Annali di Matematica (2017) 196:479–517
DOI 10.1007/s10231-016-0582-7

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Superlinear Neumann problems with the p-Laplacian plus an indefinite potential

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Received: 4 December 2015 / Accepted: 25 May 2016 / Published online: 10 June 2016

Abstract We consider nonlinear Neumann problems driven by the p-Laplacian plus an indefinite potential and with a superlinear reaction which need not satisfy the Ambrosetti–Rabinowitz condition. First, we prove an existence theorem, and then, under stronger conditions on the reaction, we prove a multiplicity theorem producing three nontrivial solutions. Then, we examine parametric problems with competing nonlinearities (concave and convex terms). We show that for all small values of the parameter $\lambda > 0$, the problem has five nontrivial solutions and if $p = 2$ (semilinear equation), there are six nontrivial solutions. Finally, we prove a bifurcation result describing the set of positive solutions as the parameter $\lambda > 0$ varies.

Keywords p-Laplacian · Superlinear reaction · Multiple solutions · Critical groups · Competing nonlinearities · Bifurcation theorem · Indefinite potential · Neumann problem

G. Fragnelli: Member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità a e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and supported by the INdAM-GNAMPA Project 2016 *Control, regularity and viability for some types of diffusive equations*.

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Mathematics Subject Classification 35J20 · 35J65 · 58E05

1 Introduction

In this paper, we study the following nonlinear Neumann problem

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = f(z, u(z)), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In this problem, $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2– boundary $\partial\Omega$ and $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$. By Δ_p, we denote the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega)\ (1 < p < \infty).$$

The potential function $\beta(\cdot)$ may be sign-changing. So, in problem (1) the differential operator is not in general coercive. The reaction $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the map $z \mapsto f(z, x)$ is measurable, and for a.e. $z \in \Omega$ the map $x \mapsto f(z, x)$ is continuous).

The aim of this work is to study the existence and multiplicity of nontrivial solutions for problem (1), when the reaction $x \mapsto f(z, x)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$. A special feature of our work is that the superlinearity of $f(z, x)$ is not expressed using the usual (in such cases) Ambrosetti–Rabinowitz condition (the AR-condition for short). In fact, we employ an alternative condition which includes superlinear reactions with "slower" growth near $\pm\infty$ and which fail to satisfy the AR-condition.

Then, we consider parametric equations with a reaction having the competing effects of "concave" (sublinear) and "convex" (superlinear) terms, and we prove multiplicity results for small values of a parameter $\lambda > 0$. Finally, we focus on positive solutions and prove a bifurcation-type result near zero, describing the set of positive solutions as the parameter $\lambda > 0$ varies.

Equations with the Neumann p-Laplacian plus an indefinite potential were studied recently by Mugnai–Papageorgiou [31], who developed the spectral properties of the indefinite differential operator $u \mapsto \Delta_p u + \beta(z)|u|^{p-2}u$ and studied resonant equations driven by such operators. Analogous Dirichlet problems were investigated by Cuesta [5], Cuesta–Ramos Quoirin [6], Del Pezzo–Fernandez Bonder–Rossi [7], Fernandez Bonder–Del Pezzo [10], Leadi–Yechoui [17] and Lopez Gomez [21]. We mention also the semilinear work of Gasinski–Papageorgiou [15]. However, none of these works prove multiplicity results producing five or six nontrivial solutions or, in the case of parametric problems, provide the precise dependence on the solutions on the parameter. More precise comparisons with the existing results in the literature will be given as we develop our existence and multiplicity results.

Our approach uses variational methods based on critical point theory, together with Morse theory (critical groups) and truncation, perturbation and comparison techniques. For the reader's convenience, in the next section we recall the main mathematical tools which we will use in the sequel. Finally, we remark that we use Morse theory also to prove uniqueness results (see Sect. 3), while in general it is used to provide multiplicity results.

2 Mathematical background

Let X be a Banach space, and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$, we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$; we say that φ satisfies the "Cerami condition" (the "C-condition" for short) if the following holds:

$$\begin{gathered}\text{every sequence } \{x_n\}_{n\geq 1} \subseteq X \text{ such that } \{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R} \text{ is bounded} \\ \text{and } (1+\|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n\to\infty, \\ \text{admits a strongly convergent subsequence.}\end{gathered}$$

Using this condition, we can prove the following theorem, known as the "mountain pass theorem," due to Ambrosetti–Rabinowitz [3].

Theorem 1 *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, $x_0,\ x_1 \in X$ satisfy*

$$\max\left\{\varphi(x_0), \varphi(x_1)\right\} \leq \inf\left\{\varphi(x) \,:\, \|x-x_0\| = \rho\right\} = \eta_\rho, \qquad \|x_1 - x_0\| > \rho > 0,$$

set $\Gamma := \left\{\gamma \in C([0,1], X) \,:\, \gamma(0) = x_0,\ \gamma(1) = x_1\right\}$ *and*

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)),$$

then $c \geq \eta_\rho$ and c is a critical value for φ.

In the analysis of problem (1), we will use the Sobolev space $W^{1,p}(\Omega)$. By $\|\cdot\|$, we denote the norm of this space. So, we have

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}}, \quad \text{for all } u \in W^{1,p}(\Omega).$$

In addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\bar{\Omega})$. The latter is an ordered Banach space with positive cone

$$C_+ := \left\{u \in C^1(\bar{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\right\}.$$

This cone above has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \bar{\Omega}\right\}.$$

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which has subcritical growth, i.e.,

$$|f_0(z,x)| \leq a_0(z)(1+|x|^{r-1}) \quad \text{for a.e. } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+ = \left\{u \in L^\infty(\Omega) \,:\, u \geq 0\right\}$, and

$$1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

Let $F_0(z,x) = \int_0^x f_0(z,s)\,\mathrm{d}s$, and consider the C^1 functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega F_0(z, u(z))\,\mathrm{d}z \qquad \text{for all } u \in W^{1,p}(\Omega).$$

The next result can be found in Motreanu–Papageorgiou [25], and it is the outgrowth of the nonlinear regularity theory of Lieberman [18].

Proposition 2 *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\bar{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in C^1(\bar{\Omega}) \textit{ with } \|h\|_{C^1(\bar{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ and it is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\|_{W^{1,p}(\Omega)} \leq \rho_1.$$

Remark 1 We mention that the first such a result relating local minimizers of functionals was proved by Brezis–Nirenberg [4] for the space $H_0^1(\Omega)$.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega |Du|^{p-2}(Du, Dv)_{\mathbb{R}^N} \mathrm{d}z \qquad \text{for all } u,\ v \in W^{1,p}(\Omega). \tag{2}$$

The next proposition summarizes the main properties of this map (see, for example, Gasinski–Papageorgiou [14]).

Proposition 3 *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined above is bounded (that is, maps bounded sets to bounded sets), semicontinuous, monotone (thus maximal monotone) and of type $(S)_+$, i.e.,*

$$\textit{if } u_n \rightharpoonup u \textit{ in } W^{1,p}(\Omega) \textit{ and } \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,$$
$$\textit{then} \quad u_n \to u \textit{ in } W^{1,p}(\Omega).$$

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. As usual, we set $\varphi^c := \{u \in X : \varphi(u) \leq c\}$, $K_\varphi := \{u \in X : \varphi'(u) = 0\}$ and $K_\varphi^c := \{u \in K_\varphi : \varphi(u) = c\}$.

Let (Y_1, Y_2) be a topological pair s.t. $Y_2 \subset Y_1 \subset X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$, we denote the kth relative singular homology group with coefficients in a fixed field $\mathbb{F}$ of characteristic zero (for example, $\mathbb{F} = \mathbb{R}$). Then the singular homology groups $H_k(Y_1, Y_2)$ are in fact $\mathbb{F}$-vector spaces, and we denote by $\dim H_k(Y_1, Y_2)$ their dimensions. Moreover, the boundary homomorphism ∂ and the homomorphisms f_* induced by maps f of pairs are $\mathbb{F}$-linear.

Consider an isolated element $u \in K_\varphi^c$. The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\}) \quad \text{for all} \quad k \geq 0,$$

where U is a neighborhood of u s.t. $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of the singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all} \quad k \geq 0.$$

The second deformation theorem (see, for example, Gasinski–Papageorgiou [13](p. 628)), implies that the above definition of critical groups at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define

$$M(t, u) = \sum_{k \geq 0} \dim C_k(\varphi, u) t^k \text{ for all } t \in \mathbb{R} \text{ and all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \geq 0} \dim C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}.$$

Then the Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t), \tag{3}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Next we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Assuming that $\beta \in L^q(\Omega)$ with $q > \dfrac{Np}{p-1} = Np'$, we know (see Mugnai–Papageorgiou [31]) that (4) admits a smallest eigenvalue $\hat{\lambda}_1(\beta)$ which is isolated and simple and admits the following variational characterization:

$$\hat{\lambda}_1(\beta) = \inf \left\{ \Psi(u) : u \in W^{1,p}(\Omega), \|u\|_p = 1 \right\}, \tag{5}$$

where $\Psi(u) = \|Du\|_p^p + \int_\Omega \beta(z)|u(z)|^p dz$ for all $u \in W^{1,p}(\Omega)$. The infimum in (5) is realized on the corresponding one-dimensional eigenspace. Then, from (5) it is clear that the eigenfunctions corresponding to $\hat{\lambda}_1(\beta)$ do not change sign. By $\hat{u}_1(\beta)$, we denote the positive, L^p-normalized (that is $\|\hat{u}_1(\beta)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(\beta)$. We know that $\hat{u}_1(\beta) \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ and if $\beta \in L^\infty(\Omega)$, then $\hat{u}_1(\beta) \in \text{int}C_+$ (see Mugnai–Papageorgiou [31]). Since $\hat{\lambda}_1(\beta)$ is isolated and the set of eigenvalues is closed, the second eigenvalue is well defined by

$$\hat{\lambda}_2(\beta) = \inf \left\{ \hat{\lambda} > \hat{\lambda}_1(\beta) : \hat{\lambda} \text{ is an eigenvalue of (4)} \right\}.$$

Let $V := \left\{ u \in W^{1,p}(\Omega) : \int_\Omega \hat{u}_1(\beta) u \mathrm{d}z = 0 \right\}$. We define

$$\hat{\lambda}_V(\beta) := \inf \left\{ \Psi(u) : u \in V, \|u\|_p = 1 \right\}.$$

We know that (see Mugnai–Papageorgiou [31, Proposition 3.8])

$$\hat{\lambda}_1(\beta) < \hat{\lambda}_V(\beta) \leq \hat{\lambda}_2(\beta).$$

If $\beta \geq 0$, $\beta \neq 0$, then $\hat{\lambda}_1(\beta) > 0$.

If $\beta \equiv 0$, then we write $\hat{\lambda}_1(0) = \hat{\lambda}_1$, $\hat{\lambda}_V(0) = \hat{\lambda}_V$ and $\hat{\lambda}_2(0) = \hat{\lambda}_2$. In this case, we have

$$\hat{\lambda}_1 = 0 \text{ and the corresponding eigenspace is } \mathbb{R},$$
$$V = \left\{u \in W^{1,p}(\Omega) : \int_\Omega u(z)\mathrm{d}z = 0\right\} \quad \text{and} \quad \hat{u}_1(0) = \hat{u}_1 = \frac{1}{|\Omega|_N^{\frac{1}{p}}},$$

where we have denoted by $|\cdot|_N$ the Lebesgue measure in $\mathbb{R}^N$. We also mention that any eigenfunction corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(\beta)$ of (4) must be nodal (sign-changing).

Finally, let us fix our notations. Given $x \in \mathbb{R}$, we set $x^\pm := \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$, and we know that $u^\pm \in W^{1,p}(\Omega)$, $|u| = u^+ + u^-$, $u = u^+ - u^-$.

If $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then we set

$$N_h(u)(\cdot) := h(\cdot, u(\cdot)), \text{ for all } u \in W^{1,p}(\Omega),$$

the Nemytski map associated with the function h. Evidently, if h is a Carathéodory function, the map $z \mapsto N_h(u)(z) = h(z, u(z))$ is measurable.

3 Existence theorem

In this section, we prove an existence theorem for problem (1). In the special case where $\beta \equiv 0$, our result illustrates the difference between the superlinear Dirichlet and Neumann problems.

The hypotheses on the reaction $f(z, x)$ are the following:

Hypothesis 1 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.e. $z \in \Omega$ and

(1) $|f(z, x)| \leq a(z)(1 + |x|^{r-1})$ for a.e. $z \in \Omega$, all $x \in \mathbb{R}$ and with $a \in L^\infty(\Omega)_+$, $1 < p < r < p^*$;

(2) if $F(z, x) = \int_0^x f(z, s)\mathrm{d}s$, then

$$\lim_{x \to \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty \text{ uniformly for a.e. } z \in \Omega;$$

(3) if $\xi(z, x) = f(z, x)x - pF(z, x)$, then there exists $\beta^* \in L^1(\Omega)_+ := \{u \in L^1(\Omega) : u \geq 0\}$ such that

$$\xi(z, x) \leq \xi(z, y) + \beta^*(z) \text{ for a.e. } z \in \Omega \text{ and all } 0 \leq x \leq y \text{ or } y \leq x \leq 0;$$

(4) there exist $\delta > 0$ and $\theta \in (\hat{\lambda}_1(\beta), \hat{\lambda}_V(\beta))$ such that

$$\frac{\hat{\lambda}_1(\beta)}{p}|x|^p \leq F(z, x) \leq \frac{\theta}{p}|x|^p \text{ for a.e. } z \in \Omega, \text{ all } |x| \leq \delta.$$

Remark 2 Hypothesis 1.(2) implies that the primitive $F(z, \cdot)$ is p-superlinear near $\pm\infty$. This hypothesis together with 1.(3) implies that the reaction $f(z, \cdot)$ is $(p-1)$-superlinear near $\pm\infty$. However, note that we do not employ the usual AR-condition. We recall that the AR-condition says that there exist $q > p$ and $M > 0$ such that

$$0 < qF(z, x) \leq f(z, x)x \text{ for a.e. } z \in \Omega \text{ and for all } |x| \geq M \tag{6}$$
$$\text{and } \operatorname*{essinf}_\Omega F(\cdot, \pm M) > 0 \tag{7}$$

(see Ambrosetti–Rabinowitz [3] and Mugnai [26,27]). Integrating (6) and using (7), we obtain

$$c_1|x|^q \leq F(z,x) \text{ for a.e. } z \in \Omega, \text{ all } |x| \geq M, \text{ with } c_1 > 0. \tag{8}$$

The AR-condition ensures that the C-condition holds for the energy functional associated with problem (1). From (8), we see that the AR-condition implies that $F(z,\cdot)$ has at least q-polynomial growth near $\pm\infty$. This fact excludes from consideration p-superlinear potential functions which have "slower" growth near $\pm\infty$ (see the examples below). So, instead of the AR-condition, we employ Hypothesis 1.(3), which fits such nonlinearities in our framework. An analogous, but global condition, was first used by Jeanjean [16]. More precisely, Jeanjean [16] assumed the following:

$$\begin{aligned}&\text{there exists } \theta > 1 \text{ such that}\\ &\xi(z,\lambda x) \leq \theta\xi(z,x) \text{ for all } (z,x) \in \Omega \times \mathbb{R} \text{ and all } \lambda \in [0,1].\end{aligned} \tag{9}$$

The drawback of condition (9) is that it is global, and so many nonlinearities of interest fail to satisfy it. More recently, Miyagaki–Souto [22], working on a parametric, semilinear (that is $p = 2$) Dirichlet problem, assumed the following:

$$\text{for all } z \in \Omega, \xi(z,\cdot) \text{ is increasing on } [M,+\infty) \text{ and decreasing on } (-\infty,-M]. \tag{10}$$

In fact, one can show that (10) above is equivalent to saying that

$$\begin{aligned}&\text{for all } z \in \Omega \text{ the map } x \mapsto \frac{f(z,x)}{|x|^{p-2}x}\\ &\text{is increasing on } [M,+\infty) \text{ and decreasing on } (-\infty,-M].\end{aligned} \tag{11}$$

Of course, our Hypothesis 1.(3) is weaker than (10) and (11). It is also weaker than the condition used by Li–Yang [20], where $f(z,x)$ is continuous on $\bar{\Omega} \times \mathbb{R}$ and β^* is constant. In Li–Yang [20], the interested reader can find a nice survey of different generalizations of the AR-condition which exist in the literature and how they are related to each other.

