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On strongly condensing operators

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Abstract Given a set-function ψ defined on bounded subsets of a Banach space with certain properties, necessary and sufficient criteria for $\psi(A(U)) = 0$ are given, when A is positively homogeneous of some order and U is bounded. The results are applied to give necessary and sufficient criteria for the compactness and weak compactness of a Fréchet derivative (in some point or at ∞) and when an operator is improving.

Keywords Positively homogeneous map · Measure of noncompactness · Fréchet derivative · Asymptotic derivative · Compactness · Weak compactness · Superposition operator · Ideal space of measurable functions · Regular space of measurable functions · Lebesgue space · Lorentz space · Orlicz space

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1 Introduction

Let X and Y be Banach spaces, and let $\mathfrak{B}(Y)$ be the family of all bounded subsets of Y. For $U, V \subseteq Y$ and $\Lambda \subseteq \mathbb{R}$, we use the notations $U + V := \{u + v : u \in U, v \in V\}$ and $\Lambda U := \{\lambda u : \lambda \in \Lambda, u \in U\}$, and similarly, we put for $u \in U$ and $\lambda \in \mathbb{R}$ also $u + U := \{u\} + U$ and $\lambda U := \{\lambda\} U$. We call a function $\psi : \mathfrak{B}(Y) \to [0, \infty)$

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(1) *monotone*, if $U \subseteq V \in \mathfrak{B}(Y)$ implies $\psi(U) \leqslant \psi(V)$.
(2) *even*, if $\psi(-U) = \psi(U)$ for all $U \in \mathfrak{B}(Y)$.
(3) *algebraically semi-additive*, if

$$\psi(U + V) \leqslant \psi(U) + \psi(V) \quad \text{for all} \quad U, V \in \mathfrak{B}(Y).$$

This will be essentially the only properties which we use in our main result. For explanations and examples, we also recall some further notions which are used in the literature, see, e.g., [1, 1.2], [3, 2.1], or [4, 3.1]. We call ψ

(4) *positively homogeneous*, if $\psi(\rho U) = \rho\psi(U)$ for all $\rho > 0$, $U \in \mathfrak{B}(Y)$.
(5) *translation invariant*, if $\psi(u + U) = \psi(U)$ for all $u \in Y$, $U \in \mathfrak{B}(Y)$.
(6) *conical invariant*, if $\psi([0, 1]U) = \psi(U)$ for all $U \in \mathfrak{B}(Y)$.
(7) *nonsingular*, if $\psi(U \cup \{u\}) = \psi(U)$ for all $u \in Y$, $U \in \mathfrak{B}(Y)$.
(8) *regular*, if $\psi(U) = 0$ iff U is precompact (in the uniform structure of Y).
(9) *convex invariant*, if $\psi(\operatorname{conv} U) = \psi(U)$ for all $U \in \mathfrak{B}(Y)$.
(10) *closed convex invariant* or *measure of noncompactness*, if $\psi(\overline{\operatorname{conv}}\, U) = \psi(U)$ for all $U \in \mathfrak{B}(Y)$.

Remark 1 If ψ is monotone, convex invariant, and nonsingular, then ψ is conical invariant, because $[0, 1]U \subseteq \operatorname{conv}(U \cup \{0\})$

A trivial example of a function ψ on Y satisfying *most* of the above properties is given by

$$\operatorname{diam}_Y(U) := \sup\{\|u - v\| : u, v \in U\}, \quad \operatorname{diam}_Y(\emptyset) := 0.$$

The only properties which fail to be satisfied by diam_Y are conical invariance, nonsingularity, and regularity. Another example, also satisfying the conical invariance but not the translation invariance, is $\operatorname{diam}_{Y,0}(U) := \operatorname{diam}_Y(U \cup \{0\})$.

An example of a function ψ on Y which satisfies *all* of the above properties is the *Hausdorff measure of noncompactness* χ_Y, defined as

$$\chi_Y(U) := \inf\left\{r > 0 \mid \exists N \subseteq Y \text{ finite with } U \subseteq \bigcup_{u \in N}(u + B_r)\right\}, \tag{1.1}$$

see, e.g., [1, 1.1.4], [3, 2.3], [4, 3.1.2], or [23, 3.2]. Here and throughout, we denote the balls and spheres with center 0 in a normed space X by

$$B_\rho := \{u \in X : \|u\| \leqslant \rho\} \quad \text{and} \quad S_\rho := \{u \in X : \|u\| = \rho\}.$$

Given $k \in [0, \infty)$, we call a map $A\colon X \to Y$ *positively homogeneous of order* k if

$$A(\rho u) = \rho^k A(u) \quad \text{for all} \quad \rho \geqslant 0, \quad u \in X.$$

Recall that a bounded linear map $A\colon X \to Y$ is called a Fréchet derivative of $F\colon X \to Y$ at $u_0 \in X$ if

$$F(u_0 + u) = F(u_0) + A(u) + r(u) \quad \text{for all} \quad u \in X \tag{1.2}$$

with $r(u)/\|u\| \to 0$ as $u \to 0$; in particular,

$$\lim_{\rho \to 0} \frac{\psi\big(F(u_0) + r(B_\rho)\big)}{\rho} = 0 \tag{1.3}$$

when, e.g., $\psi = \chi_Y$ or $\psi = \operatorname{diam}_Y$. With the choice $A_0(u) := F(u_0)+r(u)$, the equality (1.2) means

$$F(u_0 + u) = A(u) + A_0(u) \quad \text{for all} \quad u \in X. \tag{1.4}$$

We will consider a more general situation: We require only (1.4) with A being positively homogeneous of some order k, actually even only positively homogeneous with respect to ψ in a sense which will be made precise below. Moreover, we require that the remainder function A_0 is only "small" near 0 (or near ∞) with "smallness" being measured with respect to ψ similar to (1.3); the precise requirement will be formulated in the next section. (Note that we keep u_0 fixed, throughout; the functions A and A_0 will depend on u_0, in general.)

Under this hypothesis, we are interested in necessary and sufficient criteria for $\psi(A(U)) = 0$ for every $U \in \mathfrak{B}(X)$.

For instance, if F is Fréchet differentiable at u_0, and if ψ denotes the Hausdorff measure of noncompactness, then the mentioned smallness hypothesis will be satisfied due to (1.3), and so, we obtain in particular necessary and sufficient criteria for the compactness of the Fréchet derivative $A = F'(u_0)$.

