mathrm{ad}e_{-\alpha})^j e_{\beta+i\alpha}\right) \\ &\quad= \left(0, \sum_{i=0}^{p} \binom{-q-1}{i} \sum_{j\ge 0} \binom{p+j-i}{j} (-t)^{i-j} e_{\beta+(i-j)\alpha}\right). \end{aligned}$$

Here, the sum is taken over

$$\begin{aligned} &\{(i, j) \in \mathbb{Z}^2_{\ge 0} \mid 0 \le i \le p,\ 0 \le j \le q + i\} \\ &\quad= \{(i, j) \in \mathbb{Z}^2_{\ge 0} \mid j \le i \le p\} \cup \{(i, j) \in \mathbb{Z}^2_{\ge 0} \mid 0 \le i \le p,\ 0 < j - i \le q\}. \end{aligned}$$

We consider the change of variables in each domain of the right-hand side as follows:

$\underline{m = i - j \text{ and } n = i}$ on the first and $\underline{m = j - i \text{ and } n = i}$ on the second. We have

$$\begin{aligned}
&= \left(0, \sum_{m=0}^{p} \left(\sum_{n=m}^{p} \binom{p-m}{p-n} \binom{-q-1}{n} \right) (-t)^m e_{\beta+m\alpha} \right. \\
&\quad \left. + \sum_{m=1}^{q} \left(\sum_{n=0}^{p} \binom{-q-1}{n} \binom{p+m}{p-n} \right) (-t)^{-m} e_{\beta-m\alpha} \right) \\
&= \left(0, \sum_{m=-q}^{p} \binom{p-q-1-m}{p} (-t)^m e_{\beta+m\alpha} \right) \\
&= \left(0, \sum_{m=p-q}^{p} \binom{p-q-1-m}{p} (-t)^m e_{\beta+m\alpha} \right) \\
&= \left(0, \sum_{m=0}^{q} \binom{-m-1}{p} (-t)^{p-q+m} e_{\beta+(p-q+m)\alpha} \right).
\end{aligned}$$

Here, in the second equality, we used the formula

$$\sum_{\substack{i,j \geq 0 \\ i+j=N}} \binom{\alpha}{i} \binom{\beta}{j} = \binom{\alpha+\beta}{N}$$

which can be proved by looking at the coefficient of t^N in $(1+t)^\alpha \cdot (1+t)^\beta = (1+t)^{\alpha+\beta}$. Hence, we obtain

$$L_{\exp(-t^{-1}e_{-\alpha})} \circ L_{\exp(te_\alpha)}(0, e_\beta) = \left(0, (-1)^p \sum_{m=0}^{q} \binom{p+m}{p} (-t)^{p-q+m} e_{\beta+(p-q+m)\alpha} \right).$$

Finally, we compute $L_{\sigma_\alpha(t)}(0, e_\beta) = L_{\exp(te_\alpha)} \circ L_{\exp(-t^{-1}e_{-\alpha})} \circ L_{\exp(te_\alpha)}(0, e_\beta)$. We have

$$\begin{aligned}
&L_{\sigma_\alpha(t)}(0, e_\beta) \\
&\quad = \left(0, (-1)^p \sum_{m=0}^{q} \binom{p+m}{p} (-t)^{p-q+m} \sum_{n=0}^{q-m} \frac{1}{n!} t^n (\mathrm{ad} e_\alpha)^n e_{\beta+(p-q+m)\alpha} \right) \\
&\quad = \left(0, (-1)^p \sum_{m=0}^{q} \binom{p+m}{p} (-t)^{p-q+m} \sum_{n=0}^{q-m} \binom{p+m+n}{n} t^n e_{\beta+(p-q+m+n)\alpha} \right).
\end{aligned}$$

Now, we compute the summation

$$\sum_{m=0}^{q} \binom{p+m}{p} (-t)^{p-q+m} \sum_{n=0}^{q-m} \binom{p+m+n}{n} t^n.$$

By the change of variable $k = m + n$, we obtain

$$\begin{aligned}
&\sum_{m=0}^{q} \binom{p+m}{p} (-t)^{p-q+m} \sum_{n=0}^{q-m} \binom{p+m+n}{n} t^n \\
&= (-t)^{p-q} \sum_{0 \le m \le k \le q} (-1)^m \binom{p+m}{m} \binom{p+k}{k-m} t^k \\
&= (-t)^{p-q} \sum_{k=0}^{q} \sum_{m=0}^{k} \binom{-p-1}{m} \binom{p+k}{k-m} t^k \\
&= (-t)^{p-q}.
\end{aligned}$$

Hence, the only non-trivial contribution in the above summation comes from the term corresponding to $(m, n) = (0, 0)$. Thus, we obtain

$$L_{\sigma_\alpha(t)}(0, e_\beta) = (0, (-1)^q t^{p-q} e_{\beta+(p-q)\alpha}) = (0, (-1)^q t^{-\beta(\alpha^\vee)} e_{w_\alpha(\beta)}).$$

