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Time periodic traveling curved fronts of bistable reaction–diffusion equations in $\mathbb{R}^3$

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Abstract This paper is concerned with the existence and stability of time periodic traveling curved fronts for reaction–diffusion equations with bistable nonlinearity in $\mathbb{R}^3$. We first study the existence and other qualitative properties of time periodic traveling fronts of polyhedral shape. Furthermore, for any given $g \in C^\infty(S^1)$ with $\min_{0\le\theta\le 2\pi} g(\theta) = 0$ that gives a convex bounded domain with smooth boundary of positive curvature everywhere, which is included in a sequence of convex polygons, we show that there exists a three-dimensional time periodic traveling front by taking the limit of the solutions corresponding to the convex polyhedrons as the number of the lateral surfaces goes to infinity.

Keywords Time periodic · Traveling fronts · Reaction–diffusion equations · Bistable

Mathematics Subject Classification 35C07 · 35K57 · 35B10

1 Introduction

Traveling fronts have been extensively studied since the last decade, one can refer to [2,3,12,40] and references therein for the study of planar traveling fronts to the following autonomous reaction–diffusion equation

$$\frac{\partial u(\mathbf{y}, t)}{\partial t} = \Delta u(\mathbf{y}, t) + f(u(\mathbf{y}, t)), \quad \mathbf{y} \in \mathbb{R}^N, \quad t > 0 \tag{1.1}$$

in one or multidimensional space. Recently, the study on nonplanar traveling fronts among mathematicians has attracted an increasing attention and many new types of nonplanar traveling fronts have been observed of (1.1). For instance, Brazhnik and Tyson [6], Hamel and

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Nadirashsili [15], Hamel and Roquejoffre [16], El Smaily et al. [11] considered nonplanar traveling fronts of (1.1) with monostable nonlinearity for $N \geq 2$ (i.e., $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$); Bonnet and Hamel [5], Hamel et al. [17] examined the V-shaped traveling front of (1.1) with the combustion nonlinearity (i.e., $f(s) = 0$ for $s \in [0, \theta] \cup \{1\}$, $f(s) > 0$ in $(\theta, 1)$); Chen et al. [9], Gui [13] studied the existence and qualitative properties of cylindrically symmetric traveling fronts of (1.1) with the balanced bistable nonlinearity (i.e., $f(0) = f(a) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $f'(a) > 0$ and $\int_0^1 f(s)\mathrm{d}s = 0$); Hamel et al. [18,19], Ninomiya and Taniguchi [28,29] studied V-shaped traveling fronts of (1.1) for the unbalanced bistable case (i.e., $f(0) = f(a) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $f'(a) > 0$ and $\int_0^1 f(s)\mathrm{d}s \neq 0$). Additionally, Hamel et al. [18,19] considered cylindrically symmetric traveling fronts of (1.1) for $N \geq 3$; Taniguchi [35,36], Kurokawa and Taniguchi [23] studied pyramidal-shaped traveling fronts of (1.1) for $N \geq 3$. More recently, Wang [41] and Wang et al. [45] developed the arguments of [28,29,35,36] to reaction–diffusion systems. See also Haragus and Scheel [20,21] for the study of almost planar traveling fronts. Very recently, Taniguchi [37] studied multidimensional traveling fronts of (1.1) for $N = 3$. For the study on the nonconvex and nonconnected traveling fronts, we refer to del Pino et al. [10]. Other related works on traveling fronts for autonomous reaction–diffusion equations can be referred to [7,8,14,26,27,32,34,38,39,43].

It is well known that in population dynamics interactive species may live in a fluctuating environment, for instance, physical environment conditions such as temperature, humidity and the available of food, water and other resources usually varies in time with seasonal or daily changes [46]. Therefore, in nature, another more realistic model might be of the following form

$$\frac{\partial u(\mathbf{y}, t)}{\partial t} = \Delta u(\mathbf{y}, t) + f(u(\mathbf{y}, t), t), \quad \mathbf{y} \in \mathbb{R}^N, \quad t > 0. \tag{1.2}$$

When the data of (1.2) are functions with commensurate time period, we call (1.2) a periodic equation. Recently, there are a lot of works devote to the study of traveling fronts of (1.2). One can refer to [1,4,22,25,30,31,47] for the study of time almost periodic and time periodic planar traveling fronts. Nevertheless, a very little attention has been paid to the study of nonplanar traveling fronts for nonautonomous reaction–diffusion equations, even for the time periodic case. As far as we know, Wang and Wu [44] proved that there exists a two-dimensional time periodic V-shaped traveling front. Moreover, they showed that such a traveling curved front is asymptotically stable. Sheng et al. [33] addressed the existence and asymptotic stability of time periodic pyramidal-shaped traveling fronts. Very recently, Wang [42] showed the existence of time periodic cylindrically symmetric traveling fronts by appealing to the method of comparison principle and the asymptotic speed of propagation.

However, the issue of the existence and stability of multidimensional time periodic traveling curved fronts for reaction–diffusion equation with bistable nonlinearity is still open. The main contribution of the current study is to give an affirmative answer to this issue. Actually, motivated by [33,37,42,44], we first consider the existence, uniqueness and stability of three-dimensional time periodic traveling fronts of polyhedral shape. Then, for any given $g \in C^\infty(S^1)$ with $\min_{0 \leq \theta \leq 2\pi} g(\theta) = 0$ that defines a convex domain with smooth boundary of positive curvature everywhere, which is included in a sequence of convex polygons, we show that there exists a three-dimensional time periodic traveling front by taking the limit of the solutions associated with the convex polyhedrons as the number of the lateral surfaces goes to infinity.

In this paper, we study traveling curved fronts of the following reaction–diffusion equation

$$\frac{\partial u(x,y,z,t)}{\partial t} = \Delta u(x,y,z,t) + f(u(x,y,z,t),t), \quad (x,y,z) \in \mathbb{R}^3, \quad t > 0 \tag{1.3}$$

under the following hypotheses:

(H1) There exists $T > 0$ such that $f(u,t) = f(u,t+T)$ for all $(u,t) \in \mathbb{R}^2$.

(H2) The period map $P(\alpha) := \omega(\alpha, T)$ has exactly three fixed points α^-, α^0, α^+ such that $\alpha^- < \alpha^0 < \alpha^+$, where $\omega(\alpha, t)$ is the solution of

$$\omega_t = f(\omega, t), \quad t \in \mathbb{R}, \quad \omega(\alpha, 0) = \alpha \in \mathbb{R}.$$

Furthermore, they are nondegenerate and $\alpha^\pm$ are stable, i.e.,

$$\frac{d}{\mathrm{d}\alpha} P(\alpha^\pm) < 1 < \frac{d}{\mathrm{d}\alpha} P(\alpha^0).$$

(H3) There exists $\nu_0 > 0$ such that $\nu^+ + \nu^- + f_u(W^\pm(t), t) > \nu_0$ for any $t \in [0, T]$, where

$$\nu^\pm := -\frac{1}{T}\int_0^T f_u(W^\pm(\lambda), \lambda)\mathrm{d}\lambda,$$

and

$$W^\pm(t) := \omega(\alpha^\pm, t), \quad W^0(t) := \omega(\alpha^0, t).$$

(H4) There exist constants $r_0 > 0$ and $\epsilon \in \big(0, \min_{t\in[0,T]}(W^0(t) - W^-(t))\big)$ such that $\overline{f}(u,t) \geq r_0 u(\epsilon - u)$ for any $u \in (0, \epsilon)$ and $t \in [0, T]$, where $\overline{f}(u,t) := f(W^0(t), t) - f(W^0(t) - u, t)$.

A typical example of f satisfying (H1)–(H3) is the cubic potential $f = (1-u^2)(2u - \rho(t))$, where $\rho(t) \in (-2, 2)$ is T-periodic, which is the particular case of the following more general example (see Alikakos et al. [1])

$$f(u,t) = p(u)(p'(u) - \rho(t)),$$

where $\rho \in C^1$ and $p \in C^3$ satisfy $\rho(\cdot + T) = \rho(\cdot)$, and $p(\pm 1) = 0$, $p(\cdot) > 0$ in $(-1, 1)$. Moreover, by taking $|\rho(t)| \leq 2\sqrt{5}/5$, then such a function f satisfies (H4) (see Wang [42]).

It is known from [1] that when $f(u,t) \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ satisfies (H1) and (H2), there exists a unique solution pair (c, U) of (1.3) such that

$$\begin{cases} U_t + cU_\xi - U_{\xi\xi} - f(U,t) = 0, \quad (\xi, t) \in \mathbb{R}^2, \\ U(\pm\infty, t) = \lim_{\xi\to\pm\infty} U(\xi, t) = W^\pm(t), \quad t \in \mathbb{R}, \\ U(\xi, t+T) = U(\xi, t), \quad U(0,0) = \alpha^0, \end{cases} \tag{1.4}$$

where the function $U(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the wave profile and the constant $c \in \mathbb{R}$ is the wave speed. In addition, (c, U) enjoys the following properties:

(i) $U(\xi, t)$ is monotone increasing with respect to the moving coordinate for each t. Namely, $U_\xi(\xi, t) > 0$ in $\mathbb{R}\times\mathbb{R}$.

(ii) There exist positive constants C_1 and β_1 satisfying

$$|U(\pm\xi, t) - W^\pm(t)| + |U_\xi(\pm\xi, t)| + |U_{\xi\xi}(\pm\xi, t)| \leq C_1 \mathrm{e}^{-\beta_1 \xi}, \quad \xi \geq 0, t \in \mathbb{R}.$$

That is, U exponentially approaches its limits as $\xi \to \pm\infty$.