Example 1 The following functions satisfy Hypothesis 1. For the sake of simplicity, we drop the z-dependence:

$$f_1(x) = \hat{\lambda}_1(\beta)|x|^{p-2}x + |x|^{r-2}x, \quad \text{with } 1 < p < r < p^*,$$

$$f_2(x) = \begin{cases} \hat{\lambda}_1(\beta)|x|^{p-2}x, & \text{if } |x| \leq 1,\\ |x|^{p-2}x\left(\ln|x| + \dfrac{\hat{\lambda}_1(\beta)}{p}\right), & \text{if } |x| > 1.\end{cases}$$

Note that f_2 fails to satisfy the AR-condition.

Let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\Psi(u) - \int_\Omega F(z,u(z))\mathrm{d}z \text{ for all } u \in W^{1,p}(\Omega),$$

where we recall that

$$\Psi(u) = \|Du\|_p^p + \int_\Omega \beta(z)|u(z)|^p \mathrm{d}z \text{ for all } u \in W^{1,p}(\Omega).$$

Under our assumptions, it is standard to show that $\varphi \in C^1(W^{1,p}(\Omega))$.

Proposition 4 *If Hypothesis 1.(1–3) holds and $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$, then the functional φ satisfies the C-condition.*

Proof Let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \quad \text{all } n \geq 1 \tag{12}$$

and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \quad \text{as } n \to \infty. \tag{13}$$

From (12) and (13), we obtain

$$\int_\Omega \xi(z, u_n)\mathrm{d}z \leq M_2 \quad \text{for all } n \geq 1. \tag{14}$$

Claim $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. We argue by contradiction. Thus, suppose that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{15}$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \rightharpoonup y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^r(\Omega) \quad \text{as } n \to \infty. \tag{16}$$

First suppose that $y \neq 0$ and let $\Omega_0 := \{z \in \Omega : y(z) = 0\}$. Then

$$|u_n(z)| \to \infty \quad \text{for a.e. } z \in \Omega_0^c.$$

Then, Hypothesis 1.(2) and Fatou's Lemma imply that

$$\lim_{n\to\infty} \int_\Omega \frac{F(z, u_n(z))}{\|u_n\|^p}\mathrm{d}z = \infty. \tag{17}$$

On the other hand, from (12), we have that

$$\int_\Omega \frac{F(z, u_n(z))}{\|u_n\|^p}\mathrm{d}z \leq M_3 \quad \text{for some } M_3 > 0 \quad \text{and all } n \geq 1, \tag{18}$$

(note that $\{\Psi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded). Comparing (17) and (18), we reach a contradiction.

Next suppose that $y = 0$. We fix $\eta > 0$ and define

$$v_n = (2\eta)^{\frac{1}{p}} y_n \in W^{1,p}(\Omega) \quad \text{for all } n \geq 1.$$

Evidently

$$v_n \to 0 \quad \text{in } L^r(\Omega)$$

(see (16) and recall that $y = 0$). Using Krasnoselskii's Theorem (see, for example, Gasinski–Papageorgiou [13, Theorem 3.4.4]), we have

$$\int_\Omega F(z, v_n)dz \to 0, \quad \text{as } n \to \infty. \tag{19}$$

Because of (15), we can find $n_0 \geq 1$ such that

$$0 < (2\eta)^{\frac{1}{p}} \frac{1}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{20}$$

Let $t_n \in [0, 1]$ be such that

$$\varphi(t_n u_n) = \max_{0 \le t \le 1} \varphi(t u_n).$$

From (20), it follows that

$$\begin{aligned}\varphi(t_n u_n) &\ge \varphi(v_n) \\ &= 2\eta \Psi(y_n) - \int_\Omega F(z, v_n)\mathrm{d}z \quad \text{for all } n \ge 1.\end{aligned} \tag{21}$$

Since, by assumption, $q > Np' = \dfrac{Np}{p-1}$, we have $q' < (Np')' = \dfrac{Np}{Np - p + 1}$ (recall for any $\tau \in (1, \infty)$, $\dfrac{1}{\tau} + \dfrac{1}{\tau'} = 1$). So, it follows that $pq' < p^*$. For $u \in W^{1,p}(\Omega)$, from the Sobolev embedding theorem, we have $|u|^p \in L^{q'}(\Omega)$. Using Hölder's inequality, we obtain

$$\left| \int_\Omega \beta(z)|u|^p \mathrm{d}z \right| \le \|\beta\|_q \|u\|_{pq'}^p. \tag{22}$$

We have

$$W^{1,p}(\Omega) \hookrightarrow L^{pq'}(\Omega) \hookrightarrow L^p(\Omega)$$

and the first embedding is compact (recall that $pq' < p^*$). So by Ehrling's inequality (see, for example, Papageorgiou–Kyritsi [34] (p. 698)), given $\epsilon > 0$, we can find $c(\epsilon) > 0$ such that

$$\|u\|_{pq'}^p \le \epsilon \|u\|^p + c(\epsilon)\|u\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega). \tag{23}$$

Then, from (22) and (23), we have

$$(1 - \epsilon\|\beta\|_q)\|u\|^p \le \Psi(u) + (1 + c(\epsilon)\|\beta\|_q)\|u\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega). \tag{24}$$

Now, we return to (21) and use (24). Then

$$\varphi(t_n u_n) \ge 2\eta[(1 - \epsilon\|\beta\|_q) - (1 + c(\epsilon)\|\beta\|_q)\|y_n\|_p^p] - \int_\Omega F(z, v_n)\mathrm{d}z \quad \text{for all } n \ge n_0 \tag{25}$$

(recall that $\|y_n\| = 1$ for all $n \ge 1$).

Choose $\epsilon \in \left(0, \dfrac{1}{\|\beta\|_q}\right)$ and note that

$$\|y_n\|_p^p \to 0 \quad \text{as } n \to \infty$$

(see (16) and recall $y = 0$), and

$$\int_\Omega F(z, v_n)\mathrm{d}z \to 0 \quad \text{as } n \to \infty$$

from (19). So, by (25), it follows that given $\delta \in \big(0, 2\eta(1 - \epsilon\|\beta\|_q)\big)$, we can find $n_1 = n_1(\delta) \ge n_0$ such that

$$\varphi(t_n u_n) \ge 2\eta(1 - \epsilon\|\beta\|_q) - \delta \quad \text{for all } n \ge n_0.$$

Since $\eta > 0$ and $\delta > 0$ are arbitrary, by letting $\eta \to \infty$ and $\delta \to 0^+$, we conclude that

$$\varphi(t_n u_n) \to \infty \quad \text{as } n \to \infty. \tag{26}$$

Note that

$$\varphi(0) = 0 \text{ and } \varphi(u_n) \le M_1 \quad \text{for all } n \ge 1$$

by (12). Therefore, (26) implies that $t_n \in (0, 1)$ for all $n \ge n_1$. Hence

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu_n)\mid_{t=t_n} = 0 \quad \text{for all } n \ge n_1, \\ &\implies \Psi(t_n u_n) = \int_\Omega f(z, t_n u_n) t_n u_n \mathrm{d}z \quad \text{for all } n \ge n_1. \end{aligned} \tag{27}$$

Since u_n^+ and $-u_n^-$ have disjoint interior supports and $\xi(z, 0) = 0$ for a.e. $z \in \Omega$, using Hypothesis 1.(3) we have

$$\int_\Omega \xi(z, t_n u_n)\mathrm{d}z \le \int_\Omega \xi(z, u_n)\mathrm{d}z + 2\|\beta^*\|_1 \quad \text{for all } n \ge n_1.$$

Using the definition of ξ, (27) and (14), we obtain

$$p\varphi(t_n u_n) \le \int_\Omega \xi(z, u_n)\mathrm{d}z + \|\beta^*\|_1 \le M_4 \tag{28}$$

for some $M_4 > 0$ and all $n \ge n_1$. Comparing (26) and (28), we reach a contradiction. This proves the claim.

By virtue of the claim, we may assume that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega) \quad \text{and in } L^{q'}(\Omega). \tag{29}$$

We return to (13), choosing $u_n - u \in W^{1,p}(\Omega)$ as test function, we pass to the limit as $n \to \infty$ and use (29). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

and by Proposition 3, $u_n \to u \in W^{1,p}(\Omega)$ as $n \to \infty$.

Therefore, we conclude that the functional φ satisfies the C-condition. □

Proposition 5 *If Hypothesis 1.(1–3) holds, $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$ and $\inf \varphi(K_\varphi) > -\infty$, then $C_k(\varphi, \infty) = 0$ for all $k \ge 0$.*

Proof By virtue of Hypothesis 1.(1, 2), given any $\eta > 0$, we can find $c_\eta > 0$ such that

$$F(z, x) \ge \frac{\eta}{p}|x|^p - c_\eta \quad \text{for a.e. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{30}$$

For $u \in W^{1,p}(\Omega)$, $u \ne 0$ and $t > 0$, choosing $\eta > 0$ big, we have

$$\varphi(tu) \to -\infty \quad \text{as } t \to \infty. \tag{31}$$

Hypothesis 1.(3) implies that for all $u \in W^{1,p}(\Omega)$, we have

$$0 = \xi(z, 0) \le \xi(z, u^+(z)) + \beta^*(z) \text{ and } 0 = \xi(z, 0) \le \xi(z, -u^-(z)) + \beta^*(z) \text{ for a.e. } z \in \Omega.$$

Since u^+ and u^- have disjoint interior supports, we have

$$0 = \xi(z, 0) \le \xi(z, u(z)) + \beta^*(z) \text{ for a.e. } z \in \Omega.$$

Hence,

$$pF(z, u(z)) - f(z, u(z))u(z) \le \beta^*(z) \text{ for a.e. } z \in \Omega. \tag{32}$$

For $u \in W^{1,p}(\Omega)$, $u \neq 0$ and $t > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) \leq \frac{1}{t}[p\varphi(tu) + \|\beta^*\|_1] \text{ (see (32)).} \tag{33}$$

Because of (31), choosing $\mu_0 < -\frac{\|\beta^*\|_1}{p}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) < 0 \text{ for } t > 0 \text{ large enough.} \tag{34}$$

Let $\partial B_1 := \{u \in W^{1,p}(\Omega) : \|u\| = 1\}$. For $u \in \partial B_1$, we can find a maximal $\theta(u) > 0$ such that $\varphi(\theta(u)u) = \mu_0$, and the implicit function theorem implies that $\theta \in C(\partial B_1)$. We extend θ to all of $W^{1,p}(\Omega)\backslash\{0\}$ by setting

$$\theta_0(u) = \frac{1}{\|u\|}\theta\left(\frac{u}{\|u\|}\right) \quad \text{for all } u \in W^{1,p}(\Omega)\backslash\{0\}.$$

Clearly $\theta_0 \in C(W^{1,p}(\Omega)\backslash\{0\})$ and $\varphi(\theta_0(u)u) = \mu_0$ for all $u \in W^{1,p}(\Omega)\backslash\{0\}$. Moreover, $\varphi(u) = \mu_0$ implies that $\theta(u) = 1$. We set

$$\hat{\theta}_0(u) := \begin{cases} 1, & \text{if } \varphi(u) \leq \mu_0, \\ \theta_0(u), & \text{if } \varphi(u) > \mu_0. \end{cases} \tag{35}$$

Then $\hat{\theta}_0 \in C(W^{1,p}(\Omega)\backslash\{0\})$.

Now, we consider the homotopy $h : [0,1] \times (W^{1,p}(\Omega)\backslash\{0\}) \to W^{1,p}(\Omega)\backslash\{0\}$ defined by

$$h(t,u) = (1-t)u + t\hat{\theta}_0(u)u \quad \text{for all } (t,u) \in [0,1] \times (W^{1,p}(\Omega)\backslash\{0\}).$$

We have

$$h(0,u) = u,\ h(1,u) = \hat{\theta}_0(u)u \in \varphi^{\mu_0} \quad \text{for all } u \in W^{1,p}(\Omega)\backslash\{0\}$$

and

$$h(t,\cdot)\,|_{\varphi^{\mu_0}} = id\,|_{\varphi^{\mu_0}} \quad \text{for all } t \in [0,1] \quad \text{(see (35)).}$$

This shows that φ^{μ_0} is a strong deformation retract of $W^{1,p}(\Omega)\backslash\{0\}$.

Of course, ∂B_1 is a retract of $W^{1,p}(\Omega)\backslash\{0\}$ by the radial retraction $r_0(u) = \frac{u}{\|u\|}$. Moreover, using the deformation

$$\hat{h}(t,u) = (1-t)u + tr_0(u) \quad \text{for all } (t,u) \in [0,1] \times (W^{1,p}(\Omega)\backslash\{0\}),$$

we see that $W^{1,p}(\Omega)\backslash\{0\}$ is deformable onto ∂B_1. Then [9, Theorem 6.5, p. 325] implies that

$$\partial B_1 \quad \text{is a deformation retract of } W^{1,p}(\Omega)\backslash\{0\}.$$

Thus, we infer that φ^{μ_0} and ∂B_1 are homotopy equivalent, so that

$$H_k(W^{1,p}(\Omega), \varphi^{\mu_0}) = H_k(W^{1,p}(\Omega), \partial B_1) \quad \text{for all } k \geq 0, \tag{36}$$

see [24, Proposition 6.11].

Since $W^{1,p}(\Omega)$ is infinite dimensional, ∂B_1 is contractible in itself. So, from Motreanu–Motreanu–Papageorgiou [24, Propositions 6.24 and 6.25], we have

$$H_k(W^{1,p}(\Omega), \partial B_1) = 0 \quad \text{for all } k \geq 0 \Longrightarrow H_k(W^{1,p}(\Omega), \varphi^{\mu_0}) = 0 \quad \text{for all } k \geq 0.$$

Choosing $\mu_0 < \inf \varphi(K_\varphi)$ even more negative if necessary, we conclude that

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0 \text{ (see Section 2).}$$

$\square$

Next we look at the critical groups of φ at $u = 0$.

