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Asymptotically linear fractional *p*-Laplacian equations

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Abstract In this paper we study the multiplicity of weak solutions to (possibly resonant) nonlocal equations involving the fractional p-Laplacian operator. More precisely, we consider a Dirichlet problem driven by the fractional p-Laplacian operator and involving a subcritical nonlinear term which does not satisfy the technical Ambrosetti–Rabinowitz condition. By framing this problem in an appropriate variational setting, we prove a multiplicity theorem.

Keywords Fractional p-Laplacian · Integro-differential operator · Variational methods · Asymptotically linear problem · Resonant problem · Pseudo-genus

Mathematics Subject Classification Primary 49J35 · 35A15 · 35S15 · 58E05; Secondary 47G20 · 45G05

1 Introduction

In the standard framework of the p-Laplacian operator, there are a lot of interesting problems widely studied in the literature. A natural question is whether or not the results got in this classical context can be extended to the nonlocal case in the presence of fractional-type operators. Nonlocal fractional equations appear in many fields, and after the seminal works by Caffarelli and Silvestre [8–10], an increasing interest has been devoted to these topics; we refer, for instance, the recent papers [7,12], the monograph [21] for several results on

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fractional variational problems and related topics, and [25–27] as well as references therein for some existence and multiplicity results involving the fractional p-Laplacian operator.

In most of the papers concerning with fractional Laplacian equation, it is assumed that the right-hand side has a superlinear, but subcritical growth (cf., e.g., [5,22,28,29] and references therein). In the recent papers [3,14] it has been firstly studied in the nonlocal setting the so-called asymptotically linear case: as it is well known, the lack of the classical Ambrosetti–Rabinowitz assumption requires more efforts in order to obtain a compactness Palais–Smale-type condition.

Further difficulties arise in the so-called resonant case (cf., e.g., [1] and references therein): indeed, the resonance affects both the compactness property and the geometry of the Euler–Lagrange functional arising in a suitable variational approach. For the local case we recall the contributions [2,18,24].

Motivated by this interest in the current literature, we would like to focus here our attention on p-fractional "asymptotically linear" problems, also in the presence of resonance. Indeed, we look for solutions of the nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s_p u \; = \; \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $1 < p < +\infty$, $s \in]0,1[$, $(-\Delta)^s_p$ denotes the fractional p-Laplacian which (up to normalization factors) may be defined for any $x \in \mathbb{R}^N$ as

$$(-\Delta)^s_p \varphi(x) = 2 \lim_{\varepsilon \searrow 0^+} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}\,(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \mathrm{d}y$$

along any $\varphi \in C^\infty_0(\mathbb{R}^N)$, where $B_\varepsilon(x) := \left\{ y \in \mathbb{R}^N \; : \; |x-y| < \varepsilon \right\}$, Ω is an open bounded domain of $\mathbb{R}^N$ ($N > sp$) with Lipschitz boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

(h_1) $\sup_{|t| \le a} |f(\cdot, t)| \in L^\infty(\Omega) \quad \forall\, a > 0$;

(h_2) *there exist*

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{p-2}t} \; = \; 0 \tag{1.2}$$

and

$$\lim_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t} \; = \alpha \in \mathbb{R} \tag{1.3}$$

uniformly with respect to a.e. $x \in \Omega$.

Here we also deal with the resonant case, under the following further assumption (cf., e.g., [18]):

(h_3) *there exists*

$$\lim_{|t| \to +\infty} \Big(f(x,t)t - pF(x,t) \Big) \; = \; +\infty$$

uniformly with respect to a.e. $x \in \Omega$, *where, as usual, we set*

$$F(x,t) := \int_0^t f(x,\tau)\, \mathrm{d}\tau.$$

In order to state our multiplicity result we introduce some notations. Recalling that the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

the fractional Sobolev space is defined by

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty\}$$

and is equipped with the norm

$$\|u\|_{s,p} := \left(|u|_p^p + [u]_{s,p}^p \right)^{\frac{1}{p}},$$

where $|\cdot|_p$ denotes the norm on $L^p(\mathbb{R}^N)$.

Our problem is set in the closed linear space

$$X(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}$$

endowed with the norm $\|\cdot\| = [\cdot]_{s,p}$. The space $(X(\Omega), \|\cdot\|)$ is uniformly convex and setting

$$p_s^* := \frac{pN}{N - sp}$$

it results:

$$X(\Omega) \hookrightarrow L^\mu(\Omega) \text{ continuously for } \mu \in [1, p_s^*]$$

and

$$X(\Omega) \hookrightarrow\hookrightarrow L^\mu(\Omega) \text{ compactly for } \mu \in [1, p_s^*[. \tag{1.4}$$

Let us recall that $\lambda \in \mathbb{R}$ is an *eigenvalue* for $(-\Delta)_p^s$ if there exists a nontrivial weak solution on $X(\Omega)$ of the problem

$$\begin{cases} (-\Delta)_p^s u \;=\; \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

that is $u \in X(\Omega) \backslash \{0\}$ such that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y - \lambda \int_\Omega |u(x)|^{p-2} u(x)\varphi(x)\mathrm{d}x = 0,$$

for every $\varphi \in X(\Omega)$.

As usual, the set of the eigenvalues is named spectrum and it is denoted by $\sigma((-\Delta)_p^s)$.

It is well known that if $p = 2$ the spectrum of $(-\Delta)^s$ in $X(\Omega)$ consists of a diverging sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenvalues, repeated according to their multiplicity, satisfying $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots$ (see [29]). Moreover, for $p = 2$ the structure of $\sigma((-\Delta)^s)$ is very closed to that of the local operator $-\Delta$ and also provides a decomposition by means of the eigenfunctions.

On the other hand, when $p \neq 2$ the full spectrum of $(-\Delta)_p^s$ is still almost unknown, even if some important properties of the first eigenvalue and of the higher-order (variational) eigenvalues have been established in [15,19]. We also point out that a sequence of eigenvalues has been introduced in [17] by means of the cohomological index.