Such criteria have a long tradition. Sufficient criteria for the compactness of the Fréchet derivative have already been obtained by Krasnosel'skiĭ [14]. Further sufficient conditions were obtained in [9,17], and they have been extended to various necessary and sufficient criteria in [11,12]. Our results contain both of these criteria, and moreover, they apply also in a sense if A is not necessarily linear but only positively homogeneous of some order $k \in [0, \infty)$. Actually, we do not even require that A is positively homogeneous, but only either

$$\psi\,(A(\rho U)) = \rho^k \psi(A(U)) \quad \text{for all} \quad \rho > 0, \quad A(U) \in \mathfrak{B}(Y), \tag{1.5}$$

or sometimes additionally

$$\psi\,(A\,([0,r]U)) = r^k \psi(A(U)) \quad \text{for all} \quad r > 0, \quad A(U) \in \mathfrak{B}(Y), \tag{1.6}$$

which are indeed both weaker requirements:

Proposition 1 (*i*) *Suppose that ψ is positively homogeneous. If A is positively homogeneous of some order $k \in [0, \infty)$, then (1.5) is satisfied.*
(*ii*) *Suppose that ψ is positively homogeneous and conical invariant. If A is positively homogeneous of some order $k \in [0, \infty)$, then (1.6) holds.*
None of these two implications can be reverted.

Proof If A is positively homogeneous of order k, then $A(\rho U) = \rho^k A(U)$ for every $\rho > 0$. Moreover, for every $r > 0$ there holds $A([0,r]U) = r^k[0,1]A(U)$. This implies (1.5) or (1.6), respectively, if ψ has the required properties. For a counterexample, consider $\psi = \chi_Y$ and redefine a positively homogeneous map A of order k at only one point. □

Remark 2 If $A\colon X \to Y$ satisfies (1.5) and $\psi(A(B_r)) = 0$ for some $r > 0$, then $\psi(A(B_\rho)) = (\rho/r)^k \psi(A(B_r)) = 0$ for all $\rho > 0$. In particular, if ψ is also monotone, then $\psi(A(U)) = 0$ for every $U \in \mathfrak{B}(X)$.

Similarly, if $A\colon X \to Y$ satisfies (1.5) and $\psi(A(S_r)) = 0$ for some $r > 0$, then $\psi(A(S_\rho)) = (\rho/r)^k \psi(A(S_r)) = 0$ for all $\rho > 0$.

The principal aim of the paper is to apply these notions to obtain necessary and sufficient conditions for $\psi(A(U)) = 0$ for $U \in \mathfrak{B}(X)$ if A satisfies (1.5).

2 Locally strongly condensing maps

The following criteria had been used in [11] if $k = 1$. Given $k \in (0, \infty)$, we call $F\colon X \to Y$ *locally strongly ψ-condensing of order k for balls at $u_0 \in X$*, if

$$\lim_{r \to 0} \frac{\psi\,(F(u_0 + B_r))}{r^k} = 0, \tag{2.1}$$

and *locally strongly ψ-condensing of order k for balls at ∞* if

$$\lim_{r \to \infty} \sup_{R > r} \frac{\psi\,(F(B_R \backslash B_r))}{R^k} = 0. \tag{2.2}$$

For simplicity, we assume here and throughout that $\psi(U) := \infty$ if $U \subseteq Y$ is unbounded, so that, e.g., the limit (2.1) contains the requirement that $F(u_0 + B_r) \in \mathfrak{B}(Y)$ for small $r > 0$.

Remark 3 If $\varphi\colon \mathfrak{B}(X) \to [0, \infty)$ is positively homogeneous and translation invariant with $\varphi(B_1) > 0$, then (2.1) means that for every $\varepsilon > 0$ there is $R > 0$ with

$$\psi(F(M)) \leqslant \varepsilon\varphi(M)^k \quad \text{if} \quad M = u_0 + B_r, \quad 0 < r < R.$$

Similarly, if φ is positively homogeneous and conical invariant with $\varphi(B_1) > 0$, then (2.2) means that for every $\varepsilon > 0$ there is $r_0 > 0$ with

$$\psi(F(M)) \leqslant \varepsilon\varphi(M)^k \quad \text{if} \quad M = B_R \backslash B_r, \quad r_0 < r < R.$$

This explains the terminology "condensing of order k for balls" and the equivalence with the notion from [11] for $k = 1$.

There is actually a stronger relation to condensing maps:

Remark 4 Suppose that $\psi = \chi_Y$ and $\varphi = \chi_X$, and $\chi_X(B_1) > 0$. If (2.1) holds with $k = 1$ uniformly in u_0 for all $u_0 \in X$ in a neighborhood of some u_1, then for every $\varepsilon > 0$ there is a neigborhood $N \subseteq X$ of u_1 such that

$$\psi(F(M)) \leqslant \varepsilon\varphi(M) \quad \text{for every} \quad M \subseteq N.$$

In particular, F is condensing in a neighborhood of u_1, and so, if F is continuous and u_1 is an isolated fixed point of F, the topological fixed point index of F can be defined in a standard way. This has been shown in [12, Theorem 2].

The following criteria, similar to the above ones, had been used in [12]. Given $k > 0$, we call a map $F\colon X \to Y$ *locally strongly ψ-condensing of order k for spheres at $u_0 \in X$*, if

$$\lim_{r \to 0} \frac{\psi\,(F(u_0 + S_r))}{r^k} = 0, \tag{2.3}$$

and *locally strongly ψ-condensing of order k for spheres at ∞*, if

$$\lim_{r \to \infty} \frac{\psi\,(F(S_r))}{r^k} = 0. \tag{2.4}$$

In [9,11,12], the above classes of maps were studied thoroughly, and many examples have been given in case $k = 1$. Let us just give one remark which follows rather immediately from the definition:

Remark 5 Suppose that ψ is monotone, positively homogeneous, and algebraically semi-additive, $k > 0$, and $u_0 \in X$. Then the four families of maps $F: X \to Y$ satisfying (2.1)–(2.4), respectively, are linear vector spaces.

Even in case $k = 1$, these families contain maps which are not necessarily (φ, ψ)-condensing and not even necessarily (φ, ψ)-bounded in the sense of [1] in any neighborhood of u_0 (or ∞), respectively.

3 Main results

Let $\psi: \mathfrak{B}(Y) \to [0, \infty)$ satisfy (1)–(3).

Theorem 1 *Let* $k_0 \in (0, \infty)$.

(i) *Suppose that* $F, A, A_0: X \to Y$ *and* $u_0 \in X$ *satisfy* (1.4). *Suppose also that* (1.5) *holds with some* $k \in [0, k_0]$, *and that* A_0 *is locally strongly* ψ*-condensing of order* k_0 *for balls at* 0.
Then $\psi(A(B_1)) = 0$ *if and only if* F *is locally strongly* ψ*-condensing of order* k_0 *for balls at* u_0.