Proposition 6 *If Hypothesis 1 holds, $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$ and 0 is an isolated critical point for φ, then $C_1(\varphi, 0) \neq 0$.*

Proof Let $V = \left\{u \in W^{1,p}(\Omega) : \int_\Omega \hat{u}_1(\beta) u dz = 0\right\}$. Then we have the following direct sum decomposition:

$$W^{1,p}(\Omega) = \mathbb{R}\hat{u}_1(\beta) \oplus V.$$

Since the norms on $\mathbb{R}\hat{u}_1(\beta)$ are equivalent, we can find $\hat{\delta} > 0$ such that if $u \in \mathbb{R}\hat{u}_1(\beta)$ and $\|u\| \leq \hat{\delta}$, then $|u(z)| \leq \delta$ for all $z \in \bar{\Omega}$. So, for such a $u \in \mathbb{R}\hat{u}_1(\beta)$ $(u = \sigma \hat{u}_1(\beta))$, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\Psi(u) - \int_\Omega F(z, u)dz \leq \frac{1}{p}\Psi(u) - \frac{\hat{\lambda}_1(\beta)}{p}\|u\|_p^p \quad \text{(see Hypothesis 1.(4))} \\
&= \frac{|\sigma|^p}{p}[\Psi(\hat{u}_1(\beta)) - \hat{\lambda}_1(\beta)] = 0 \quad \text{(recall that } \|\hat{u}_1(\beta)\|_p = 1).
\end{aligned}$$

On the other hand, from Hypotheses 1.(1) and 1.(4) we have

$$F(z, x) \leq \frac{\theta}{p}|x|^p + c_3|x|^r \quad \text{for a.e. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_3 > 0. \tag{37}$$

If $u \in V$, then, from Hypothesis 1.(4),

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\Psi(u) - \int_\Omega F(z, u)dz \\
&\geq \frac{1}{p}[\Psi(u) - \theta\|u\|_p^p] - c_4\|u\|^r \quad \text{for some } c_4 > 0 \\
&\geq c_5\|u\|^p - c_6\|u\|^r \quad \text{for some } c_5, c_6 > 0.
\end{aligned}$$

Since $r > p$, it follows that for $\rho \in (0, 1)$ small enough, we have

$$\varphi(u) \geq 0 \quad \text{for all } u \in V \text{ with } \|u\| \leq \rho.$$

Then, from Motreanu–Motreanu–Papageorgiou [24, Corollary 6.88], we conclude that $C_1(\varphi, 0) \neq 0$. $\square$

Now, we are ready for our first existence theorem concerning problem (1). As usual, in what follows we assume that K_φ is finite (otherwise we already have infinitely many solutions for problem (1).

Theorem 7 *If Hypothesis 1 holds and $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$, then problem (1) admits a nontrivial solution $u_0 \in C^{1,\alpha}(\Omega)$, $\alpha \in (0, 1)$.*

Proof Let $\epsilon > 0$ be so small that $\varphi(K_\varphi) \cap [-\epsilon, \epsilon] = \{0 = \varphi(0)\}$. Pick $c < \inf \varphi(K_\varphi)$, $c < -\epsilon$. We have

$$\dim C_k(\varphi, 0) = \dim H_k(\varphi^\epsilon, \varphi^{-\epsilon}) \quad \text{for all } k \geq 0. \tag{38}$$

Moreover, by definition, we have

$$H_k(W^{1,p}(\Omega), \varphi^c) = C_k(\varphi, \infty) \quad \text{for all } k \geq 0. \tag{39}$$

We consider the following quadruple of sets

$$\varphi^c \subseteq \varphi^{-\epsilon} \subseteq \varphi^\epsilon \subseteq W^{1,p}(\Omega).$$

From Motreanu–Motreanu–Papageorgiou [24, Lemma 6.90], we have

$$\begin{aligned}
&\dim H_k(\varphi^\epsilon, \varphi^{-\epsilon}) \leq \dim H_{k-1}(\varphi^{-\epsilon}, \varphi^c) \\
&\qquad + \dim H_{k+1}(W^{1,p}(\Omega), \varphi^\epsilon) + \dim H_k(W^{1,p}(\Omega), \varphi^c) \quad \text{for all } k \geq 0, \\
&\Longrightarrow \dim C_k(\varphi, 0) \leq \dim H_{k-1}(\varphi^{-\epsilon}, \varphi^c) + \dim H_{k+1}(W^{1,p}(\Omega), \varphi^\epsilon) \\
&\qquad + \dim C_k(\varphi, \infty) \quad \text{for all } k \geq 0
\end{aligned}$$

(see (38) and (39)).

In particular, for $k = 1$, by Propositions 5 and 6, we have

$$1 \leq \dim H_0(\varphi^{-\epsilon}, \varphi^c) + \dim H_2(W^{1,p}(\Omega), \varphi^\epsilon). \tag{40}$$

From (40) it follows that at least one between $H_0(\varphi^{-\epsilon}, \varphi^c)$ and $H_2(W^{1,p}(\Omega), \varphi^\epsilon)$ is nontrivial. But $H_0(\varphi^{-\epsilon}, \varphi^c)$ is trivial, since φ satisfies the C-condition and it is not bounded below, see Motreanu–Motreanu–Papageorgiou [24, Proposition 6.64(b)].

Then, $H_2(W^{1,p}(\Omega), \varphi^\epsilon)$ is nontrivial, so that there is

$$u_0 \in K_\varphi \text{ with } \varphi(u_0) > 0 = \varphi(0),$$

see [24, Proposition 6.53].

Thus, $u_0 \neq 0$ and it is a solution problem (1). Moreover, the local nonlinear regularity result of Di Benedetto [8] implies that $u_0 \in C_0^{1,\alpha}(\Omega)$ with $\alpha \in (0, 1)$. □

Remark 3 If $\beta \in L^\infty(\Omega)$, then $u_0 \in C^{1,\alpha}(\bar{\Omega})$ (see Lieberman [18]). Suppose that $\beta \equiv 0$ and $f(z, x) = f(x) = |x|^{r-2}x$ with $p < r < p^*$. This reaction satisfies Hypothesis 1. If we consider the Dirichlet problem with this special reaction, then we know that it has at least three nontrivial solutions, two of which have constant sign. We refer to Wang [38] ($p = 2$, semilinear problem) and to Mugnai–Papageorgiou [32] (for $p \neq 2$ and even for nonhomogeneous equations). Note that the Neumann problem cannot have constant sign solutions, since necessarily we have $\int_\Omega |u(z)|^{r-2}u(z)\mathrm{d}z = 0$. So, we see that the superlinear Dirichlet and Neumann problems differ considerably.

4 Multiple solutions

In this section, we look for multiple solutions to problem (1). More precisely, our aim is to have the "Neumann" analogue of the three solutions theorem of Wang [38], where the problem is Dirichlet, semilinear (that is $p = 2$), $f(z, x) = f(x)$ with $f \in C^1(\mathbb{R})$, the AR-condition holds and $\beta \equiv 0$. The result of Wang [38] was extended to linearly perturbed problems by Mugnai [27] and Rabinowitz–Su–Wang [36] and to nonlinear and nonhomogeneous equations by Mugnai–Papageorgiou [32].

Now the hypotheses of the reaction f are the following:

Hypothesis 2 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.e. $z \in \Omega$, Hypothesis 2.(1–3) is the same as the corresponding Hypothesis 1.(1–4). There exist $\theta_0 \in L^\infty(\Omega)$ and $\eta_0 > 0$ such that

$$\theta_0(z) \le \hat{\lambda}_1(\beta) \quad \text{for a.e. } z \in \Omega, \theta_0 \ne \hat{\lambda}_1(\beta)$$

and

$$-\eta_0 \le \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \limsup_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \theta_0(z) \text{ uniformly for a.e. } z \in \Omega.$$

First we produce nontrivial solutions of constant sign.

To this end, we introduce the following truncations-perturbations of the reaction f:

$$\begin{aligned} \hat{f}_+(z,x) &= \begin{cases} 0, & \text{if } x \le 0, \\ f(z,x) + \gamma x^{p-1}, & \text{if } 0 < x \end{cases} \quad \text{and} \\ \hat{f}_-(z,x) &= \begin{cases} f(z,x) + \gamma |x|^{p-2}x, & \text{if } x < 0, \\ 0, & \text{if } 0 \le x, \end{cases} \end{aligned} \tag{41}$$

where $\gamma > (1 + c(\epsilon)\|\beta\|_q)$ (see the proof of Proposition 4), once ϵ is chosen. We set

$$\hat{F}_\pm(z,x) = \int_0^x \hat{f}_\pm(z,s)\mathrm{d}s.$$

Then, set $\hat{\beta}(z) = \beta(z) + \gamma$ and define

$$\hat{\Psi}(u) = \|Du\|_p^p + \int_\Omega \hat{\beta}(z)|u(z)|^p \mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

Finally, we consider the C^1-functional $\hat{\varphi}_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{p}\hat{\Psi}(u) - \int_\Omega \hat{F}_\pm(z,u(z))\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 8 *If Hypothesis 2 holds and $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$, then $u = 0$ is a strict local minimizer for functionals $\hat{\varphi}_\pm$ and φ.*

Proof We do the proof for the functional $\hat{\varphi}_+$, the proofs for $\hat{\varphi}_-$ and φ being similar. By virtue of Hypothesis 2.(1, 4), given $\epsilon > 0$, we can find $c_7 = c_7(\epsilon) > 0$ such that

$$F(z,x) \le \frac{1}{p}(\theta_0(z) + \epsilon)|x|^p + c_7|x|^r \quad \text{for a.e.} \quad z \in \Omega, \ \text{all } x \in \mathbb{R}. \tag{42}$$

Then for $u \in W^{1,p}(\Omega)$ and $\varepsilon > 0$ small, we have

$$\hat{\varphi}_+(u) \ge c_{10}\|u\|^p - c_8\|u\|^r \quad \text{for some } c_{10} > 0, \tag{43}$$

see Mugnai–Papageorgiou [31, Lemma 4.11]. Since $r > p$, from (43) we infer that there exists $\rho > 0$ such that

$$\hat{\varphi}_+(u) > 0 = \hat{\varphi}_+(0) \quad \text{for all } 0 < \|u\| \le \rho.$$

This proves that $u = 0$ is a (strict) local minimizer of $\hat{\varphi}_+$. Similarly for the functionals $\hat{\varphi}_-$ and φ. ◻

Using Hypothesis 2.(2), as in the proof of Proposition 5 (see (31)), we show that:

Proposition 9 *If Hypothesis 2 holds and $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$, then for every $u \in C_+ \backslash \{0\}$, we have $\hat{\varphi}_{\pm}(tu) \to -\infty$ as $t \to \pm\infty$.*

Since in Proposition 4 we only used Hypothesis 1.(1–3), we immediately have:

Proposition 10 *If Hypothesis 2 holds and $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$, then functionals $\hat{\varphi}_{\pm}$ satisfy the C-condition.*

Now, we are ready to produce two nontrivial solutions of constant sign.

Proposition 11 *If Hypothesis 2 holds and $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$, then problem (1) admits at least two nontrivial solutions of constant sign $\hat{u}, \hat{v} \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ with $\alpha \in (0, 1)$ such that*

$$\hat{v}(z) < 0 < \hat{u}(z) \quad \textit{for all } z \in \Omega.$$

Proof By virtue of Proposition 8 (see (43)), we can find $\rho \in (0, 1)$ so small that

$$\hat{\varphi}_+(0) = 0 < \inf \left\{ \hat{\varphi}_+(u) : \|u\| = \rho \right\} := \hat{m}_+. \tag{44}$$

Then (44), together with Propositions 9 and 10, implies that we can use Theorem 1 (the mountain pass theorem). So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(0) = 0 < \hat{m}_+ \le \hat{\varphi}_+(\hat{u}) \tag{45}$$

and

$$\hat{\varphi}'_+(\hat{u}) = 0. \tag{46}$$

From (45), we see that $\hat{u} \neq 0$, while from (46), we have

$$A(\hat{u}) + \hat{\beta}(z)|\hat{u}|^{p-2}\hat{u} = N_{\hat{f}_+}(\hat{u}). \tag{47}$$

On (47), we act with $-\hat{u}^- \in W^{1,p}(\Omega)$. Then

$$\Psi(\hat{u}^-) + \gamma \|\hat{u}^-\|_p^p = 0 \quad \text{(see (41))}. \tag{48}$$

From (24) with $\epsilon > 0$ small, we have

$$(1 - \epsilon\|\beta\|_q)\|\hat{u}^-\|^p - (1 + c(\epsilon)\|\beta\|_q)\|\hat{u}^-\|_p^p \le \Psi(\hat{u}^-). \tag{49}$$

Since $\gamma > (1 + c(\epsilon)\|\beta\|_q)$, from (48) and (49) it follows that

$$c_{11}\|\hat{u}^-\|^p \le 0 \tag{50}$$

for some $c_{11} > 0$, which implies

$$\hat{u} \ge 0, \quad \hat{u} \neq 0.$$

So, (47) becomes

$$A(\hat{u}) + \beta(z)\hat{u}^{p-1} = N_f(\hat{u}),$$

that is

$$\begin{cases} -\Delta_p \hat{u}(z) + \beta(z)\hat{u}(z)^{p-1} = f(z, \hat{u}(z)), & \text{for a.e. } z \in \Omega, \\ \dfrac{\partial \hat{u}}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

Using the Moser iteration technique, we have that $\hat{u} \in L^\infty(\Omega)$ (see Winkert [39]). Therefore, the local regularity result of Di Benedetto [8], implies that $\hat{u} \in C^{1,\alpha}(\Omega)$ with $\alpha \in (0, 1)$. Moreover, invoking the Harnack inequality of Pucci–Serrin [35, Theorem 7.2.1], we have

$$0 < \hat{u}(z) \quad \text{for all } z \in \Omega.$$

In a similar fashion, working this time with the functional $\hat{\varphi}_-$, we obtain another nontrivial constant sign solution $\hat{v} \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ with $\hat{v}(z) < 0$ for all $z \in \Omega$. □

If we strengthen the hypothesis on the potential function β, we can improve the conclusion of Proposition 11.

Proposition 12 *If Hypothesis 2 holds and $\beta \in L^\infty(\Omega)$, then problem (1) has at least two nontrivial solutions of constant sign*

$$\hat{u} \in \mathrm{int}C_+ \quad \textit{and} \quad \hat{v} \in -\mathrm{int}C_+.$$

Proof From Proposition 11, we already have two solutions $\hat{u}, \hat{v} \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$, $\alpha \in (0, 1)$, such that

$$\hat{v}(z) < 0 < \hat{u}(z) \quad \text{for all } z \in \Omega.$$

Using Lieberman [18, Theorem 2], we have that

$$\hat{u}, \hat{v} \in C^{1,\alpha}(\bar{\Omega}).$$

Hypothesis 2.(1, 4) implies that given $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$f(z, x)x + \xi_\rho |x|^p \geq 0 \quad \text{for a.e. } z \in \Omega, \ \text{all } |x| \leq \rho. \tag{51}$$

Then, for $\rho = \|\hat{u}\|_\infty$ and $\xi_\rho > 0$ as in (51), we have

$$\Delta_p \hat{u}(z) \leq (\|\beta\|_\infty + \xi_\rho)\hat{u}(z)^{p-1} \quad \text{a.e. in } \Omega,$$

so that $\hat{u} > 0$ in Ω by Pucci–Serrin [35, Theorem 5.3.1], and then, using the Neumann condition, $\hat{u} \in \mathrm{int}C_+$, see Pucci–Serrin [35, Theorem 5.5.1].

