Annali di Matematica (2016) 195:2171–2181
DOI 10.1007/s10231-016-0557-8

Rigidity of four-dimensional compact manifolds with harmonic Weyl tensor

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Received: 21 September 2014 / Accepted: 15 February 2016 / Published online: 27 February 2016

Abstract We prove that a 4-dimensional compact manifold M^4 with harmonic Weyl tensor must be either locally conformally flat or isometric to a complex projective space $\mathbb{CP}^2$, provided that the biorthogonal (sectional) curvature satisfies a suitable pinching condition. In particular, we improve the pinching constants considered by some preceding works on a rigidity result for 4-dimensional compact manifolds.

Keywords Einstein manifolds · Biorthogonal curvature · 4-Manifolds

Mathematics Subject Classification Primary 53C21 · 53C20; Secondary 53C25

1 Introduction

It plays an important role in geometry to classify 4-dimensional compact manifolds in the category of either topology, diffeomorphism or isometry. This is because dimension four enjoys a privileged status. For instance, the bundle of 2-forms can be invariantly decomposed as a direct sum; further relevant facts may be found in [2,24]. There has been a considerable amount of research on 4-dimensional manifolds involving some pinching curvature condition. We underline the next ones: nonnegative or positive sectional curvature (cf. [22,25]), nonnegative or positive Ricci curvature (cf. [19,28]), nonnegative or positive scalar curvature (cf. [12,13,17]), nonnegative or positive isotropic curvature (cf. [4,5,21,23,26]) and nonnegative or positive biorthogonal (sectional) curvature (cf. [3,7,25]). In order to set up the notation, M^4 will denote a compact oriented 4-dimensional manifold and g is a Riemannian

E. Ribeiro acknowledges partial support by grants from CNPq/Brazil.

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metric on M^4 with scalar curvature s_g, or simply s, sectional curvature K and volume form dV_g.

We recall that $(M^4,\ g)$ is called *Einstein* if the Ricci curvature is given by

$$\mathrm{Ric} = \lambda g,$$

for some constant λ. This means that M^4 has constant Ricci curvature. In such a case, a classical result due to Berger [1] combined with Synge's theorem allows us to conclude that if M^4 has positive sectional curvature, then it satisfies

$$2 \leq \chi(M) \leq 9,$$

where $\chi(M)$ stands for the Euler characteristic of M^4. Furthermore, Gursky and LeBrun showed that M^4 must satisfy

$$\chi(M) \geq \frac{15}{4}|\tau(M)|,$$

where $\tau(M)$ denotes the signature of M^4. This result improves the famous Hitchin–Thorpe inequality (cf. [16,30]). On the basis of these comments we may conclude that most 4-dimensional manifolds cannot carry any Einstein structure with positive or nonnegative sectional curvature.

Tachibana [29] asserts that a compact Einstein manifold with positive curvature operator is isometric to a spherical space form, while Micallef and Wang [21] extended Tachibana's result for dimension 4 by showing that a 4-dimensional compact Einstein manifold with nonnegative isotropic curvature is locally symmetric. Recently, Brendle [4] proved that a compact Einstein manifold with positive isotropic curvature must be isometric to a spherical space form. In 2000, Yang [32] has shown a rigidity result for Einstein structures with positive sectional curvature on 4-dimensional manifolds. More precisely, he proved the following result.

Theorem 1.1 (Yang [32]) *Let $(M^4,\ g)$ be a 4-dimensional complete Einstein manifold with normalized Ricci curvature Ric* $= 1$. *Suppose that*

$$K \geq \frac{(\sqrt{1249} - 23)}{120} \approx 0.102843. \tag{1.1}$$

Then, M^4 is isometric to either

(1) $\mathbb{S}^4$ *with its canonical metric, or*
(2) $\mathbb{CP}^2$ *with the Fubini–Study metric.*

As it was pointed out by Yang 0.102843 is apparently not the best possible lower bound on the sectional curvature to get this conclusion. In fact, from a convergence argument it is possible to show that there is a constant $0 < \beta < 0.102843$ such that the conclusion of Theorem 1.1 even is true for $K \geq \beta$. Motivated by this information, in 2004, Costa [6] was able to show that Yang's result remains true under weaker condition

$$K \geq \frac{(2 - \sqrt{2})}{6} \approx 0.09763. \tag{1.2}$$

It has been conjectured that:

Every 4-dimensional Einstein manifold with normalized Ricci curvature $Ric = 1$ and positive sectional curvature must be isometric to either $\mathbb{S}^4$ or $\mathbb{CP}^2$ with their normalized standard metrics. For more details, see [32].