As we shall see in the proof of our main result, the definition of the quasi-eigenvalues proposed here and inspired to that in [11] fits in with our purposes, as a suitable decomposition of $X(\Omega)$ can be introduced and it turns out to be the known one for $p = 2$ (cf. Sect. 2 for the details).

Thus, the interaction of the nonlinearity f with the spectrum $\sigma((-\Delta)^s_p)$ of the fractional p-Laplacian operator must be taken into account. To our knowledge the only contribution in this direction is by [17], where the authors obtain existence results via Morse theory.

Now we state our main result.

Theorem 1.1 *Let $s \in]0, 1[$, $N > sp$ and Ω be an open bounded subset of $\mathbb{R}^N$ with continuous boundary. Assume that (h_1)–(h_2) hold, $f(x, \cdot)$ is odd for a.e. $x \in \Omega$ and that*

(h_4) *there exist $h, k \in \mathbb{N}$ with $k \geq h$ such that*

$$\alpha + \lambda < \beta_h \leq \gamma_k < \lambda,$$

where $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$ are, respectively, as in (2.11) and (2.13) below.

Then, problem (1.1) has at least $k-h+1$ distinct pairs of nontrivial weak solutions, provided either

- (a) $\lambda \notin \sigma((-\Delta)^s_p)$ *or*
- (b) $\lambda \in \sigma((-\Delta)^s_p)$ *and* (h_3) *holds true.*

For the sake of completeness we recall here that a function $u \in X(\Omega)$ is a *weak solution* of problem (1.1) if

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y - \lambda \int_\Omega |u(x)|^{p-2} u(x)\varphi(x)\mathrm{d}x - \int_\Omega f(x, u(x))\varphi(x)\mathrm{d}x = 0,$$

for every $\varphi \in X(\Omega)$.

This paper is organized as follows. In Sect. 2 we introduce the sequences of quasi-eigenvalues and depict some properties; then, in Sect. 3 we prove Theorem 1.1 by making use of a pseudo-index result recalled in Appendix as well as other classical tools.

2 Splitting of the fractional space $X(\Omega)$

In this section we adapt the arguments in [11, Section 5] to the nonlocal setting, thus constructing a first sequence of quasi-eigenvalues for $(-\Delta)^s_p$ on $X(\Omega)$. Let us define the subset

$$\mathcal{S} := \left\{ v \in X(\Omega) : \int_{\mathbb{R}^N} |v(x)|^p \mathrm{d}x = 1 \right\}, \tag{2.1}$$

the functional $\Phi : X(\Omega) \to \mathbb{R}$ by

$$\Phi(u) := \|u\|^p = [u]^p_{s,p} \tag{2.2}$$

and

$$\beta_1 := \inf_{v\in\mathcal{S}} \Phi(v) > 0$$

(β_1 is indeed the first eigenvalue of the p-fractional operator). Let us remark that

- $X(\Omega)$ is a reflexive Banach space;
- $\mathcal{S}$ is weakly closed in $X(\Omega)$ (by (1.4));
- Φ is coercive on $X(\Omega)$;
- Φ is weakly lower semicontinuous on $X(\Omega)$.

Then, by the generalized Weierstrass theorem, there exists a function $\psi_1 \in X(\Omega)$ such that

$$\int_{\mathbb{R}^N} |\psi_1(x)|^p \mathrm{d}x = 1, \quad \Phi(\psi_1) = [\psi_1]_{s,p}^p = \beta_1; \tag{2.3}$$

therefore,

$$\beta_1 |u|_p^p \leq \Phi(u) \quad \text{for all}\, u \in X(\Omega).$$

Let us consider the linear operator $\mathcal{L}_1 : L^p(\mathbb{R}^N) \to \mathbb{R}$ related to ψ_1 ($\psi_1 \in W^{s,p}(\mathbb{R}^N) \Rightarrow \psi_1 \in L^p(\mathbb{R}^N)$ thus $|\psi_1|^{p-1} \in L^{p'}(\mathbb{R}^N)$, where p' is the conjugate of p) defined by

$$\mathcal{L}_1 u := \int_{\mathbb{R}^N} |\psi_1(x)|^{p-2} \psi_1(x) u(x)\, \mathrm{d}x.$$

By definition, $\mathcal{L}_1 \in L^{p'}(\mathbb{R}^N)$, while (2.3) implies $\mathcal{L}_1 \psi_1 = 1$, then $\|\mathcal{L}_1\|_{L^{p'}} = 1$. Denoting again by $\mathcal{L}_1$ the restriction to the subspace $X(\Omega)$, it is also $\mathcal{L}_1 \in (X(\Omega))'$.

Now, we define the new constraint

$$\mathcal{S}_1 := \{v \in \mathcal{S} : \mathcal{L}_1 v = 0\} = \ker(\mathcal{L}_1|_{\mathcal{S}})$$

and the corresponding constrained infimum

$$\beta_2 := \inf_{v \in \mathcal{S}_1} \Phi(v).$$

We have that $\beta_2 > \beta_1$ (the first eigenvalue is isolated). We claim that also $\mathcal{S}_1$ is weakly closed in $X(\Omega)$. In fact, taking a sequence $\{v_m\}_{m\in\mathbb{N}} \subset \mathcal{S}_1$ and $v \in X(\Omega)$ such that

$$v_m \rightharpoonup v \;\; \text{weakly in}\, X(\Omega),$$

by (1.4) it follows that

$$v_m \to v \;\; \text{strongly in}\, L^p(\mathbb{R}^N)$$

and, since $\mathcal{L}_1 \in L^{p'}$ and $v_m \in \mathcal{S}_1$ for each $m \in \mathbb{N}$, we get that $\int_{\mathbb{R}^N} |v(x)|^p \mathrm{d}x = 1$ and $\mathcal{L}_1 v_m \to \mathcal{L}_1 v$; therefore, $v \in \mathcal{S}_1$.
Thus, the generalized Weierstrass theorem applies again and there exists $\psi_2 \in \mathcal{S}_1$ such that $\Phi(\psi_2) = \beta_2$, i.e.,

$$\int_{\mathbb{R}^N} |\psi_2(x)|^p \mathrm{d}x = 1, \quad \mathcal{L}_1 \psi_2 = 0, \quad \Phi(\psi_2) = \beta_2.$$