(ii) *Suppose that* $F, A, A_0: X \to Y$ *satisfy*

$$F(u) = A(u) + A_0(u), \quad \text{for all large} \quad u \in X. \tag{3.1}$$

Suppose also that (1.5) *holds with some* $k \in [k_0, \infty)$, *and that* A_0 *is locally strongly* ψ*-condensing of order* k_0 *for balls at* ∞.
Then $\psi(A(B_1 \backslash B_\rho)) = 0$ *for every* $\rho \in (0, 1)$ *if and only if* F *is locally strongly* ψ*-condensing of order* k_0 *for balls at* ∞.

Proof (i) Suppose $\psi(A(B_1)) = 0$. By Remark 2, we have $\psi(A(B_\rho)) = 0$ for all $\rho > 0$. Using (1.4), monotonicity and algebraic semi-additivity of ψ, we obtain for small $\rho > 0$

$$\psi\left(F(u_0 + B_\rho)\right) \leqslant \psi\left(A(B_\rho) + A_0(B_\rho)\right) \leqslant \psi\left(A(B_\rho)\right) + \psi\left(A_0(B_\rho)\right) = \psi\left(A_0(B_\rho)\right),$$

hence

$$\frac{\psi\left(F(u_0 + B_\rho)\right)}{\rho^{k_0}} \leqslant \frac{\psi\left(A_0(B_\rho)\right)}{\rho^{k_0}}.$$

Since A_0 is locally strongly ψ-condensing of order k_0 for balls at 0, this implies that F is locally strongly ψ-condensing of order k_0 for balls at u_0. The proof of the necessity of (i) is completed.

It remains to prove the sufficiency of (i). By (1.4), there holds $A(B_\rho) \subseteq F(u_0 + B_\rho) - A_0(B_\rho)$. Using that ψ is monotone, algebraic semi-additive, and even, we obtain

$$\psi\left(A(B_\rho)\right) \leqslant \psi\left(F(u_0 + B_\rho)\right) + \psi\left(A_0(B_\rho)\right).$$

Using (1.5), we have thus shown that

$$\rho^{k-k_0}\psi\left(A(B_1)\right) = \frac{\psi\left(A(B_\rho)\right)}{\rho^{k_0}} \leqslant \frac{\psi\left(F(u_0 + B_\rho)\right)}{\rho^{k_0}} + \frac{\psi\left(A_0(B_\rho)\right)}{\rho^{k_0}}.$$

Since F and A_0 are locally strongly ψ-condensing of order k_0 for balls at u_0 or 0, respectively, the right-hand side converges to 0 as $\rho \to 0$. But $\rho^{k-k_0} \geqslant 1$ for small $\rho \leqslant 1$, because $k \leqslant k_0$, so we must have $\psi(A(B_1)) = 0$. Part (i) of Theorem 1 is proved.

(ii) Let $\psi(A(B_1\backslash B_r)) = 0$ for all $0 < r < 1$. By (1.5), we have $\psi(A(B_\rho\backslash B_{\rho r})) = \rho^k\psi(A(B_1\backslash B_r)) = 0$ for all $0 < r < 1$, $\rho > 0$ and thus $\psi(A(B_R\backslash B_r))$ for all $0 < r < R$. By (3.1), we obtain $F(B_R\backslash B_r) \subseteq A(B_R\backslash B_r)+A_0(B_R\backslash B_r)$ for $R > r$ with sufficiently large r. Using that ψ is monotone and algebraic semi-additive, we conclude

$$\psi\,(F(B_R\backslash B_r)) \leqslant \psi\,(A(B_R\backslash B_r)) + \psi\,(A_0(B_R\backslash B_r)) = \psi\,(A_0(B_R\backslash B_r).$$

Dividing this inequality by R^{k_0}, taking the supremum over all $R > r$ and letting $r \to \infty$, we obtain that F is locally strongly ψ-condensing of order k_0 for balls at ∞, because A_0 has this property.

Conversely, suppose that F is locally strongly ψ-condensing of order k_0 for balls at ∞. We obtain from (3.1) with $M := B_R\backslash B_r$ for $R > r$ and sufficiently large $r > 0$ that $A(M) \subseteq F(M) - A_0(M)$. Since ψ is monotone, algebraic semi-additive, and even, we thus find

$$\psi(A(M)) \leqslant \psi(F(M)) + \psi(A_0(M)).$$

Using (1.5), we thus have shown for all $R > r$ with large r that

$$R^{k-k_0}\psi\left(A(B_1\backslash B_{r/R})\right) = \frac{\psi\,(A(B_R\backslash B_r))}{R^{k_0}} \leqslant \frac{\psi\,(F(B_R\backslash B_r))}{R^{k_0}} + \frac{\psi\,(A_0(B_R\backslash B_r))}{R^{k_0}}. \tag{3.2}$$

Since F and A_0 are locally strongly ψ-condensing of order k_0 for balls at ∞ and since $R^{k-k_0} \geqslant 1$ for $R \geqslant 1$, because $k \geqslant k_0$, we conclude

$$\lim_{r\to\infty}\sup_{R>r}\psi\left(A(B_1\backslash B_{r/R})\right) = 0.$$

Given $0 < \rho < 1$ and choosing $R := r/\rho$ in this limit, we obtain

$$\lim_{r\to\infty}\psi\left(A(B_1\backslash B_\rho)\right) = 0.$$

Since the expression under this limit is independent of r, we conclude $\psi(A(B_1\backslash B_\rho)) = 0$, and Theorem 1 is proved. □

Remark 6 Since ψ is monotone, we have $\psi(A(S_1)) \leqslant \psi(A(B_1))$ and $\psi(A(S_1)) \leqslant \psi(A(B_1\backslash B_\rho))$ for every $\rho > 0$. In particular, Theorem 1 thus provides sufficient criteria for $\psi(A(S_1)) = 0$.

Moreover, if (1.6) holds, then

$$\psi\,(A(S_1)) = \psi\,(A([0,1]S_1)) = \psi\,(A(B_1)),$$

so that in view of the monotonicity of ψ there holds for $0 < r < 1$ even

$$\psi\,(A(S_1)) = 0 \quad \text{iff} \quad \psi\,(A(B_1)) = 0 \quad \text{iff} \quad \psi\,(A(B_1\backslash B_r)) = 0.$$

Thus, Theorem 1 provides for such A and ψ even necessary and sufficient criteria for $\psi(A(S_1)) = 0$ which are simultaneously necessary and sufficient for $\psi(A(U)) = 0$ for every $U \in \mathfrak{B}(X)$.

A further necessary and sufficient condition for $\psi(A(S_1)) = 0$ is provided by the following result.

Theorem 2 *Let $k_0 \in (0, \infty)$.*

(*i*) *Suppose that $F, A, A_0 \colon X \to Y$ and $u_0 \in X$ satisfy* (1.4). *Suppose also that* (1.5) *holds with some $k \in [0, k_0]$ and that A_0 is locally strongly ψ-condensing of order k_0 for spheres at* 0.
Then $\psi(A(S_1)) = 0$ if and only if F is locally strongly ψ-condensing of order k_0 for spheres at u_0.