Without loss of generality, we assume that the solutions travel toward z-direction. Set

$$u(x, y, z + lt, t) = v(x, y, z_1, t), \quad z_1 = z + lt.$$

For simplicity, we denote $v(x, y, z_1, t)$ by $v(x, y, z, t)$. Substituting v into (1.3), we have

$$\begin{aligned} &\mathcal{L}[v] := v_t - v_{xx} - v_{yy} - v_{zz} + l v_z - f(v, t) = 0, \quad (x, y, z) \in \mathbb{R}^3, t > 0, \\ &v(x, y, z, 0) = v_0(x, y, z), \quad (x, y, z) \in \mathbb{R}^3. \end{aligned} \tag{1.5}$$

Hereafter, we always assume $l > c > 0$. The objective of this paper is to seek for the solution $V(x, y, z, t)$ of

$$\mathcal{L}[V] := V_t - V_{xx} - V_{yy} - V_{zz} + l V_z - f(V, t) = 0, \quad (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.6}$$

$$V(x, y, z, t) = V(x, y, z, t + T), \quad (x, y, z) \in \mathbb{R}^3, \quad t \in [0, T]. \tag{1.7}$$

Let

$$\tau := \frac{\sqrt{l^2 - c^2}}{c} > 0.$$

Given $n \geq 3$ be an integer. Assume that $\{\theta_j\}_{1 \leq j \leq n}$ satisfy

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi \quad \text{and} \quad \max_{1 \leq j \leq n} (\theta_{j+1} - \theta_j) < \pi,$$

where $\theta_{n+1} = \theta_1 + 2\pi$. Given, l_j with

$$\min_{1 \leq j \leq n} l_j \geq 0 \quad \text{for } 1 \leq j \leq n.$$

Then,

$$\nu_j = \frac{1}{\sqrt{1 + \tau^2}} \begin{pmatrix} \tau \cos \theta_j \\ \tau \sin \theta_j \\ 1 \end{pmatrix}$$

is the unit normal vector of a surface $\{z = \tau(x \cos \theta_j + y \sin \theta_j)\}$. Putting

$$\begin{aligned} &h_j(x, y) := \tau(x \cos \theta_j + y \sin \theta_j - l_j), \\ &h(x, y) = \max_{1 \leq j \leq n} h_j(x, y) = \tau \max_{1 \leq j \leq n} (x \cos \theta_j + y \sin \theta_j - l_j), \end{aligned} \tag{1.8}$$

then $\{(x, y, z) \in \mathbb{R}^3 | -z \geq h(x, y)\}$ is a convex polyhedron. If $(l_1, l_2, \ldots, l_n) = (0, 0, \ldots, 0)$, the polyhedron becomes a pyramid in $\mathbb{R}^3$.

Denote

$$\Theta := \max_{2 \leq j \leq n-1} \frac{l_j \sin(\theta_{j+1} - \theta_{j-1}) - l_{j-1} \sin(\theta_{j+1} - \theta_j) - l_{j+1} \sin(\theta_j - \theta_{j-1})}{\sin(\theta_{j+1} - \theta_j) + \sin(\theta_j - \theta_{j-1}) - \sin(\theta_{j+1} - \theta_{j-1})}. \tag{1.9}$$

For $j = 1, 2 \ldots, n$, define

$$\Omega_j := \left\{(x, y) \in \mathbb{R}^2 | h(x, y) = h_j(x, y), h(x, y) \geq \tau\Theta\right\}.$$

We note that $\Omega_j \neq \emptyset$ for all $1 \leq j \leq n$. Here $\Omega_1, \Omega_2, \ldots, \Omega_n$ are located counterclockwise. Set

$$S_j = \{(x, y, z) \in \mathbb{R}^3 | -z = h_j(x, y), \quad (x, y) \in \Omega_j\}, \quad j = 1, \ldots, n.$$

Let

$$\Gamma_j = \{(x, y, z) \in \mathbb{R}^3 | -z = h_j(x, y) = h_{j+1}(x, y) \geq \tau\Theta\}, \quad j = 1, \ldots, n$$

be a part of an edge of a polyhedron $\{(x, y, z) \in \mathbb{R}^3 | -z \geq h(x, y)\}$. If $(l_1, l_2, \ldots, l_n) = (0, 0, \ldots, 0)$ and $\Theta = 0$, then Γ_j and $\bigcup_{j=1}^n \Gamma_j$ stand for an edge and the set of all edges of a pyramid, respectively. For each $\gamma > 0$, we define

$$D(\gamma) := \left\{ (x, y, z) \in \mathbb{R}^3 \middle| \mathrm{dist}\left((x, y, z), \bigcup_{j=1}^n \Gamma_j \right) > \gamma \right\}.$$

Theorem 1.1 *Let $l > c > 0$ and $h(x, y)$ be given by* (1.8)*. Under the assumptions* (H1)–(H3)*, there exists a solution $V(x, y, z, t)$ of* (1.6)–(1.7) *such that*

$$\lim_{\gamma \to \infty} \sup_{(x,y,z) \in D(\gamma), t \in [0,T]} \left| V(x, y, z, t) - U\left(\frac{c}{l}(z + h(x, y)), t \right) \right| = 0, \tag{1.10}$$

$$\lim_{R \to \infty} \sup_{|\mathbf{x}| > R, t \in [0,T]} \left| V(x, y, z, t) - \max_{1 \leq j \leq n} E_j(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \right| = 0, \tag{1.11}$$

$$W^+(t) > V(x, y, z, t) > U\left(\frac{c}{l}(z + h(x, y)), t \right) > W^-(t), \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0, \tag{1.12}$$

$$\inf_{W^-(t)+\delta \leq V(x,y,z,t) \leq W^+(t)-\delta, t \in [0,T]} V_z(x, y, z, t) > 0 \quad \text{for } \delta > 0 \text{ small enough}, \tag{1.13}$$

$$\lim_{R \to \infty} \sup_{|z + h(x,y)| > R, t \in [0,T]} |V_z(x, y, z, t)| = 0. \tag{1.14}$$

Moreover, if

$$\begin{aligned} &\max_{1 \leq j \leq n} \widetilde{V}\left(x - X_j(-\hat{l}), y - Y_j(-\hat{l}), z - \tau\hat{l}, 0 \right) \\ &\leq v_0(x, y, z) \\ &\leq \min_{1 \leq j \leq n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, 0), \end{aligned} \tag{1.15}$$

then, the solution $v(x, y, z, t; v_0)$ of (1.5) *satisfies*

$$\lim_{k \to \infty} \sup_{(x,y,z) \in \mathbb{R}^3, t \in [0,T]} |v(x, y, z, t + kT; v_0) - V(x, y, z, t)| = 0, \tag{1.16}$$

where $\hat{l} := \max_{1 \leq j \leq n} l_j \geq 0$, E_j is the two-dimensional V-shaped traveling front defined in (2.6)*, $\widetilde{V}$ is the pyramidal traveling front given in Theorem 2.2, $X_j(-\hat{l})$, $Y_j(-\hat{l})$ and $X_j(\rho)$, $Y_j(\rho)$ satisfy $h(X_j(-\hat{l}), Y_j(-\hat{l})) = -\tau\hat{l}$ and $h(X_j(\rho), Y_j(\rho)) = \tau\rho$, respectively. Furthermore, V enjoys the following properties:*

- **(i)** *Let $h(x, y)$ be defined in* (1.8)*, $\overline{h}(x, y) := \tau \max_{1 \leq j \leq n}(x \cos\theta_j + y \sin\theta_j - \overline{l}_j)$ with $\min_{1 \leq j \leq n} \overline{l}_j \geq 0$ and $V(\mathbf{x}, t)$ be given in Theorem 1.1, $\overline{V}(\mathbf{x}, t)$ be the traveling fronts of polyhedral-shape associated with $\overline{h}$. If $\overline{h}(x, y) \geq h(x, y)$ for any $(x, y) \in \mathbb{R}^2$, then it holds $\overline{V}(\mathbf{x}, t) \geq V(\mathbf{x}, t)$ for all $(\mathbf{x}, t) \in \mathbb{R}^4$.*
- **(ii)** *There holds*

$$\frac{\partial V}{\partial \nu} > 0 \quad \text{in } \mathbb{R}^4$$

for

$$\nu = \frac{1}{\sqrt{1 + t_1^2 + t_2^2}} \begin{pmatrix} t_1 \\ t_2 \\ 1 \end{pmatrix} \quad \text{with} \sqrt{t_1^2 + t_2^2} \leq \frac{1}{\tau}.$$

(iii) *If $h(x, y) = h(|x|, |y|)$, then one has*

$$
\begin{aligned}
&V(x, y, z, t) = V(|x|, |y|, z, t), \quad (x, y, z, t) \in \mathbb{R}^4,\\
&V_x(x, y, z, t) > 0 \ \text{ for } (x, y, z, t) \in (0, \infty) \times \mathbb{R}^3,\\
&V_x(0, y, z, t) = 0 \ \text{ for } (y, z, t) \in \mathbb{R}^3,\\
&V_y(x, y, z, t) > 0 \ \text{ for } (x, y, z, t) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}^2,\\
&V_y(x, 0, z, t) = 0 \ \text{ for } (x, z, t) \in \mathbb{R}^3.
\end{aligned}
$$

An immediate consequence of this theorem is the following corollary.