The procedure can be repeated, so fixing any $n \in \mathbb{N}$ we can define some positive numbers

$$\beta_1 < \beta_2 \leq \cdots \leq \beta_n$$

and some functions

$$\psi_1, \psi_2, \ldots, \psi_n \in \mathcal{S}$$

such that, for each $i \in \{1, \dots, n\}$, related to ψ_i we can consider the linear operator $\mathcal{L}_i \in L^{p'}$ defined by

$$\mathcal{L}_i u := \int_{\mathbb{R}^N} |\psi_i(x)|^{p-2}\psi_i(x)u(x)\mathrm{d}x \tag{2.4}$$

such that

$$[\psi_i]^p_{s,p} = \beta_i \tag{2.5}$$

and

$$\mathcal{L}_i\psi_i = \int_{\mathbb{R}^N} |\psi_i(x)|^p \mathrm{d}x = 1, \tag{2.6}$$

hence

$$\|\mathcal{L}_i\|_{L^{p'}} = 1,$$

while $\mathcal{L}_j\psi_i = 0$ for all $j \in \{1, \dots, i-1\}$, thus

$$\psi_i \in \bigcap_{j=1}^{i-1} \ker(\mathcal{L}_j|_{\mathcal{S}}), \quad \text{if } i \geq 2.$$

Therefore, we can define $\mathcal{S}_0 := \mathcal{S}$,

$$\mathcal{S}_n := \{v \in \mathcal{S} : \mathcal{L}_1 v = \dots = \mathcal{L}_n v = 0\} = \bigcap_{i=1}^{n} \ker(\mathcal{L}_i|\mathcal{S}) \quad \text{if } n \in \mathbb{N},$$

and the corresponding constrained infimum

$$\beta_{n+1} := \inf_{v\in\mathcal{S}_n} \Phi(v) \quad \text{if } n \geq 0. \tag{2.7}$$

We claim that there exists $\psi_{n+1} \in \mathcal{S}_n$ such that

$$[\psi_{n+1}]^p_{s,p} = \beta_{n+1}. \tag{2.8}$$

To this aim, arguing as above, it is enough proving that each $\mathcal{S}_n$ is weakly closed. Indeed, if $\{v_m\}_{m\in\mathbb{N}} \subset \mathcal{S}_n$ weakly converges to $v \in X(\Omega)$, by (1.4) and $\mathcal{L}_i \in L^{p'}$, it follows that $|v|_p = 1$ and $\mathcal{L}_i v_m \to \mathcal{L}_i v$ for all $i \in \{1, \dots, n\}$, hence $v \in \mathcal{S}_n$. Thus, Φ attains its infimum on $\mathcal{S}_n$ and (2.8) holds.

Summing up, by induction, we construct a sequence of positive numbers $\{\beta_n\}_{n\in\mathbb{N}}$, of functions $\{\psi_n\}_{n\in\mathbb{N}} \subset X(\Omega)$ and of linear operators $\{\mathcal{L}_n\}_{n\in\mathbb{N}} \subset L^{p'}$ such that (2.4)–(2.6) hold for all $n \in \mathbb{N}$; furthermore, it is

$$0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_n \leq \cdots$$

and $\psi_n \neq \psi_m$ if $n \neq m$.

Now we recall that if $V \subseteq X$ is a closed subspace of a Banach space X, a subspace $W \subseteq X$ is a *topological complement* of V, briefly $X = V \oplus W$, if W is closed and every $x \in X$ can be uniquely written as $v + w$, with $v \in V$ and $w \in W$; furthermore, the projection operators onto V and W are linear and continuous; hence, there exists $L := L(V, W) > 0$ such that

$$\|v\| + \|w\| \leq L\|v + w\|. \tag{2.9}$$

When $X = V \oplus W$ and V has finite dimension, we say that W has finite *codimension*, with codim $W = \dim V$.

By using again (1.4), the proofs of [11, Lemmas 5.2 and 5.3 and Proposition 5.4] can be adapted with minor changes to our setting; thus, the following properties can be stated:

- the increasing sequence $\{\beta_n\}_{n\in\mathbb{N}}$ diverges positively;
- fixing any $n \geq 1$ and setting

$$X_n := \text{span}\{\psi_1, \dots, \psi_n\} = \left\{ v \in X(\Omega) : \exists\, b_1, \dots, b_n \in \mathbb{R} \text{ s.t. } v = \sum_{i=1}^{n} b_i \psi_i \right\},$$

$$Y_n := \bigcap_{i=1}^{n} \ker(\mathcal{L}_i) = \{w \in X(\Omega) : \mathcal{L}_1 w = \cdots = \mathcal{L}_n w = 0\},$$

  we have

$$X(\Omega) = X_n \oplus Y_n; \tag{2.10}$$

- taking $n \geq 1$, for all $w \in Y_n$ we get

$$\beta_{n+1} \int_{\mathbb{R}^N} |w(x)|^p \mathrm{d}x \leq [w]_{s,p}^p; \tag{2.11}$$

- the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ generates the whole space $X(\Omega)$.

