Locally conformal calibrated G_2 -manifolds

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Abstract We study conditions for which the mapping torus of a 6-manifold endowed with an $SU(3)$ -structure is a locally conformal calibrated G_2 -manifold, that is, a 7-manifold endowed with a G_2 -structure φ such that $d\varphi = -\theta \wedge \varphi$ for a closed nonvanishing 1-form θ . Moreover, we show that if (M, φ) is a compact locally conformal calibrated G_2 -manifold with $\mathcal{L}_{\theta^\#}\varphi = 0$, where $\theta^\#$ is the dual of θ with respect to the Riemannian metric g_φ induced by φ , then M is a fiber bundle over S^1 with a coupled $SU(3)$ -manifold as fiber.

Keywords Locally conformal calibrated G_2 -structure · $SU(3)$ -structure · Mapping torus

Mathematics Subject Classification Primary 53C10; Secondary 53C15

1 Introduction

A G_2 -structure on a 7-manifold M can be characterized by the existence of a globally defined 3-form φ , called the fundamental 3-form, which can be written at each point as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \quad (1)$$

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with respect to some local coframe $\{e^1, \dots, e^7\}$ on M . A G_2 -structure φ induces a Riemannian metric g_φ and a volume form dV_{g_φ} on M given by

$$g_\varphi(X, Y)dV_{g_\varphi} = \frac{1}{6}i_X\varphi \wedge i_Y\varphi \wedge \varphi, \quad (2)$$

for any pair of vector fields X, Y on M .

The classes of G_2 -structures can be described in terms of the exterior derivatives of the fundamental 3-form φ and the 4-form $*\varphi$ [1–3], where $*$ is the Hodge operator defined from g_φ and dV_{g_φ} . In this paper, we focus our attention on the class of *locally conformal calibrated* G_2 -structures, which are characterized by the condition

$$d\varphi = -\theta \wedge \varphi,$$

for a closed nonvanishing 1-form θ also known as the *Lee form* of the G_2 -structure. We refer to a manifold endowed with such a structure as a *locally conformal calibrated G_2 -manifold*.

A differential complex for locally conformal calibrated G_2 -manifolds was introduced in [4], where such manifolds were characterized as the ones endowed with a G_2 -structure φ for which the space of differential forms annihilated by φ becomes a differential subcomplex of the de Rham's complex.

Locally conformal calibrated G_2 -structures φ whose underlying Riemannian metric g_φ is Einstein were studied in [5], where it was shown that in the compact case, the scalar curvature of g_φ cannot be positive. Moreover, in contrast to a result obtained in the compact homogeneous case, a noncompact example of homogeneous manifold endowed with a locally conformal calibrated G_2 -structure whose associated Riemannian metric is Einstein and non-Ricci-flat was given. The homogeneous Einstein metric is a rank-one extension of a Ricci soliton induced on the three-dimensional complex Heisenberg group by a left-invariant coupled $SU(3)$ -structure (ω, ψ_+) , i.e., such that $d\omega = c\psi_+$, with $c \in \mathbb{R} - \{0\}$ (see Definition 2.2 for details on coupled $SU(3)$ -structures).

In the general case, it is not difficult to show that the product of a coupled $SU(3)$ -manifold by S^1 admits a natural locally conformal calibrated G_2 -structure.

In [6], a characterization of compact locally conformal parallel G_2 -manifolds as fiber bundles over S^1 with compact nearly Kähler fiber was obtained (see also [7]).

Banyaga in [8] showed that special types of exact locally conformal symplectic manifolds are fibered over S^1 with each fiber carrying a contact form. In this context, exact means that the locally conformal symplectic structure Ω is d_θ -exact, i.e., $\Omega = d\alpha + \theta \wedge \alpha$, for a 1-form α and a closed 1-form θ .

Exact locally conformal symplectic structures are precisely the structures called *of the first kind* in Vaisman's paper [9], where he showed that a manifold endowed with such a structure is a 2-contact manifold and has a vertical two-dimensional foliation. Moreover, if this foliation is regular, then the manifold is a T^2 -principal bundle over a symplectic manifold.