Corollary 1.2 *Let $V(\mathbf{x}, t)$ be the time periodic traveling curved front defined in Theorem 1.1. If there is a time periodic solution $w(\mathbf{x}, t)$ of (1.6) and (1.7) satisfying (1.10), then it holds*

$$w(\mathbf{x}, t) \equiv V(\mathbf{x}, t) \quad \text{for all } (\mathbf{x}, t) \in \mathbb{R}^4.$$

In what follows, we treat the three-dimensional traveling fronts of (2.1) for any given $g \in C^\infty(S^1)$, where

$$C^\infty(S^1) := \left\{g \in C^\infty(\mathbb{R}) | g(\theta + 2\pi) = g(\theta)\right\}$$

for any $\theta \in \mathbb{R}$. We identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Let $g \in C^\infty(S^1)$ be any given function with $\min_{0\le\theta\le 2\pi} g(\theta) = 0$ and $r_* \ge 1$ be large enough such that

$$r_*^2 + r_* \min_{0\le\theta\le 2\pi} \left(2g(\theta) - g''(\theta)\right) + \min_{0\le\theta\le 2\pi} \left(g(\theta)^2 + 2g'(\theta)^2 - g(\theta)g''(\theta)\right) > 0.$$

Setting

$$R(\theta) := r_* + g(\theta) \ \text{ for } 0 \le \theta \le 2\pi,$$

then,

$$\mathcal{C} := \{(R(\theta)\cos\theta, R(\theta)\sin\theta) | 0 \le \theta \le 2\pi\}$$

is a smooth convex closed curve such that

$$R(0) = R(2\pi), \quad \min_{0\le\theta\le 2\pi} R(\theta) = r_*, \quad 0 < \min_{0\le\theta\le 2\pi} \kappa(\theta) \le \max_{0\le\theta\le 2\pi} \kappa(\theta) < \infty,$$

where $\kappa(\theta)$ is the curvature of $\mathcal{C}$ given by

$$\kappa(\theta) = \frac{R(\theta)^2 + 2R'(\theta)^2 - R(\theta)R''(\theta)}{(R(\theta)^2 + R'(\theta)^2)^{\frac{3}{2}}}, \quad \theta \in [0, 2\pi].$$

Define

$$\mathcal{D} := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 | 0 \le r < R(\theta), 0 \le \theta \le 2\pi\}. \tag{1.17}$$

Then, $\mathcal{C}$ is the corresponding boundary to $\mathcal{D}$. Let

$$\kappa_{\min} = \min_{0\le\theta\le 2\pi} \kappa(\theta) \quad \text{and} \quad \kappa_{\max} = \max_{0\le\theta\le 2\pi} \kappa(\theta).$$

Then, we have $0 < \kappa_{\min} \le \kappa_{\max} < \infty$. Set

$$R_{\max} := \max_{0\le\theta\le 2\pi} R(\theta) \in [1, \infty) \quad \text{and} \quad r_* \le R_{\max} < \infty.$$

Let $R_* \in \left(\kappa_{\min}^{-1}, \infty\right)$ be large enough such that, for each $\theta \in [0, 2\pi)$, there is a circle of radius R_* that circumscribes $\mathcal{C}$ at $(R(\theta)\cos\theta, R(\theta)\sin\theta)$. Let $(\xi_*(\theta), \eta_*(\theta))$ and $B(\theta)$ be the center and the interior of this circle for each $\theta \in [0, 2\pi)$. It is obvious that $\overline{\mathcal{D}} \subset \overline{B}(\theta)$ for all $\theta \in [0, 2\pi)$.

Theorem 1.3 *Assume that* (H1)–(H4) *hold. Let $g \in C^\infty(S^1)$ be any given function such that $\min_{0\le\theta\le2\pi} g(\theta) = 0$ and $R(\theta) = r_* + g(\theta)$. Then, there exits a solution $\widetilde{W}(\mathbf{x}, t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$ of* (1.6)–(1.7) *satisfying*

$$\max_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-\xi_*(\theta))^2+(y-\eta_*(\theta))^2},\ z-\tau R_*, t\right) \\ \le \widetilde{W}(\mathbf{x}, t) \le \min_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-R(\theta)\cos\theta)^2+(y-R(\theta)\sin\theta)^2}, z, t\right) \tag{1.18}$$

for all $(x, y, z, t) \in \mathbb{R}^4$. Morevoer, one has

$$\lim_{A\to\infty} \sup_{x^2+y^2+z^2\ge A^2, t\in[0,T]} \Big(\widetilde{W}(x, y, z, t) \\ - \min_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-R(\theta)\cos\theta)^2+(y-R(\theta)\sin\theta)^2}, z, t\right)\Big) = 0. \tag{1.19}$$

Furthermore, such $\widetilde{W}(x, y, z, t)$ is uniquely determined by (1.6) *and* (1.19)*. Moreover, if*

$$\max_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-\xi_*(\theta))^2+(y-\eta_*(\theta))^2}, z-\tau R_*, 0\right) \\ \le v_0(x, y, z) \le \min_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-R(\theta)\cos\theta)^2+(y-R(\theta)\sin\theta)^2}, z, 0\right) \tag{1.20}$$

for all $(x, y, z) \in \mathbb{R}^3$, then the solution $v(\mathbf{x}, t; v_0)$ of (1.5) *with initial value v_0 satisfies*

$$\lim_{t\to\infty} \sup_{\mathbf{x}\in\mathbb{R}^3} |v(\mathbf{x}, t; v_0) - \widetilde{W}(\mathbf{x}, t)| = 0 \tag{1.21}$$

or equivalently

$$\lim_{k\to\infty} \sup_{\mathbf{x}\in\mathbb{R}^3, t\in[0,T]} |v(\mathbf{x}, t+kT; v_0) - \widetilde{W}(\mathbf{x}, t)| = 0, \tag{1.22}$$

where Ψ is the cylindrically symmetric traveling front defined in Theorem 2.4.

The rest of this paper is organized as follows. In Sect. 2, we state some preliminaries including two-dimensional time periodic V-shaped traveling fronts, three-dimensional time periodic pyramidal-shaped traveling fronts and cylindrically symmetric time periodic traveling fronts. Section 3 is devoted to the existence and stability of time periodic traveling curved fronts with polyhedral shape, that is, we prove Theorem 1.1. In Sect. 4, we show Theorem 1.3.

2 Preliminaries

In this section, we recall some results established by Wang and Wu [44], Sheng et al. [33] and Wang [42] about two-dimensional time periodic V-shaped taveling fronts, three-dimensional time periodic pyramidal traveling fronts and time periodic cylindrically symmetric in $\mathbb{R}^3$, respectively. In the sequel, we write (c, U) be the planar traveling front defined by (1.4).

2.1 Two-dimensional V-shaped fronts

Let $\hat{v}(\xi, \eta, t; \hat{v}_0)$ be the solution of the following equation

$$\hat{v}_t - \hat{v}_{\xi\xi} - \hat{v}_{\eta\eta} + l\hat{v}_\eta - f(\hat{v}, t) = 0 \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, \quad t > 0,$$
$$\hat{v}(\xi, \eta, 0) = \hat{v}_0(\xi, \eta) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2.$$

Then, from [44, Theorem 1.1], we have the following theorem.

Theorem 2.1 *Assume that* (H1)–(H3) *hold and* $l > c > 0$. *Then, there exists a unique* $\widehat{V}(\xi, \eta, t)$ *such that*

$$\widehat{V}_t - \widehat{V}_{\xi\xi} - \widehat{V}_{\eta\eta} + l\widehat{V}_\eta - f(\widehat{V}, t) = 0 \quad \text{for } (\xi, \eta) \in \mathbb{R}^2 \quad \text{and} \quad t \in \mathbb{R},$$

and

$$\widehat{V}(\xi, \eta, t + T) = \widehat{V}(\xi, \eta, t) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2 \quad \text{and} \quad t \in \mathbb{R}.$$

Moreover, there holds

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2, t\in[0,T]} \left|\widehat{V}(\xi, \eta, t) - U\left(\frac{c}{l}(\eta + \tau|\xi|), t\right)\right| = 0. \tag{2.1}$$

One also has

$$U\left(\frac{c}{l}(\eta + \tau|\xi|), t\right) < \widehat{V}(\xi, \eta, t) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, t \in \mathbb{R} \tag{2.2}$$

$$\inf_{W^-(t)+\delta \le \widehat{V}(\xi,\eta,t) \le W^+(t)-\delta, t\in[0,T]} \widehat{V}_\eta(\xi, \eta, t) > 0 \quad \text{for } \delta > 0 \text{ small enough},$$
$$\widehat{V}_\xi(\xi, \eta, t) > 0, \quad \forall(\xi, \eta, t) \in (0, \infty) \times \mathbb{R}^2,$$
$$\widehat{V}(\xi + \xi_0, \eta, t) \le \widehat{V}(\xi, \eta + \eta_0, t), \quad \forall(\xi, \eta, t) \in \mathbb{R}^3, \quad \xi_0, \eta_0 \in \mathbb{R} \quad \text{with } \eta_0 \ge \tau|\xi_0|. \tag{2.3}$$

2.2 Three-dimensional pyramidal traveling fronts

Consider the following problem:

$$\widetilde{v}_t - \widetilde{v}_{xx} - \widetilde{v}_{yy} - \widetilde{v}_{zz} + l\widetilde{v}_z - f(\widetilde{v}, t) = 0, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0,$$
$$\widetilde{v}(x, y, z, 0) = \widetilde{v}_0(x, y, z), \quad (x, y, z) \in \mathbb{R}^3. \tag{2.4}$$

Set

$$p_j(x, y) := \tau(x\cos\theta_j + y\sin\theta_j),$$
$$p(x, y) := \tau \max_{1\le j\le n}(x\cos\theta_j + y\sin\theta_j) \tag{2.5}$$

and

$$k_j := \cos\left(\frac{\theta_{j+1} - \theta_j}{2}\right) > 0, \quad \phi_j := \frac{\theta_{j+1} + \theta_j}{2}, \quad 1 \le j \le n.$$

Define

$$E_j(x, y, z, t) = \widehat{V}\left(x\sin\phi_j - y\cos\phi_j, \frac{z - \tau k_j(x\cos\phi_j + y\sin\phi_j)}{\sqrt{1 + \tau^2 k_j^2}}, t\right). \tag{2.6}$$

Substituting $E_j(x, y, z, t)$ into (2.4), we obtain that every $E_j(x, y, z, t)$ is a time periodic V-shaped traveling front with speed $\frac{l}{\sqrt{1+\tau^2 k_j^2}} > c$. By Sheng et al. [33, Theorems 1.1–1.2 and Lemma 4.6], we get the following theorem.