Similarly, we show that $\hat{v} \in -\mathrm{int}C_+$. □

To produce a third solution, we use Morse theory. So, first we compute the critical groups of functionals $\hat{\varphi}_\pm$ at infinity. The proof follows the steps in the proof of Proposition 5, with some necessary modifications, and assuming that 0 is the lowest critical value.

Proposition 13 *If Hypothesis 2 holds, $\beta \in L^q(\Omega)$ with $q > Np' = \dfrac{Np}{p-1}$ and $\varphi(K_\varphi) \geq 0$, then $C_k(\hat{\varphi}_+, \infty) = C_k(\hat{\varphi}_-, \infty) = 0$ for all $k \geq 0$.*

Proof We do the proof for the functional $\hat{\varphi}_+$, the proof for $\hat{\varphi}_-$ being similar. Recall that $\partial B_1 = \{u \in W^{1,p}(\Omega) : \|u\| = 1\}$ and let $\partial B_1^+ := \{u \in \partial B_1 : u^+ \neq 0\}$. We consider the deformation $h_+ : [0, 1] \times \partial B_1^+ \to \partial B_1^+$ defined by

$$h_+(t,u)=\frac{(1-t)u+t\hat{u}_1(\beta)}{\|(1-t)u+t\hat{u}_1(\beta)\|}\quad \text{for all } (t,u)\in[0,1]\times\partial B_1^+.$$

Note that

$$h_+(1,u)=\frac{\hat{u}_1(\beta)}{\|\hat{u}_1(\beta)\|}\in\partial B_1^+\quad \text{for all } u\in\partial B_1^+,$$

so that

$$\partial B_1^+ \text{ is contractible in itself.} \tag{52}$$

Hypothesis 2.2 implies that for all $u\in\partial B_1^+$, we have

$$\hat{\varphi}_+(tu)\to-\infty\quad \text{as } t\to\infty. \tag{53}$$

For every $u\in\partial B_1^+$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_+(u)\le\frac{1}{t}[p\hat{\varphi}_+(tu)+\|\beta^*\|_1]. \tag{54}$$

From (53) and (54), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_+(tu)<0\quad \text{for all } u\in\partial B_1^+ \text{ and for all } t>0 \text{ large enough.} \tag{55}$$

Now, choose $\theta\in\mathbb{R}^-$ such that

$$\theta<\min\left\{-\frac{\|\beta^*\|_1}{p},\inf_{\bar{B}_1}\hat{\varphi}_+\right\}\quad (\bar{B}_1=\{u\in W^{1,p}(\Omega):\|u\|\le1\}). \tag{56}$$

Using (55) and reasoning as in the proof of Proposition 5, via the implicit function theorem, we can find a unique function $\Lambda\in C(\partial B_1),\ \Lambda\ge1$ such that

$$\hat{\varphi}_+(tu)\begin{cases}>\theta, & \text{if } t\in[0,\Lambda(u)),\\ =\theta, & \text{if } t=\Lambda(u),\\ <\theta, & \text{if } t>\Lambda(u).\end{cases} \tag{57}$$

From (56) and (57), we have

$$\hat{\varphi}_+^\theta=\big\{tu:u\in\partial B_1,\ t\ge\Lambda(u)\big\}.$$

We introduce

$$D_+=\big\{tu:u\in\partial B_1^+,\ t\ge1\big\}.$$

Since $\Lambda\ge1$, it follows that $\hat{\varphi}_+^\theta\subseteq D_+$. We consider the deformation $\hat{h}_+:[0,1]\times D_+\to D_+$ defined by

$$\hat{h}_+(s,tu)=\begin{cases}(1-s)tu+s\Lambda(u)tu, & \text{if } t\in[1,\Lambda(u)],\\ tu, & \text{if } t>\Lambda(u).\end{cases}$$

Note that, using the definition of Λ, one has

$$\hat{h}_+(0,tu)=tu,\ \hat{h}_+(1,tu)=\Lambda(u)tu\in\hat{\varphi}_+^\theta\quad (\text{see } (57))$$

and

$$\hat{h}(s,\cdot)\mid_{\hat{\varphi}_+^\theta}=id\mid_{\hat{\varphi}_+^\theta}\quad \text{for all } s\in[0,1].$$

This means that $\hat{\varphi}_+^\theta$ is a strong deformation retract of D_+ and so we have

$$H_k(W^{1,p}(\Omega), D_+) = H_k(W^{1,p}(\Omega), \hat{\varphi}_+^\theta) \quad \text{for all } k \geq 0, \tag{58}$$

see [24, Corollary 6.15(a)].

Therefore, we consider the deformation $\tilde{h}_+ : [0,1] \times D_+ \to D_+$ defined by

$$\tilde{h}_+(s, tu) = (1-s)(tu) + s\frac{tu}{\|tu\|}.$$

This implies that D_+ is deformable into ∂B_1^+ and clearly the latter is a retract of D_+. Therefore, [9, Theorem 6.5] implies that ∂B_1^+ is a deformation retract of D_+ and so we have

$$H_k(W^{1,p}(\Omega), D_+) = H_k(W^{1,p}(\Omega), \partial B_1^+) \quad \text{for all } k \geq 0. \tag{59}$$

From (58) and (59), it follows that

$$H_k(W^{1,p}(\Omega), \hat{\varphi}_+^\theta) = H_k(W^{1,p}(\Omega), \partial B_1^+) = 0 \quad \text{for all } k \geq 0,$$

see [24, Propositions 6.24 and 6.25], since ∂B_1^+ is contractible in itself by (52). Hence, from the choice of θ in (56), we get

$$C_k(\hat{\varphi}_+, \infty) = 0 \quad \text{for all } k \geq 0.$$

Similarly, we show that

$$C_k(\hat{\varphi}_-, \infty) = 0 \quad \text{for all } k \geq 0.$$

□

Using this proposition, we can compute exactly the critical groups of the two constant sign solutions $\hat{u} \in \text{int}C_+$ and $\hat{v} \in -\text{int}C_+$ produced in Proposition 12. As always, we assume that K_φ is finite.

Proposition 14 *If Hypothesis 2 holds and $\beta \in L^\infty(\Omega)$, then $C_k(\varphi, \hat{u}) = C_k(\varphi, \hat{v}) = \delta_{k,1}\mathbb{F}$ for all $k \geq 0$.*

Proof We do the proof for $\hat{u} \in \text{int}C_+$, the proof for $\hat{v} \in -\text{int}C_+$ being similar. Note that $\varphi\,|_{C_+} = \hat{\varphi}_+\,|_{C_+}$ (see (41)). Consider the homotopy $\hat{h}(t,u) = (1-t)\varphi(u) + t\hat{\varphi}_+(u)$ for all $(t,u) \in [0,1] \times W^{1,p}(\Omega)$. Suppose we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t \in [0,1],\ u_n \to \hat{u} \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \hat{h}'_u(t_n, u_n) = 0 \text{ for all } n \geq 1. \tag{60}$$

We have

$$\begin{cases} -\Delta_p u_n(z) + \beta|u_n|^{p-2}u_n - t_n\gamma u_n^-(z)^p \\ \quad = (1-t_n)f(z, u_n(z)) + t_n f(z, u_n^+(z)) & \text{for a.e. } z \in \Omega, \\ \dfrac{\partial u_n}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

From Lieberman [18, Theorem 2], we know that we can find $\alpha \in (0,1)$ and $M_5 > 0$ such that

$$u_n \in C^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\bar{\Omega})} \leq M_5 \quad \text{for all } n \geq 1.$$

Exploiting the compact embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, from the convergence in (60), we have

$$u_n \to \hat{u} \text{ in } C^1(\bar{\Omega}).$$

Since $\hat{u} \in \mathrm{int}C_+$, we have $u_n \in C_+\backslash\{0\}$ for all $n \geq n_0$; hence, $\{u_n\}_{n\geq n_0}$ is a sequence of distinct solutions of (1), a contradiction to the assumption that K_φ is finite. So, (60) cannot happen and then the homotopy invariance of critical groups implies that

$$C_k(\varphi, \hat{u}) = C_k(\hat{\varphi}_+, \hat{u}) \quad \text{for all } k \geq 0. \tag{61}$$

It is easy to check that $K_{\hat{\varphi}_+} \subseteq C_+$ (see (50)). Hence we may assume that $K_{\hat{\varphi}_+} = \{0, \hat{u}\}$ or otherwise we already have a third nontrivial solution distinct from $\hat{u}$ and $\hat{v}$ (in fact this third solution is positive and belongs to $\mathrm{int}C_+$).

From the proof of Proposition 11, we know that

$$\hat{\varphi}_+(0) = 0 < \hat{m}_+ \leq \hat{\varphi}_+(\hat{u}) \quad \text{(see (45))}.$$

Let $\theta < 0 < \lambda < \hat{\varphi}_+(\hat{u})$ and consider the triple of sets

$$\hat{\varphi}_+^\theta \subseteq \hat{\varphi}_+^\lambda \subseteq W^{1,p}(\Omega).$$

For this triple, we consider the corresponding long exact sequence of homology groups

$$\cdots \to H_k(W^{1,p}(\Omega), \hat{\varphi}_+^\theta) \xrightarrow{i_*} H_k(W^{1,p}(\Omega), \hat{\varphi}_+^\lambda) \xrightarrow{\partial_*} H_{k-1}(\hat{\varphi}_+^\lambda, \hat{\varphi}_+^\theta) \to \cdots \tag{62}$$

for all $k \geq 1$, with i_* being the group homomorphism induced by the inclusion $i : \hat{\varphi}_+^\theta \to \hat{\varphi}_+^\lambda$ and ∂_* is the boundary homomorphism. From (62) and the well-known rank theorem, we have

$$\begin{aligned} \mathrm{dim} H_k(W^{1,p}(\Omega), \hat{\varphi}_\lambda^\theta) &= \dim \ker \partial_* + \dim \mathrm{im}\partial_* \\ &= \dim \mathrm{im} i_* + \dim \mathrm{im}\partial_* \quad \text{(since (62) is exact)}. \end{aligned} \tag{63}$$

Since $\theta < 0$ and $K_{\hat{\varphi}_+} = \{0, \hat{u}\}$, we see that

$$H_k(W^{1,p}(\Omega), \hat{\varphi}_+^\theta) = C_k(\hat{\varphi}_+, \infty) = 0 \quad \text{for all } k \geq 0 \text{ (see Proposition 13)}. \tag{64}$$

Therefore, from the choice of $\lambda > 0$ and since $K_{\hat{\varphi}_+} = \{0, \hat{u}\}$, we have

$$H_{k-1}(\hat{\varphi}_+^\lambda, \hat{\varphi}_+^\theta) = C_{k-1}(\hat{\varphi}_+, 0) \text{ and } H_k(W^{1,p}(\Omega), \hat{\varphi}_+^\lambda) = C_k(\hat{\varphi}_+, \hat{u}) \text{ for all } k \geq 1 \tag{65}$$

(see Motreanu–Motreanu–Papageorgiou [24, Lemma 6.55]).

But from Proposition 8, we have

$$C_{k-1}(\hat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{F} = \delta_{k,1}\mathbb{F} \quad \text{for all } k \geq 1. \tag{66}$$

Then from (65) and (66), it follows that in the chain (62), only the tail $k = 1$ is nontrivial. From (63), (64), (65) and (66), we have

$$\mathrm{dim} C_1(\hat{\varphi}_+, \hat{u}) = \dim \mathrm{im}\partial_* \leq 1. \tag{67}$$

But recall that $\hat{u} \in \mathrm{int}C_+$ is a critical point of mountain pass type for the functional $\hat{\varphi}_+$. Hence

$$\mathrm{dim} C_1(\hat{\varphi}_+, \hat{u}) \geq 1 \tag{68}$$

(see Motreanu–Motreanu–Papageorgiou [24, Proposition 6.100]).

From (67) and (68), we infer that

$$\begin{aligned} &C_k(\hat{\varphi}_+, \hat{u}) = \delta_{k,1}\mathbb{F} \quad \text{for all } k \geq 0, \\ &\implies C_k(\varphi, \hat{u}) = \delta_{k,1}\mathbb{F} \quad \text{for all } k \geq 0 \text{ (see (61))}. \end{aligned}$$

In a similar fashion, using this time the functional $\hat{\varphi}_-$, we show that

$$C_k(\varphi, \hat{v}) = \delta_{k,1}\mathbb{F} \quad \text{for all } k \geq 0.$$

□

Now we are ready to produce a third nontrivial solution for problem (1).

Theorem 15 *If Hypothesis 2 holds and $\beta \in L^\infty(\Omega)$, then problem (1) has at least three nontrivial solutions*

$$\hat{u} \in \mathrm{int}C_+, \ \hat{v} \in -\mathrm{int}C_+ \ \textit{and} \ \hat{y} \in C^1(\bar{\Omega})\backslash\{0\}.$$

Proof From Proposition 12, we already have two nontrivial constant sign solutions

$$\hat{u} \in \mathrm{int}C_+ \quad \text{and} \quad \hat{v} \in -\mathrm{int}C_+.$$

Suppose that $K_\varphi = \{0, \hat{u}, \hat{v}\}$. From Proposition 14, we have

$$C_k(\varphi, \hat{u}) = C_k(\varphi, \hat{v}) = \delta_{k,1}\mathbb{F} \quad \text{for all } k \geq 0. \tag{69}$$

Moreover, from Proposition 5, we have

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0. \tag{70}$$

Finally, from Proposition 8, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{F} \quad \text{for all } k \geq 0. \tag{71}$$

From (66), (69), (70), (71), and the Morse relation with $t = -1$ (see (3)), we have

$$2(-1)^1 + (-1)^0 = 0,$$

a contradiction. So, there exists $\hat{y} \in K_\varphi$, $\hat{y} \notin \{0, \hat{u}, \hat{v}\}$. Therefore, $\hat{y}$ is the third nontrivial solution of (1), and the nonlinear regularity theory (see Lieberman [18]) implies that $\hat{y} \in C^1(\bar{\Omega})$. □

5 Parametric problems with competing nonlinearities

In this section, we study the following parametric nonlinear Neumann problem:

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda g(z, u(z)) + f(z, u(z)), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

$\lambda > 0$ being a parameter.

We impose the following conditions on the functions g and f involved in the reaction of problem (P_λ).