Now, following [18] we introduce another sequence of positive numbers. For all $n \in \mathbb{N}$, taking ψ_1 as in (2.3), we set

$$\mathbb{W}_n = \{Z : Z \text{ is a subspace of } X(\Omega),\ \psi_1 \in Z \text{ and } \dim Z \geq n\} \tag{2.12}$$

and

$$\gamma_n := \inf_{Z\in\mathbb{W}_n} \sup_{u\in\mathcal{S}\cap Z} \Phi(u), \tag{2.13}$$

with $\mathcal{S}$ as in (2.1). By the previous definitions, it follows that $\beta_1 = \gamma_1$ and $\mathbb{W}_{n+1} \subseteq \mathbb{W}_n$; hence, $\{\gamma_n\}_{n\in\mathbb{N}}$ is an increasing sequence of quasi-eigenvalues.

Remark 2.1 For $p = 2$ the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ reduce to the known sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $(-\Delta)^s$ (see, for instance, the paper [29]).

By using the genus we can construct a sequence of eigenvalues $\{\mu_n\}_{n\in\mathbb{N}}$ for the nonlinear operator $(-\Delta)_s^p$ on $X(\Omega)$, alike in the case of local p-Laplacian as in [16,20].

Let us consider the nonlinear eigenvalue problem (1.5) and set

$$\mu_n := \inf_{A\in\Sigma_n} \sup_{u\in A\setminus\{0\}} \frac{\Phi(u)}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^p \,\mathrm{d}x}, \quad n \in \mathbb{N} \tag{2.14}$$

where (cf. Sect. 1) $\Sigma_n := \{A \in \Sigma : \gamma(A) \geq n\}$ with Φ as in (2.2),

$$\Sigma := \{A \subseteq X(\Omega), \text{ closed and symmetric w.r.t. the origin}\} \tag{2.15}$$

and consider the even functional

$$\Psi(u) := \frac{\Phi(u)}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^p \,\mathrm{d}x} \quad \text{on } X(\Omega)\setminus\{0\}.$$

The critical values and the critical points of Ψ restricted to the manifold $\mathcal{S}$ defined in (2.1) are eigenvalues and eigenfunctions of $(-\Delta)_p^s$ on $X(\Omega)$, respectively. We can state the following proposition.

Proposition 2.2 *For every $n \in \mathbb{N}$ the numbers μ_n in (2.14) are eigenvalues for the nonlinear operator $(-\Delta)_p^s$ on $X(\Omega)$.*

Proof By [13, Lemma 4] and (1.4) the functional $\Psi\big|_{\mathcal{S}}$ satisfies the Palais–Smale condition. Then, by using a suitable version of the deformation lemma (cf., e.g., [6]), standard mini–max arguments give the result. □

Furthermore, we point out that $\mu_1 = \beta_1 = \gamma_1$. Slight changes in the proof of [2, Proposition 2.9] and in [18, Remark 1.1(4)] provide the following proposition stating a comparison among the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ and the sequences of quasi-eigenvalues $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ of $(-\Delta)_p^s$.

Proposition 2.3 *For all $n \in \mathbb{N}$ we have that $\beta_n \le \mu_n \le \gamma_n$.*

Remark 2.4 The properties of $\{\beta_n\}_{n\in\mathbb{N}}$ and Proposition 2.3 imply that $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are diverging sequences. Moreover, as $\{\gamma_n\}_{n\in\mathbb{N}}$ is increasing, we have also $\beta_h \le \gamma_k$ for $k \ge h \ge 1$; therefore, this inequality is not an assumption in (h_4).

3 Proof of Theorem 1.1

From (h_1) and (1.2) for all $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon |t|^{p-1} + K_\varepsilon \quad \text{for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{3.1}$$

The weak solutions of problem (1.1) are the critical points of the C^1-functional

$$J_\lambda(u) := \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega |u(x)|^p \,\mathrm{d}x - \int_\Omega F(x,u(x))\mathrm{d}x \tag{3.2}$$

on $X(\Omega)$ whose derivative is given by

$$\begin{aligned} \mathrm{d}J_\lambda(u)[\varphi] = &\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &- \lambda\int_\Omega |u(x)|^{p-2}u(x)\varphi(x)\mathrm{d}x \\ &- \int_\Omega f(x,u(x))\varphi(x)\mathrm{d}x, \end{aligned}$$

for any $\varphi \in X(\Omega)$.

For the sake of simplicity we introduce the operator $A : X(\Omega) \to (X(\Omega))^*$, defined for all $u, \varphi \in X(\Omega)$ by

$$\langle A(u), \varphi\rangle := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y.$$

In next proposition we prove that the functional J satisfies the (C) condition (cf. Appendix) both in the nonresonant case and in the resonant one, up to assume also assumption (h_3).

Proposition 3.1 *Assume that (h_1)–(h_2) hold. Then*

(i) if $\lambda \notin \sigma((-\Delta)_p^s)$, the functional J_λ in (3.2) satisfies (C) *in $\mathbb{R}$;*
(ii) if $\lambda \in \sigma((-\Delta)_p^s)$ and (h_3) holds, the functional J_λ in (3.2) satisfies (C) *in $\mathbb{R}$.*

Proof (i) Let $c \in \mathbb{R}$ and $\{u_m\}_{m\in\mathbb{N}}$ be a sequence in $X(\Omega)$ such that (3.30) holds; then in particular

$$\begin{aligned}\langle A(u_m), \varphi\rangle - \lambda \int_\Omega |u_m(x)|^{p-2}u_m(x)\varphi(x)\mathrm{d}x \\ - \int_\Omega f(x, u_m(x))\varphi(x)\mathrm{d}x = o(1),\end{aligned} \tag{3.3}$$

for every $\varphi \in X(\Omega)$, where $o(1)$ denotes an infinitesimal sequence.