Theorem 2.2 *Assume that $l > c > 0$ and (H1)–(H3) hold. Let $p(x, y)$ be given by (2.5). Then, there exists a solution $\widetilde{V}(x, y, z, t)$ of (2.4) such that*

$$U\left(\frac{c}{l}(z + p(x, y)), t\right) < \widetilde{V}(x, y, z, t) < W^+(t), \quad (x, y, z, t) \in \mathbb{R}^3 \times [0, T],$$
$$\widetilde{V}(\mathbf{x}, t) = \widetilde{V}(\mathbf{x}, t + T), \quad \widetilde{V}_z(\mathbf{x}, t) > 0 \quad \text{for all } (\mathbf{x}, t) \in \mathbb{R}^4$$

and

$$\lim_{\gamma\to+\infty} \sup_{(x,y,z)\in D(\gamma), t\in[0,T]} \left|\widetilde{V}(x, y, z, t) - U\left(\frac{c}{l}(z + p(x, y)), t\right)\right| = 0, \tag{2.7}$$

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|\geq R, t\in[0,T]} \left|\widetilde{V}(\mathbf{x}, t) - \max_{1\leq j\leq n} E_j(\mathbf{x}, t)\right| = 0, \quad \lim_{R\to\infty} \sup_{|z+p(x,y)|\geq R, t\in[0,T]} |\widetilde{V}_z(\mathbf{x}, t)| = 0, \tag{2.8}$$

$$\inf_{W^-(t)+\delta\leq \widetilde{V}(x,y,z,t)\leq W^+(t)-\delta, t\in[0,T]} \widetilde{V}_z(x, y, z, t) > 0 \quad \text{for } \delta > 0 \text{ small enough},$$

$$U\left(\frac{c}{l}(z + p(x, y)), t\right) < \max_{1\leq j\leq n} E_j(\mathbf{x}, t) < \widetilde{V}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^3, \quad t \in [0, T], \tag{2.9}$$

$$\lim_{\gamma\to\infty} \sup_{\mathbf{x}\in D(\gamma), t\in[0,T]} \left|\max_{1\leq j\leq n} E_j(\mathbf{x}, t) - U\left(\frac{c}{l}(z + p(x, y)), t\right)\right| = 0. \tag{2.10}$$

Lemma 2.3 *There exists positive constants δ_0, σ and β ($\beta < \frac{\nu_0}{4}$) such that, for any $\delta \in (0, \delta_0)$, the functions $w^\pm$ defined by*

$$w^\pm(\mathbf{x}, t) := \widetilde{V}\left(x, y, z \pm \sigma\delta\left(1 - e^{-\beta t}\right), t\right) \pm \delta a(t)$$

are a supersolution and a subsolution of (2.4) on $\mathbf{x} \in \mathbb{R}^3$ and $t \in [0, \infty)$, respectively, where

$$a(t) = \exp\left\{\left(\nu^+ + \nu^- - \frac{\nu_0}{4}\right) t + \int_0^t f_u(W^+(\tau), \tau)\mathrm{d}\tau + \int_0^t f_u(W^-(\tau), \tau)\mathrm{d}\tau\right\}$$

and

$$K_0 = \max_{t\in[0,T]} \exp\left\{(\nu^+ + \nu^-)t + \int_0^t f_u(W^+(\tau), \tau)\mathrm{d}\tau + \int_0^t f_u(W^-(\tau), \tau)\mathrm{d}\tau\right\} \geq 1$$

with the constants ν_0, ν^+ and ν^- are defined as in (H3).

2.3 Cylindrically symmetric traveling fronts

Let

$$p^{(m)}(x, y) = \tau \max_{1\leq j\leq 2^m} \left\{x \cos\frac{2(j-1)\pi}{2^m} + y \sin\frac{2(j-1)\pi}{2^m}\right\}, \quad m = 1, 2, \ldots. \tag{2.11}$$

Clearly,

$$z = \tau\left(x \cos\frac{2(j-1)\pi}{2^m} + y \sin\frac{2(j-1)\pi}{2^m}\right)$$

is tangent to

$$z = \tau\sqrt{x^2 + y^2}$$

for any $m \in \mathbb{N}$ and $1 \le j \le 2^m$. Replacing $p(x, y)$ by $p^{(m)}(x, y)$ in Theorem 2.2, we obtain a sequence of time periodic pyramidal traveling fronts of (2.4), namely,

$$\widetilde{V}^1, \widetilde{V}^2, \ldots, \widetilde{V}^m, \ldots,$$

where

$$\widetilde{V}^m(\mathbf{x}, t) = \lim_{k\to\infty} \widetilde{v}\left(\mathbf{x}, t + kT; \widetilde{v}_0^{m,-}\right), \quad \widetilde{v}_0^{m,-}(\mathbf{x}, 0) = U\left(\frac{c}{l}(z + p^{(m)}(x, y)), 0\right)$$

Denote the edge of the pyramid $-z = p^{(m)}(x, y)$ by Γ^m and put

$$D^m(\gamma) := \left\{(x, y, z) \in \mathbb{R}^3 \middle| \mathrm{dist}\left((x, y, z), \bigcup_{j=1}^{n} \Gamma_j^m\right) > \gamma\right\} \quad \text{for } \gamma > 0.$$

Owing to $\widetilde{v}_0^{m,-}(\mathbf{x}, 0)$ is nondecreasing on $x \in (0, \infty)$ and $y \in (0, \infty)$ and is even on $x \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively. It then follows from Theorem 2.2 that

$$\begin{aligned}
&\widetilde{V}^1 \le \widetilde{V}^2 \le \cdots \le \widetilde{V}^m \le \cdots, \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad t \in \mathbb{R},\\
&\frac{\partial}{\partial x}\widetilde{V}^m > 0, \quad \forall \mathbf{x} \in (0, \infty) \times \mathbb{R}^2, \quad t \in \mathbb{R},\\
&\frac{\partial}{\partial y}\widetilde{V}^m > 0, \quad \forall \mathbf{x} \in \mathbb{R} \times (0, \infty) \times \mathbb{R}, \quad t \in \mathbb{R},\\
&\frac{\partial}{\partial \nu}\widetilde{V}^m > 0, \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad t \in \mathbb{R},
\end{aligned}$$

where $\nu = \frac{1}{\sqrt{1+\nu_1^2+\nu_2^2}}\begin{pmatrix}\nu_1\\ \nu_2\\ 1\end{pmatrix}$ with $\sqrt{\nu_1^2 + \nu_2^2} \le \frac{1}{\tau}$. Thanks to

$$p^{(m)}(x, y) = p^{(m)}\left(x\cos\frac{\pi}{2^{m-1}} + y\sin\frac{\pi}{2^{m-1}}, -x\sin\frac{\pi}{2^{m-1}} + y\cos\frac{\pi}{2^{m-1}}\right),$$

one infers

$$\widetilde{V}^m(\mathbf{x}, t) = \widetilde{V}^m(\mathbf{x}, t) = \widetilde{V}^m(\mathbf{B}_m \cdot (x, y), z, t), \quad \forall (\mathbf{x}, t) \in \mathbb{R}^4,$$

where

$$\mathbf{B}_m = \begin{pmatrix}\cos\frac{\pi}{2^{m-1}} & \sin\frac{\pi}{2^{m-1}}\\ -\sin\frac{\pi}{2^{m-1}} & \cos\frac{\pi}{2^{m-1}}\end{pmatrix}.$$

Let $z^m \in \mathbb{R}$ be such that

$$z^m \ge z^{m+1} \quad \text{and} \quad \widetilde{V}^m(0, 0, z^m, 0) = \theta_0 \tag{2.12}$$

for a given constant $\theta_0 \in (\alpha^-, \alpha^0)$. Denote

$$\overline{V}^m(x, y, z, t) = \widetilde{V}^m(x, y, z + z^m, t), \quad \forall (\mathbf{x}, t) \in \mathbb{R}^4.$$

It then deduces from the parabolic estimate [24] and Theorem 2.2 that there exists a solution $W(\mathbf{x}, t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$ of (2.4) (even if up to an extraction of some subsequence) such that

$$\overline{V}^m(\mathbf{x}, t) \to W(\mathbf{x}, t) \text{ in } \|\cdot\|_{C^{2,1}_{loc}(\mathbb{R}^3\times\mathbb{R})} \text{ as } m \to \infty.$$

Define

$$\Psi(r, z, t) = \Psi(\sqrt{x^2 + y^2}, z, t) := W(\mathbf{x}, t), \quad r = \sqrt{x^2 + y^2} \tag{2.13}$$

for any $(x, y, z, t) \in \mathbb{R}^4$. Then, one has

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial z^2} - \frac{1}{r}\frac{\partial}{\partial r} + l\frac{\partial}{\partial z}\right)\Psi - f(\Psi(r, z, t), t) = 0, \quad \forall r > 0, \quad z \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{2.14}$$

For a given constant $\theta_0 \in (\alpha^-, \alpha^0)$, we define $\phi(r) \in \mathbb{R}$ by

$$\Psi(r, \phi(r), 0) = \theta_0. \tag{2.15}$$

By a shift, one can assume $U(0, 0) = \theta_0$ without loss of generality. Then, Wang [42] established the following results.

Theorem 2.4 *Assume that* (H1)–(H4) *hold. Suppose* $c > 0$. *Then,* $\Psi(r, z, t)$ *defined by* (2.13) *satisfies* (2.14) *and* $\Psi(r, z, t + T) = \Psi(r, z, t)$ *for all* $(r, z, t) \in \mathbb{R}^3$. *Moreover, there hold*

- **(i)** $\frac{\partial}{\partial r}\Psi(r, z, t) > 0$ *and* $\frac{\partial}{\partial z}\Psi(r, z, t) > 0$ *for any* $(r, z, t) \in (0, \infty) \times \mathbb{R}^2$.
- **(ii)** $\lim_{z\to+\infty} \|\Psi(\cdot, z, t) - W^+(t)\|_{C(\mathbb{R}^2)} = 0$, $\lim_{z\to-\infty} \|\Psi(\cdot, z, t) - W^-(t)\|_{C_{loc}(\mathbb{R}^2)} = 0$.
- **(iii)** $\frac{\partial}{\partial \nu}\Psi(r, z, t) > 0$ *for any* $r > 0$, $z > 0$, $t \in \mathbb{R}$, *where*

$$\nu = \frac{1}{\sqrt{1 + \nu'^2}}\begin{pmatrix}\nu' \\ 1\end{pmatrix} \quad \text{with } \nu' \geq -\frac{1}{\tau}.$$

- **(iv)** $\lim_{r\to\infty} \phi'(r) = -\tau$.
- **(v)** $\lim_{r\to\infty} \|\Psi(x + r, z + \phi(r), t) - U(\frac{c}{l}(z + \tau x), t)\|_{C^{2,1}_{loc}(\mathbb{R}^2\times\mathbb{R})} = 0$.