Hypothesis 3 $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z, 0) = 0$ for a.e. $z \in \Omega$ and

(1) there exist $b \in L^\infty(\Omega)_+$ and $\mathcal{P} \in (p, p^*)$ such that

$$|g(z,x)| \le b(z)(1 + |x|^{\mathcal{P}-1}) \quad \text{for a.e. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(2) $\displaystyle\lim_{x\to\pm\infty} \frac{g(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.e. $z \in \Omega$;

(3) if $G(z,x) = \int_0^x g(z,s)\mathrm{d}s$, then there exist $\tau, q \in (1,p)$, $\delta > 0$ and $\hat{\eta}_0, \eta_0 > 0$ such that

$$\begin{aligned}
&0 < g(z,x)x \le qG(z,x) \ \text{ for a.e. } z \in \Omega, \text{ all } 0 < |x| \le \delta,\\
&\operatorname*{essinf}_{\Omega} G(\cdot, \pm\delta) > 0,\\
&\limsup_{x\to 0} \frac{g(z,x)}{|x|^{q-2}x} \le \hat{\eta}_0 \quad \text{uniformly for a.e. } z \in \Omega \text{ and}\\
&\eta_0|x|^\tau \le g(z,x)x \quad \text{for a.e. } z \in \Omega, \text{ all } x \in \mathbb{R}.
\end{aligned}$$

Remark 4 According to Hypothesis 3.(3), $g(z,\cdot)$ exhibits a "superlinear" growth near zero (concave nonlinearity). In fact, we have $\hat{c}|x|^q \le G(z,x)$ for a.e. $z \in \Omega$, all $|x| \le \delta$, with $\hat{c} > 0$, see Mugnai [26,27].

Hypothesis 4 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.e. $z \in \Omega$ and

(1) $|f(z,x)| \le a(z)(1+|x|^{r-1})$ for a.e. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(2) $\displaystyle\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \infty$ uniformly for a.e. $z \in \Omega$;

(3) $\displaystyle\lim_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.e. $z \in \Omega$.

Remark 5 Evidently $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$ and 0 (convex nonlinearity). So, problem (P_λ) exhibits the competing effects of concave and convex nonlinearities. Such problems were investigated in the context of Dirichlet problems with $\beta \equiv 0$ by Ambrosetti–Brezis–Cerami [2] (semilinear equations) and by Garcia Azorero–Manfredi–Peral Alonso [12], Gasinski–Papageorgiou [14] (nonlinear equations driven by the p-Laplacian). The first two works focus on positive solutions, and the authors prove bifurcation-type results (see Sect. 6 of this paper). In [14], the authors produce nodal solutions. We mention that in all the above works the reaction is more restrictive than ours.

Let $F(z,x) := \int_0^x f(z,s)\mathrm{d}s$ and set

$$\xi_\lambda(z,x) := \lambda g(z,x)x + f(z,x)x - \lambda p G(z,x) - pF(z,x).$$

As in the previous sections, instead of the AR-condition, we impose a quasi-monotonicity condition on $\xi_\lambda(z,\cdot)$.

Hypothesis 5 For every $\lambda > 0$, there exists $\beta_\lambda^* \in L^1(\Omega)$ such that

$$\xi_\lambda(z,x) \le \xi_\lambda(z,y) + \beta_\lambda^*(z)$$

for a.e. $z \in \Omega$ and all $0 \le x \le y$ or $y \le x \le 0$.

Remark 6 A simple reaction satisfying the hypotheses above with (β_λ^* =constant) is

$$\lambda g(x) + f(x) = \lambda|x|^{q-2}x + |x|^{r-2}x$$

for all $x \in \mathbb{R}$, with $1 < q < p < r < p^*$. This is the reaction employed in [2,12].

In what follows for every $\lambda > 0$, by $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\Psi(u) - \lambda \int_\Omega G(z, u(z))\mathrm{d}z - \int_\Omega F(z, u(z))\mathrm{d}z$$

for all $u \in W^{1,p}(\Omega)$. Evidently, $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

As in Sect. 4, in order to generate nontrivial solutions of constant sign, we introduce certain truncation perturbations of the map $x \mapsto \lambda g(z, x) + f(z, x)$. So, let $\beta \in L^\infty(\Omega)$ and, fixed $\varepsilon > 0$, let

$$\gamma > (1 + c(\epsilon)\|\beta\|_\infty |\Omega|^{\frac{1}{q}}) \geq (1 + c(\epsilon)\|\beta\|_q) \quad \text{with } c(\epsilon) \geq 1,$$

see (23). So, we define

$$\begin{aligned} \hat{h}^+_\lambda(z, x) &= \begin{cases} 0, & \text{if } x \leq 0, \\ \lambda g(z, x) + f(z, x) + \gamma x^{p-1}, & \text{if } x > 0, \end{cases} \quad \text{and} \\ \hat{h}^-_\lambda(z, x) &= \begin{cases} \lambda g(z, x) + f(z, x) + \gamma |x|^{p-2}x, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0. \end{cases} \end{aligned} \tag{72}$$

Both $\hat{h}^\pm_\lambda$ are Carathéodory functions. We set

$$\hat{H}^\pm_\lambda(z, x) = \int_0^x \hat{h}^\pm_\lambda(z, s)\mathrm{d}s$$

and consider the C^1-functionals $\hat{\varphi}^\pm_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}^\pm_\lambda(u) = \frac{1}{p}\Psi(u) + \frac{\gamma}{p}\|u\|_p^p - \int_\Omega \hat{H}^\pm_\lambda(z, u(z))\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

Note that, using Hypotheses 3, 4 and 5 the reaction $(z, x) \mapsto \lambda g(z, x) + f(z, x)$ satisfies Hypothesis 1.(1–3), and so from Propositions 4 and 10, we have:

Proposition 16 *If Hypotheses 3, 4 and 5 hold, $\lambda > 0$ and $\beta \in L^\infty(\Omega)$, then functionals φ_λ and $\hat{\varphi}^\pm_\lambda$ satisfy the C-conditions.*

The next two propositions show that for $\lambda > 0$ small, the functionals $\hat{\varphi}^\pm_\lambda$ satisfy the mountain pass geometry.

Proposition 17 *1. There exists $\lambda^*_+ > 0$ such that for all $\lambda \in (0, \lambda^*_+)$ there exists $\rho^+_\lambda > 0$ for which we have*

$$\inf\left\{\hat{\varphi}^+_\lambda(u) : \|u\| = \rho^+_\lambda\right\} := m^+_\lambda > 0.$$

2. There exists $\lambda^*_- > 0$ such that for every $\lambda \in (0, \lambda^*_-]$ there exists $\rho^-_\lambda > 0$ for which we have

$$\inf\left\{\hat{\varphi}^-_\lambda(u) : \|u\| = \rho^-_\lambda\right\} = m^-_\lambda > 0.$$

Proof Without loss of generality, we assume $\mathcal{P} \leq r$ (otherwise r is replaced by $\mathcal{P}$ in the calculations below).
1. Hypotheses 3 and 4.(1, 3) imply that given $\theta > 0$, we can find $c_{12} = c_{12}(\theta) > 0$ and $c_{13} = c_{13}(\theta) > 0$ such that

$$\hat{H}_\lambda^+(z,x) \leq \frac{\theta}{p}(x^+)^p + \lambda c_{12}(x^+)^q + c_{13}(1+\lambda)(x^+)^r \quad \text{for a.e.} z \in \Omega \text{ and all } x \in \mathbb{R}, \tag{73}$$

since $|x|^p \leq |x|^q + |x|^r$ for every $x \in \mathbb{R}$.

Then, for all $u \in W^{1,p}(\Omega)$, choosing $\theta > 0$ small and using (24) we have

$$\hat{\varphi}_\lambda^+(u) \geq \left(c_{16} - \lambda c_{14}\|u\|^{q-p} - c_{15}(1+\lambda)\|u\|^{r-p}\right)\|u\|^p. \tag{74}$$

for some $c_{14}, c_{15}, c_{16} > 0$.

Now, we consider the function

$$y_\lambda(t) = \lambda c_{14} t^{q-p} + c_{15}(1+\lambda)t^{r-p} \quad \text{for all } t > 0.$$

Evidently, $y_\lambda \in C^1(0,\infty)$ and since $q < p < r$ (see Hypotheses 3 and 4), we have

$$y_\lambda(t) \to \infty \text{ as } t \to 0^+ \text{ and as } \ t \to \infty.$$

So, we can find $t_0 \in (0,\infty)$ such that

$$y_\lambda(t_0) = \min\left\{y_\lambda(t) : t > 0\right\} \Longrightarrow y_\lambda'(t_0) = 0,$$

that is $\lambda c_{14}(p-q) = c_{15}(1+\lambda)(r-p)t_0^{r-q}$, and so

$$t_0(\lambda) = \left[\frac{\lambda c_{14}(p-q)}{c_{15}(1+\lambda)(r-p)}\right]^{\frac{1}{r-q}}.$$

Then $y_\lambda(t_0) \to 0^+$ as $\lambda \to 0^+$ and so we can find $\lambda_+^* > 0$ such that for every $\lambda \in (0,\lambda_+^*)$ we have

$$y_\lambda(t_0) < c_{16}.$$

So, from (74) it follows that

$$\inf\left\{\hat{\varphi}_\lambda^+(u) : \|u\| = \rho_\lambda^+ = t_0(\lambda)\right\} = m_\lambda^+ > 0 \quad \text{for all } \lambda \in (0,\lambda_+^*).$$

2. In a similar fashion, we show the corresponding result for functional $\hat{\varphi}_\lambda^-$. □

For the next result, we set

$$\lambda^* := \min\{\lambda_+^*, \lambda_-^*\}.$$

Proposition 18 *If Hypotheses 3, 4, 5 hold, $\lambda \in (0,\lambda^*)$ and $\beta \in L^\infty(\Omega)$, then for every $u \in C_\pm$ with $\|u\|_p = 1$, we have $\hat{\varphi}_\lambda^\pm(tu) \to -\infty$ as $t \to \infty$.*

Proof Hypothesis 3.(1, 2) implies that, given $\theta > 0$, there exists $c_{17} = c_7(\theta) > 0$ such that

$$\lambda G(z,x) \geq -\frac{\theta}{p}|x|^p - c_{17} \quad \text{for a.e. } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{all } \lambda \in (0,\lambda^*). \tag{75}$$

Similarly, Hypothesis 4.(1, 2) implies that given $\mu > 0$, we can find $c_{18} = c_{18}(\mu) > 0$ such that

$$F(z,x) \geq \frac{\mu}{p}|x|^p - c_{18} \quad \text{for a.e. } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{76}$$

Let $u \in C_+$ with $\|u\|_p = 1$ and let $t > 0$. Then, from (72), (75) and (76), we have

$$\hat{\varphi}_\lambda^+(tu) \le \frac{t^p}{p}[\|Du\|_p^p + (\|\beta\|_\infty + \theta - \mu + c_{19})] \quad (\text{since } \|u\|_p = 1) \tag{77}$$

for some $c_{19} > 0$.

Since $\theta > 0$ and $\mu > 0$ are arbitrary, we can choose $\theta > 0$ so small and $\mu > 0$ so large that $\mu - \theta > \|\beta\|_\infty + \|Du\|_p^p + c_{19}$. Then, from (77), we infer that

$$\hat{\varphi}_\lambda^+(tu) \to -\infty \quad \text{as } t \to \infty.$$

In a similar fashion, we show that if $u \in C_-$ with $\|u\|_p = 1$, then

$$\hat{\varphi}_\lambda^-(tu) \to -\infty \quad \text{as } t \to \infty.$$

□

Next we will produce two ordered pairs of nontrivial constant sign solutions. To this end, we will need an additional hypothesis which will allow us to compare solutions:

Hypothesis 6 For every $\rho > 0$ and $\lambda > 0$, there exists $\xi_\rho^\lambda > 0$ such that for a.e. $z \in \Omega$, the function

$$x \mapsto \lambda g(z, x) + f(z, x) + \xi_\rho^\lambda |x|^{p-2} x$$

is nondecreasing on $[-\rho, \rho]$.

Using this hypothesis, we prove the following multiplicity result for nontrivial constant sign solutions of problem (P_λ).

Proposition 19 *If Hypotheses 3, 4, 5 and 6 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then problem (P_λ) admits at least four nontrivial solutions of constant sign*

$$u_0, \hat{u} \in \text{int} C_+, \;\; u_0 \le \hat{u}, \;\; u_0 \ne \hat{u}$$
$$v_0, \hat{v} \in -\text{int} C_+, \;\; \hat{v} \le v_0, \;\; v_0 \ne \hat{v}.$$

Proof Let $\mu \in (\lambda, \lambda^*)$ and consider the problem (P_μ). Propositions 16, 17 and 18 imply that we can use Theorem 1 (the Mountain Pass Theorem) and obtain $u_\mu \in W^{1,p}(\Omega)$ such that

$$u_\mu \in K_{\hat{\varphi}_\mu^+} \quad \text{and} \quad \hat{\varphi}_\mu^+(0) = 0 < m_\mu^+ \le \hat{\varphi}_\mu^+(u_\mu),$$

so that $u_\mu \ne 0$. Since $K_{\hat{\varphi}_\mu^+} \subseteq C_+$, as we obtain from (50), by (72) we have

$$A(u_\mu) + \beta(z) u_\mu^{p-1} = \mu N_g(u_\mu) + N_f(u_\mu), \tag{78}$$

that is, u_μ is a nontrivial positive solution of problem (P_μ). The nonlinear regularity theory implies that $u_\mu \in C_+ \setminus \{0\}$. Note that, since $g \ge 0$ (see Hypothesis 3.(3)), we get

$$-\Delta_p u_\mu(z) + \beta(z) u_\mu(z)^{p-1} \ge f(z, u_\mu(z)) \quad \text{for a.e. } z \in \Omega. \tag{79}$$

Hypothesis 4.(1, 3) implies that, given $\epsilon > 0$, there exists $c_{20} = c_{20}(\epsilon) > 0$ such that

$$f(z, x)x \ge -\epsilon |x|^p - c_{20}|x|^r \quad \text{for a.e. } z \in \Omega, \;\; \text{all } x \in \mathbb{R}.$$

Using this unilateral estimate in (79), we obtain

$$\begin{aligned}\Delta_p u_\mu(z) &\le (\|\beta\|_\infty + \epsilon)u_\mu(z)^{p-1} + c_{20}u_\mu(z)^{r-1}\\ &\le \left(\|\beta\|_\infty + \epsilon + c_{20}\|u_\mu\|_\infty^{r-p}\right)u_\mu(z)^{p-1} \quad \text{for a.e. } z \in \Omega.\end{aligned}$$

Hence $u_\mu \in \mathrm{int}C_+$ (see, for example, Gasinski–Papageoriou [13, p. 738]).