In order to prove the statement, it is enough to show that $\{\|u_m\|\}_{m\in\mathbb{N}}$ is bounded (cf. [23, Proposition 1.3]). Then, arguing by contradiction, let us assume that

$$\|u_m\| \to +\infty \quad \text{as } m \to +\infty. \tag{3.4}$$

Setting $w_m := \dfrac{u_m}{\|u_m\|}$, $\{w_m\}_{m\in\mathbb{N}}$ is bounded in $X(\Omega)$ and there exists $w \in X(\Omega)$ such that, up to subsequences, we have

$$w_m \rightharpoonup w \quad \text{weakly in } X(\Omega) \tag{3.5}$$

and

$$w_m \to w \quad \text{strongly in } L^p(\Omega). \tag{3.6}$$

Evaluating (3.3) in $w_m - w$ and dividing by $\|u_m\|^{p-1}$, we get

$$\begin{aligned}\langle A(w_m), w_m - w\rangle = \lambda \int_\Omega |w_m(x)|^{p-2}w_m(x)\,(w_m - w)(x)\,\mathrm{d}x \\ + \int_\Omega \frac{f(x, u_m(x))}{\|u_m\|^{p-1}}(w_m - w)(x)\,\mathrm{d}x + o(1).\end{aligned} \tag{3.7}$$

Let us analyze this last equation. Firstly, by (3.6) it follows that

$$\left|\int_\Omega |w_m(x)|^{p-2}w_m(x)(w_m - w)(x)\mathrm{d}x\right| \le |w_m|_p^{p-1}|w_m - w|_p = o(1).$$

Furthermore, (3.1), (3.4) and (3.6) imply that

$$\begin{aligned}\left|\int_\Omega \frac{f(x, u_m(x))}{\|u_m\|^{p-1}}(w_m - w)(x)\mathrm{d}x\right| \le \varepsilon|w_m|_p^{p-1}|w_m - w|_p \\ + \frac{K_\varepsilon}{\|u_m\|^{p-1}}|w_m - w|_1 = o(1).\end{aligned} \tag{3.8}$$

Hence, by (3.7)

$$\langle A(w_m), w_m - w\rangle = o(1)$$

and by [23, Proposition 1.3]

$$w_m \to w \quad \text{strongly in } X(\Omega). \tag{3.9}$$

Thus, by the definition of w_m it follows $w \neq 0$.

Now, dividing (3.3) by $\|u_m\|^{p-1}$, for all $\varphi \in X(\Omega)$ we have that

$$\begin{aligned}\langle A(w_m), \varphi\rangle = \lambda \int_\Omega |w_m(x)|^{p-2}w_m(x)\varphi(x)\mathrm{d}x \\ + \int_\Omega \frac{f(x, u_m(x))}{\|u_m\|^{p-1}}\varphi(x)\mathrm{d}x + o(1).\end{aligned} \tag{3.10}$$

Again (3.1), (3.4) and (3.6) give

$$\lim_{m\to+\infty}\int_\Omega \frac{f(x,u_m(x))}{\|u_m\|^{p-1}}\varphi(x)\mathrm{d}x = 0 \quad \text{for all } \varphi \in X(\Omega). \tag{3.11}$$

Therefore, by (3.9) and (3.11), passing to the limit in (3.10), we get

$$\langle A(w), \varphi\rangle = \lambda \int_\Omega |w(x)|^{p-2}w(x)\varphi(x)\mathrm{d}x \quad \text{for all } \varphi \in X(\Omega).$$

But this means that $\lambda \in \sigma((-\Delta)^s_p)$, against our assumption; thus, the proof is complete.
(ii) Let $c \in \mathbb{R}$ and $\{u_m\}_{m\in\mathbb{N}}$ be a sequence in $X(\Omega)$ such that (3.30) holds. Set

$$g(x,t) := \lambda t + f(x,t) \quad \text{for a.e. } x \in \Omega,\ \forall t \in \mathbb{R}. \tag{3.12}$$

By using (3.12) we have that

$$\frac{1}{p}\|u_m\|^p - \int_\Omega G(x,u_m(x))\mathrm{d}x = c + o(1) \tag{3.13}$$

and

$$\|u_m\|^p - \int_\Omega g(x,u_m(x))u_m(x)\mathrm{d}x = o(1), \tag{3.14}$$

with $G(x,t) := \int_0^t g(x,\tau)\mathrm{d}\tau$.
By assumption (h_3) there exists $\eta_1 > 0$ such that

$$g(x,t)t - p\,G(x,t) \geq 0 \quad \text{if } |t| \geq \eta_1, \text{for a.e.} x \in \Omega. \tag{3.15}$$

On the other hand, by using condition (h_1) there exists $C_1 = C_1(\eta_1) > 0$ such that

$$\int_{\{|u_m|\leq\eta_1\}} (g(x,u_m(x))u_m(x) - p\,G(x,u_m(x)))\,\mathrm{d}x \geq -C_1, \tag{3.16}$$

for every $m \in \mathbb{N}$. Fixing $\varepsilon > 0$, by (3.12) in addition to (1.2) of (h_2), there exists $\eta_\varepsilon > 0$ such that

$$|g(x,t)| \leq (|\lambda| + \varepsilon)|t| \quad \text{if } |t| > \eta_\varepsilon, \text{for a.e.} x \in \Omega. \tag{3.17}$$

Now, taking $q \in]p, p_s^*[$, there exists $C > 0$ such that

$$|u|_q \leq C^{\frac{1}{p}}\|u\| \quad \text{for all } u \in X(\Omega) \tag{3.18}$$

(cf. (1.4)). Hence, let us set

$$\kappa := (2c + C_1)(2(|\lambda| + \varepsilon)C)^{\frac{q}{q-p}}, \tag{3.19}$$

with c as in (3.13) and C_1 as in (3.16).
Again by (h_3) we get the existence of $\eta_2 := \eta_2(\kappa) > \max\{\eta_1, \eta_\varepsilon\}$ such that

$$g(x,t)t - p\,G(x,t) \geq \kappa \quad \text{if } |t| \geq \eta_2, \text{for a.e.} x \in \Omega. \tag{3.20}$$

Then, for η_2 as above, we define

$$A_m := \{x \in \Omega : |u_m(x)| \geq \eta_2\}$$

and

$$B_m := \{x \in \Omega : |u_m(x)| \leq \eta_2\},$$

for every $m \in \mathbb{N}$.