As an attempt to better understand 4-dimensional manifolds with positive sectional curvature, it is natural to investigate other curvature positivity conditions. In order to explain our assumption in the main result to follow let us recall briefly the concept of biorthogonal curvature. For each plane $P \subset T_x M$ at a point $x \in M^4$, we define *the biorthogonal (sectional) curvature* of P by the following average of the sectional curvatures

$$K^{\perp}(P) = \frac{K(P) + K(P^{\perp})}{2}, \tag{1.3}$$

where $P^{\perp}$ is the orthogonal plane to P. For our purposes, for each point $x \in M^4$, we take the minimum of biorthogonal curvature to obtain the following function

$$K^{\perp}_{\min}(x) = \min\left\{K^{\perp}(P); \quad P \text{ is a 2-plane in } T_x M\right\}. \tag{1.4}$$

It should be emphasized that the sum of two sectional curvatures on two orthogonal planes, which was perhaps first observed by Chern [8], plays an interesting role on 4-dimensional manifolds. This notion also appeared in works due to Seaman [25], Noronha [22] and LeBrun [18] (cf. Section 5). Interestingly enough, $\mathbb{S}^1 \times \mathbb{S}^3$ with its canonical metric shows that the positivity of the biorthogonal curvature is an intermediate condition between positive sectional curvature and positive scalar curvature. Moreover, a 4-dimensional Riemannian manifold $(M^4,\ g)$ is Einstein if and only if $K^{\perp}(P) = K(P)$ for any plane $P \subset T_x M$ at any point $x \in M^4$ (cf. Corollary 6.26 [10,27]). From Costa and Ribeiro [7], $\mathbb{S}^4$ and $\mathbb{CP}^2$ are the only compact simply connected four-dimensional manifolds with positive biorthogonal curvature that can have (weakly) 1/4-pinched biorthogonal curvature, or nonnegative isotropic curvature, or satisfy $K^{\perp} \geq \frac{s}{24} > 0$. For more details on this subject we address to [3,7,22,23,25].

In order to proceed, we recall that the Weyl tensor W which is defined by the following decomposition formula

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) + \frac{R}{(n-1)(n-2)}\big(g_{jl}g_{ik} - g_{il}g_{jk}\big), \tag{1.5}$$

where R_{ijkl} stands for the Riemann curvature operator Rm. Moreover, we say that the Weyl tensor W is harmonic if $\delta W = 0$, where δ is the formal divergence defined for any $(0, 4)$-tensor T by

$$\delta T(X_1, X_2, X_3) = -\text{trace}_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

for some metric g on M^4. We remark that metrics with harmonic curvature as well as conformally flat metrics with constant scalar curvature are real analytic in harmonic coordinates; see [9]. One should be emphasized that every Einstein manifold has harmonic Weyl tensor W (cf. 16.65 in [2], see also Lemma 6.14 in [10]). Therefore, it is natural to ask which geometric implications have the assumption of the harmonicity of the tensor W on 4-dimensional manifolds.

In [7], inspired by Yang's work, Costa and Ribeiro proved the following result.