By (78), we have

$$\begin{aligned}A(u_\mu) + \beta(z)u_\mu^{p-1} &= \mu N_g(u_\mu) + N_f(u_\mu)\\ &\ge \lambda N_g(u_\mu) + N_f(u_\mu) \quad \text{in } W^{1,p}(\Omega)^* \quad (\text{recall that } g \ge 0).\end{aligned} \tag{80}$$

With $\gamma > 1 + c(\epsilon)\|\beta\|_\infty|\Omega|^{\frac{1}{q}} \ge 1 + c(\epsilon)\|\beta\|_q > 0$ as before, we introduce the following truncation perturbation of the reaction in problem (P_λ):

$$\hat{e}_\lambda^+(z,x) = \begin{cases} 0, & \text{if } x < 0,\\ \lambda g(z,x) + f(z,x) + \gamma x^{p-1}, & \text{if } 0 \le x \le u_\mu(z),\\ \lambda g(z,u_\mu(z)) + f(z,u_\mu(z)) + \gamma u_\mu(z)^{p-1}, & \text{if } x > u_\mu(z).\end{cases} \tag{81}$$

Of course, $\hat{e}_\lambda^+$ is a Carathéodory function. We set $\hat{E}_\lambda^+(z,x) = \int_0^x \hat{e}_\lambda^+(z,s)\mathrm{d}s$ and then consider the C^1-functional $\hat{\psi}_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda^+(u) = \frac{1}{p}\Psi(u) + \frac{\gamma}{p}\|u\|_p^p - \int_\Omega \hat{E}_\lambda^+(z,u(z))\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

By (24), the choice of $\gamma > 0$ implies that

$$\Psi(u) + \gamma\|u\|_p^p \ge c_{21}\|u\|^p \quad \text{for some } c_{21} > 0. \tag{82}$$

Moreover, from (81) and (82), we see that $\hat{\varphi}_\lambda^+$ is coercive. Finally, using the Sobolev embedding theorem, we can easily check that $\hat{\varphi}_\lambda^+$ is sequentially weakly lower semicontinuous. Thus, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda^+(u_0) = \inf\left\{\hat{\psi}_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\} := \hat{m}_\lambda^+. \tag{83}$$

By virtue of Hypothesis 4.(3), given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) \in (0, \min_{\bar{\Omega}} u_\mu)$ (recall that $u_\mu \in \mathrm{int}C_+$) such that

$$-F(z,x) \le \frac{\epsilon}{p}|x|^p \quad \text{for a.e. } z \in \Omega,\ \text{all } |x| \le \delta. \tag{84}$$

For $t \in (0,1)$ small enough, we have that $t\hat{u}_1(\beta)(z) \in (0,\delta]$ for all $z \in \bar{\Omega}$ (recall that $\hat{u}_1(\beta) \in \mathrm{int}C_+$). Then

$$\hat{\psi}_\lambda^+(t\hat{u}_1(\beta)) \le \frac{t^p}{p}[\hat{\lambda}_1(\beta) + \gamma + \epsilon] - \frac{\lambda\eta_0 t^\tau}{\tau}\|\hat{u}_1(\beta)\|_\tau^\tau,$$

see (5), (81), (84) and Hypothesis 3.(3).

Since $\tau < p$ (see Hypothesis 3.(3)), by choosing $t \in (0, 1)$ even smaller if necessary, we have

$$\hat{\psi}_\lambda^+(t\hat{u}_1(\beta)) < 0,$$

so that from (83)

$$\hat{\psi}_\lambda^+(u_0) = \hat{m}_\lambda^+ < 0 = \hat{\psi}_\lambda^+(0), \tag{85}$$

and hence $u_0 \neq 0$.

From (83), we have $(\hat{\psi}_\lambda^+)'(u_0) = 0$, that is

$$A(u_0) + (\beta(z) + \gamma)|u_0|^{p-2}u_0 = N_{\hat{e}_\lambda^+}(u_0). \tag{86}$$

On (86), we act with $-u_0^- \in W^{1,p}(\Omega)$ and by (81) we obtain

$$\Psi(u_0^-) + \gamma\|u_0^-\|_p^p = 0,$$

and by (82)

$$c_{21}\|u_0^-\|^p \le 0,$$

and hence $u_0 \ge 0$.

Now, on (86) we act with $(u_0 - u_\mu)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(u_0), (u_0 - u_\mu)^+\rangle + \int_\Omega (\beta(z) + \gamma)u_0^{p-1}(u_0 - u_\mu)^+ \mathrm{d}z \\ &\quad \le \langle A(u_\mu), (u_0 - u_\mu)^+\rangle + \int_\Omega (\beta(z) + \gamma)u_\mu^{p-1}(u_0 - u_\mu)^+ \mathrm{d}z \end{aligned}$$

since $g \ge 0$ and $\lambda < \mu$. Then there exists $c_{22} > 0$ such that

$$\langle A(u_0) - A(u_\mu), (u_0 - u_\mu)^+\rangle + c_{22}\int_\Omega (u_0^{p-1} - u_\mu^{p-1})(u_0 - u_\mu)^+ \mathrm{d}z \le 0,$$

by the choice of $\gamma > 0$. By Proposition 3, we get

$$|\{u_0 > u_\mu\}|_N = 0, \quad \text{hence } u_0 \le u_\mu.$$

So, we have proved that

$$u_0 \in [0, u_\mu] = \Big\{u \in W^{1,p}(\Omega) : 0 \le u(z) \le u_\mu(z) \text{ a.e. in } \Omega\Big\}.$$

Therefore, from (81), (86) becomes

$$A(u_0) + \beta(z)u_0^{p-1} = \lambda N_g(u_0) + N_f(u_0), \tag{87}$$

that is

$$u_0 \text{ is a positive solution of } (P_\lambda).$$

The nonlinear regularity theory and the nonlinear maximum principle imply that $u_0 \in [0, u_\mu] \cap \mathrm{int}C_+$. Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho^\mu > 0$ be as postulated by Hypothesis 6. Let

$\hat{\xi}_\rho^\mu > \max\{\xi_\rho^\mu, \|\beta\|_\infty\}$. For $\theta > 0$, we set $u_0^\theta = u_0 + \theta \in \mathrm{int}C_+$. We have

$$
\begin{aligned}
&-\Delta_p u_0^\theta(z) + (\beta(z) + \hat{\xi}_\rho^\mu) u_0^\theta(z)^{p-1} \text{ (by the Mean Value Theorem)}\\
&= \lambda g(z, u_0(z)) + f(z, u_0(z)) + \hat{\xi}_\rho^\mu u_0(z)^{p-1} + \chi(\theta)\\
&\quad \text{with } \chi(\theta) = (p-1)(u_0 + t\theta)^{p-2}\theta \to 0^+ \text{ as } \theta \to 0^+,\ t \in (0,1),\\
&= \mu g(z, u_0(z)) + f(z, u_0(z)) - (\mu - \lambda) g(z, u_0(z)) + \hat{\xi}_\rho^\mu u_0(z)^{p-1} + \chi(\theta)\\
&\le \mu g(z, u_0(z)) + f(z, u_0(z)) - (\mu - \lambda)\eta_0 u_0(z)^{\tau-1} + \hat{\xi}_\rho^\mu u_0(z)^{p-1} + \chi(\theta)\\
&\quad \text{(see Hypothesis 3.(3))}\\
&\le \mu g(z, u_0(z)) + f(z, u_0(z)) + \hat{\xi}_\rho^\mu u_0(z)^{p-1} - (\mu - \lambda)\eta_0 m_0^{\tau-1} + \chi(\theta)\\
&\quad \text{with } m_0 = \min u_0 > 0 \text{ (recall that } u_0 \in \text{ int } C_+)\\
&\le \mu g(z, u_0(z)) + f(z, u_0(z)) + \hat{\xi}_\rho^\mu u_0(z)^{p-1}\\
&\quad \text{for } \theta > 0 \text{ small enough, since } \xi \to 0^+ \text{ and } \mu > \lambda,\\
&\le \mu g(z, u_\mu(z)) + f(z, u_\mu(z)) + \hat{\xi}_\rho^\mu(z)^{p-1} \quad \text{for a.e. } z \in \Omega\\
&\quad \text{(see Hypothesis 6)}\\
&= -\Delta_p u_\mu(z) + (\beta(z) + \hat{\xi}_\rho^\mu) u_0(z)^{p-1}.
\end{aligned}
$$

Thus, by the choice of $\hat{\xi}_\rho^\mu$ and the comparison principle (see Pucci–Serrin [35, Theorem 2.4.1]), we get

$$
u_0^\theta \le u_\mu \quad \text{for } \theta > 0 \text{ small, and hence } u_\mu - u_0 \in \mathrm{int}C_+.
$$

So, we have proved that

$$
u_0 \in \mathrm{int}_{C^1(\bar{\Omega})}[0, u_\mu]. \tag{88}
$$

Now, observe that, by

$$
\begin{aligned}
&\hat{\varphi}_\lambda^+ \mid_{[0,u_\mu]} = \hat{\psi}_\lambda^+ \mid_{[0,u_\mu]} \quad \text{(see (72) and (81))}\\
&\implies u_0 \text{ is a local } C_0^1(\bar{\Omega}) - \text{minimizer of } \hat{\varphi}_\lambda^+,\\
&\implies u_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \hat{\varphi}_\lambda^+ \text{(see Proposition 2)}.
\end{aligned} \tag{89}
$$

We introduce the following truncation of $\hat{h}_\lambda^+$ (see (72)):

$$
\tilde{h}_\lambda^+(z, x) = \begin{cases} \hat{h}_\lambda^+(z, u_0(z)), & \text{if } x \le u_0(z),\\ \hat{h}_\lambda^+(z, x), & \text{if } x > u_0(z). \end{cases} \tag{90}
$$

This is a Carathéodory function. We also set $\tilde{H}_\lambda^+(z, x) = \int_0^x \tilde{h}_\lambda^+(z, s) ds$ and consider the C^1- functional $\tilde{\varphi}_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
\tilde{\varphi}_\lambda^+(u) = \frac{1}{p}\Psi(u) + \frac{\gamma}{p}\|u\|_p^p - \int_\Omega \tilde{H}_\lambda^+(z, u(z))\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).
$$

From (72) and (90) we see that there exists $\xi_\lambda^+ = \int_\Omega [\tilde{H}_\lambda^+(u) - \hat{H}_\lambda^+(u)]$ such that

$$
\hat{\varphi}_\lambda^+ = \tilde{\varphi}_\lambda^+ + \xi_\lambda^+. \tag{91}
$$

From this, we claim that

$$
K_{\tilde{\varphi}_\lambda^+} \subseteq [u_0) := \Big\{ u \in W^{1,p}(\Omega) : u_0(z) \le u(z) \text{ a.e. in } \Omega \Big\}. \tag{92}
$$

Indeed, let $u \in K_{\tilde{\varphi}_\lambda^+}$. Then

$$A(u) + (\beta(z) + \gamma)|u|^{p-2}u = N_{\tilde{h}_\lambda^+}(u).$$

We act on the previous equation with $(u_0 - u)^+ \in W^{1,p}(\Omega)$ and obtain

$$\langle A(u_0) - A(u), (u_0 - u)^+ \rangle + c_{23} \int_\Omega (u_0^{p-1} - u^{p-1})(u_0 - u)^+ \mathrm{d}z \leq 0$$

for some $c_{23} > 0$, from the choice of γ. As a consequence, by Proposition 3,

$$|\{u_0 > u\}|_N = 0, \quad \text{hence } u_0 \leq u.$$

This proves (92).

From (89) and (91), we see that $u_0 \in \mathrm{int}C_+$ is a local minimizer of $\tilde{\varphi}_\lambda^+$. We may assume that u_0 is isolated (otherwise we already have a whole sequence of distinct solutions of (P_λ) all belonging to $\mathrm{int}C_+$, see (72), (90) and (92). Therefore, we can find $\rho \in (0, 1)$ so small that

$$\tilde{\varphi}_\lambda^+(u_0) < \tilde{\eta}_\lambda^+ := \inf \left\{ \tilde{\varphi}_\lambda^+(u) : \|u - u_0\| = \rho \right\}, \tag{93}$$

see Aizicovici–Papageorgiou–Staicu [1].

From (91) and Proposition 16, we infer that

$$\tilde{\varphi}_\lambda^+ \text{ satisfies the C-condition.} \tag{94}$$

Therefore, if $u \in C_+$ with $\|u\|_p = 1$, then

$$\tilde{\varphi}_\lambda^+(tu) \to -\infty \text{ as } t \to \infty \tag{95}$$

see (91) and Proposition 18.

From (93), (94) and (95), we see that we can apply Theorem 1 (the mountain pass theorem) and so we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\tilde{\varphi}_\lambda^+} \quad \text{and} \quad \tilde{\varphi}_\lambda^+(u_0) < \tilde{\eta}_\lambda^+ \leq \tilde{\varphi}_\lambda^+(\hat{u}). \tag{96}$$

From (92) and (96), we have

$$u_0 \leq \hat{u}, u_0 \neq \hat{u} \ \text{ and } \ \hat{u} \ \text{ is a solution of } \ (P_\lambda) \tag{97}$$

(see (72) and (90)). The nonlinear regularity theory implies that $\hat{u} \in \mathrm{int}C_+$.

Similarly, working with $\hat{\varphi}_\lambda^-$, we generate two negative solutions $v_0, \hat{v} \in -\mathrm{int}C_+$ such that $v_0 \neq \hat{v}$ and $\hat{v} \leq v_0$. □

In the next proposition, we produce a fifth nontrivial solution for problem (P_λ) when $\lambda \in (0, \lambda^*)$.

Proposition 20 *If Hypotheses 3, 4, 5, 6 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then problem (P_λ) has a fifth nontrivial solution*

$$y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega}).$$

Proof Let $u_0 \in \mathrm{int}C_+$ and $v_0 \in -\mathrm{int}C_+$ be the two constant sign solutions from Proposition 19. With $\gamma > 0$ as before, we consider the following truncation perturbation of the reaction of problem (P_λ):

$$d_\lambda(z,x)=\begin{cases}\lambda g(z,v_0(z))+f(z,v_0(z))+\gamma|v_0(z)|^{p-1}v_0(z), & \text{if } x<v_0(z),\\ \lambda g(z,x)+f(z,x)+\gamma|x|^{p-2}x, & \text{if } v_0(z)\le x\le u_0(z),\\ \lambda g(z,u_0(z))+f(z,u_0(z))+\gamma u_0(z)^{p-1}, & \text{if } x>u_0(z).\end{cases} \tag{98}$$

This is a Carathéodory function. Set $D_\lambda(z,x) = \int_0^x d_\lambda(z,s)\mathrm{d}s$ and consider the C^1-functional $\Xi_\lambda : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\Xi_\lambda(u)=\frac{1}{p}\Psi(u)+\frac{\gamma}{p}\|u\|_p^p-\int_\Omega D_\lambda(z,u(z))\mathrm{d}z \quad \text{for all } u\in W^{1,p}(\Omega).$$

From (82) and (98), it is clear that Ξ_λ is coercive. Moreover, it is sequentially weakly lower semicontinuous. So, we can find $y_0\in W^{1,p}(\Omega)$ such that

$$\Xi_\lambda(y_0)=\inf\left\{\Xi_\lambda(u):u\in W^{1,p}(\Omega)\right\}. \tag{99}$$

As in the proof of Proposition 19, we have

$$\Xi_\lambda(y_0)<0=\Xi_\lambda(0),\ \text{hence } y_0\neq 0.$$

From (99), we have $\Xi'_\lambda(y_0)=0$, that is

$$A(y_0)+(\beta(z)+\gamma)|y_0|^{p-2}y_0=N_{d_\lambda}(u_0). \tag{100}$$

On (100) first we act with $(v_0-y_0)^+\in W^{1,p}(\Omega)$ and then with $(y_0-u_0)^+\in W^{1,p}(\Omega)$ and obtain $y_0\in[v_0,u_0]=\{u\in W^{1,p}(\Omega):v_0(z)\le u(z)\le u_0(z)\text{ a.e. in }\Omega\}$ (as in the proof of Proposition 19).