3 Proof of Theorem 1.1

In this section, we study the existence and asymptotic stability of traveling fronts with convex polyhedral shapes, that is, we prove Theorem 1.1.

Firstly, note that $\{(x, y, z) \in \mathbb{R}^3 | -z \geq h(x, y)\}$ is a convex polyhedron. Indeed, if $-z_i \geq h(x_i, y_i)$ for $i = 1, 2$, then $-z_i \geq h_j(x_i, y_i)$ for all $1 \leq j \leq n$ and $i = 1, 2$. It then follows from (1.8) that

$$\begin{aligned} -az_1 - (1-a)z_2 &\geq ah_j(x_1, y_1) + (1-a)h_j(x_2, y_2) \\ &= h_j(ax_1 + (1-a)x_2, ay_1 + (1-a)y_2) \end{aligned}$$

for all $1 \leq j \leq n$ and any $a \in (0, 1)$. Then, one has

$$\begin{aligned} -(az_1 + (1-a)z_2) &\geq \max_{1\leq j\leq n} h_j(ax_1 + (1-a)x_2, ay_1 + (1-a)y_2) \\ &= h(ax_1 + (1-a)x_2, ay_1 + (1-a)y_2). \end{aligned}$$

Hence, $\{(x, y, z) \in \mathbb{R}^3 | -z \geq h(x, y)\}$ is a convex polyhedron.

On the other hand, for any $\zeta \in \mathbb{R}$ and $1 \le j \le n$, let $(X_j(\zeta), Y_j(\zeta))$ be such that

$$h_j(X_j(\zeta), Y_j(\zeta)) = h_{j+1}(X_j(\zeta), Y_j(\zeta)) = \tau\zeta.$$

Direct computations give

$$\begin{pmatrix} X_j(\zeta) \\ Y_j(\zeta) \end{pmatrix} = \frac{1}{\sin(\theta_{j+1} - \theta_j)} \begin{pmatrix} (\zeta + c_j)\sin\theta_{j+1} - (\zeta + c_{j+1})\sin\theta_j \\ -(\zeta + c_j)\cos\theta_{j+1} + (\zeta + c_{j+1})\cos\theta_j \end{pmatrix}.$$

Here, we would like to point out that, for every $\zeta \in \mathbb{R}$, the set $\{(x, y) \in \mathbb{R}^2 | h(x, y) \le \zeta\}$ is either an empty set or a nonempty convex closed set in $\mathbb{R}^2$. Indeed, if (x_i, y_i) satisfies $h(x_i, y_i) \le \zeta$ for $i = 1, 2$, then $h_j(x_i, y_i) \le \zeta$ for all $1 \le j \le n$ and $i = 1, 2$. In view of (1.8), we have

$$ah_j(x_1, y_1) + (1 - a)h_j(x_2, y_2) = h_j(ax_1 + (1 - a)x_2, ay_1 + (1 - a)y_2) \le \zeta$$

for all $1 \le j \le n$ and any $a \in (0, 1)$, whence

$$h(ax_1 + (1 - a)x_2, ay_1 + (1 - a)y_2) = \max_{1\le j\le n} h_j(ax_1 + (1 - a)x_2, ay_1 + (1 - a)y_2) \le \zeta.$$

If $\zeta < -\max_{1\le j\le n} l_j$ with l_j given in (1.8), then $\{(x, y) \in \mathbb{R}^2 | h(x, y) \le \zeta\}$ is an empty set. Moreover, it derives from [37, Lemma 3.1] that the set $\{(x, y) \in \mathbb{R}^2 | h(x, y) \le \tau\rho\}$ is a convex n-polygon in the x-y plane with vertices $\{(X_j(\rho), Y_j(\rho))\}_{1\le j\le n}$ for any fixed number $\rho \in (\Theta, \infty)$.

Proof of Theorem 1.1 Thanks to $h(X_j(\rho), Y_j(\rho)) = \tau\rho$ for all $1 \le j \le n$, one infers that

$$h(x, y) \le \tau\rho + p(x - X_j(\rho), y - Y_j(\rho)) \quad \text{for all } (x, y) \in \mathbb{R}^2, 1 \le j \le \text{n},$$

where h and p are defined in (1.8) and (2.5), respectively. Set

$$v^-(x, y, z, t) := U\left(\frac{c}{l}(z + h(x, y)), t\right) = \max_{1\le j\le n} U\left(\frac{c}{l}(z + h_j(x, y)), t\right). \tag{3.1}$$

Write the solution of (1.5) with $v_0(x, y, z) = v^-(x, y, z, 0)$ by $v(\mathbf{x}, t; v^-(x, y, z, 0))$. Call

$$V(\mathbf{x}, t) = \lim_{k\to\infty} v(\mathbf{x}, t + kT;\ v^-(x, y, z, 0)) \quad \text{in } C^{2,1}_{\text{loc}}(\mathbb{R}^4).$$

Then, the function $V(\mathbf{x}, t) \in C^{2,1}(\mathbb{R}^4)$ is a solution of (1.6). Since

$$v^-(x, y, z, t) = v^-(x, y, z, t + T),$$

then, we have

$$\begin{aligned} v(\mathbf{x}, t + kT;\ v^-(x, y, z, 0)) &= v(\mathbf{x}, t + kT;\ v^-(x, y, z, T)) \\ &= v(\mathbf{x}, t + T + kT;\ v^-(x, y, z, 0)). \end{aligned}$$

Letting $k \to \infty$, we arrive at

$$V(\mathbf{x}, t) = V(\mathbf{x}, t + T), \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \quad \text{and} \quad t \in [0, T].$$

Moreover, a similar discussion to Sheng [33, Lemma 3.1] yields that there is a supersolution v^+ that converges to v^- far away from the set of edges of the given polyhedron. Thus, (1.10) follows.

Notice that the function v^- defined in (3.1) is a subsolution of (1.6) and the pyramidal traveling front $\widetilde{V}$ defined in Theorem 2.2 is solution of (1.6). As a result of the comparison principle, we have

$$v^-(x,y,z,t) < \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \tag{3.2}$$

for all $(x,y,z,t) \in \mathbb{R}^4$ and $1 \le j \le n$. This shows that

$$\min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t)$$

is a supersolution of (1.6) for any $(x,y,z) \in \mathbb{R}^3$ and $t \in [0,T]$. It then follows from the comparison principle that

$$v^-(x,y,z,t) < V(x,y,z,t) \le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \tag{3.3}$$

for all $(x,y,z) \in \mathbb{R}^3$ and $t \in [0,T]$.

On the other hand, since

$$\max\{h_j(x,y), h_{j+1}(x,y)\} \le h(x,y) \quad \text{in } \mathbb{R}^2$$

for all $1 \le j \le n$, then

$$U\left(\frac{c}{l}(z + \max\{h_j(x,y), h_{j+1}(x,y)\}, 0)\right) \le v^-(x,y,z,0), \quad (x,y,z) \in \mathbb{R}^3, \quad 1 \le j \le n.$$

We consider the left-hand side and the right-hand side as an initial value of (1.5), respectively. Then the comparison principle yields that

$$v\left(\mathbf{x}, t+kT;\ U\left(\frac{c}{l}(z + \max\{h_j(x,y), h_{j+1}(x,y)\}, 0)\right)\right) \le v\left(\mathbf{x}, t+kT;\ v^-(x,y,z,0)\right) \tag{3.4}$$

for all $1 \le j \le n$. Notice that

$$h_j(x,y) = p_j(x - X_j(\rho), \quad y - Y_j(\rho)) + \tau\rho.$$

Sending $k \to \infty$ in (3.4), it then follows from Theorem 2.1, (2.10) and (2.6) that

$$E_j(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \le V(x,y,z,t), \quad (x,y,z) \in \mathbb{R}^3, \quad t \in [0,T].$$

This together with (3.3), we arrive at

$$\begin{aligned} \max_{1\le j\le n} E_j(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) &\le V(x,y,z,t) \\ &\le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \end{aligned} \tag{3.5}$$

for all $(x,y,z) \in \mathbb{R}^3$ and $t \in [0,T]$. As a consequence, (1.11) and (1.12) follow from (2.8), (2.2) and (3.5).

By (2.3) and (1.11) and applying the Schauder interior estimate to the following equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + l\frac{\partial}{\partial z}\right)(V - E_j) = f(V,t) - f(E_j,t),$$

we get

$$\inf_{W^-(t)+\delta \le V(\mathbf{x},t) \le W^+(t)-\delta,\, t\in[0,T]} V_z(\mathbf{x},t) > 0 \quad \text{for any } \delta > 0 \text{ small enough},$$

that is, (1.13) holds.