Theorem 1.2 (Costa–Ribeiro [7]) *Let $(M^4,\ g)$ be a 4-dimensional compact oriented Riemannian manifold with harmonic Weyl tensor and positive scalar curvature. We assume that g is analytic and*

$$K^{\perp}_{\min} \geq \frac{s^2}{8(3s + 5\lambda_1)}, \tag{1.6}$$

where λ_1 *stands for the first eigenvalue of the Laplacian with respect to* g*. Then,* M^4 *is either*

(1) *Diffeomorphic to a connected sum* $\mathbb{S}^4\sharp(\mathbb{R}\times\mathbb{S}^3)/G_1\sharp\ldots\sharp(\mathbb{R}\times\mathbb{S}^3)/G_n$*, where each* G_i *is a discrete subgroup of the isometry group of* $\mathbb{R}\times\mathbb{S}^3$*. In this case, g is locally conformally flat; or*
(2) *Isometric to a complex projective space* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

Motivated by the historical development on the study of the rigidity of 4-dimensional manifolds, in this paper, we use the notion of biorthogonal curvature to obtain a rigidity result for 4-dimensional compact manifold with harmonic Weyl tensor under a pinching condition weaker than (1.6).

After these settings we may state our main result as follows.

Theorem 1.3 *Let* $(M^4,\ g)$ *be a 4-dimensional compact oriented Riemannian manifold with harmonic Weyl tensor and positive scalar curvature. We assume that g is analytic and*

$$K^{\perp}_{\min}\geq\frac{s^2}{24(3\lambda_1+s)},\tag{1.7}$$

where λ_1 *stands for the first eigenvalue of the Laplacian with respect to* g*. Then,* M^4 *is either*

(1) *Diffeomorphic to a connected sum* $\mathbb{S}^4\sharp(\mathbb{R}\times\mathbb{S}^3)/G_1\sharp\ldots\sharp(\mathbb{R}\times\mathbb{S}^3)/G_n$*, where each* G_i *is a discrete subgroup of the isometry group of* $\mathbb{R}\times\mathbb{S}^3$*. In this case, g is locally conformally flat; or*
(2) *Isometric to a complex projective space* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

It is not difficult to check that

$$\frac{s^2}{8(3s+5\lambda_1)}>\frac{s^2}{24(s+3\lambda_1)}.$$

For this, our condition (1.7) considered in Theorem 1.3 is weaker that the former considered in Theorem 1.2. At the same time, we already know that Einstein metrics are analytic (cf. Theorem 5.26 in [2]). Also, as we have mentioned a 4-dimensional compact manifold M^4 is Einstein if and only if $K^{\perp}=K$. Furthermore, every Einstein metric has harmonic Weyl tensor. From these comments, we may conclude that Theorem 1.3 generalizes Theorem 1.1 and Theorem 1.2.

One should be emphasized that our arguments designed for the proof of Theorem 1.3 differ significantly from [7,32]. Our strategy is to use a more refined technique outlined previously in [14] to assure a lower estimate to the minimum of biorthogonal curvature.

In the sequel, it is well known that $\mathrm{Ric}\geq\rho>0$ implies $\lambda_1\geq\frac{4\rho}{3}$ and $s\geq 4\rho$, see, e.g., [11,20] for details. Then we can combine Theorem 1.3 with Tani's theorem [28] as well as Bonnet–Myers theorem to establish the following result.

Corollary 1 *Let* (M^4,g) *be a 4-dimensional complete oriented Riemannian manifold with harmonic Weyl tensor and metric g analytic. We assume that* $Ric\geq\rho>0$ *and* $K^{\perp}_{\min}\geq\frac{s^2}{192\rho}$*. Then,* M^4 *is isometric to either*

(1) $\mathbb{S}^4$ *with its canonical metric, or*
(2) $\mathbb{CP}^2$ *with the Fubini–Study metric.*

This improves Corollary 3 in [7]. In particular, Corollary 1 shows that the pinching constants used by Yang in (1.1) as well as by Costa in (1.2) can be improved to ≈0.08333. In other words, as an immediate consequence of Theorem 1.3, we conclude that Theorem 1.1 remains true under the rather weak pinching condition:

$$K \geq \frac{1}{12} \approx 0.08333.$$

The paper is organized as follows. In Sect. 2, we review some classical facts on 4-dimensional manifolds that will be used here. Moreover, we briefly outline some useful information on biorthogonal (sectional) curvature. In Sect. 3, we prove the main result.