In particular, we have

$$L_{s'_{\delta_\sharp - \alpha_f}} \circ L_{s'_{\alpha_f}}(0, e_\beta) = L_{s'_{\delta_\sharp - \alpha_f}}(0, (-1)^q e_{\beta - \beta(\alpha_f^\vee)\alpha_f}) = (0, e_{\beta - \beta(\alpha_f^\vee)\delta_\sharp})$$

for $\alpha = \alpha_f \in \Delta_f$ and $\sharp \in \{x, y\}$, and for any $\alpha \in \Delta_{ell}^{re}$

$$L_{\sigma_\alpha(t)^2}(0, e_\beta) = L_{\sigma_\alpha(t)}(0, (-1)^q t^{-\beta(\alpha^\vee)} e_{w_\alpha(\beta)}) = (0, (-1)^{p+q} e_\beta),$$

i.e.,

$$\sigma_\alpha(t)^2 = (-1)^{\alpha^\vee}. \tag{9}$$

Appendix 2: Normalization on exterior product

Here, for a differentiable manifold M, we fix a normalization on differential p-form on M. For $\omega_1, \ldots, \omega_r \in \Omega_M^1$ and $X_1, \ldots, X_r \in \Theta_M$, we set

$$(\omega_1 \wedge \cdots \wedge \omega_r)(X_1, \ldots, X_r) := \det(\omega_i(X_j))_{1 \le i,j \le r}.$$

Hence, for $\omega \in \Omega_M^p$, $\omega' \in \Omega_M^q$ and $X_1, \ldots, X_{p+q} \in \Theta_M$, the exterior product $\omega \wedge \omega'$ is defined by

$$\begin{aligned}
&(\omega \wedge \omega')(X_1, \ldots, X_{p+q}) \\
:=& \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (\mathrm{sgn}\sigma) \omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \omega'(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).
\end{aligned}$$

With this normalization, it is natural to define the de Rham differential d as follows: for $\omega \in \Omega_M^r$ and $X_0, \ldots, X_r \in \Theta_M$,

$$\begin{aligned}
(d\omega)(X_0, \ldots, X_r) =& \sum_{i=0}^{r} (-1)^i X_i(\omega(X_0, \ldots, \overset{\vee}{X_i}, \ldots, X_r) \\
&+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots, \overset{\vee}{X_i}, \ldots, \overset{\vee}{X_j}, \ldots, X_r).
\end{aligned}$$

References

1. Bernstein, J., Schwarzman, O.: Chevalley's theorem for the complex crystallographic groups. J. Nonlinear Math. Phys. **13**, 323–351 (2006)
2. Helmke, S., Slodowy, P.: Loop groups, elliptic singularities and principal bundles over elliptic curves, Banach Center Publ. 62, Polish Acad. Sci. Warsaw, pp. 87–99 (2004)
3. Kac, V.G.: Infinite Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
4. Kac, V., Peterson, D.: Infinite-dimensional Lie algebras, theta-functions and modular forms. Adv. Math. **53**, 125–264 (1984)
5. Kassel, C., Loday, J.-L.: Extension centeales d'algèbres de Lie. Ann. Inst. Fourier **32**, 119–142 (1982)
6. Mickelsson, J.: Current Algebras and Groups. Plenum Press, New York, Plenum Mongr. Nonlinear Phys. (1989)
7. Moody, R., Shi, Z.: Toroidal Weyl groups. Nova J. Algebra Geom. **1**, 314–337 (1992)
8. Neeb, K.-H.: Towards a Lie theory of locally convex groups. Jpn. J. Math. **1**, 291–468 (2006)
9. Neher, E.: Extended affine Lie algebras and other generalizations of the affine Lie algebras—a survey. In: Developments and Trends in Infinite-Dimensional Lie Theory, pp. 53–126, Progr. Math. **288**. Birkhäuser Boston (2011)
10. Omori, H.: Infinite-Dimensional Lie Groups, Transl. Math. Monographs **158**. Am. Math. Soc. (1997)
11. Polyakov, A.M., Wiegmann, P.B.: Goldstone fields in two dimensions with multivalued actions. Phys. Lett. B **151**, 223–228 (1984)
12. Saito, K.: Einfach Elliptische Singularitäten. Invent. Math. **23**, 289–325 (1974)
13. Saito, K.: Period mapping associated to a primitive form. Publ. RIMS Kyoto Univ. **19**, 1231–1264 (1983)
14. Saito, K.: Extended affine root systems I (coxeter transformations). Publ. RIMS Kyoto Univ. **21**, 75–179 (1985)
15. Saito, K., Takebayashi, T.: Extended affine root systems III (elliptic Weyl group). Publ. RIMS Kyoto Univ. **33**, 301–329 (1997)
16. Saito, K., Yoshii, D.: Extended affine root system IV (simply-laced elliptic Lie algebras). Publ. RIMS Kyoto Univ. **36**, 385–421 (2000)
17. Slodowy, P.: Four Lectures on Simple Groups and Singularities. Communications of the Mathematical Institute Rijksuniversiteit, Utrecht (1980)
18. Slodowy, P.: Beyond Kac-Moody algebras, and inside, Lie algebras and related topics (Windsor, Ont., 1984), 361–371, CMS Conf. Proc. **5**, Am. Math. Soc., Providence, RI (1986)
19. Wakimoto M.: Extended affine Lie algebras and a certain series of Hermitian representations, preprint (1985)
20. Wirthmüller, K.: Root systems and Jacobi forms. Compos. Math. **82**, 293–354 (1992)
21. Yamada, H.K.: Extended affine Lie algebras and their vertex representations. Publ. RIMS **25**, 587–603 (1989)
22. Yamada, H.: Lie group theoretical construction of period mapping. Math. Z. **220**, 231–255 (1995)