Notice that $|z + h(x, y)| \to \infty$ implies $\text{dist}(\mathbf{x}, \Gamma_j) \to \infty$ for $1 \le j \le n$. In light of (1.10), we have

$$\lim_{R_1 \to \infty} \sup_{|z+h(x,y)| \ge R_1, t \in [0,T]} \left| V(x, y, z, t) - U\left(\frac{c}{l}(z + h(x, y)), t\right) \right| \to 0.$$

By the interpolation $\| \cdot \|_{C^1} \le 2\sqrt{\| \cdot \|_{C^0} \| \cdot \|_{C^2}}$, we get (1.14) due to

$$\lim_{R_1 \to \infty} \sup_{|z+h(x,y)| \ge R_1, t \in [0,T]} \left| U_z\left(\frac{c}{l}(z + h(x, y)), t\right) \right| \to 0.$$

Now, we show that the time periodic curved front V is asymptotically stable. Set

$$\hat{l} := \max_{1 \le j \le n} l_j \ge 0.$$

Then, there holds

$$-\tau \hat{l} + p(x - X_j(-\hat{l}), y - Y_j(-\hat{l})) \le h(x, y) \quad \text{for } 1 \le j \le n,$$

whence

$$U\left(\frac{c}{l}(z - \tau\hat{l} + p(x - X_j(-\hat{l}), y - Y_j(-\hat{l}))), 0\right) \le U\left(\frac{c}{l}(z + h(x, y)), 0\right) \quad \text{for } 1 \le j \le n. \tag{3.6}$$

Considering the left-hand side and the right-hand side of (3.6) as initial values of (1.5), we have

$$\begin{aligned} &v\left(\mathbf{x}, t + kT;\ U\left(\frac{c}{l}(z - \tau\hat{l} + p(x - X_j(-\hat{l}),\ y - Y_j(-\hat{l})), 0)\right)\right) \\ &\le v\left(\mathbf{x}, t + kT;\ U\left(\frac{c}{l}(z + h(x, y)), 0\right)\right) \end{aligned}$$

from the comparison principle. Passing $k \to \infty$, one derives that

$$\widetilde{V}(x - X_j(-\hat{l}), y - Y_j(-\hat{l}), z - \tau\hat{l}, t) \le V(x, y, z, t), \quad (x, y, z) \in \mathbb{R}^3, \quad t \in [0, T] \tag{3.7}$$

for $1 \le j \le n$. Combining (3.5) and (3.7), we have

$$\begin{aligned} &\max_{1 \le j \le n} \widetilde{V}(x - X_j(-\hat{l}), y - Y_j(-\hat{l}), z - \tau\hat{l}, t) \\ &\le V(x, y, z, t) \le \min_{1 \le j \le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \end{aligned} \tag{3.8}$$

for all $(x, y, z) \in \mathbb{R}^3$ and $t \in [0, T]$.

In view of (3.8), we have

$$V(x, y, z, 0) \le \min_{1 \le j \le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, 0), \quad (x, y, z) \in \mathbb{R}^3.$$

For all $(x, y, z) \in \mathbb{R}^3$, and $t \in [0, T]$, set

$$V^*(\mathbf{x}, t) := \lim_{k \to \infty} v\left(\mathbf{x}, t + kT;\ \min_{1 \le j \le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, 0)\right).$$

Then, the comparison principle gives that

$$V(x, y, z, t) \le V^*(x, y, z, t) \le \min_{1 \le j \le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + \tau\rho, t) \tag{3.9}$$

and

$$V^*(x, y, z, t) = V^*(x, y, z, t + T)$$

for all $(x, y, z) \in \mathbb{R}^3$, and $t \in [0, T]$. By (1.11) and (2.8), we infer

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|>R, t\in[0,T]} \left|V^*(x, y, z, t) - V(x, y, z, t)\right| = 0.$$

On the other hand, owing to (1.13), there exists a positive constant σ such that

$$\beta\sigma \inf_{W^-(t)+\delta\le V(\mathbf{x},t)\le W^+(t)-\delta, t\in[0,T]} V_z(x, y, z, t) > 2K_0 \sup_{W^-(t)+\delta\le u\le W^+(t)-\delta, t\in[0,T]} |f_u(u, t)|,$$

where $0 < \delta < \delta_0$ is small enough, δ_0, β and K_0 are given in Lemma 2.3, $f_u(u, t)$ denotes the partial derivative of f with respect to u. For $0 < \delta < \delta_0$, it follows from Lemma 2.3 that

$$V\left(x, y, z + \sigma\delta\left(1 - \mathrm{e}^{-\beta(t+kT)}\right), t + kT\right) + \delta a(t + kT)$$

is a supersolution to (1.6) on $\mathbb{R}^3 \times [0, \infty)$. In light of the boundedness of $V^*(x, y, z, t)$ and the monotonicity of $V(x, y, z, t)$ with respect to z, we can take $\lambda > 0$ large enough such that

$$V^*(x, y, z, 0) \le V(x, y, z + \lambda, 0) + \delta, \quad (x, y, z) \in \mathbb{R}^3.$$

Then, the comparison principle implies

$$V^*(x, y, z, t + kT) \le V\left(x, y, z + \lambda + \sigma\delta\left(1 - \mathrm{e}^{-\beta(t+kT)}\right), t + kT\right) + \delta a(t + kT)$$

for $(x, y, z) \in \mathbb{R}^3$, $t \in [0, T]$ and $k \in \mathbb{N}$. Sending $k \to \infty$, we get

$$V^*(x, y, z, t) \le V(x, y, z + \lambda + \sigma\delta, t)$$

for all $(x, y, z) \in \mathbb{R}^3$ and $t \in [0, T]$.

Define

$$\Lambda := \inf\left\{\lambda \in (-\infty, +\infty) | V^*(x, y, z, 0) \le V(x, y, z + \lambda, 0) \quad \text{for all } (x, y, z) \in \mathbb{R}^3\right\}.$$

It is evident that $\Lambda \ge 0$ from (3.9). If $\Lambda = 0$, then $V^* \equiv V$ in $\mathbb{R}^4$. We prove $\Lambda = 0$ by a contradiction argument. Assume that $\Lambda > 0$. Then, we have

$$V^*(x, y, z, 0) \le V(x, y, z + \Lambda, 0), \quad (x, y, z) \in \mathbb{R}^3.$$

It then follows from the strong maximum principle that

$$V^*(x, y, z, 0) < V(x, y, z + \Lambda, 0), \quad (x, y, z) \in \mathbb{R}^3.$$

By (1.14), there exists a constant $R_0 > 0$ large enough such that

$$2\sigma \sup_{|z+h(x,y)|\ge R_0-\Lambda-1, t\in[0,T]} |V_z(x, y, z, t)| < 1.$$

Let

$$k_1 \in \left(0, \min\left\{\delta_0, \frac{1}{4\sigma}, \frac{\Lambda}{4\sigma}\right\}\right)$$

be small enough. If $|z + h(x, y)| \le R_0 - \Lambda - 1$, we have

$$V^*(x, y, z, 0) \le V(x, y, z + \Lambda - 2\sigma k_1, 0). \tag{3.10}$$

If $|z + h(x, y)| \geq R_0 - \Lambda - 1$, we have

$$\begin{aligned} &V(x, y, z + \Lambda, 0) - V(x, y, z + \Lambda - 2\sigma k_1, 0) \\ &= 2\sigma k_1 \int_0^1 V_z(x, y, z + \Lambda - 2\theta\sigma k_1, 0)\mathrm{d}\theta \leq k_1. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$V^*(x, y, z, 0) \leq V(x, y, z + \Lambda - 2\sigma k_1, 0) + k_1$$

for all $(x, y, z) \in \mathbb{R}^3$. It then follows from the comparison principle and Lemma 2.3 that

$$\begin{aligned} V^*(x, y, z, t + kT) \leq V\left(x, y, z + \Lambda - 2\sigma k_1 + \sigma k_1 \left(1 - \mathrm{e}^{-\beta(t+kT)}\right), t + kT\right) \\ + k_1 a(t + kT) \end{aligned}$$

for all $(x, y, z) \in \mathbb{R}^3$, $t \in [0, T]$ and $k \in \mathbb{N}$. Letting $k \to \infty$, we have

$$V^*(x, y, z, t) \leq V(x, y, z + \Lambda - \sigma k_1, t), \quad (x, y, z) \in \mathbb{R}^3, \quad t \in [0, T].$$

This contradicts with the definition of Λ. Thus, $V \equiv V^*$ in $\mathbb{R}^4$. Namely,

$$\lim_{k\to\infty} \left\| v\left(\mathbf{x}, t+kT; \min_{1\leq j\leq n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z+\tau\sigma, 0)\right) - V(x, y, z, t)\right\|_{L^\infty(\mathbb{R}^4)} = 0$$

By a similar argument to

$$v\left(\mathbf{x}, t + kT; \max_{1\leq j\leq n} \widetilde{V}(x - X_j(-\hat{l}), y - Y_j(-\hat{l}), z - \tau\hat{l}, 0)\right),$$

we have

$$\lim_{k\to\infty} \left\| v\left(\mathbf{x}, t+kT; \max_{1\leq j\leq n} \widetilde{V}(x - X_j(-\hat{l}), y - Y_j(-\hat{l}), z - \tau\hat{l}, 0)\right) - V(x, y, z, t)\right\|_{L^\infty(\mathbb{R}^4)} = 0.$$

Note that for any fixed $\mathbf{x} \in \mathbb{R}^3$ and $t \geq 0$, $v(\mathbf{x}, t; \cdot)$ is a continuous mapping in $BU(\mathbb{R}^3)$ with $BU(\mathbb{R}^3)$ standing for the set of bounded and continuous functions. By this continuity, Theorem 2.2 and the comparison principle, we obtain

$$\lim_{k\to\infty} \|v(\mathbf{x}, t + kT; v_0) - V(\mathbf{x}, t)\|_{L^\infty(\mathbb{R}^3\times[0,T])} = 0.$$

In other words, the desired result (1.16) holds. The proof is complete.