Finally, by the nonlinear regularity theory, we conclude that $y_0\in C^1(\bar{\Omega})$. ◻

So, summarizing the situation for problem (P_λ), we can state the following multiplicity theorem:

Theorem 21 *If Hypotheses 3, 4, 5, 6 hold, and $\beta\in L^\infty(\Omega)$, then there exists $\lambda^*>0$ such that for all $\lambda\in(0,\lambda^*)$ problem (P_λ) has at least five nontrivial solutions*

$$\begin{aligned}&u_0,\hat{u}\in \mathrm{int}C_+,\ u_0\le\hat{u},\ u_0\neq\hat{u},\\ &v_0,\hat{v}\in -\mathrm{int}C_+,\ \hat{v}\le v_0,\ v_0\neq\hat{v},\\ &y_0\in[v_0,u_0]\cap C^1(\bar{\Omega}).\end{aligned}$$

In the semilinear case (that is $p=2$) and under stronger regularity conditions on the functions $g(z,\cdot)$, we can improve the conclusion of Theorem 21 in two distinct ways:

1. We produce six nontrivial solutions;
2. We allow the potential function β to be unbounded.

So, the problem under consideration, is the following:

$$\begin{cases}-\Delta u(z)+\beta(z)u(z)=\lambda g(z,uz(z))+f(z,u(z)) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n}=0, & \text{on } \partial\Omega.\end{cases} \tag{S_λ}$$

We remark that for problem (S_λ) we do not need to assume that β is bounded, since we can use the regularity result of Wang [37] and infer that the solutions of problem (S_λ) belong in $C^1(\bar{\Omega})$. On the other hand, the bound on β^+ is needed in order to apply Theorem 21, which is valid also in this case, as it is clear from its proof, and to prove a stronger order relation between solutions (see Theorem 22 below).

The hypotheses on the functions f and g are now the following:

Hypothesis 7 $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.e. $z \in \Omega$, $g(z, 0) = 0$, $g(z, \cdot) \in C^1(\mathbb{R}\backslash\{0\})$ and

(1) there exists $\mu \in (1, 2^* - 1)$ and $a \in L^\infty(\Omega)$ such that

$$|g'_x(z, x)| \leq a(z)(1 + |x|^{\mu-1}) \text{ for a.e. } z \in \Omega, \text{ all } x \in \mathbb{R}\backslash\{0\};$$

(2) same as Hypothesis 3.(2) with $p = 2$;
(3) same as Hypothesis 3.(3) with $p = 2$.

Hypothesis 8 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.e. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(1) there exists $r \in (2, 2^*)$ such that $|f'_x(z, x)| \leq a(z)(1+|x|^{r-2})$ for a.e. $z \in \Omega$, all $x \in \mathbb{R}$;
(2) $\displaystyle\lim_{x\to\pm\infty} \frac{f(z, x)}{x} = \infty$ uniformly for a.e. $z \in \Omega$;
(3) $\displaystyle f'_x(z, 0) = \lim_{x\to 0} \frac{f(z, x)}{x} = 0$ uniformly for a.e. $z \in \Omega$.

Remark 7 Observe that in this case the differentiability hypotheses on $g(z, \cdot)$ and $f(z, \cdot)$ and Hypotheses 7.(1) and 8.(1) imply that given $\rho > 0$ and $\lambda > 0$, we can find $\xi_\rho^\lambda > 0$ such that for a.e. $z \in \Omega$, the function $x \to \lambda g(z, x) + f(z, x) + \xi_\rho^\lambda x$ is nondecreasing on $[-\rho, \rho]$. So, Hypothesis 6 is automatically satisfied.

For every $\lambda > 0$, we introduce the energy functional $\varphi_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{2}\|Du\|_2^2 + \frac{1}{2}\int_\Omega \beta(z)u^2 \mathrm{d}z - \lambda\int_\Omega G(z, u)\mathrm{d}z - \int_\Omega F(z, u)\mathrm{d}z \quad \text{for all } u \in H^1(\Omega).$$

We have $\varphi_\lambda \in C^{2-0}(H^1(\Omega))$ (see also Li–Li–Liu [19]).

Theorem 22 *If Hypotheses 5, 7, 8, hold, $\beta \in L^\infty(\Omega)$, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (S_λ) admits at least six nontrivial solutions*

$$\begin{aligned} &u_0, \hat{u} \in \mathrm{int}C_+, \hat{u} - u_0 \in \mathrm{int}C_+, \\ &v_0, \hat{v} \in -\mathrm{int}C_+, v_0 - \hat{v} \in \mathrm{int}C_+, \\ &y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega}) \quad \textit{and} \quad \hat{y} \in C^1(\bar{\Omega}). \end{aligned}$$

Proof From Theorem 21, we know that there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (S_λ) has at least five nontrivial solutions

$$\begin{aligned} &u_0, \hat{u} \in \mathrm{int}C_+,\ u_0 \leq \hat{u},\ u_0 \neq \hat{u}, \\ &v_0, \hat{v} \in -\mathrm{int}C_+,\ \hat{v} \leq v_0,\ v_0 \neq \hat{v}, \\ &y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega}). \end{aligned}$$

Now, we assume by contradiction that $K_{\varphi_\lambda} = \{0, u_0, \hat{u}, v_0, \hat{v}, y_0\}$.

Now, thanks to the bound on β^+, we find a stronger order relation between $\{u_0, \hat{u}\}$ and between $\{v_0, \hat{v}\}$. To see this recall that $u_0 \leq \hat{u}$. Let $\rho = \|\hat{u}\|_\infty$ and let $\xi_\rho^\lambda > 0$ be such that for a.e. $z \in \Omega$ the map

$$x \mapsto \lambda g(z, x) + f(z, x) + \xi_\rho^\lambda x$$

is nondecreasing on $[-\rho, \rho]$. Then, we have

$$\Delta(\hat{u} - u_0)(z) \leq (\beta(z) + \xi_\rho^\lambda)(\hat{u} - u_0)(z) \leq (\|\beta^+\|_\infty + \xi_\rho^\lambda)(\hat{u} - u_0)(z) \ \text{ for a.e. } z \in \Omega,$$

and thus

$$\hat{u} - u_0 \in \text{int}C_+ \quad \text{(see [13, Corollary 6.1.47])}.$$

Similarly, we show that

$$v_0 - \hat{v} \in \text{int}C_+ \quad \text{and } y_0 \in \ \text{int}_{C^1(\bar{\Omega})}[v_0, u_0]. \tag{101}$$

From the proof of Proposition 19, we know that

$$u_0 \in \text{int}C_+ \ \text{ is a local minimizer of } \hat{\varphi}_\lambda^+,$$
$$v_0 \in -\text{int}C_+ \ \text{ is a local minimizer of } \hat{\varphi}_\lambda^-.$$

Since $\hat{\varphi}_\lambda^+ \ |_{C_+} = \varphi_\lambda \ |_{C_+}$ and $\hat{\varphi}_\lambda^- \ |_{C_+} = \varphi_\lambda \ |_{C_+}$ and $u_0 \in \text{int}C_+$, $v_0 \in -\text{int}C_+$, from Proposition 2, it follows that u_0 and v_0 are both local minimizers of φ_λ. Hence

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,0}\mathbb{F} \quad \text{for all } k \geq 0, \tag{102}$$

see [24, Example 6.45].

From the proof of Proposition 19, we know that $\hat{u} \in \text{int}C_+$ is a critical point of mountain pass type for functional $\tilde{\varphi}_\lambda^+$ and $\hat{v} \in -\text{int}C_+$ is a critical point of mountain pass type for functional $\tilde{\varphi}_\lambda^-$. From (91), recalling that $\hat{u} > u_0$, we see that ξ_λ^+ is constant near $\hat{u}$ (precisely $\int_\Omega [\hat{h}(z, u_0)u_0 - \hat{H}(z, u_0)]$), so that

$$C_k(\tilde{\varphi}_\lambda^+, \hat{u}) = C_k(\hat{\varphi}_\lambda^+, \hat{u}) \ \text{ and } \ C_k(\tilde{\varphi}_\lambda^-, \hat{v}) = C_k(\hat{\varphi}_\lambda^-, \hat{v}) \ \text{ for all } k \geq 0. \tag{103}$$

Since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ and $\hat{u} \in \text{int}C_+$, $\hat{v} \in -\text{int}C_+$, we have

$$C_k(\hat{\varphi}_\lambda^+, \hat{u}) = C_k(\varphi_\lambda, \hat{u}) \ \text{ and } \ C_k(\hat{\varphi}_\lambda^-, \hat{v}) = C_k(\varphi_\lambda, \hat{v}) \ \text{ for all } k \geq 0 \tag{104}$$

(see [24]). Then (103) and (104) with [24, Proposition 6.100] imply that

$$C_1(\varphi_\lambda, \hat{u}), C_1(\varphi_\lambda, \hat{v}) \neq 0,$$

so that

$$C_k(\varphi_\lambda, \hat{u}) = C_k(\varphi_\lambda, \hat{v}) = \delta_{k,1}\mathbb{F} \quad \text{for all } k \geq 0, \tag{105}$$

since clearly $0 \notin \sigma(A)$, see Li–Li–Liu [19].

Moreover, recall that y_0 is a minimizer of functional Ξ_λ (see the proof of Proposition 20) and $\Xi_\lambda \ |_{[v_0,u_0]} = \varphi_\lambda \ |_{[v_0,u_0]}$ (see (98)). From (101), it follows that y_0 is a local minimizer of φ_λ (see Proposition 2). Hence

$$C_k(\varphi_\lambda, y_0) = \delta_{k,0}\mathbb{F} \quad \text{for all } k \geq 0. \tag{106}$$

Finally, from Proposition 5, we have

$$C_k(\varphi_\lambda, \infty) = 0 \quad \text{for all } k \geq 0, \tag{107}$$

while Hypotheses 7.(3) and 8.(3) imply that

$$C_k(\varphi_\lambda, 0) = 0 \quad \text{for all } k \geq 0 \quad \text{(see Moroz [23])}. \tag{108}$$

Since we had supposed that $K_{\varphi_\lambda} = \{0, u_0, \hat{u}, v_0, \hat{v}, y_0\}$, from (102), (105), (106), (107), (108) and the Morse relation with $t = -1$ (see (3)), we have

$$2(-1)^1 + 3(-1)^0 = 0,$$

a contradiction. So, there exists $\hat{y} \in K_{\varphi_\lambda}$, $\hat{y} \notin \{0, u_0, \hat{u}, v_0, \hat{v}, y_0\}$. Then, $\hat{y}$ is a nontrivial solution of (S_λ) with $\lambda \in (0, \lambda^*)$, and the nonlinear regularity theory implies that $\hat{y} \in C^1(\bar{\Omega})$. □

6 Bifurcation theorem for positive solutions

In this section, we focus on the positive solutions of problem (P_λ), and we prove a bifurcation-type result describing in a precise way the set of positive solutions of (P_λ) as the parameter λ varies in $(0, \infty)$.

So, let

$$\mathcal{L} := \Big\{\lambda > 0 : \ \text{problem } (P_\lambda) \text{ admits a positive solution}\Big\}$$

and, for every $\lambda > 0$, let

$$\mathscr{S}(\lambda) = \ \text{set of positive solutions of problem } (P_\lambda).$$

We introduce the following hypotheses on functions g and f:

Hypothesis 9 $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z, 0) = 0$ for a.e. $z \in \Omega$ and

(1) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$g(z, x) \leq a_\rho(z) \quad \text{for a.e. } z \in \Omega \text{ and all } x \in [0, \rho];$$

(2) $\displaystyle\lim_{x\to\infty} \frac{g(z, x)}{x^{p-1}} = 0$ uniformly for a.e. $z \in \Omega$;

(3) if $G(z, x) = \int_0^x g(z, s)\mathrm{d}s$ for a.e. $z \in \Omega$ and all $x \geq 0$, then there exist $1 < q \leq \tau < p$, $\delta > 0$ and $\hat{\eta}_0$, $\eta_0 > 0$ such that

$$0 < g(z, x)x \leq qG(z, x) \quad \text{for a.e. } z \in \Omega, \ \text{all } x \in (0, \delta],$$

$$\limsup_{x\to 0^+} \frac{g(z, x)}{x^{q-1}} \leq \hat{\eta}_0 \quad \text{uniformly for a.e. } z \in \Omega$$

$$\eta_0 x^\tau \leq g(z, x)x \quad \text{for a.e. } z \in \Omega \text{ and all } x \geq 0.$$

Hypothesis 10 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.e. $z \in \Omega$ and

(1) $|f(z, x)| \leq a(z)(1 + x^{r-1})$ for a.e. $z \in \Omega$, all $x \geq 0$ with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(2) $\displaystyle\lim_{x\to\infty} \frac{f(z, x)}{x^{p-1}} = \infty$ uniformly for a.e. $z \in \Omega$;

(3) $\displaystyle\lim_{x\to 0^+} \frac{f(z, x)}{x^{p-1}} = 0$ uniformly for a.e. $z \in \Omega$.

At this point, we introduce the following unilateral versions of Hypotheses 5 and 6:

Hypothesis 11 For every $\lambda > 0$, there exists $\beta_\lambda^* \in L^1(\Omega)_+$ such that

$$\xi_\lambda(z, x) \leq \xi_\lambda(z, y) + \beta_\lambda^*(z) \text{ for a.e. } z \in \Omega \text{ and all } 0 \leq x \leq y.$$

Hypothesis 12 For every $\rho > 0$ and $\lambda > 0$, there exists $\xi_\rho^\lambda > 0$ such that for a.e. $z \in \Omega$, the function

$$x \to \lambda g(z, x) + f(z, x) + \xi_\rho^\lambda x^{p-1}$$

is nondecreasing on $[0, \rho]$.

We will also need the following hypothesis:

Hypothesis 13 For a.e. $z \in \Omega$ and all $x \geq 0$, $g(z, x)x \leq pG(z, x)$.

Remark 8 Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis, without any loss of generality we may assume that $g(z, x) = f(z, x) = 0$ for a.e. $z \in \Omega$, all $x \leq 0$. The hypotheses above incorporate in our framework the classical "concave–convex" reaction

$$x \mapsto \lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geq 0 \ \text{ with } \ 1 < q < p < r.$$

However, they are also satisfied by the nonlinearity

$$x \mapsto \lambda x^{q-1} + x^{r-1} \ln(x + 1) \quad \text{for all } x \geq 0,$$

which fails to satisfy the AR-condition.

Finally, let us recall that reversed AR-conditions like the one in Hypothesis 13 have already been used also to treat superlinear problems, for instance see [28–30].

Under these conditions, we know that

$$\mathcal{L} \neq \emptyset \quad \text{and} \quad \mathcal{S}(\lambda) \subseteq \mathrm{int} C_+ \quad \text{(see Section 5);}$$

thus, set $\hat{\lambda}^* = \sup \mathcal{L}$.