By (3.13)–(3.16) and (3.20) it follows that

$$\begin{aligned} pc + o(1) &= \int_{\Omega} (g(x, u_m(x))u_m(x) - p\,G(x, u_m(x)))\,\mathrm{d}x \\ &= \int_{A_m} (g(x, u_m(x))u_m(x) - p\,G(x, u_m(x)))\,\mathrm{d}x \\ &\quad + \int_{\{|u_m|\le \eta_1\}} (g(x, u_m(x))u_m(x) - p\,G(x, u_m(x)))\,\mathrm{d}x \\ &\quad + \int_{\{\eta_1 \le |u_m| \le \eta_2\}} (g(x, u_m(x))u_m(x) - p\,G(x, u_m(x)))\,\mathrm{d}x \\ &\ge \kappa \operatorname{meas}(A_m) - C_1. \end{aligned}$$

Hence, from the above inequality, one has

$$\operatorname{meas}(A_m) \le \frac{2c + C_1}{\kappa} + o(1) \quad \text{for all } m \in \mathbb{N}. \tag{3.21}$$

Taking $r > p$, by (3.13) and (3.14) we have that

$$\begin{aligned} \left(\frac{1}{p} - \frac{1}{r}\right) \|u_m\|^p - \int_{\Omega} \left(G(x, u_m(x)) - \frac{1}{r}\, g(x, u_m(x))u_m(x)\right) \mathrm{d}x \\ = c + o(1). \end{aligned} \tag{3.22}$$

Moreover, by (h_1) there exists $C_2 := C_2(\Omega, g, \eta_2, r) > 0$ such that

$$\left| \int_{B_m} \left(G(x, u_m(x)) - \frac{1}{r}\, g(x, u_m(x))u_m(x)\right) \mathrm{d}x \right| \le C_2, \quad \forall m \in \mathbb{N}. \tag{3.23}$$

Hence, by (3.22) and (3.23) we infer that

$$\begin{aligned} c + o(1) \ge & \left(\frac{1}{p} - \frac{1}{r}\right) \|u_m\|^p \\ & - \int_{A_m} \left(G(x, u_m(x)) - \frac{1}{r}\, g(x, u_m(x))u_m(x)\right) \mathrm{d}x - C_2. \end{aligned}$$

Further, by (3.15) and (3.17) it follows that

$$\begin{aligned} c + o(1) \ge & \left(\frac{1}{p} - \frac{1}{r}\right) \|u_m\|^p \\ & - \int_{A_m} \left(\frac{1}{p} g(x, u_m(x))u_m(x) - \frac{1}{r}\, g(x, u_m(x))u_m(x)\right) \mathrm{d}x - C_2 \\ \ge & \left(\frac{1}{p} - \frac{1}{r}\right) \left(\|u_m\|^p - \int_{A_m} (|\lambda| + \varepsilon)|u_m(x)|^p\, \mathrm{d}x\right) - C_2. \end{aligned}$$

Now, by the Hölder inequality, (3.18), (3.19) and (3.21) we have that

$$\begin{aligned} c + o(1) &\ge \left(\frac{1}{p} - \frac{1}{r}\right) \left(\|u_m\|^p - (\lambda + \varepsilon)\, |u_m|_q^p \operatorname{meas}(A_m)^{\frac{q-p}{q}}\right) - C_2 \\ &\ge \left(\frac{1}{p} - \frac{1}{r}\right) \|u_m\|^p \left(1 - (|\lambda| + \varepsilon) C \left(\frac{1}{(2(\lambda + \varepsilon)C)^{\frac{q}{q-p}}} + o(1)\right)^{\frac{q-p}{q}}\right) \\ &\quad - C_2. \end{aligned}$$

Thus, the sequence $\{\|u_m\|\}_{m\in\mathbb{N}}$ is bounded in $X(\Omega)$. □

Lemma 3.2 *Assume that* (h_1)–(h_2) *hold. Let* β_h *be as in* (h_4) *and* Y_{h-1} *as in* (2.10)*. Then, there exist* $\rho > 0$ *and* $c_0 > 0$ *such that, setting* $S_\rho := \{u \in X(\Omega) : \|u\| = \rho\}$*, the functional* J_λ *in* (3.2) *verifies*

$$J_\lambda(u) \geq c_0 \quad \textit{for all } u \in S_\rho \cap Y_{h-1}. \tag{3.24}$$

Proof By (h_2) it follows that, uniformly with respect to almost every $x \in \Omega$, there exist

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = 0$$

and

$$\lim_{t\to 0} \frac{F(x,t)}{|t|^p} = \frac{\alpha}{p}.$$

Therefore, for every $\varepsilon > 0$ there exist $R_\varepsilon, \delta_\varepsilon > 0$ such that, for almost every $x \in \Omega$,

$$|F(x,t)| \leq \frac{\varepsilon}{p}|t|^p \qquad \text{if } |t| > R_\varepsilon \tag{3.25}$$

and

$$\left|F(x,t) - \frac{\alpha}{p}|t|^p\right| \leq \frac{\varepsilon}{p}|t|^p \qquad \text{if } |t| < \delta_\varepsilon, \tag{3.26}$$

without loss of generality with $R_\varepsilon \geq 1$. On the other hand, by (h_1), taking any $l \in [0, p_s^* - p[$, there exists $k_{R_\varepsilon} > 0$ such that, for almost every $x \in \Omega$,

$$|F(x,t)| \leq k_{R_\varepsilon}|t|^{l+p} \qquad \text{if } \delta_\varepsilon \leq |t| \leq R_\varepsilon. \tag{3.27}$$