4 Proof of Theorem 1.3

Set

$$\begin{aligned} \mathcal{A}_0 &:= \left\{(\xi, \eta) \,\middle|\, \overline{\mathcal{D}} \subset \overline{B((\xi, \eta); R_*)}\right\} \\ &= \left\{(\xi, \eta) \in \mathbb{R}^2 \,\middle|\, \sqrt{(x - \xi)^2 + (y - \eta)^2} \leq R_* \quad \text{if } (x, y) \in \mathcal{C}\right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_m &:= \left\{(\xi, \eta) \,\middle|\, \overline{\mathcal{D}} \subset \left\{(x, y) \,\middle|\, p^{(m)}(x - \xi, y - \eta) \leq \tau R_*\right\}\right\} \\ &= \left\{(\xi, \eta) \in \mathbb{R}^2 \,\middle|\, p^{(m)}(x - \xi, y - \eta) \leq \tau R_* \quad \text{if } (x, y) \in \mathcal{C}\right\} \end{aligned}$$

for every $m \geq 2$, where $\overline{\mathcal{D}}$ and $p^{(m)}$ are defined as in (1.17) and (2.11), respectively. Since $(\xi_*(\theta), \eta_*(\theta)) \in \mathcal{A}_0$ for all $\theta \in [0, 2\pi)$, and $\partial B(0; R_*)$ is the inscribed circle of a polygon $\{(x, y) \in \mathbb{R}^2 | p^{(m)}(x, y) = \tau R_*\}$, then we have

$$\lim_{m\to\infty} \operatorname{dist}(\mathcal{A}_m, \mathcal{A}_0) = 0.$$

For any $\varepsilon > 0$ there holds

$$\operatorname{dist}((\xi, \eta), \mathcal{C}) \leq R_* + \varepsilon \quad \text{for all } (\xi, \eta) \in \mathcal{A}_m, \text{ and sufficiently large } m,$$

namely,

$$\operatorname{dist}(\mathcal{A}_m, \mathcal{C}) \leq R_* + \varepsilon$$

as m large enough. It then follows that

$$\sqrt{\xi^2 + \eta^2} \leq R_{\max} + R_* + \varepsilon \quad \text{for all } (\xi, \eta) \in \mathcal{A}_m$$

for m sufficiently large.

Proof of Theorem 1.3 Define

$$h^{(m)}(x, y) := \sup_{(\xi,\eta)\in\mathcal{A}_m} p^{(m)}(x - \xi, y - \eta) - \tau R_*.$$

Since $\partial B(0; R_*)$ is the inscribed circle of a polygon $\{(x, y) \in \mathbb{R}^2 | p^{(m)}(x, y) = \tau R_*\}$, then one arrives at

$$h^{(m)}(x, y) = \tau \max_{1\leq j\leq 2^m} \left(x \cos \frac{2\pi j}{2^m} + y \sin \frac{2\pi j}{2^m} - R\left(\frac{2\pi j}{2^m}\right) - R_* \right).$$

Moreover, it holds

$$p^{(m)}(x - \xi, y - \eta) - \tau R_* \leq p^{(m)}(x - R(\theta)\cos\theta, y - R(\theta)\sin\theta)$$

for all $(\xi, \eta) \in \mathcal{A}_m$ and $\theta \in [0, 2\pi]$, whence

$$\begin{aligned} &\sup_{(\xi,\eta)\in\mathcal{A}_m} p^{(m)}(x - \xi, y - \eta) - \tau R_* \\ &\leq \inf_{\theta\in[0,2\pi]} p^{(m)}(x - R(\theta)\cos\theta, y - R(\theta)\sin\theta) \quad \text{for all } (x, y) \in \mathbb{R}^2. \end{aligned}$$

It then follows that

$$p^{(m)}(x - \xi, y - \eta) - \tau R_* \leq h^{(m)}(x, y) \leq p^{(m)}(x - R(\theta)\cos\theta, y - R(\theta)\sin\theta)$$

for all $(\xi, \eta) \in \mathcal{A}_m$, $\theta \in [0, 2\pi]$ and $(x, y) \in \mathbb{R}^2$. As a result, one infers that

$$\begin{aligned} &\widetilde{V}^m \left(x - \xi, y - \eta, z + z^m - \tau R_*, t\right) \leq V(x, y, z + z^m, t) \\ &\leq \widetilde{V}^m \left(x - R(\theta)\cos\theta, y - R(\theta)\sin\theta, z + z^m, t\right) \end{aligned} \tag{4.1}$$

for all $(\xi, \eta) \in \mathcal{A}_m$ and $\theta \in [0, 2\pi]$, where z^m is defined as in (2.12) and $\widetilde{V}^m$ is the pyramidal-shaped traveling fronts corresponding to $p^{(m)}$. Combining this inequality with parabolic estimates [24] and Sobolev imbedding theorem, we obtain that the function $\widetilde{W}(\mathbf{x}, t)$ defined by

$$\widetilde{W}(\mathbf{x}, t) := \lim_{m\to\infty} V(x, y, z + z^m, t) \quad \text{in } C^{2,1}_{\text{loc}}(\mathbb{R}^4).$$

is a solution of (1.6). Moreover, it holds $\widetilde{W}(\mathbf{x}, t) = \widetilde{W}(\mathbf{x}, t + T)$. On the other hand, by letting $m \to \infty$ in (4.1), one derives from Theorem 2.4 that

$$\begin{aligned}\Psi\left(\sqrt{(x-\xi)^2+(y-\eta)^2}, z-\tau R_*, t\right) &\le \widetilde{W}(x, y, z, t)\\ &\le \Psi\left(\sqrt{(x-R(\theta)\cos\theta)^2+(y-R(\theta)\sin\theta)^2}, z, t\right)\end{aligned} \tag{4.2}$$

for all $(\xi, \eta) \in \mathcal{A}_m$ and $\theta \in [0, 2\pi]$. Thus (1.18) follows. Furthermore, we have

$$\widetilde{W}_z(x, y, z, t) > 0 \quad \text{for all } (x, y, z, t) \in \mathbb{R}^4$$

and

$$\frac{\partial \widetilde{W}}{\partial \nu} > 0 \quad \text{for } \nu = (\nu_1, 1) \quad \text{with } |\nu_1| \le \tau^{-1}$$

from the definition of $\widetilde{W}(x, y, z, t)$.

In view of Theorem 2.4 (ii), (iv) and (v), we get

$$\begin{aligned}\lim_{\sqrt{r^2+z^2}\to\infty} &\left(\Psi\left(\sqrt{(r\cos\theta-R(\theta)\cos\theta)^2+(r\cos\theta-R(\theta)\sin\theta)^2}, z, t\right)\right.\\ &\left.-\Psi\left(\sqrt{(r\cos\theta-\xi_*(\theta))^2+(r\cos\theta-\eta_*(\theta))^2}, z-\tau R_*, t\right)\right)=0 \quad \text{uniformly in } t\in[0, T]\end{aligned}$$

for all $\theta \in [0, 2\pi]$. It then follows that

$$\begin{aligned}\lim_{A\to\infty} \sup_{x^2+y^2+z^2\ge A^2, t\in[0,T]} &\left|\min_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-R(\theta)\cos\theta)^2+(y-R(\theta)\sin\theta)^2}, z, t\right)\right.\\ &\left.-\max_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x-\xi_*(\theta))^2+(y-\eta_*(\theta))^2}, z-\tau R_*, t\right)\right| = 0.\end{aligned}$$

Thus, (1.19) follows.

It remains to prove that such $\widetilde{W}$ is the unique solution of (1.6) under (1.19) which is also stable. We first show that

$$\inf_{W^-(t)+\delta\le\widetilde{W}(\mathbf{x},t)\le W^+(t)-\delta, t\in[0,T]} \widetilde{W}_z(x, y, z, t) > 0 \tag{4.3}$$

for $\delta > 0$ small enough. We only sketch the proof here, for details one can refer to [33, Lemma 4.6]. Indeed, we have $\widetilde{W}_z > 0$ in $\mathbb{R}^4$. Hence, $\widetilde{W}_z$ has a positive minimum on any compact subset of $\mathbb{R}^4$. Thus, we need only to study $\widetilde{W}_z(\mathbf{x}, t)$ as $|\mathbf{x}| \to \infty$. Assume that $\mathbf{x}_i = (x_i, y_i, z_i)$ satisfies $\lim_{i\to\infty} |\mathbf{x}_i| = \infty$ and $W^-(t) + \delta \le \widetilde{W}(\mathbf{x}_i, t) \le W^+(t) - \delta$ for all $t \in [0, T]$. It suffices to prove $\liminf_{i\to\infty, t\in[0,T]} \widetilde{W}_z(\mathbf{x}_i, t) > 0$. It then follows from (1.19) that

$$\begin{aligned}\lim_{A\to\infty} \sup_{x^2+y^2+z^2\ge A^2, t\in[0,T]} &\left|\widetilde{W}(x_i, y_i, z_i, t)\right.\\ &\left.-\min_{0\le\theta\le2\pi} \Psi\left(\sqrt{(x_i-R(\theta)\cos\theta)^2+(y_i-R(\theta)\sin\theta)^2}, z_i, t\right)\right| = 0.\end{aligned}$$

Namely,

$$\lim_{i\to\infty}\sup_{|\mathbf{x_i}|\in B(\mathbf{x}_i;2),t\in[0,T]}\Bigg|\widetilde{W}(x_i,y_i,z_i,t) - \min_{0\le\theta\le 2\pi}\Psi\left(\sqrt{(x_i-R(\theta)\cos\theta)^2+(y_i-R(\theta)\sin\theta)^2},z_i,t\right)\Bigg| = 0.$$

By the interpolation $\|\cdot\|_{C^1}\le 2\sqrt{\|\cdot\|_{C^0}\|\cdot\|_{C^2}}$, we have

$$\left\|\frac{\partial}{\partial z}\widetilde{W}(\mathbf{x}_i,t) - \min_{0\le\theta\le 2\pi}\frac{\partial}{\partial z}\Psi\left(\sqrt{(x_i-R(\theta)\cos\theta)^2+(y_i-R(\theta)\sin\theta)^2},z_i,t\right)\right\|_{C^0(B(\mathbf{x}_i;2)\times[0,T])}\to 0.$$

This together with (2.9) and Theorem 2.4 yields

$$\lim_{i\to\infty}\inf_{t\in[0,T]}\widetilde{W}_z(\mathbf{x}_i,t)>0.$$

Consequently, (4.3) holds. For $0<\delta<\delta_0$ with δ_0 given in Lemma 2.3, it follows from (4.3) and Lemma 2.3 that

$$\widetilde{W}\left(x,y,z+\sigma\delta\left(1-e^{-\beta t}\right),t\right)+\delta a(t)$$

is a supersolution of (1.6), where $\beta>0$ sufficiently small and $\sigma>0$ large enough such that

$$\sigma\beta\inf_{W^-(t)+\delta\le\widetilde{W}(\mathbf{x},t)\le W^+(t)-\delta}\widetilde{W}_z(x,y,z,t)>2K_0\sup_{W^-(t)+\delta\le u\le W^+(t)-\delta,t\in[0,T]}|f_u(u,t)|$$

with K_0 given in Lemma 2.3.