2 Background

Throughout this section we recall some information and important results that will be useful in the proof of our main result. In what follows M^4 will denote an oriented 4-dimensional manifold and g is a Riemannian metric on M^4. As it was previously pointed out 4-manifolds are fairly special. In fact, many peculiar features are directly attributable to the fact that the bundle of 2-forms on a 4-dimensional oriented Riemannian manifold can be invariantly decomposed as a direct sum

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{2.1}$$

where $\Lambda^{\pm}$ is the (± 1) eigenspace of Hodge star operator. The decomposition (2.1) is conformally invariant. Moreover, it allows us to conclude that the Weyl tensor W is an endomorphism of Λ^2 such that

$$W = W^+ \oplus W^-. \tag{2.2}$$

For more details see [10,31] p. 46.

Proceeding, since the Riemann curvature tensor $\mathcal{R}$ of M^4 can be seen as a linear map on Λ^2, we have the following decomposition

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} Id & B \\ \hline B^* & W^- + \frac{s}{12} Id \end{array}\right), \tag{2.3}$$

where $B : \Lambda^- \rightarrow \Lambda^+$ stands for the Ricci traceless operator of M^4 given by $B = \mathrm{Ric} - \frac{s}{4} g$. For more details see [2,31].

We now fix a point and diagonalize $W^{\pm}$ such that $w_i^{\pm}$, $1 \leq i \leq 3$, are their respective eigenvalues. We stress that the eigenvalues of $W^{\pm}$ satisfy

$$w_1^{\pm} \leq w_2^{\pm} \leq w_3^{\pm} \quad \text{and} \quad w_1^{\pm} + w_2^{\pm} + w_3^{\pm} = 0. \tag{2.4}$$

In particular, (2.4) allows us to infer

$$|W^{\pm}|^2 \leq 6 \left(w_1^{\pm}\right)^2. \tag{2.5}$$

In fact, from (2.4) it is easy to see that

$$\left(w_2^{\pm}\right)^2 + \left(w_3^{\pm}\right)^2 = \left(w_1^{\pm}\right)^2 - 2 w_2^{\pm} w_3^{\pm}.$$

Therefore, we achieve

$$|W^+|^2 = 2 \left(w_1^+\right)^2 - 2 w_2^+ w_3^+.$$

Taking into account that $w_1^+ w_3^+ \leq w_2^+ w_3^+$ and $(w_1^+)^2 = -w_1^+ w_3^+ - w_1^+ w_2^+$ we deduce $|W^+|^2 \leq 6(w_1^+)^2$. In the same way we obtain $|W^-|^2 \leq 6(w_1^-)^2$.

For the sake of completeness let us briefly outline the construction of the minimum of biorthogonal curvature; more details can be found in [7], see also [18] (cf. Section 5). To do so, we consider a point $p \in M^4$ and $X, Y \in T_pM$ orthonormal. Whence, the unitary 2-form $\alpha = X \wedge Y$ can be uniquely written as $\alpha = \alpha^+ + \alpha^-$, where $\alpha^\pm \in \Lambda^\pm$ with $|\alpha^+|^2 = \frac{1}{2}$ and $|\alpha^-|^2 = \frac{1}{2}$. From these settings, the sectional curvature $K(\alpha)$ can be written as

$$K(\alpha) = \frac{s}{12} + \langle \alpha^+, W^+(\alpha^+)\rangle + \langle \alpha^-, W^-(\alpha^-)\rangle + 2\langle \alpha^+, B\alpha^-\rangle. \tag{2.6}$$

Moreover, we immediately have

$$K(\alpha^\perp) = \frac{s}{12} + \langle \alpha^+, W^+(\alpha^+)\rangle + \langle \alpha^-, W^-(\alpha^-)\rangle - 2\langle \alpha^+, B\alpha^-\rangle, \tag{2.7}$$

where $\alpha^\perp = \alpha^+ - \alpha^-$. Combining (2.6) with (2.7) we arrive at

$$\frac{K(\alpha) + K(\alpha^\perp)}{2} = \frac{s}{12} + \langle \alpha^+, W^+(\alpha^+)\rangle + \langle \alpha^-, W^-(\alpha^-)\rangle. \tag{2.8}$$