More in general, by [10, Proposition 3.3], every compact manifold admitting a generalized contact pair of type $(k, 0)$ fibers over the circle with fiber a contact manifold and the monodromy acting by a contactomorphism. Note that a contact pair (α, β) of type $(k, 0)$ induces a locally conformal symplectic form defined by $d\alpha + \alpha \wedge \beta$. Conversely, every mapping torus of a contactomorphism admits a generalized contact pair of type $(k, 0)$ and an induced locally conformal symplectic form.

A theorem by Tischler [11] asserts that the existence of a nonvanishing closed 1-form on a compact manifold M is equivalent to the condition that M is a mapping torus. In the last years, mapping tori of certain diffeomorphisms have been very useful to study some geometric structures. For example, Li in [12] proved that any compact cosymplectic manifold is the

mapping torus of a symplectomorphism. Furthermore, formality of mapping tori was studied in [13].

A natural problem is then to find a characterization of d_θ -exact locally conformal calibrated G_2 -manifolds with Lee form θ and to see under which conditions a locally conformal calibrated G_2 -manifold is a mapping torus of a special type of $SU(3)$ -manifold or, more in general, it is fibered over S^1 with fiber endowed with a special type of $SU(3)$ -structure.

In Sect. 3, we will show that if N is a six-dimensional, compact, connected coupled $SU(3)$ -manifold, and $\nu: N \rightarrow N$ is a diffeomorphism preserving the $SU(3)$ -structure of N , then the mapping torus N_ν of ν admits a locally conformal calibrated G_2 -structure. Moreover, in the same section, we will show that a result of this kind also holds for a compact nearly Kähler $SU(3)$ -manifold, which is a particular case of coupled $SU(3)$ -manifold. In detail, we will prove that the mapping torus of a nearly Kähler $SU(3)$ -manifold with respect to a diffeomorphism preserving the nearly Kähler structure is endowed with a locally conformal parallel G_2 -structure.

In a similar way as in the paper [8] on locally conformal symplectic manifolds, in Sect. 4 we find some characterizations for locally conformal calibrated G_2 -structures φ which are d_θ -exact, that is, such that $\varphi = d_\theta\omega = d\omega + \theta \wedge \omega$, where θ is the Lee form of φ and ω is a 2-form on M . In fact, for a locally conformal calibrated G_2 -manifold (M, φ) , we prove in Proposition 4.4 that if X is the g_φ -dual vector field of θ and ω is the 2-form given by $\omega = i_X\varphi$, then X is an infinitesimal automorphism of φ ($\mathcal{L}_X\varphi = 0$) if and only if $\theta(X)\varphi = d_\theta\omega$.

Section 5 is devoted to the construction of Lie algebras admitting a locally conformal calibrated G_2 -structure from six-dimensional Lie algebras endowed with a coupled $SU(3)$ -structure. As a consequence, new examples of compact manifolds with a locally conformal calibrated G_2 -structure are constructed.

Finally, in the last section, we obtain a characterization for compact locally conformal calibrated G_2 -manifolds (M, φ) under the assumption $\mathcal{L}_{\theta^\#}\varphi = 0$, where $\theta^\#$ is the dual of the Lee form θ with respect to the Riemannian metric g_φ induced by φ . More precisely, in Theorem 6.4, we show that M is a fiber bundle over S^1 such that each fiber is equipped with a coupled $SU(3)$ -structure.

2 Preliminaries

Given a seven-dimensional manifold M endowed with a G_2 -structure φ , let g_φ and dV_{g_φ} denote, respectively, the Riemannian metric and the volume form on M induced by φ via the relation (2).