Proposition 23 *If Hypotheses 9, 10, 12 hold and $\beta \in L^\infty(\Omega)$, then $\hat{\lambda}^* < \infty$.*

Proof From Hypotheses 9 and 10, we see that we can find $\bar{\lambda} > 0$ such that

$$\bar{\lambda} g(z, x) + f(z, x) \geq \hat{\lambda}_1(\beta) x^{p-1} \quad \text{for a.e. } z \in \Omega, \ \text{ all } x \geq 0. \tag{109}$$

We claim that, if $\lambda > \bar{\lambda}$, then $\lambda \notin \mathcal{L}$. Arguing by contradiction, suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in \mathcal{S}(\lambda) \subseteq \mathrm{int} C_+$. Let $t > 0$ be the biggest positive number such that

$$t\hat{u}_1(\beta) \leq u_\lambda \tag{110}$$

(see Filippakis–Kristaly–Papageorgiou [11, Lemma 3.3]).

Set $\rho = \|u_\lambda\|_\infty$ and let $\xi_\rho^\lambda > 0$ be as postulated by Hypothesis 12, and $\hat{\xi}_\rho^\lambda > \max\{\xi_\rho^\lambda, \|\beta\|_\infty\}$. For $\delta > 0$ we set $\hat{u}_1^\delta := \hat{u}_1(\beta) + \delta \in \mathrm{int} C_+$. Then, since $-\Delta_p \hat{u}_1^\delta(\beta) =$

$-\Delta_p \hat{u}_1(\beta)$, we have

$$
\begin{aligned}
&- \Delta_p(t\hat{u}_1^\delta(z)) + (\beta(z) + \hat{\xi}_\rho^\lambda)(t\hat{u}_1^\delta(z))^{p-1}\\
&\quad = \hat{\lambda}_1(\beta)(t\hat{u}_1(\beta)(z))^{p-1} + \hat{\xi}_\rho^\lambda(t\hat{u}_1(\beta)(z))^{p-1} + \chi(\delta)\\
&\quad\ \ \text{with } \ \chi(\delta) = (p-1)(\beta + \hat{\xi}_\rho^\lambda)(t\hat{u}_1(\beta) + \eta)^{p-2}\delta \to 0^+ \ \text{ as } \delta \to 0^+,\ 0 < \eta < \delta,\\
&\quad \le \bar{\lambda} g(z, t\hat{u}_1(\beta)(z)) + f(z, t\hat{u}_1(\beta)(z)) + \hat{\xi}_\rho^\lambda(t\hat{u}_1(\beta)(z))^{p-1} + \chi(\delta) \quad \text{(see (109)}\\
&\quad \le \bar{\lambda} g(z, u_\lambda(z)) + f(z, u_\lambda(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} + \chi(\delta) \quad \text{(see Hypothesis 12 and (110))}\\
&\quad = \lambda g(z, u_\lambda(z)) + f(z, u_\lambda(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} - (\lambda - \bar{\lambda}) g(z, u_\lambda(z)) + \chi(\delta)\\
&\quad \le \lambda g(z, u_\lambda(z)) + f(z, u_\lambda(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} - (\lambda - \bar{\lambda})\eta_0 u_\lambda(z)^\tau + \chi(\delta)\\
&\quad\ \ \text{(see Hypothesis 9.(3) and recall that } \lambda > \bar{\lambda})\\
&\quad \le \lambda g(z, u_\lambda(z)) + f(z, u_\lambda(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} - (\lambda - \bar{\lambda})\eta_0 m_\lambda^{\tau-1} + \chi(\delta)\\
&\quad\ \ \text{with } m_\lambda = \min_{\bar{\Omega}} u_\lambda > 0\\
&\quad \le \lambda g(z, u_\lambda(z)) + f(z, u_\lambda(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1}\\
&\quad\ \ \text{for } \delta > 0 \text{ small}\\
&\quad = -\Delta_p u_\lambda(z) + (\beta(z) + \hat{\xi}_\rho^\lambda) u_\lambda(z)^{p-1} \text{ for a.e. } z \in \Omega.
\end{aligned}
$$

Thus, $u_\lambda \ge t\hat{u}_1^\delta$ for $\delta > 0$ small; hence $u_\lambda - t\hat{u}_1 \in \text{int}C_+$, which contradicts the maximality of t. Therefore, $\lambda \notin \mathcal{L}$ and so $\hat{\lambda}^* \le \bar{\lambda} < \infty$. □

Proposition 24 *If Hypotheses 9, 10, 11, 12 and $\beta \in L^\infty(\Omega)$, then $(0, \hat{\lambda}^*) \subseteq \mathcal{L}$.*

Proof Let $\lambda \in (0, \lambda^*)$; then, we can find $\mu \in (\lambda, \lambda^*) \cap \mathcal{L}$. Let $u_\mu \in \mathscr{S}(\mu) \subseteq \text{int}C_+$. With $\gamma > 0$ as before, we consider the truncation perturbation of the reaction of problem (P_λ), given by $\hat{e}_\lambda(z, x)$, see (81). We consider the corresponding C^1-functional $\hat{\psi}_\lambda^+$ as in the proof of Proposition 19 and via the direct methods, we obtain $u_\lambda \in [0, u_\mu] \cap \mathscr{S}(\lambda)$. Therefore, $\lambda \in \mathcal{L}$ and so $(0, \hat{\lambda}^*) \subseteq \mathcal{L}$. □

Actually, following the argument in the proof of Proposition 19, we can say more:

Proposition 25 *If Hypotheses 9, 10, 11, 12 hold, $\beta \in L^\infty(\Omega)$ and $\lambda \in (0, \hat{\lambda}^*)$, then problem (P_λ) has at least two positive solutions*

$$u_0, \hat{u} \in \text{int}C_+,\ u_0 \le \hat{u},\ u_0 \ne \hat{u}.$$

Finally, we examine what happens in the critical case $\lambda = \hat{\lambda}^*$:

Proposition 26 *If Hypotheses 9, 10, 11, 12, 13 hold and $\beta \in L^\infty(\Omega)$, then $\hat{\lambda}^* \in \mathcal{L}$.*

Proof Let $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{L}$ be such that $\lambda_n \uparrow \hat{\lambda}^*$. From the proof of Proposition 19 (see (85)), we know that we can find $u_n \in \mathscr{S}(\lambda_n),\ n \ge 1$, such that $\varphi_{\lambda_n}(u_n) < 0$ for all $n \ge 1$, so that

$$\Psi(u_n) - \lambda_n \int_\Omega pG(z, u_n)\mathrm{d}z - \int_\Omega pF(z, u_n)\mathrm{d}z < 0 \quad \text{for all } n \ge 1. \tag{111}$$

Also, we have

$$A(u_n) + \beta(z)u_n^{p-1} = \lambda_n N_g(u_n) + N_f(u_n) \quad \text{for all } n \ge 1. \tag{112}$$

Acting on (112) with $u_n \in W^{1,p}(\Omega)$, we obtain

$$-\Psi(u_n) + \lambda_n \int_\Omega g(z, u_n)u_n \mathrm{d}z + \int_\Omega f(z, u_n)u_n \mathrm{d}z = 0 \quad \text{for all } n \geq 1. \tag{113}$$

We add (111) and (113) and obtain

$$\int_\Omega \xi_{\lambda_n}(z, u_n)\mathrm{d}z < 0 \quad \text{for all } n \geq 1;$$

by Hypothesis 13 and recalling that $\lambda_n < \hat{\lambda}^*$ for all $n \geq 1$, we get

$$\int_\Omega \xi_{\hat{\lambda}^*}(z, u_n)\mathrm{d}z < 0 \quad \text{for all } n \geq 1. \tag{114}$$

Using (114) and reasoning as in the proof of Proposition 4 (see the claim), we obtain that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightharpoonup u_* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^r(\Omega) \ \text{ as } n \to \infty. \tag{115}$$

On (112), we act with $u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (115). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

and by Proposition 3 we get

$$u_n \to u_* \text{ in } W^{1,p}(\Omega) \ \text{ as } n \to \infty. \tag{116}$$

Finally, if in (112) we pass to the limit as $n \to \infty$ and use (116), then we have

$$A(u_*) + \beta(z)u_*^{p-1} = \hat{\lambda}^* N_g(u_*) + N_f(u_*).$$

We need to show that $u_* \neq 0$.

Hypotheses 9.(3) and 10.(1),(2),(3) imply that we can find $\eta_1 > 0$ such that

$$\lambda_1 g(z, x) + f(z, x) \geq \lambda_1 \eta_0 x^{\tau-1} - \eta_1 x^{r-1} \quad \text{for a.e. } z \in \Omega, \ \text{ all } x \geq 0. \tag{117}$$

We consider the following auxiliary Neumann problem

$$\begin{cases} -\Delta_p u(z) + \beta(z)u^{p-1}(z) = \lambda_1 \eta_0 u^{\tau-1}(z) - \eta_1 u^{r-1}(z), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \\ u > 0. \end{cases} \tag{118}$$

We show that (118) admits a solution $\bar{u} \in \operatorname{int}C_+$. To this end, let $\Phi : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\begin{aligned} \Phi(u) &= \frac{1}{p}\Psi(u) + \frac{\gamma}{p}\|u\|_p^p - \frac{\lambda_1\eta_0}{\tau}\|u^+\|_\tau^\tau + \frac{1}{r}\eta_1\|u^+\|_r^r - \frac{\gamma}{p}\|u^+\|_p^p \\ &\geq \frac{c_{21}}{p}\|u^-\|^p + \frac{1}{p}\Psi(u^+) + \frac{1}{r}\eta_1\|u^+\|_r^r - \frac{\lambda_1}{\tau}\eta_0\|u^+\|_\tau^\tau, \end{aligned}$$

where γ is as in the previous sections, so that (82) holds.

Since $\tau < p < r$ and $\beta \in L^\infty(\Omega)$, it follows that Φ is coercive. Therefore, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$\Phi(\bar{u}) = \inf\left\{\Phi(u) : u \in W^{1,p}(\Omega)\right\}. \tag{119}$$

Exploiting the fact that $\tau < p$, for $t \in (0, 1)$ small enough, from the very definition of Φ, we have that

$$\Phi(t\hat{u}_1(\beta)) < 0,$$

and from (119)

$$\Phi(\bar{u}) < 0 = \Phi(0),$$

so that

$$\bar{u} \neq 0.$$

From (119), we have $\Phi'(\bar{u}) = 0$, that is

$$A(\bar{u}) + (\beta(z) + \gamma)|\bar{u}|^{p-2}\bar{u} = \lambda_1\eta_0(\bar{u}^+)^{\tau-1} - \eta_1(\bar{u}^+)^{r-1} + \gamma(\bar{u}^+)^{p-1}. \tag{120}$$

On (120) we act with $-\bar{u}^- \in W^{1,p}(\Omega)$ and obtain

$$\Psi(\bar{u}^-) + \gamma\|\bar{u}^-\|_p^p = 0,$$

so that, by (82), $\bar{u} \geq 0$, and $\bar{u} \neq 0$. Therefore, (120) becomes

$$A(\bar{u}) + \beta(z)\bar{u}^{p-1} = \lambda_1\eta_0\bar{u}^{\tau-1} - \eta_1\bar{u}^{r-1}.$$

Thus, $\bar{u}$ is a nontrivial positive solution of auxiliary problem (118). The nonlinear regularity theory implies that $\bar{u} \in C_+\backslash\{0\}$. In addition,

$$\Delta_p\bar{u}(z) \leq (\|\beta\|_\infty + \eta_1\|\bar{u}\|_\infty^{r-p})\bar{u}(z)^{p-1} \quad \text{a.e. in } \Omega,$$

so that

$$\bar{u} \in \mathrm{int}C_+,$$

see [35].

For every $n \geq 1$, let $t_n > 0$ be the biggest positive real such that

$$t_n\bar{u} \leq u_n \quad \text{for all } n \geq 1 \quad \text{(see [11])}, \tag{121}$$

and suppose that $t_n \in (0, 1)$, $n \geq 1$.

By Hypotheses 9.(1),(2) and 10.(1), we can apply Winkert's regularity result in [39] and find $M_* > 0$ such that

$$\|u_n\|_\infty \leq M_* \quad \text{for all } n \geq 1. \tag{122}$$

Let $\rho = M_*$ and let $\xi_\rho^{\lambda_n} > 0$ be as postulated by Hypothesis 12. Set $\hat{\xi}_\rho^{\lambda_n} > \max\{\xi_\rho^{\lambda_n}, \|\beta\|_\infty\}$ and for $\delta > 0$, $\bar{u}_n^\delta = t_n\bar{u} + \delta \in \mathrm{int}C_+$. Then, as before, for $\delta > 0$ small we have

$$\begin{aligned}
&-\Delta_p\bar{u}_n^\delta(z) + (\beta(z) + \hat{\xi}_\rho^{\lambda_n})\bar{u}_n^\delta(z)^{p-1} \text{ (see the proof of Proposition 23)}\\
&\leq -\Delta_p u_n(z) + (\beta(z) + \hat{\xi}_\rho^{\lambda_n})u_n(z)^{p-1} \quad \text{a.e. in } \Omega\\
&\quad \text{(see (112) and recall that } \lambda_1 < \lambda_n \text{ for all } n \in \mathbb{N}).
\end{aligned}$$

As a consequence, $u_n^\delta \leq u_n$ for every $\delta > 0$ small enough (see Pucci–Serrin [35, Theorem 2.4.1]), so that

$$u_n - t_n\bar{u} \in \mathrm{int}C_+,$$

which contradicts the maximality of t_n. Then $t_n \geq 1$ and so, from (121),

$$\bar{u} \leq u_n \quad \text{for all } n \geq 1;$$

by (116) we get that

$$\bar{u} \leq u_*,$$

so that

$$u_* \neq 0 \quad \text{and so } \lambda_* \in \mathcal{L}.$$

□

So, summarizing the situation for the positive solutions of problem (P_λ), we can state the following bifurcation near zero result.

Theorem 27 *If Hypotheses 9, 10, 11, 12, 13 hold and $\beta \in L^\infty(\Omega)$, then there exists $\hat{\lambda}^* > 0$ such that*

1. *for all $\lambda \in (0, \hat{\lambda}^*)$ problem (P_λ) has at least two positive solutions*

$$u_0, \hat{u} \in intC_+, u_0 \leq \hat{u}, u_0 \neq \hat{u};$$

2. *for $\lambda = \hat{\lambda}^*$ problem (P_λ) has at least one positive solution*

$$u_* \in intC_+;$$

3. *for $\lambda > \hat{\lambda}^*$ problem (P_λ) has no positive solution.*

Remark 9 Theorem 27 extends the results of Ambrosetti–Brezis–Cerami [2] and Garcia Azorero–Manfredi–Peral Alonso [12] which deal with Dirichlet problems and the reaction has the form

$$x \mapsto \lambda x^{\tau-1} + x^{r-1} \quad \text{for all } x \geq 0 \ \text{ with } \ \tau < p < r,$$

and moreover, in [2] only the case $p = 2$ (semilinear equations) was considered. We mention also the recent work of Mugnai–Papageorgiou [33], where a bifurcation result is proved for p-logistic equations in $\mathbb{R}^N$ with indefinite weight.

Acknowledgements The authors wish to thank a knowledgeable referee for her/his corrections and remarks.

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