Hence, we may use (1.4) to infer

$$K^\perp_{\min} = \frac{s}{12} + \min\left\{\langle \alpha^+, W^+(\alpha^+)\rangle;\ |\alpha^+|^2 = \frac{1}{2}\right\} + \min\left\{\langle \alpha^-, W^-(\alpha^-)\rangle;\ |\alpha^-|^2 = \frac{1}{2}\right\}.$$

Furthermore, as it was explained in [7,25], Equation (1.4) provides the following useful identity

$$K^\perp_{\min} = \frac{w_1^+ + w_1^-}{2} + \frac{s}{12}. \tag{2.9}$$

Proceeding, given a section $T \in \Gamma(E)$, where $E \to M$ is a vector bundle over M, we already know that

$$|\nabla |T|| \leq |\nabla T|$$

away from the zero locus of T. This inequality is known as *Kato's inequality*. In [15], Gursky and LeBrun proved a *refined Kato's inequality*. More precisely, if W^+ is harmonic, then away from the zero locus of W^+ we have

$$|\nabla |W^+|| \leq \sqrt{\frac{3}{5}}|\nabla W^+|, \tag{2.10}$$

for more details see Lemma 2.1 in [14].

We also recall that if $\delta W^+ = 0$, then the Weitzenböck formula (cf. 16.73 in [2], see also [10]) is given by

$$\Delta |W^+|^2 = 2|\nabla W^+|^2 + s|W^+|^2 - 36 \det W^+. \tag{2.11}$$

We highlight that our Laplacian differs from "Besse's book" by a sign. Finally, by a simple Lagrange multiplier argument it is not difficult to check that

$$\det W^+ \leq \frac{\sqrt{6}}{18}|W^+|^3. \tag{2.12}$$

3 Proof of the main result

3.1 Proof of Theorem 1.3

Proof To begin with, we assume that $(M^4,\ g)$ is a compact oriented Riemannian manifold with positive scalar curvature and harmonic Weyl tensor. Since g is analytic, we deduce that $|W^{\pm}|^2$ are analytic. Moreover, supposing $W^{\pm} \not\equiv 0$ it is easy to see that the set

$$\Sigma = \Big\{ p \in M;\, |W^+|(p) = 0 \text{ or } |W^-|(p) = 0 \Big\}$$

is finite.

We now suppose by contradiction that (M^4, g) is not half conformally flat. For this, for some $\alpha > 0$ (to be chosen later) and any $\varepsilon > 0$, there exists $t = t(\alpha, \varepsilon) > 0$ such that

$$\int_M \Big(\big(|W^+|^2 + \varepsilon\big)^{\frac{\alpha}{2}} - t\,\big(|W^-|^2 + \varepsilon\big)^{\frac{\alpha}{2}} \Big) dV_g = 0.$$

On the other hand, we notice that

$$\begin{aligned} &\Delta\Big(\big(|W^+|^2 + \varepsilon\big)^{\alpha} + t^2\big(|W^-|^2 + \varepsilon\big)^{\alpha}\Big) \\ &= \alpha\big(|W^+|^2 + \varepsilon\big)^{\alpha-2}\Big(\big(|W^+|^2 + \varepsilon\big)\Delta|W^+|^2 + (\alpha-1)|\nabla|W^+|^2|^2\Big) \\ &\quad + t^2\alpha\big(|W^-|^2 + \varepsilon\big)^{\alpha-2}\Big(\big(|W^-|^2 + \varepsilon\big)\Delta|W^-|^2 + (\alpha-1)|\nabla|W^-|^2|^2\Big). \end{aligned}$$

We recall that in dimension 4 we have

$$|\delta W|^2 = |\delta W^+|^2 + |\delta W^-|^2.$$

Therefore, since $\delta W^{\pm} = 0$, we may invoke the Weitzenböck formula (2.11) to arrive at

$$\begin{aligned} &\Delta\Big(\big(|W^+|^2 + \varepsilon\big)^{\alpha} + t^2\big(|W^-|^2 + \varepsilon\big)^{\alpha}\Big) \\ &= \alpha\big(|W^+|^2 + \varepsilon\big)^{\alpha-2}\Big(\big(|W^+|^2 + \varepsilon\big)\big(2|\nabla W^+|^2 + s|W^+|^2 - 36\det W^+\big) \\ &\quad + (\alpha-1)|\nabla|W^+|^2|^2\Big) + t^2\alpha\big(|W^-|^2 + \varepsilon\big)^{\alpha-2}\Big(\big(|W^-|^2 + \varepsilon\big)\big(2|\nabla W^-|^2 + s|W^-|^2 \\ &\quad - 36\det W^-\big) + (\alpha-1)|\nabla|W^-|^2|^2\Big). \end{aligned}$$