A manifold endowed with a G_2 -structure φ is said to be *locally conformal calibrated* G_2 -manifold if

$$d\varphi = -\theta \wedge \varphi,$$

where θ is a nonvanishing 1-form which has to be closed. θ is called the *Lee form* associated with the locally conformal calibrated G_2 -structure φ and can be defined as

$$\theta = \frac{1}{4} * (*d\varphi \wedge \varphi),$$

where $*$ is the Hodge star operator defined from the metric g_φ and the volume form dV_{g_φ} .

If the Lee form vanishes, then the 3-form φ is closed and the G_2 -structure is called *calibrated*.

A G_2 -structure φ is said to be *locally conformal parallel* if

$$d\varphi = -\theta \wedge \varphi, \quad d*\varphi = -\frac{4}{3}\theta \wedge *\varphi.$$

By [14, Theorem 3.1], given a compact manifold M admitting a G_2 -structure φ , there exists a unique (up to homothety) conformal G_2 -structure $e^{3f}\varphi$ such that the corresponding Lee form is co-closed. A G_2 -structure with co-closed Lee form is also called a *Gauduchon G_2 -structure*.

Given a closed 1-form τ on M , we say that a G_2 -structure φ is d_τ -exact if there exists a 2-form $\alpha \in \Lambda^2(M)$ such that $\varphi = d\alpha + \tau \wedge \alpha$. In this case, a simple computation shows that the G_2 -structure is locally conformal calibrated with Lee form τ . The converse might not be true in general.

Remark 2.1 Given a locally conformal calibrated G_2 -structure φ , we can consider the class

$$[\varphi] = \{f\varphi : f \in C^\infty(M) \text{ and } f > 0\}$$

of locally conformal calibrated G_2 -structures which are conformally equivalent to φ . If $d\varphi = -\theta \wedge \varphi$, then $d(f\varphi) = (d(\ln f) - \theta) \wedge f\varphi$ and φ is d_θ -exact if and only if $f\varphi$ is $d_{\theta-d(\ln f)}$ -exact. Thus, being d_θ -exact is a conformal property for locally conformal calibrated G_2 -structures.

Recall that an $SU(3)$ -structure (ω, ψ_+) on a 6-manifold N is said to be *half-flat* if both $\omega^2 = \omega \wedge \omega$ and ψ_+ are closed. By [15], if a half-flat $SU(3)$ -structure on a six-dimensional connected manifold N is such that $d\omega = f\psi_+$ for some nonvanishing function $f \in C^\infty(N)$, then f has to be a constant function. This motivates the following

Definition 2.2 Let N be a six-dimensional connected manifold, and let (ω, ψ_+) be an $SU(3)$ -structure on it. We say that (ω, ψ_+) is a *coupled $SU(3)$ -structure* if $d\omega = c\psi_+$ for some nonzero real constant c called the *coupled constant*.

Given an $SU(3)$ -structure (ω, ψ_+) , denote by J the almost complex structure induced by ψ_+ , by $\psi_- = J\psi_+$ and by $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$ the induced Riemannian metric. By [16], since an $SU(3)$ -structure on a six-dimensional manifold N is characterized by a pair of stable, compatible ad normalized forms $(\omega, \psi_+) \in \Lambda^2(N) \times \Lambda^3(N)$ inducing a Riemannian metric and since the construction of J , ψ_- and h is invariant, a diffeomorphism preserving the forms ω and ψ_+ is actually an automorphism of the $SU(3)$ -structure itself and, in particular, an isometry. As a special case of this, if we have a coupled $SU(3)$ -structure with $d\omega = c\psi_+$ for some nonzero constant c , then every diffeomorphism of N preserving ω is an automorphism of the $SU(3)$ -structure and an isometry.

Remark 2.3 Note that if (ω, ψ_+) is a *nearly Kähler $SU(3)$ -structure* ($d\omega = 3\psi_+$, $d\psi_- = -2\omega^2$), then it is in particular a coupled $SU(3)$ -structure. Moreover, if (ω, ψ_+) is a coupled $SU(3)$ -structure with $d\omega = c\psi_+$, then the pair $\hat{\omega} := k^2\omega$, $\hat{\psi}_+ := k^3\psi_+$, where k is a nonzero real constant, is still a coupled $SU(3)$ -structure with $d\hat{\omega} = \frac{c}{k}\hat{\psi}_+$. As a consequence, it is always possible to find a coupled structure having coupled constant $c = 1$. Changing the constant c of a coupled $SU(3)$ -structure in the way just described, the almost complex structure J is preserved, while the Riemannian metric h is rescaled by k^2 .