(*ii*) *Suppose that $F, A, A_0 \colon X \to Y$ satisfy* (3.1). *Suppose also that* (1.5) *holds with some $k \in [k_0, \infty)$, and that A_0 is locally strongly ψ-condensing of order k_0 for spheres at ∞.*
Then $\psi(A(S_1)) = 0$ if and only if F is locally strongly ψ-condensing of order k_0 for spheres at ∞.

Proof The proof of (i) is analogous to that of Theorem 1(i) if one replaces B_r by S_r. Similarly, (ii) follows essentially by replacing $B_R \backslash B_r$ by S_R in the proof of Theorem 1 (ii), noting that (3.2) becomes

$$R^{k-k_0}\psi\left(A(S_1)\right) = \frac{\psi\left(A(S_R)\right)}{R^{k_0}} \leqslant \frac{\psi\left(F(S_R)\right)}{R^{k_0}} + \frac{\psi\left(A_0(S_R)\right)}{R^{k_0}}.$$

□

Remark 7 All results in this section hold true if X and Y are not necessarily complete. Moreover, it suffices that Y is a topological vector space and that X is quasi-normed, that is instead of the triangle inequality we need only

$$\|u + v\| \leqslant c_X \left(\|u\| + \|v\|\right) \quad \text{for all} \quad u, v \in X$$

with some $c_X \in [1, \infty)$. Moreover, instead of the algebraic semi-additivity of ψ, it suffices to assume that

$$\psi(U + V) \leqslant c_\psi \left(\psi(U) + \psi(V)\right) \quad \text{for all} \quad U, V \in \mathfrak{B}(Y) \tag{3.3}$$

with a constant $c_\psi \in [1, \infty)$. In particular, the Hausdorff measure of noncompactness in a quasi-normed space has the latter property, see [19]. All of our previous assertions and remarks hold in quasi-normed spaces with the following exception: If Y is quasi-normed, none of diam_Y, $\operatorname{diam}_{Y,0}$, or χ_Y is conical invariant, convex invariant, or closed convex invariant, in general.

Remark 8 We write $F \colon X \multimap Y$ if F is a multivalued map from X into Y, that is if $F(u) \subseteq Y$ for every $u \in X$. For such maps, we use the customary notations $F(U) := \bigcup_{u \in U} F(u)$ for $U \subseteq X$, and $D(F) := \{u \in X : F(u) \neq \emptyset\}$.

All previous results (and their proof) hold unchanged for multivalued maps if $D(F) = D(A) = D(A_0) = X$. However, with slight modifications, they hold even in the case $D(F), D(A), D(A_0) \subseteq X$ as we discuss now. Note that even for single-valued maps the subsequent discussion includes the case that the maps are only defined on subsets of X.

We first extend the definition of a positively homogeneous map $A \colon X \multimap Y$ to such a case. We call A *positively homogeneous of order* $k \in [0, \infty)$ if

$$A(\rho u) = \rho^k A(u) \quad \text{for all} \quad \rho > 0, \quad u \in X, \quad \text{and} \quad A(0) = \{0\}.$$

Note that the equality is not required for $\rho = 0$ so that $D(A)$ can indeed be a nontrivial (conical) set. If $D(A) \neq X$, we have to require for Proposition 1(ii) in addition that $\psi(\{0\}) = \psi(\emptyset)$ (because $A([0, 1]M) = \{0\}$ if $A(M) = \emptyset \neq M$). We have to require (1.4) and (3.1) only for all small or large $\|u\|$, respectively, but we have to endow them with the additional requirement that

$$u_0 + u \in D(F) \implies u \in D(A) \quad \text{for all small} \quad \|u\| \tag{3.4}$$

or

$$u \in D(F) \implies u \in D(A) \quad \text{for all large} \quad \|u\|\,, \tag{3.5}$$

respectively. With the above changes, all previous results (and proofs) carry over.

4 Applications to Fréchet derivatives and "homogenizations"

Given $k > 0$, we call a map $F\colon X \to Y$ *k-homogenizable at* $u_0 \in X$ if there are maps $A, \omega\colon X \to Y$ with A being positively homogeneous of order k such that

$$F(u_0 + u) = F(u_0) + A(u) + \omega(u) \tag{4.1}$$

and

$$\lim_{\|u\|\to 0} \frac{\omega(u)}{\|u\|^k} = 0. \tag{4.2}$$

We call then A the *k-homogenization* of F at $u_0 \in X$. For the case that A is a bounded linear operator and $k = 1$, this extends the usual definition of the Fréchet derivative $A = F'(u_0)$, see, e.g.,[14, Chapter II, 4.8]. Note that A is indeed uniquely defined:

Remark 9 We obtain from (4.1)

$$\frac{F\,(u_0 + ru) - F(u_0)}{r^k} = A(u) + \frac{\omega(ru)}{r^k} \quad \text{if} \quad u \neq 0. \tag{4.3}$$

Letting $r \to 0^+$, we obtain in view of (4.2) that $A(u)$ is uniquely determined.

For $k > 0$, we say that $F\colon X \to Y$ is *k-homogenizable at* ∞ with *k-homogenization* $A\colon X \to Y$ if A is positively homogeneous of order k and if there is $\omega\colon X \to Y$ with

$$F(u) = A(u) + \omega(u) \tag{4.4}$$

and

$$\lim_{\|u\|\to \infty} \frac{\omega(u)}{\|u\|^k} = 0. \tag{4.5}$$

Again, if A is a bounded linear operator and $k = 1$, this extends the usual definition of an *asymptotically linear* operator with derivative A at ∞, see, e.g.,[1, 3.3.3].

We recall that two functions $\psi, \widetilde{\psi}\colon \mathfrak{B}(Y) \to [0, \infty)$ are *equivalent* if there exist constants $c_1, c_2 > 0$ such that

$$c_1\psi(U) \leqslant \widetilde{\psi}(U) \leqslant c_2\psi(U) \quad \text{for all} \quad U \in \mathfrak{B}(Y).$$

We call $A\colon X \to Y$ *compact* if $A(U)$ is precompact for every $U \in \mathfrak{B}(X)$.