The inequalities (3.25)–(3.27) imply that for any $\varepsilon > 0$ there exists $k_\varepsilon > 0$ such that

$$F(x,t) \leq \frac{\alpha+\varepsilon}{p}|t|^p + k_\varepsilon|t|^{l+p} \qquad \text{for a.e.} x \in \Omega, \text{ for all } t \in \mathbb{R}.$$

We infer that

$$\int_\Omega F(x,u(x))\mathrm{d}x \leq \frac{\alpha+\varepsilon}{p}|u|_p^p + k_\varepsilon|u|_{l+p}^{l+p} \quad \text{for all } u \in X(\Omega).$$

For a suitable $k'_\varepsilon > 0$ we have

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p - \frac{\lambda+\alpha+\varepsilon}{p}|u|_p^p - k'_\varepsilon\|u\|^{l+p} \quad \text{for all } u \in X(\Omega). \tag{3.28}$$

Let us recall that by the decomposition (2.10) it is $X(\Omega) = X_{h-1} \oplus Y_{h-1}$, where $X_{h-1} := \operatorname{span}\{\psi_1, \ldots, \psi_{h-1}\}$ and Y_{h-1} is its complement. Thus by (2.11) and (3.28) it follows that

$$J_\lambda(u) \geq \frac{1}{p}\left(1 - \frac{\lambda+\alpha+\varepsilon}{\beta_h}\right)\|u\|^p - k'_\varepsilon\|u\|^{l+p} \quad \text{for all } u \in Y_{h-1}$$

and by (h_4), for a suitable ε, there exists $k''_\varepsilon > 0$ such that

$$J_\lambda(u) \geq k''_\varepsilon\|u\|^p - k'_\varepsilon\|u\|^{l+p} \quad \text{for all } u \in Y_{h-1}.$$

Thus we conclude that if ρ is small enough there exists $c_0 > 0$ such that (3.24) holds. □

Lemma 3.3 *Assume that* (h_1) *and* (1.2) *hold. Let* γ_k *as in* (h_4), $\mathbb{W}_k$ *as in* (2.12) *and* c_0 *as in Lemma 3.2. Then, there exist a k-dimensional space* $V \in \mathbb{W}_k$ *and* $c_\infty > c_0$ *such that the functional* J_λ *in* (3.2) *verifies*

$$J_\lambda(u) \le c_\infty \quad \text{for all } u \in V. \tag{3.29}$$

Proof By (3.1), fixing any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$J_\lambda(u) \le \frac{1}{p}\|u\|^p - \frac{\lambda}{p}|u|^p + \frac{\varepsilon}{2p}|u|_p^p + C_\varepsilon|u|_p \quad \text{for all } u \in X(\Omega).$$

Let γ_k be as in (h_4) and take $\varepsilon > 0$ such that $\gamma_k + \varepsilon < \lambda$. From definition (2.13), for such a fixed $\varepsilon > 0$ there exists a subspace V_k^ε in $\mathbb{W}_k$, with dim $V_k^\varepsilon \ge k$, such that

$$\gamma_k \le \sup_{u \in V_k^\varepsilon \setminus \{0\}} \frac{\|u\|^p}{|u|^p} < \gamma_k + \frac{\varepsilon}{2}.$$

Thus it results that

$$J_\lambda(u) \le \frac{1}{p}(\gamma_k + \varepsilon - \lambda)\,|u|^p + C_\varepsilon|u|_p \quad \text{for all } u \in V_k^\varepsilon$$

and, as without loss of generality we can assume that V_k^ε is a k-dimensional subspace, the functional J_λ tends to $-\infty$ as $\|u\|$ diverges in V_k^ε, so there exists $c_\infty = c_\infty(\varepsilon)$ (with $c_\infty > c_0$), such that (3.29) holds. □

Proof of Theorem 1.1. (a) Firstly, by Proposition 3.1—part (i) the functional J_λ in (3.2) satisfies (C) in $\mathbb{R}$, and by assumption, it is even.

Let us consider β_h, Y_{h-1}, ρ, c_0 as in Lemma 3.2 and γ_k, $\mathbb{W}_k$, V_k^ε, c_∞ as in Lemma 3.3.

Then, we consider the pseudo-index theory $(S_\rho \cap Y_{h-1}, \mathcal{H}^*, \gamma^*)$ related to the genus and $S_\rho \cap Y_{h-1}$. By Remark 3.7 applied to $V := V_k^\varepsilon$, $\partial B := S_\rho$ and $W := Y_{h-1}$, we get

$$\gamma\left(V_k^\varepsilon \cap h\left(S_\rho \cap Y_{h-1}\right)\right) \ge \dim V_k^\varepsilon - \operatorname{codim} Y_{h-1} \quad \text{for all } h \in \mathcal{H}^*,$$

which implies

$$\gamma^*(V_k^\varepsilon) \ge k - h + 1.$$

The proof is then complete: in fact Theorem 3.6 applies with $\tilde{A} := V_k^\varepsilon$ and $S := S_\rho \cap Y_{h-1}$ and J has at least $k - h + 1$ distinct pairs of critical points corresponding to at most $k - h + 1$ distinct critical values c_i, where c_i is as in (3.32).

(b) In the resonant case, by Proposition 3.1—part (ii) the functional J_λ satisfies (C), and we can proceed as above. □

Remark 3.4 We point out that Theorem 1.1 holds also with slight changes in the proof when (h_4) is replaced by

$$\lambda < \beta_h \le \gamma_k < \alpha + \lambda.$$

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Appendix: Abstract framework

Throughout this paper $(X, \|\cdot\|_X)$ is a Banach space, $(X', \|\cdot\|_{X'})$ its dual, I a C^1 functional on X, $I^b := \{u \in X : I(u) \leq b\}$ the sublevel of I corresponding to $b \in \mathbb{R}$ and

$$K_c := \{u \in X : \ I(u) = c, \ \mathrm{d}I(u) = 0\}$$

the set of the critical points of I in X at the critical level $c \in \mathbb{R}$.