Upon integrating of the above expression over M^4 we obtain

$$\begin{aligned} 0 &= \alpha\int_M \big(|W^+|^2 + \varepsilon\big)^{\alpha-1}\big(s|W^+|^2 - 36\det W^+\big)\,dV_g \\ &\quad + \alpha t^2\int_M \big(|W^-|^2 + \varepsilon\big)^{\alpha-1}\big(s|W^-|^2 - 36\det W^-\big)\,dV_g \\ &\quad + \alpha\int_M \big(|W^+|^2 + \varepsilon\big)^{\alpha-2}\Big(2\big(|W^+|^2 + \varepsilon\big)|\nabla W^+|^2 + (\alpha-1)|\nabla|W^+|^2|^2\Big)dV_g \\ &\quad + \alpha t^2\int_M \big(|W^-|^2 + \varepsilon\big)^{\alpha-2}\Big(2\big(|W^-|^2 + \varepsilon\big)|\nabla W^-|^2 + (\alpha-1)|\nabla|W^-|^2|^2\Big)dV_g. \end{aligned} \tag{3.1}$$

Next, we use the refined Kato's inequality (2.10) as well as (2.12) into (3.1) to infer

$$\begin{aligned}0 \geq\ & \alpha \int_M \left(|W^+|^2+\varepsilon\right)^{\alpha-1}(s|W^+|^2-2\sqrt{6}|W^+|^3)dV_g\\ &+\alpha t^2\int_M \left(|W^-|^2+\varepsilon\right)^{\alpha-1}(s|W^-|^2-2\sqrt{6}|W^-|^3)dV_g\\ &+\alpha\int_M \left(|W^+|^2+\varepsilon\right)^{\alpha-2}\left(\frac{10}{3}|W^+|^2|\nabla|W^+||^2+\frac{10}{3}\varepsilon|\nabla|W^+||^2+(\alpha-1)|\nabla|W^+|^2|^2\right)dV_g\\ &+\alpha t^2\int_M \left(|W^-|^2+\varepsilon\right)^{\alpha-2}\left(\frac{10}{3}|W^-|^2|\nabla|W^-||^2+\frac{10}{3}\varepsilon|\nabla|W^-||^2+(\alpha-1)|\nabla|W^-|^2|^2\right)dV_g.\end{aligned}$$

Easily one verifies that

$$\frac{1}{4}|\nabla|W^\pm|^2|^2=|W^\pm|^2|\nabla|W^\pm||^2.$$

From this, we deduce

$$\begin{aligned}0 \geq\ & \alpha \int_M |W^+|^2\left(|W^+|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^+|)dV_g\\ &+\alpha t^2\int_M |W^-|^2\left(|W^-|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^-|)dV_g\\ &+\alpha\int_M \left(|W^+|^2+\varepsilon\right)^{\alpha-2}\left(\frac{5}{6}|\nabla|W^+|^2|^2+(\alpha-1)|\nabla|W^+|^2|^2\right)dV_g\\ &+\alpha t^2\int_M \left(|W^-|^2+\varepsilon\right)^{\alpha-2}\left(\frac{5}{6}|\nabla|W^-|^2|^2+(\alpha-1)|\nabla|W^-|^2|^2\right)dV_g\\ =\ & \alpha \int_M |W^+|^2\left(|W^+|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^+|)dV_g\\ &+\alpha t^2\int_M |W^-|^2\left(|W^-|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^-|)dV_g\\ &+\alpha\left(\alpha-\frac{1}{6}\right)\int_M\left(\left(|W^+|^2+\varepsilon\right)^{\alpha-2}|\nabla|W^+|^2|^2+t^2\left(|W^-|^2+\varepsilon\right)^{\alpha-2}|\nabla|W^-|^2|^2\right)dV_g.\end{aligned} \tag{3.2}$$