Theorem 3 *Let $\psi\colon \mathfrak{B}(Y) \to [0, \infty)$ be equivalent to χ_Y. Let $F\colon X \to Y$ and $k \in (0, \infty)$*

(*i*) *Suppose that F is k-homogenizable at $u_0 \in X$ with k-homogenization A. Then the following assertions are equivalent:*

(*a*) *A is compact.*

(*b*) *F is locally strongly ψ-condensing of order k for balls at u_0.*
(*c*) *F is locally strongly ψ-condensing of order k for spheres at u_0.*

(*ii*) *Suppose that F is k-homogenizable at ∞ with k-homogenization A. Then the following assertions are equivalent:*

(*a*) *A is compact.*
(*b*) *F is locally strongly ψ-condensing of order k for balls at ∞.*
(*c*) *F is locally strongly ψ-condensing of order k for spheres at ∞.*

Proof Since none of the assertions changes when we pass to an equivalent ψ, we can assume without loss of generality that $\psi = \chi_Y$. Now we observe that Remarks 2 and 6 imply that

$$A \text{ is compact} \quad \text{iff} \quad \psi\,(A(B_1)) = 0 \quad \text{iff} \quad \psi\,(A(B_1 \backslash B_r)) = 0 \quad \text{iff} \quad \psi\,(A(S_1)) = 0$$

for every $0 < r < 1$. Hence, the assertion (i) follows from Theorems 1(i) and 2(i) with $k_0 = k$ if we can show that $A_0(u) := F(u_0) + \omega(u)$ with ω from (4.1) is locally strongly ψ-condensing of order k for balls and spheres at u_0. However, in view of (4.2) this follows straightforwardly from

$$\psi\,(A_1(U)) \leqslant \psi\,(F(u_0)) + \psi\,(\omega(U)) = \psi\,(\omega(U)) \leqslant \sup\{\|v\| : v \in \omega(U)\},$$

where for the last inequality we have chosen $N := \{0\}$ in (1.1).

In a similar manner, one obtains assertion (ii) from Theorems 1(ii) and 2(ii) with $k_0 = k$, because $A_0 := \omega$ with ω from (4.5) is locally strongly ψ-condensing of order k for balls/spheres at ∞. The latter follows from (4.5) and

$$\psi\,(A_1(U)) \leqslant \sup\{\|v\| : v \in \omega(U)\},$$

where we used again the choice $N := \{0\}$ in (1.1). □

Remark 10 All above results hold if X and Y are only quasi-normed and not necessarily complete. However, be aware that in case of incomplete Y the precompactness of $A(U)$ means by definition only that the *completion* of the metric space $A(U)$ is compact.

There are many "natural" examples of regular measures of noncompactness known, for instance, the Kuratowski or the Istrăţescu measures of noncompactness (see, e.g., [1,23]), but all of these are easily seen to be equivalent to the Haudorff measure of noncompactness.

This is different when we pass to weak topologies. Recall that the *De Blasi measure of noncompactness* in Y is defined as

$$\beta_Y(U) = \inf\left\{r > 0 \mid \exists\, N \subseteq Y \text{ weakly compact with } U \subseteq \bigcup_{u \in N} (u + B_r)\right\},$$

see, e.g., [6]. In contrast to the case of the norm topology, there are "natural" measures of noncompactness known which possess the regularity property for the weak topology

$$\varphi(U) = 0 \quad \text{iff} \quad U \text{ is relatively weakly compact}$$

but which fail to be equivalent to β_Y, see [2].

We call an operator $A\colon X \to Y$ *weakly compact*, if $A(U)$ is relatively weakly compact for every $U \in \mathfrak{B}(X)$.

With the existence of nonequivalent "weak" measures of noncompactness in mind, the special role of the De Blasi measure of noncompactness in the following characterization of operators with weakly compact derivatives is rather remarkable.

Theorem 4 *Let $\psi : \mathfrak{B}(Y) \to [0, \infty)$ be equivalent to β_Y. Let $F : X \to Y$ and $k \in (0, \infty)$.*

(*i*) *Suppose that F is k-homogenizable at $u_0 \in X$ with k-homogenization A. Then the following assertions are equivalent:*

(*a*) *A is weakly compact.*
(*b*) *F is locally strongly ψ-condensing of order k for balls at u_0.*
(*c*) *F is locally strongly ψ-condensing of order k for spheres at u_0.*

(*ii*) *Suppose that F is k-homogenizable at ∞ with k-homogenization A. Then the following assertions are equivalent:*

(*a*) *A is weakly compact.*
(*b*) *F is locally strongly ψ-condensing of order k for balls at ∞.*
(*c*) *F is locally strongly ψ-condensing of order k for spheres at ∞.*

Proof The proof is completely analogous to that of Theorem 3. □

Remark 11 All results in this section hold also if $F, A, \omega : X \multimap Y$ are multivalued, and in this case, we have to require (4.1) or (4.4) only for all $u \in X$ with small or large $\|u\|$, respectively. To include the case $D(F), D(A), D(\omega) \neq X$, we require in addition that $u \in D(\omega)$ for all small or large $\|u\|$, respectively, Moreover, (4.2) and (4.5) have to be replaced by

$$\lim_{\|u\|\to 0} \frac{\sup\{\|v\| : v \in \omega(u)\}}{\|u\|^k} = 0$$

and

$$\lim_{\|u\|\to \infty} \frac{\sup\{\|v\| : v \in \omega(u)\}}{\|u\|^k} = 0,$$

respectively. For the assertion of Remark 9, we assume in addition that $F(u_0)$ is a singleton, and we note that (4.3) holds only for sufficiently (depending on $\|u\|$) small $r > 0$ (because we require (4.1) only for small $\|u\|$ in the multivalued case). For Theorems 3(i) and 4(i), we assume that $F(u_0)$ is precompact or relatively weakly compact, respectively.

The assumption that $F(u_0)$ is a singleton holds for multivalued maps of course only at exceptional points u_0. Nevertheless, a concept of "homogenization/differentiability" of a multivalued map F at a certain distinguished point u_0 can be quite convenient and natural in some applications, see, e.g., [20–22].

5 The ψ-spherical property

For Theorems 1 and 2, we do not have to require that A is k-homogeneous: The weaker requirement (1.5) is sufficient. However, to obtain the desired conclusion $\psi(A(U)) = 0$ for every $U \in \mathfrak{B}(X) = 0$, we need a further requirement, e.g., (1.6) for Remark 6.

Following [13], we introduce now a weaker property which gives the same conclusion.

Definition 1 An operator $A : X \to Y$ has the *ψ-spherical property* if for every $R > 0$ satisfying $\psi(A(B_R)) > 0$ there is some $r \in (0, R]$ with $\psi(A(S_r)) > 0$.

This property indeed gives the conclusion of Remark 6:

Lemma 1 *Suppose that ψ is monotone. If A has the ψ-spherical property and satisfies* (1.5) *with some $k \in [0, \infty)$, then the following assertions are equivalent:*

(i) $\psi(A(U)) = 0$ *for every* $U \in \mathfrak{B}(X)$.
(ii) $\psi(A(B_1)) = 0$.
(iii) $\psi(A(B_1 \setminus B_r)) = 0$ *for every* $r \in (0, 1)$.
(iv) $\psi(A(S_1)) = 0$.