Suppose that $\widehat{W}(\mathbf{x},t)$ satisfies (1.6) and (1.19). We now prove that $\widehat{W}(\mathbf{x},t)\equiv\widetilde{W}(\mathbf{x},t)$ in $\mathbb{R}^4$. Assume that this is not true. Without loss of generality, we assume that $\widehat{W}(\mathbf{x},0)\le\widetilde{W}(\mathbf{x},0)$ but $\widehat{W}(\mathbf{x},0)\not\equiv\widetilde{W}(\mathbf{x},0)$. We can choose a constant $\lambda>0$ large enough such that

$$\widehat{W}(\mathbf{x},0)\le\widetilde{W}(x,y,z+\lambda,0)+\delta. \tag{4.4}$$

Indeed, (4.4) is obviously true if $|\mathbf{x}|^2<A^2$ for some constant $A>0$ sufficiently large. If $|\mathbf{x}|^2\ge A^2$, (4.4) follows from (1.19) and $W_z(x,y,z,t)>0$ for all $(x,y,z,t)\in\mathbb{R}^4$. It then follows from Lemma 2.3 and the comparison principle that

$$\widehat{W}(\mathbf{x},t+kT)\le\widetilde{W}\left(x,y,z+\lambda+\sigma\delta\left(1-e^{-\beta(t+kT)}\right),t+kT\right)+\delta a(t+kT)$$

for all $(x,y,z)\in\mathbb{R}^3$, $t\in[0,T]$ and $k\in\mathbb{N}$. Sending $k\to\infty$, we have

$$\widehat{W}(\mathbf{x},t)\le\widetilde{W}(x,y,z+\lambda+\sigma\delta,t)$$

for all $(x,y,z)\in\mathbb{R}^3$ and $t\in[0,T]$.

Define

$$\Lambda_1:=\inf\left\{\lambda\in(-\infty,\infty)|\widehat{W}(\mathbf{x},0)\le\widetilde{W}(x,y,z+\lambda,0)\quad\text{for all }(x,y,z)\in\mathbb{R}^3\right\}.$$

If $\Lambda_1=0$, we get $\widehat{W}(\mathbf{x},0)\le\widetilde{W}(\mathbf{x},0)$. Similarly, we can obtain $\widehat{W}(\mathbf{x},0)\ge\widetilde{W}(\mathbf{x},0)$. Thus, $\widehat{W}(\mathbf{x},t)\equiv\widetilde{W}(\mathbf{x},t)$ for all $(x,y,z)\in\mathbb{R}^3$ and $t\in[0,T]$. We prove $\Lambda_1=0$ by a contradiction argument. Assume that $\Lambda_1\ne 0$. Without loss of generality, we assume that $\Lambda_1>0$. It then follows from the strong maximum principle that

$$\widehat{W}(\mathbf{x},0)<\widetilde{W}(x,y,z+\Lambda_1,0)\quad\text{for all }(x,y,z)\in\mathbb{R}^3.$$

Fix $R_1 > 0$ sufficiently large such that

$$2\sigma \sup_{|z+\phi(\sqrt{x^2+y^2})|\geq R_1-\Lambda_1-1} |\widetilde{W}_z(\mathbf{x},0)| < 1,$$

where ϕ is defined in (2.15). Define

$$\mathcal{O} := \left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \left|z+\phi\left(\sqrt{x^2+y^2}\right)\right| \leq R_1-\Lambda_1-1\right\}$$

For $(x,y,z)\in\mathcal{O}$, we have

$$\begin{aligned}&\Psi\left(\sqrt{x^2+y^2}, z+\Lambda_1, 0\right) - \Psi\left(\sqrt{x^2+y^2}, z, 0\right)\\ &\geq \Lambda_1 \inf_{\alpha^-+\delta\leq\Psi\leq\alpha^+-\delta} \Psi_z\left(\sqrt{x^2+y^2}, z, 0\right) > 0,\end{aligned}$$

where $\alpha^{\pm}$ are given in the assumption (H2). Then, we have

$$\inf_{(x,y,z)\in\mathcal{O}}\left(\Psi\left(\sqrt{x^2+y^2}, z+\Lambda_1-2\sigma\varepsilon, 0\right) - \Psi\left(\sqrt{x^2+y^2}, z, 0\right)\right) > 0.$$

If $\mathbf{x}\in\mathcal{O}$ and $|\mathbf{x}|$ is large enough, say $|\mathbf{x}|\geq R_0$ for some $R_0>0$, by (1.20), we get

$$\widehat{W}(\mathbf{x},0) \leq \widetilde{W}(x,y,z+\Lambda_1-2\sigma\varepsilon,0)$$

for $\varepsilon>0$ small enough. If $\mathbf{x}\in\mathcal{O}\cap B(0;R_0)$, we have

$$\widehat{W}(\mathbf{x},0) \leq \widetilde{W}(x,y,z+\Lambda_1-2\sigma\varepsilon,0)$$

for sufficiently small $0<\varepsilon<\delta_0$ due to the compactness of the set $\mathcal{O}\cap B(0;R_0)$ in $\mathbb{R}^3$. Thus, we obtain that

$$\widehat{W}(\mathbf{x},0) \leq \widetilde{W}(x,y,z+\Lambda_1-2\sigma\varepsilon,0) \quad \text{in } \mathcal{O}. \tag{4.5}$$

In $\mathbb{R}^3\setminus\mathcal{O}$, we have

$$\begin{aligned}&\widetilde{W}(x,y,z+\Lambda_1-2\sigma\varepsilon,0) - \widetilde{W}(x,y,z+\Lambda_1,0)\\ &= 2\sigma\varepsilon\int_0^1 -\widetilde{W}_z(x,y,z+\Lambda_1-2\theta\sigma\varepsilon,0)\mathrm{d}\theta \geq -\varepsilon.\end{aligned}$$

This yields

$$\widehat{W}(\mathbf{x},0) \leq \widetilde{W}(x,y,z+\Lambda_1-2\sigma\varepsilon,0)+\varepsilon \quad \text{in } \mathbb{R}^3\setminus\mathcal{O}. \tag{4.6}$$

Combining(4.5) and (4.6), we have

$$\widehat{W}(\mathbf{x},0) \leq \widetilde{W}(x,y,z+\Lambda_1-2\sigma\varepsilon,0)+\varepsilon \quad \text{in } \mathbb{R}^3.$$

It then follows from Lemma 2.3 that

$$\widetilde{W}\left(x,y,z+\Lambda_1-2\sigma\varepsilon+\sigma\varepsilon\left(1-\mathrm{e}^{-\beta t}\right),t\right)+\varepsilon a(t)$$

is a supersolution of (1.6) on $\mathbb{R}^3\times[0,\infty)$. Hence, we have

$$\widehat{W}(\mathbf{x},t+kT) \leq \widetilde{W}\left(x,y,z+\Lambda_1-2\sigma\varepsilon+\sigma\varepsilon\left(1-\mathrm{e}^{-\beta(t+kT)}\right),t+kT\right)+\varepsilon a(t+kT)$$

for all $\mathbf{x} \in \mathbb{R}^3$, $t \in [0, T]$ and $k \in \mathbb{N}$. Letting $k \to \infty$, we have

$$\widehat{W}(\mathbf{x}, t) \leq \widetilde{W}(x, y, z + \Lambda_1 - \sigma\varepsilon, t)$$

for all $(x, y, z) \in \mathbb{R}^3$ and $t \in [0, T]$. This contradicts the definition of Λ_1. Thus, $\Lambda_1 = 0$ and $\widehat{W}(\mathbf{x}, t) \equiv \widetilde{W}(\mathbf{x}, t)$ in $\mathbb{R}^4$ follows.

Now we prove that (1.21) holds. Define

$$\underline{W}(x, y, z, t) := \max_{0 \leq \theta \leq 2\pi} \Psi\left(\sqrt{(x - \xi_*(\theta))^2 + (y - \eta_*(\theta))^2}, z - \tau R_*, t\right)$$

and

$$\overline{W}(x, y, z, t) := \min_{0 \leq \theta \leq 2\pi} \Psi\left(\sqrt{(x - R(\theta)\cos\theta)^2 + (y - R(\theta)\sin\theta)^2}, z, t\right)$$

for all $(x, y, z, t) \in \mathbb{R}^4$. Then, $\underline{W}(x, y, z, t)$ and $\overline{W}(x, y, z, t)$ are a subsolution and a supersolution of (1.6), respectively. Thus, $\lim_{t\to\infty} v(\mathbf{x}, t; \underline{W}(x, y, z, 0))$ and $\lim_{t\to\infty} v(\mathbf{x}, t; \overline{W}(x, y, z, 0))$ are solutions of (1.6) between $\underline{W}(x, y, z, t)$ and $\overline{W}(x, y, z, t)$. By the uniqueness of the solution bounded between $\underline{W}$ and $\overline{W}$, we have $\lim_{t\to\infty} v(\mathbf{x}, t; \underline{W}(x, y, z, 0)) = \lim_{t\to\infty} v(\mathbf{x}, t; \overline{W}(x, y, z, 0)) = \widetilde{W}(\mathbf{x}, t)$. Thanks to the continuity of $v(\mathbf{x}, t; v_0)$ with respect to v_0, we get the desired result from the assumption (1.20). The proof is complete. □

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