Moreover, a straightforward computation gives

$$\alpha\left(|W^\pm|^2+\varepsilon\right)^{\alpha-2}|\nabla|W^\pm|^2|^2=\frac{4}{\alpha}|\nabla\left(|W^\pm|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2.$$

This jointly with (3.2) yields

$$\begin{aligned}0 \geq\ & \alpha \int_M |W^+|^2\left(|W^+|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^+|)dV_g\\ &+\alpha t^2\int_M |W^-|^2\left(|W^-|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^-|)dV_g\\ &+\left(4-\frac{2}{3\alpha}\right)\int_M\left(|\nabla\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2+t^2|\nabla\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2\right)dV_g.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$|\nabla \left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2+t^2|\nabla \left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2=\frac{1}{2}\Big\{|\nabla(\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}-t\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}})|^2 + |\nabla(\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}+t\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}})|^2\Big\} \geq \frac{1}{2}|\nabla(\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}-t\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}})|^2. \tag{3.4}$$

At the same time, from Poincaré inequality we get

$$\int_M |\nabla(\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}-t\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}})|^2 dV_g \geq \lambda_1\int_M\left(\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}-t\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}}\right)^2 dV_g. \tag{3.5}$$

So, we combine (3.4) with (3.5) to infer

$$\int_M |\nabla \left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2+t^2|\nabla \left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}}|^2 dV_g \geq \frac{\lambda_1}{2}\int_M\left(\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}-t\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}}\right)^2 dV_g. \tag{3.6}$$

Hereafter, we compare (3.6) with (3.3) and pick α such that $\left(4-\frac{2}{3\alpha}\right)\geq 0$ to arrive at

$$\begin{aligned}0\geq &\int_M |W^+|^2\left(|W^+|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^+|)dV_g\\ &+\int_M t^2|W^-|^2\left(|W^-|^2+\varepsilon\right)^{\alpha-1}(s-2\sqrt{6}|W^-|)dV_g\\ &+\frac{1}{\alpha}\left(2-\frac{1}{3\alpha}\right)\lambda_1\Big[\int_M\left(\left(|W^+|^2+\varepsilon\right)^{\alpha}-2t\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha}{2}}\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha}{2}}\right)dV_g\\ &+\int_M t^2\left(|W^-|^2+\varepsilon\right)^{\alpha}dV_g\Big].\end{aligned}$$

Moreover, choosing $\alpha_0=\frac{1}{3}$, which maximizes $\frac{1}{\alpha}\left(2-\frac{1}{3\alpha}\right)$, we get

$$\begin{aligned}0\geq &\int_M\left(|W^+|^2\left(|W^+|^2+\varepsilon\right)^{\alpha_0-1}(s-2\sqrt{6}|W^+|)\right)dV_g\\ &+\int_M t^2|W^-|^2\left(|W^-|^2+\varepsilon\right)^{\alpha_0-1}(s-2\sqrt{6}|W^-|)dV_g+3\lambda_1\Big[\int_M\left(|W^+|^2+\varepsilon\right)^{\alpha_0}dV_g\\ &-2\int_M t\left(|W^+|^2+\varepsilon\right)^{\frac{\alpha_0}{2}}\left(|W^-|^2+\varepsilon\right)^{\frac{\alpha_0}{2}}dV_g+\int_M t^2\left(|W^-|^2+\varepsilon\right)^{\alpha_0}dV_g\Big].\end{aligned}$$

Therefore, when ε goes to 0 we obtain

$$\begin{aligned}0\geq \int_M\Big(&|W^-|^{2\alpha_0}(s+3\lambda_1-2\sqrt{6}|W^-|)t^2-6\lambda_1|W^+|^{\alpha_0}|W^-|^{\alpha_0}t\\ &+|W^+|^{2\alpha_0}(s+3\lambda_1-2\sqrt{6}|W^+|)\Big)dV_g.\end{aligned} \tag{3.7}$$

We now remark that the integrand of (3.7) is a quadratic function of t. Whence, for simplicity, we set

$$P(t)=|W^-|^{2\alpha_0}(a-2\sqrt{6}|W^-|)t^2-6\lambda_1|W^+|^{\alpha_0}|W^-|^{\alpha_0}t+|W^+|^{2\alpha_0}(a-2\sqrt{6}|W^+|),$$

where $a=s+3\lambda_1$.