Proof In view of Remark 2 and the monotonicity, it suffices to show that $\psi(A(S_1)) = 0$ implies $\psi(A(B_1)) = 0$. Thus, assume by contradiction that $\psi(A(B_1)) > 0$ and $\psi(A(S_1)) = 0$. By Remark 2, it follows that $\psi(A(S_r)) = 0$ for every $r > 0$, contradicting the hypothesis that A has the ψ-spherical property. □

Theorem 5 *Let* $\psi : \mathfrak{B}(Y) \to [0, \infty)$ *satisfy* (1)–(3)*, and suppose that* $A: X \to Y$ *has the* ψ*-spherical property and satisfies* (1.5) *with some* $k \in [0, \infty)$*. Let* $k_0 \in (0, \infty)$.

(i) *Suppose that* $F, A, A_0 : X \to Y$ *and* $u_0 \in X$ *satisfy* (1.4)*,* $k \leqslant k_0$*, and that* A_0 *is locally strongly* ψ*-condensing of order* k_0 *for balls at* 0*. Then the following assertions are equivalent.*

(a) $\psi(A(U)) = 0$ *for every* $U \in \mathfrak{B}(X)$.
(b) F *is locally strongly* ψ*-condensing of order* k_0 *for balls at* u_0.
(c) F *is locally strongly* ψ*-condensing of order* k_0 *for spheres at* u_0.

(i) *Suppose that* $F, A, A_0 : X \to Y$ *and* $u_0 \in X$ *satisfy* (3.1)*,* $k \geqslant k_0$*, and that* A_0 *is locally strongly* ψ*-condensing of order* k_0 *for balls at* ∞*. Then the following assertions are equivalent.*

(a) $\psi(A(U)) = 0$ *for every* $U \in \mathfrak{B}(X)$.
(b) F *is locally strongly* ψ*-condensing of order* k_0 *for balls at* u_0.
(c) F *is locally strongly* ψ*-condensing of order* k_0 *for spheres at* u_0.

Proof Since ψ is monotone and A_0 is locally strongly ψ-condensing of order k_0 for balls at 0 or ∞, A_0 is also locally strongly ψ-condensing of order k_0 for spheres at 0 or ∞, respectively. In view of Lemma 1, the assertion thus follows from Theorems 1 and 2. □

The ψ-spherical property is indeed implied by (1.6):

Proposition 2 *If* A *satisfies* (1.6) *with some* $k \in [0, \infty)$*, then* A *has the* ψ*-spherical property.*

Proof Using (1.6) with $U = S_\rho$ and $r = R/\rho$, we find $\psi(A(B_R)) = (R/\rho)^k \psi(A(S_\rho)))$. In particular, $\psi(A(B_R)) > 0$ implies $\psi(A(S_\rho)) > 0$ for every $\rho > 0$. □

Proposition 2 implies in view of Lemma 1 again the assertion of Remark 6.

It is natural to ask whether the ψ-spherical property holds under a different type of hypotheses than (1.6). Here is such a result.

Proposition 3 *Suppose that* $\psi : \mathfrak{B}(Y) \to [0, \infty)$ *has the following properties:*

(i) ψ *is monotone.*
(ii) $\psi(U) = 0$ *for every compact* $U \subseteq Y$.
(iii) *Every* $U \in \mathfrak{B}(Y)$ *with* $\psi(U) > 0$ *contains a sequence* v_n *such that every bounded sequence* $w_n \in Y$ *with* $\underline{\lim}(w_n - v_n) = 0$ *satisfies* $\psi(\{w_1, w_2, \ldots\}) > 0$.

Then every map $A: X \to Y$ *which is uniformly continuous on bounded sets has the* ψ*-spherical property.*

Proof If $\psi(A(B_R)) > 0$, let $v_n \in U := A(B_R)$ be the sequence required in (iii), and let $u_n \in B_R$ be such that $A(u_n) = v_n$. Using the compactness of $[0, R]$ we can assume, passing to a subsequence if necessary, that $\|u_n\| \to r \in [0, R]$.

We must have $r > 0$. Indeed, if $r = 0$, then $u_n \to 0$ implies in view of the continuity of A at 0 that the sequence $w_n := A(u_n)$ is convergent to $A(0)$ and thus $\psi(\{w_1, w_2, \ldots\}) = 0$ by (ii), contradicting $w_n - v_n = 0$.

In view of $r > 0$, we can assume without loss of generality that $u_n \neq 0$ for every n. Then $\widetilde{u}_n := \frac{r}{\|u_n\|} u_n \in S_r$ satisfy $\widetilde{u}_n - u_n \to 0$. Since A is uniformly continuous on bounded sets, we obtain that $A(\widetilde{u}_n) - A(u_n) \to 0$. Thus, the sequence $z_n := A(\widetilde{u}_n)$ satisfies $z_n - v_n \to 0$, and so $\psi(A(S_r)) \geqslant \psi(\{z_1, z_2, \ldots\}) > 0$.

The requirement concerning ψ in Proposition 3 is not very restrictive. A large class of examples is contained in the following observation, and a further example will be given later.

Lemma 2 *Every monotone regular function* $\psi : \mathfrak{B}(Y) \to [0, \infty)$ *has the properties required in* Proposition 3.

Proof Only property (iii) requires a proof. Thus, assume that $\psi(U) > 0$, that is U fails to be precompact. Then the Istrăţescu measures of noncompactness of U is positive (see, e.g., [23, Proposition 3.23]) which means that there is a sequence $v_n \in U$ with $\|v_n - v_m\| > r > 0$ for every $n \neq m$. Hence, for every sequence w_n satisfying $\underline{\lim}(w_n - v_n) = 0$ there is a subsequence satisfying $\|w_{n_k} - w_{n_j}\| > r$ for all $k \neq j$, and so $C = \{w_1, w_2, \ldots\}$ fails to be precompact; in particular, $\psi(C) > 0$. □

It is not sufficient to require in Proposition 3 only that A is continuous. Indeed, even under weaker hypotheses about ψ, one can construct a large class of counterexamples:

Proposition 4 *Suppose that* $\psi : \mathfrak{B}(Y) \to [0, \infty)$ *has the following properties.*

(*i*) ψ *is monotone.*
(*ii*) $\psi(U) = 0$ *for every compact* $U \subseteq Y$.
(*iii*) *There is a bounded sequence* $w_n \in Y$ *with* $\psi(\{w_1, w_2, \ldots\}) > 0$.

If X has infinite dimension, then there is a continuous bounded map $A : X \to Y$ without the ψ-spherical property.

Proof Since X has infinite dimension, Riesz's lemma implies that there is a sequence $e_n \in S_1$ satisfying $\|e_n - e_m\| \geqslant 1/2$ for all $n \neq m$.