The following proposition shows a first link between coupled $SU(3)$ -structures and locally conformal calibrated G_2 -structures.

Proposition 2.4 *Let N be a six-dimensional connected manifold endowed with a coupled $SU(3)$ -structure (ω, ψ_+) with coupled constant c ; then, the cylinder $(N \times \mathbb{R}, h + dr^2)$ over N has a locally conformal calibrated G_2 -structure given by $\varphi = \omega \wedge dr + \psi_+$ and such that $g_\varphi = h + dr^2$. Moreover, if $c \neq 3$, also the cone over N , $(N \times (0, +\infty), r^2h + dr^2)$, is endowed with a locally conformal calibrated G_2 -structure $\varphi = r^2\omega \wedge dr + r^3\psi_+$ whose associated metric g_φ is the cone's one.*

3 Mapping torus

Let N be a compact manifold and $\nu : N \rightarrow N$ a diffeomorphism. The mapping torus N_ν of ν is the quotient space of $N \times [0, 1]$ in which any point $(p, 0)$ is identified with $(\nu(p), 1)$. N_ν is naturally a smooth manifold, since it is the quotient of $N \times \mathbb{R}$ by the infinite cyclic group generated by the diffeomorphism $(p, t) \mapsto (\nu(p), t + 1)$. The natural map $\pi : N_\nu \rightarrow S^1$ defined by $\pi(p, t) = e^{2\pi it}$ is the projection of a locally trivial fiber bundle (here we think S^1 as the interval $[0, 1]$ with identified end points). Thus, any ν -invariant form α on N defines a form $\tilde{\alpha}$ on N_ν since the pullback of α to $N \times \mathbb{R}$ is invariant by the diffeomorphism $(p, t) \mapsto (\nu(p), t + 1)$. For the same reason, the 1-form dt on $\mathbb{R}$, where t is the coordinate on $\mathbb{R}$, induces a closed 1-form η on N_ν . Moreover, on N_ν , we have a distinguished vector field ξ induced by the vector field $\frac{d}{dt}$. This vector field is such that $\eta(\xi) = 1$.

In the case of a compact manifold endowed with a coupled $SU(3)$ -structure (ω, ψ_+) , we can prove the following

Proposition 3.1 *Let N be a six-dimensional, compact, connected manifold endowed with a coupled $SU(3)$ -structure (ω, ψ_+) with $d\omega = c\psi_+$ and let $\nu : N \rightarrow N$ be a diffeomorphism such that $\nu^*\omega = \omega$. Then, the mapping torus N_ν admits a locally conformal calibrated G_2 -structure φ . Moreover, $\mathcal{L}_\xi\varphi = 0$.*

Proof We have the following situation

$$\begin{array}{ccc} N & \xleftarrow{p_1} N \times [0, 1] & \xrightarrow{q} N_\nu \\ & \downarrow p_2 & \downarrow \pi \\ & [0, 1] & \xrightarrow{\Pi} S^1 \end{array}$$

where p_1 and p_2 are the projections from $N \times [0, 1]$ on the first and on the second factor, respectively, q is the quotient map, π is the fibration map, and $\Pi(t) = e^{2\pi it}$.