In Sect. 3 we have seen that problem (1.1) has a variational structure; thus, next we recall some abstract tools used before.

Firstly, we recall the so-called Cerami's variant of the Palais–Smale condition; even if it is a condition weaker than the classical one, it is enough in order to state a deformation theorem and some critical point theorems (cf. [1]).

Definition 3.5 The functional I satisfies the *Cerami's variant of the Palais–Smale condition at level* c ($c \in \mathbb{R}$), if any sequence $\{u_m\}_{m\in\mathbb{N}} \subseteq X$ such that

$$\{I(u_m)\}_{m\in\mathbb{N}} \text{ is bounded and } \lim_{m\to+\infty} \|\mathrm{d}I(u_m)\|_{X'}(1 + \|u_m\|_X) = 0 \tag{3.30}$$

converges in X, up to subsequences. In general, if $-\infty \leq a < b \leq +\infty$, I satisfies (C) in $]a, b[$ if so is at each level $c \in]a, b[$.

In the proof of our main theorem we use [1, Theorem 2.9] rewritten on Banach spaces where the index theory related to the genus acts. The proof is based on the use of a pseudo-index theory, and before introducing such a definition, we recall some notions of the index theory on Banach spaces X for an even functional with symmetry group $\mathbb{Z}_2 := \{\mathrm{id}, -\mathrm{id}\}$.

Define

$$\Sigma := \ \{A \subseteq X \text{ closed and symmetric w.r.t. the origin}\}$$

and

$$\mathcal{H} := \{h \in C(X, X) : h \text{ odd}\}.$$

Taking $A \in \Sigma$, $A \neq \emptyset$, the genus of A is

$$\gamma(A) \ := \ \inf\{k \in \mathbb{N} : \ \exists\psi \in C(A, \mathbb{R}^k\backslash\{0\}) \text{ s.t. } \psi(-u) = -\psi(u) \text{ for all } u \in A\},$$

if such an infimum exists, otherwise $\gamma(A) = +\infty$. Assume $\gamma(\emptyset) = 0$.
The index theory $(\Sigma, \mathcal{H}, \gamma)$ related to $\mathbb{Z}_2$ is also called *genus* (we refer for more details, e.g., to [30, Section II.5]).

According to [4], the pseudo-index related to the genus, an even functional $I : X \to \mathbb{R}$ and $S \in \Sigma$ is the triplet $(S, \mathcal{H}^*, \gamma^*)$ such that $\mathcal{H}^*$ is a group of odd homeomorphisms and $\gamma^* : \Sigma \longrightarrow \mathbb{N} \cup \{+\infty\}$ is the map defined by

$$\gamma^*(A) := \min_{h\in\mathcal{H}^*} \gamma(h(A) \cap S) \quad \text{for all } A \in \Sigma.$$

Since

$$\gamma(h(A) \cap S) = \gamma(A \cap h^{-1}(S)) \quad \text{for all } h \in \mathcal{H}^*,$$

then

$$\gamma^*(A) = \min_{h\in\mathcal{H}^*} \gamma(A \cap h(S)) \quad \text{for all } A \in \Sigma. \tag{3.31}$$

The following *mini–max* theorem was proved in [1, Theorem 2.9] in the setting of Hilbert spaces; the same proof holds on Banach spaces.

Theorem 3.6 *Consider $a, b, c_0, c_\infty \in \bar{\mathbb{R}}$, $-\infty \le a < c_0 < c_\infty < b \le +\infty$. Let I be an even functional, $(\Sigma, \mathcal{H}, \gamma)$ the genus theory on X, $S \in \Sigma$, $(S, \mathcal{H}^*, \gamma^*)$ the pseudo-index theory related to the genus, I and S, with*

$$\mathcal{H}^* = \{h \in \mathcal{H} : h \text{ bounded homeomorphism s.t. } h(u) = u \text{ if } u \notin I^{-1}(]a, b[)\}.$$

Assume that:

(i) *the functional I satisfies* (C) *in $]a, b[$;*
(ii) *$S \subseteq I^{-1}([c_0, +\infty[)$;*
(iii) *there exist $\tilde{k} \in \mathbb{N}$ and $\tilde{A} \in \Sigma$ such that $\tilde{A} \subseteq I^{c_\infty}$ and $\gamma^*(\tilde{A}) \ge \tilde{k}$.*

Then the numbers

$$c_i := \inf_{A \in \Sigma_i^*} \sup_{u \in A} I(u), \quad i \in \{1, \dots, \tilde{k}\}, \tag{3.32}$$

with $\Sigma_i^* := \{A \in \Sigma : \gamma^*(A) \ge i\}$, are critical values for I and

$$c_0 \le c_1 \le \cdots \le c_{\tilde{k}} \le c_\infty.$$

Furthermore, if $c = c_i = \cdots = c_{i+r}$, with $i \ge 1$ and $i + r \le \tilde{k}$, then $\gamma(K_c) \ge r + 1$.

Remark 3.7 In order to apply the theorem above, we need a lower bound for the pseudo-index of a suitable $\tilde{A}$ as in (iii) of Theorem 3.6. Thus, let us consider the genus theory $(\Sigma, \mathcal{H}, \gamma)$ on X and V, W two closed subspaces of X. If

$$\dim V < +\infty \quad \text{and} \quad \operatorname{codim} W < +\infty,$$

then, for every odd bounded homeomorphism h on X and every open bounded symmetric neighborhood B of 0 in X, it results

$$\gamma(V \cap h(\partial B \cap W)) \ge \dim V - \operatorname{codim} W$$

(cf. [1, Theorem A.2] and [2, Theorem 2.7]).

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