Put $\lambda_n := 1 - 2^{-n}$ and $r_n := 2^{-n}/8$. Then the balls $K_n = \lambda_n e_n + B_{r_n}$ are pairwise disjoint, and moreover, every sphere S_r with center 0 intersects *at most* one K_n.

Now define $A : X \to Y$ for $u \in K_n$ as $A(u) = (1 - r_n^{-1} \|u - \lambda_n e_n\|) w_n$, and $A(u) = 0$ for $u \notin \bigcup_n K_n$.

Then A is continuous with $A(B_1) \supseteq \{w_1, w_2, \ldots\}$, hence $\psi(A(B_1)) > 0$. Moreover, for every $r > 0$ there holds: Since S_r intersects at most one K_n, we have $A(S_r) \subseteq A(K_n) = [0, 1] w_n$ which is compact and thus $\psi(A(S_r)) = 0$. □

Now we intend to give another example of a function ψ satisfying the hypotheses of Proposition 3 which does not fall into the large class of examples of Lemma 2.

Let (Ω, Σ, μ) be some nonnegative measure space with $\mu(\Omega) < \infty$.

We consider an ideal space Y of (classes of) real-valued measurable functions on Ω. Recall that this means that Y is a Banach space, and $v \in Y$ implies for every measurable function $u : \Omega \to \mathbb{R}$ satisfying $|u(x)| \leqslant |v(x)|$ for almost all $x \in \Omega$ that $u \in X$ and $\|u\| \leqslant \|v\|$.

We call Y a *regular space* if additionally each function $u \in X$ has absolutely continuous norm:

$$\lim_{\mu(D)\to 0} \|P_D u\| = 0,$$

where

$$P_D u(x) := \begin{cases} u(x) & \text{if } x \in D, \\ 0 & \text{if } x \notin D. \end{cases}$$

Examples of regular spaces are, for instance, Lebesgue spaces $L_p(\Omega)$ with $1 \leqslant p < \infty$, Lorentz spaces, or Orlicz spaces generated by a Δ_2-function (see, e.g., [7,10,15,16,18]).

We consider in regular spaces the *measure of nonequiabsolute continuity* ν which is defined by

$$\nu(U) = \overline{\lim_{\mu(D)\to 0}} \sup_{u\in U} \|P_D u\| .$$

The measure ν has all properties (1)–(10) with the exception of regularity. Indeed, $U \subseteq X$ is precompact if and only if $\nu(U) = 0$ and U is precompact in measure (see, e.g., [19, Theorem 3.19]), and the latter cannot be ommitted, see, e.g., [10].

Lemma 3 *$\psi = \nu$ has all properties required in* Proposition 3. *Moreover, for every nonempty $U \in \mathfrak{B}(Y)$, there is a sequence $v_n \in U$ such that for every sequence $w_n \in Y$ satisfying $\underline{\lim}(w_n - v_n) = 0$ there holds $\nu(\{w_1, w_2, \ldots\}) \geqslant \nu(U)$.*

Proof By the previous remarks, it suffices to verify the last assertion. In case $\nu(U) = 0$, there is nothing to prove. Thus, let $0 < r < \nu(U)$. By definition of ν, there are sequences $D_n \subseteq \Omega$ and $v_n \in U$ with $\mu(D_n) \to 0$ and $\|P_{D_n} v_n\| > r$. If $\underline{\lim}(w_n - v_n) = 0$, then

$$\|P_{D_n} w_n - P_{D_n} v_n\| = \|P_{D_n}(w_n - v_n)\| \leqslant \|w_n - v_n\|$$

implies $\underline{\lim}\, P_{D_n} w_n - P_{D_n} v_n = 0$. The triangle inequality thus implies $\|P_{D_n} w_n\| > r$ for infinitely many n. The latter implies $\nu(\{w_1, w_2, \ldots\}) > r$. □

In view of Proposition 3, we thus find that every map $A\colon X \to Y$ which is uniformly continuous on bounded sets has the ν-spherical property, and so Theorem 5 provides criteria for $\nu(A(U)) = 0$ for every $U \in \mathfrak{B}(X)$ for such maps.

Note that for the case $X = L_q(\Omega)$ and $Y = L_p(\Omega)$ operators $A\colon X \to Y$ satisfying $\nu(A(U)) = 0$ for every $U \in \mathfrak{B}(X)$ are called *improving*, and it is natural to use this terminology also in the general case.

We point out that the notion of improving operators is useful for the investigation of solvability of an equation in regular spaces, see, for example [5,7,10,18].

In view of Lemma 3, Theorem 5 thus provides criteria when an operator $A\colon X \to Y$ satisfying (1.5) and uniformly continuous on bounded sets is improving. We recall that instead of the uniform continuity, we can in view of Proposition 2 also require (1.6) which holds in particular if A is positively homogeneous of order $k \in [0, \infty)$.

Remark 12 As was observed in [8], if two operators F and F_1 acting from a set $G \subseteq X$ of a regular space X into a regular space Y are comparable on the set G, i.e., there is $b \in Y$ with

$$|(F(u))(x)| \leqslant |b_1(s)| + |(F_1(u))(x)| \quad \text{for almost all} \quad x \in \Omega$$

for every $b \in Y$, then $\nu(F(U)) \leqslant \nu(F_1(U))$. Hence, if F_1 is locally strongly ν-condensing operator of order k for balls/spheres at some u_0 or ∞, then every operator F comparable to F_1 has the same property.

6 Further remarks

Proposition 4 is rather disappointing since it shows that either the hypothesis (1.6) or the *uniform* continuity of A on bounded sets is probably the only natural hypotheses if we want to have a result like Theorem 5 only under the assumption (1.5).

It turns out that if we are less ambiguous and do not insist on the last equivalence in the two assertions of Theorem 5, a mild additional assumption about ψ is enough. We formulate such a corresponding result now which also takes Remarks 7 and 8 into account.

Theorem 6 *Let Y be a topological vector space, and X be a quasi-normed space. Let $\psi\colon \mathfrak{B}(Y) \to [0,\infty)$ satisfy* (1), (2) *and* (3.3)*, and suppose that $A\colon X \multimap Y$ satisfies* (1.5) *with some $k \in [0,\infty)$. Let $k_0 \in (0,\infty)$.*

(*i*) *Suppose that $F, A_0\colon X \multimap Y$ and $u_0 \in X$ are such that* (3.4) *and* (1.4) *hold for all small $\|u\|$. Assume that $k \leqslant k_0$, and that A_0 is locally strongly ψ-condensing of order k_0 for balls at* 0*. Then the following assertions are equivalent.*

(*a*) $\psi(A(U)) = 0$ *for every $U \in \mathfrak{B}(X)$.*

(*b*) *F is locally strongly ψ-condensing of order k_0 for balls at u_0.*

(*ii*) *Suppose that $F, A_0\colon X \multimap Y$ are such that* (3.5) *and* (3.1) *hold for all large $\|u\|$. Assume that $k \geqslant k_0$, and that A_0 is locally strongly ψ-condensing of order k_0 for balls at ∞. Assume in addition that*

$$\psi(U \cup V) \leqslant c_\psi\,(\psi(U) + \psi(V)) \quad \text{for all} \quad U, V \in \mathfrak{B}(Y). \tag{6.1}$$

Then the following assertions are equivalent.