Let us observe that $p_1^*(\omega) \in \Lambda^2(N \times [0, 1])$ and $p_1^*(\psi_\pm) \in \Lambda^3(N \times [0, 1])$. Then, since $\nu^*\omega = \omega$ and (as a consequence) $\nu^*\psi_\pm = \psi_\pm$, we can glue up these pullbacks and obtain a 2-form $\tilde{\omega} \in \Lambda^2(N_\nu)$ and 3-forms $\tilde{\psi}_\pm \in \Lambda^3(N_\nu)$ satisfying the same relations that hold on N . In particular, since $d(p_1^*\omega) = cp_1^*\psi_+$, we have

$$d\tilde{\omega} = c\tilde{\psi}_+, \quad d\tilde{\omega}^2 = 0.$$

Using the closed 1-form η defined on N_ν , we have that

$$\varphi = \eta \wedge \tilde{\omega} + \tilde{\psi}_+$$

is a 3-form on N_ν defining a G_2 -structure on it. Moreover,

$$d\varphi = d\eta \wedge \tilde{\omega} - \eta \wedge d\tilde{\omega} + d\tilde{\psi}_+ = -c\eta \wedge \tilde{\psi}_+ = -c\eta \wedge \varphi,$$

that is, φ is locally conformal calibrated with Lee form $\theta = c\eta$.

Since both $\tilde{\omega}$ and $\tilde{\psi}_+$ derive from differential forms defined on N , we have $i_{\xi}\tilde{\omega} = 0$ and $i_{\xi}\tilde{\psi}_+ = 0$. From these conditions, it follows that $i_{\xi}\varphi = \tilde{\omega}$ and, consequently, $\mathcal{L}_{\xi}\varphi = 0$. $\square$

Remark 3.2 As we mentioned before, if (ω, ψ_+) is a pair of compatible, normalized, stable forms on N , then the remaining tensors appearing in the definition of an $SU(3)$ -structure, namely the almost complex structure J , the 3-form $\psi_- = J\psi_+$ and the Riemannian metric h , can be completely determined from (ω, ψ_+) . Thus, the diffeomorphism ν preserves not only the 2-form ω and the 3-form ψ_+ on N but also the Riemannian metric h of the $SU(3)$ -structure (ω, ψ_+) . Hence, (globally) the metric g_{φ} on N_{ν} is $g_{\varphi} = \tilde{h} + \eta^2$, and we have that ξ is the vector field dual to η with respect to g_{φ} .

Example 3.3 The previous proposition can be applied to compact nilmanifolds admitting a coupled $SU(3)$ -structure. Nilpotent Lie algebras admitting a coupled $SU(3)$ -structure were classified in [5]. One of these is the Iwasawa Lie algebra, i.e., the Lie algebra of the complex Heisenberg group of complex dimension 3

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}, z_i \in \mathbb{C}, i = 1, 2, 3 \right\}.$$

This complex Lie group admits a co-compact discrete subgroup Γ which is defined as the subgroup of G for which z_i are Gaussian integers. Consider the automorphism

$$\nu : G \rightarrow G, \quad \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & z_1 & -iz_3 \\ 0 & 1 & -iz_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, if we denote by

$$e^1 + ie^2 = dz_1, \quad e^3 + ie^4 = dz_2, \quad e^5 + ie^6 = -dz_3 + z_1 \wedge dz_2,$$

we have

$$\nu^*(e^1) = e^1, \quad \nu^*(e^2) = e^2, \quad \nu^*(e^3) = e^4, \quad \nu^*(e^4) = -e^3, \quad \nu^*(e^5) = e^6, \quad \nu^*(e^6) = -e^5.$$

The nilmanifold $N = \Gamma \backslash G$ admits the coupled $SU(3)$ -structure (ω, ψ_+) defined by

$$\omega = e^{12} + e^{34} - e^{56}, \quad \psi_+ = e^{136} - e^{145} - e^{235} - e^{246}.$$

Since $\nu^*\omega = \omega$, we can apply Proposition 3.1 and the mapping torus N_{ν} admits a locally conformal calibrated G_2 -structure.

Remark 3.4 Since by Remark 2.3 we can always suppose that the coupled constant is $c = 1$, the form $\varphi = \eta \wedge \tilde{\omega} + \tilde{\psi}_+$ is d_{η} -exact with $\varphi = d_{\eta}\tilde{\omega}$.