We also notice that the discriminant Δ of $P(t)$ is given by

$$\Delta=36\lambda_1^2|W^+|^{2\alpha_0}|W^-|^{2\alpha_0}-4|W^-|^{2\alpha_0}|W^+|^{2\alpha_0}(a-2\sqrt{6}|W^-|)(a-2\sqrt{6}|W^+|). \quad (3.8)$$

We now claim that Δ is less than or equal to zero. In fact, from (2.5) we already know that

$$|W^\pm|^2\leq 6(w_1^\pm)^2.$$

This together with (2.9) provides

$$|W^+|+|W^-|\leq\sqrt{6}\Big(\frac{s}{6}-2K_{\min}^\perp\Big). \quad (3.9)$$

Moreover, a straightforward computation using our assumption on $K_{\min}^\perp$ yields

$$\Big(\frac{s}{6}-2K_{\min}^\perp\Big)\leq\frac{a^2-9\lambda_1^2}{12a}. \quad (3.10)$$

Next, we combine (3.9) with (3.10) and (3.8) to deduce

$$\begin{aligned}\Delta&=|W^-|^{2\alpha_0}|W^+|^{2\alpha_0}\Big(36\lambda_1^2-4a^2+8\sqrt{6}a(|W^+|+|W^-|)-96|W^+||W^-|\Big)\\&\leq|W^-|^{2\alpha_0}|W^+|^{2\alpha_0}\Big(36\lambda_1^2-4a^2+4(a^2-9\lambda_1^2)-96|W^+||W^-|\Big)\\&=-96|W^+|^{2\alpha_0+1}|W^-|^{2\alpha_0+1}\\&\leq 0,\end{aligned}$$

which settles our claim. Hereafter, we use once more (3.7) to conclude $|W^+||W^-|=0$ in M^4. From this, since Σ is finite, we arrive at a contradiction. Therefore, this contradiction argument guarantees that $(M^4,\ g)$ is half conformally flat.

From now on it suffices to follow the arguments applied in the final steps of the proof of Theorem 6 in [7] (see also [32]). More precisely, we define the following set

$$A=\Big\{p\in M^4;\quad \mathrm{Ric}(p)\neq\frac{s(p)}{4}g\Big\},$$

where $(\mathrm{Ric}-\frac{s}{4}g)$ stands for the traceless Ricci tensor of $(M^4,\ g)$. Then, if A is empty, we use Hitchin's theorem [16] to deduce that M^4 is either isometric to $\mathbb{S}^4$ with its canonical metric or isometric to $\mathbb{CP}^2$ with the Fubini–Study metric. Otherwise, if A is not empty, we deduce that M^4 is locally conformally flat. In other words, one of the following assertions holds:

(1) M^4 is isometric to $\mathbb{S}^4$ with its canonical metric;
(2) M^4 is isometric to $\mathbb{CP}^2$ with the Fubini–Study metric;
(3) Or M^4 has positive isotropic curvature.

In this last case we can invoke Chen–Tang–Zhu theorem [5] to conclude that M^4 is diffeomorphic to a connected sum $\mathbb{S}^4\sharp(\mathbb{R}\times\mathbb{S}^3)/G_1\sharp\ldots\sharp(\mathbb{R}\times\mathbb{S}^3)/G_n$, where each G_i is a discrete subgroup of the isometry group of $\mathbb{R}\times\mathbb{S}^3$ (see also [21]).

This is what we wanted to prove. □

Acknowledgments The author would like to thank the Department of Mathematics—Lehigh University, where many results in this paper were proved. It is a pleasure to thank H.-D. Cao for the warm hospitality and his constant support. Moreover, he would like to extend his special thanks to A. Barros, E. Costa, P. Wu and R. Diógenes for many helpful conversations. Finally, the author wants to thank the referee for his careful reading and helpful suggestions.

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