(*a*) $\psi(A(U)) = 0$ *for every $U \in \mathfrak{B}(X)$.*

(*b*) *F is locally strongly ψ-condensing of order k_0 for balls at ∞, and there is some $r > 0$ with $A(B_r) \in \mathfrak{B}(Y)$.*

Proof In view of Remark 2, assertion (i) follows immediately from Theorem 1(i) (with the generalizations already observed in Remarks 7 and 8). Assertion (ii) follows similarly from Theorem 1(ii) in view of

$$\begin{aligned}\psi\,(A(B_1)) = 0 \quad \text{iff} \quad &(\text{There is } r > 0 \text{ with } A(B_r) \in \mathfrak{B}(Y), \quad \text{and}\\ &\psi\left(A(B_1 \backslash B_\rho)\right) = 0 \text{ for every } \rho \in (0,1)\big).\end{aligned} \tag{6.2}$$

To see (6.2), note that "⇒" is immediate from the monotonicity of ψ. To see the converse implication, we use (6.1) and (1.5) to obtain

$$\psi\,(A(B_1)) \leqslant c_\psi\left(\psi\left(A(B_1\backslash B_\rho)\right) + \psi\left(A(B_\rho)\right)\right) = c_\psi \psi\left(A(B_\rho)\right) = c_\psi (\rho/r)^k \psi\,(A(B_r)).$$

Letting $\rho \to 0$, and noting that $k \geqslant k_0 > 0$, we obtain $\psi(A(B_1)) = 0$, and so (6.2) is established. □

We close with some remarks.

Theorem 3 combines the two results [12, Theorem 1] and [11, Theorem 1] containing necessary and sufficient conditions for complete continuity of the Fréchet derivative at a point and the asymptotic derivative, respectively. However, unlike [11] and [12], our result applies also if the "derivative" is not necessarily linear.

The main results and Propopsition 3 generalize analogous results from [13].

We finally note that two different results on bifurcation points from [14] were generalized in [11,12] to locally strongly condensing operators.

References

1. Akhmerov, R.R., Kamenskiĭ, M.I., Potapov, A.S., Rodkina, A.E., Sadovskiĭ, B.N.: Measures of Noncompactness and Condensing Operators. Birkhäuser, Basel (1992)
2. Angosto, C., Cascales, B.: Measures of weak noncompactness in Banach spaces. Topol. Appl. **156**, 1412–1421 (2009)
3. Ayerbe Toledano, J.M., Domínguez Benavides, T., López Acedo, G.: Measures of Noncompactness in Metric Fixed Point Theory. Birkhäuser, Basel (1997)
4. Banaś, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics, vol. 60. Marcel Dekker, New York (1980)
5. Cichoń, M., Metwali, M.M.A.: On solutions of quadratic integral equations in Orlicz spaces. J. Math. Anal. Appl. **387**, 419–432 (2012)
6. De Blasi, F.S.: On a property of the unit sphere in a Banach space. Bull. Math. Soc. Sci. Math. R. S. Roum. **21**, 259–262 (1977)
7. Erzakova, N.A.: On a certain class of condensing operators in spaces of summable functions Russian. In: Belonosov, S.M., Rukavishnikov, V.A. (eds.) Applied Numerical Analysis and Modeling, pp. 65–72. Akad. Nauk SSSR., Dal'nevotochn. Otdel., Vladivostok (1989)
8. Erzakova, N.A.: On a criterion for the complete continuity of the Fréchet derivative. Izv. Vysš. Učebn. Zaved. Mat. **9**, 44–51 (2011) [Engl. transl.: Russian Math. (Iz. VUZ) **66** , no. 9, 37–42 (2011)]
9. Erzakova, N.A.: On locally condensing operators. Nonlinear Anal. **75**(8), 3552–3557 (2012)
10. Erzakova, N.A.: Measures of noncompactness in regular spaces. Can. Math. Bull. **57**, 780–793 (2014)
11. Erzakova, N.A.: Generalization of some M. A. Krasnosel'skii's results. J. Math. Anal. Appl. **428**, 1368–1376 (2015)
12. Erzakova, N.A.: On a criterion for the complete continuity of the Fréchet derivative. Funkcional. Anal. i Priložen. **49**(4), 79–82 (2015) [Engl. transl.: Funct. Anal. Appl. **49**, no. 4, 304–306 (2015)]
13. Erzakova, N.A.: On Semi-Homogeneous Maps of Degree k. arXiv:1508.04215 [math.FA] (2015)
14. Krasnoselskiĭ, M.A.: Topological Methods in the Theory of Nonlinear Integral Equations (in Russian). Gostehizdat, Moscow (1956) [Engl. transl.: Pergamon Press, Oxford (1964)]
15. Krasnoselskiĭ, M.A., Rutickiĭ, Y.B.: Convex Functions and Orlicz Spaces (in Russian). Fizmatgiz, Moscow (1958) [Engl. transl.: Noordhoff, Groningen (1961)]
16. Krasnoselskiĭ, M.A., Zabreĭko, P.P., Pustylnik, E.I., Sobolevskiĭ, P.E.: Integral Operators in Spaces of Summable Functions (in Russian). Nauka, Moscow (1958) [Noordhoff, Leyden, Engl. transl. (1976)]
17. Melamed, V.B., Perov, A.I.: A generalization of a theorem of M. A. Krasnosel'skii on the complete continuity of the Frechet derivative of a completely continuous operator. Sibirsk. Mat. Z. **4**(3), 702–704 (1963)
18. Väth, M.: Ideal Spaces. Lecture Notes in Mathematics, No. 1664. Springer, Berlin (1997)
19. Väth, M.: Volterra and Integral Equations of Vector Functions. Marcel Dekker, New York (2000)
20. Väth, M.: Bifurcation for a reaction–diffusion system with unilateral obstacles with pointwise and integral conditions. Nonlinear Anal. Real World Appl. **12**, 817–836 (2011)
21. Väth, M.: Continuity and differentiability of multivalued superposition operators with atoms and parameters I. J. Anal. Appl. **31**, 93–124 (2012)
22. Väth, M.: Continuity and differentiability of multivalued superposition operators with atoms and parameters II. J. Anal. Appl. **31**, 139–160 (2012)
23. Väth, M.: Topological Analysis from the Basics to the Triple Degree for Nonlinear Fredholm Inclusions. de Gruyter, Berlin (2012)