As we observed in Remark 2.3, a special case of coupled $SU(3)$ -structures is given by the nearly Kähler $SU(3)$ -structures. In this case, we can prove what follows.

Proposition 3.5 *Let N be a six-dimensional, compact, connected manifold endowed with a nearly Kähler $SU(3)$ -structure (ω, ψ_+) , and let $\nu : N \rightarrow N$ be a diffeomorphism such that $\nu^*\omega = \omega$. Then, the mapping torus N_{ν} admits a locally conformal parallel G_2 -structure.*

Proof As in Proposition 3.1, we can define the differential forms $\tilde{\omega} \in \Lambda^2(N_{\nu})$ and $\tilde{\psi}_{\pm} \in \Lambda^3(N_{\nu})$, which in this case satisfy the relations

$$\begin{aligned} d\tilde{\omega} &= 3\tilde{\psi}_+, \\ d\tilde{\psi}_- &= -2\tilde{\omega}^2. \end{aligned}$$

The positive 3-form

$$\varphi = \eta \wedge \tilde{\omega} + \tilde{\psi}_+$$

defines a G_2 -structure on N_ν with Hodge dual

$$*\varphi = \tilde{\psi}_- \wedge \eta + \frac{1}{2}\tilde{\omega}^2.$$

It follows from computations that

$$\begin{aligned} d\varphi &= 3(-\eta) \wedge \varphi, \\ d*\varphi &= 4(-\eta) \wedge *\varphi. \end{aligned}$$

Therefore, φ is a locally conformal parallel G_2 -structure defined on N_ν . $\square$

Example 3.6 Consider the six-dimensional compact manifold $S^3 \times S^3$. As a Lie group, it is $SU(2) \times SU(2)$, and its Lie algebra is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Let $\{e_1, e_2, e_3\}$ denote the standard basis for the first copy of $\mathfrak{su}(2)$, let $\{e_4, e_5, e_6\}$ denote it for the second one, and let $\{e^1, e^2, e^3\}$ and $\{e^4, e^5, e^6\}$ denote their dual bases. The structure equations of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are:

$$\begin{aligned} de^1 &= e^{23}, & de^2 &= e^{31}, & de^3 &= e^{12}, \\ de^4 &= e^{56}, & de^5 &= e^{64}, & de^6 &= e^{45}. \end{aligned}$$

On $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we have a pair of stable, compatible, normalized forms

$$\begin{aligned} \omega &= -\frac{\sqrt{3}}{18} (e^{14} + e^{25} + e^{36}), \\ \psi_+ &= \frac{\sqrt{3}}{54} (-e^{234} + e^{156} + e^{135} - e^{246} - e^{126} + e^{345}), \end{aligned}$$

defining an $SU(3)$ -structure which is nearly Kähler and induces a left-invariant nearly Kähler $SU(3)$ -structure on $S^3 \times S^3$.

Let $\nu : S^3 \times S^3 \rightarrow S^3 \times S^3$ be the diffeomorphism such that

$$\nu^*e^1 = e^1, \quad \nu^*e^2 = e^3, \quad \nu^*e^3 = -e^2, \quad \nu^*e^4 = e^4, \quad \nu^*e^5 = e^6, \quad \nu^*e^6 = -e^5.$$

ν preserves ω ; therefore, the mapping torus $(S^3 \times S^3)_\nu$ is endowed with a locally conformal parallel G_2 -structure by the previous proposition.

A characterization of compact locally conformal parallel G_2 -manifolds as fiber bundles over S^1 with compact nearly Kähler fiber was obtained in [6] (see also [7]). It was also shown there that for compact seven-dimensional locally conformal parallel G_2 -manifolds (M, φ) with co-closed Lee form θ , the Lee flow preserves the Gauduchon G_2 -structure, i.e., $\mathcal{L}_{\theta^\#}\varphi = 0$, where $\theta^\#$ is the dual of θ with respect to g_φ . In the next section, we will characterize the locally conformal calibrated G_2 -structures such that $\mathcal{L}_{\theta^\#}\varphi = 0$.

4 Exact locally conformal calibrated G_2 -manifolds

In a similar way as in the paper [8] on locally conformal symplectic manifolds, we can find some characterizations for d_θ -exact locally conformal calibrated G_2 -structures φ with Lee form θ .

We recall that $X \in \mathfrak{X}(M)$ is a *conformal infinitesimal automorphism* of φ if and only if there exists a smooth function ρ_X on M such that $\mathcal{L}_X\varphi = \rho_X\varphi$. If $\rho_X \equiv 0$, then X is a *conformal automorphism* of φ . We start by proving the following

Lemma 4.1 *Let (M, φ) be a locally conformal calibrated G_2 -manifold with Lee form θ . A vector field $X \in \mathfrak{X}(M)$ is a conformal infinitesimal automorphism of φ if and only if there exists a smooth function $f_X \in C^\infty(M)$ such that $d_\theta \omega = f_X \varphi$, where $\omega = i_X \varphi$. Moreover, if M is connected, f_X is constant.*

Proof Let us compute the expression of the Lie derivative of φ with respect to X

$$\begin{aligned}\mathcal{L}_X \varphi &= d(i_X \varphi) + i_X(d\varphi) \\ &= d\omega + i_X(-\theta \wedge \varphi) \\ &= d\omega - \theta(X)\varphi + \theta \wedge (i_X \varphi) \\ &= d\omega + \theta \wedge \omega - \theta(X)\varphi \\ &= d_\theta \omega - \theta(X)\varphi,\end{aligned}$$

where $\omega = i_X \varphi$. Therefore, X is a conformal infinitesimal automorphism of φ with $\mathcal{L}_X \varphi = \rho_X \varphi$ if and only if $d_\theta \omega = f_X \varphi$, where f_X is a smooth real-valued function on M such that $f_X = \rho_X + \theta(X)$.

Suppose now that M is connected, and let X be a conformal infinitesimal automorphism of φ , as we have just shown $d_\theta \omega = f_X \varphi$ for some $f_X \in C^\infty(M)$. We have

$$\begin{aligned}0 &= d_\theta(d_\theta \omega) \\ &= d_\theta(f_X \varphi) \\ &= d(f_X \varphi) + \theta \wedge (f_X \varphi) \\ &= df_X \wedge \varphi + f_X d\varphi + f_X(\theta \wedge \varphi) \\ &= df_X \wedge \varphi + f_X d\varphi - f_X d\varphi \\ &= df_X \wedge \varphi.\end{aligned}$$

Since the linear mapping $\wedge \varphi : \Lambda^1(M) \rightarrow \Lambda^4(M)$ is injective, we obtain that $df_X = 0$, and from this, the assertion follows. $\square$

Remark 4.2 It is worth emphasizing here that if X is a conformal infinitesimal automorphism of φ with f_X a nonzero constant, then φ is d_θ -exact. Indeed

$$\varphi = \frac{1}{f_X} d_\theta \omega = d_\theta \left(\frac{\omega}{f_X} \right).$$

Recall the result contained in [17]:

Lemma 4.3 ([17]) *Let M be a seven-dimensional, compact manifold. Then, for any G_2 -structure φ on M , any vector field $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, it holds*

$$\int_M \mathcal{L}_X \varphi \wedge *f\varphi = -3 \int_M df \wedge *X^\flat.$$

From this Lemma with $f \equiv 1$ and X conformal infinitesimal automorphism of φ with $\mathcal{L}_X \varphi = \rho_X \varphi$, we have

$$\int_M \rho_X dV_{g_\varphi} = 0.$$

Thus, thinking at the proof of Lemma 4.1, we get

$$\int_M \theta(X) dV_{g_\varphi} = \int_M f_X dV_{g_\varphi} = f_X \text{Vol}(M),$$

that is, the Riemannian integral of the function $\theta(X)$ over M is constant.

In conclusion, we can prove the following characterization for a d_θ -exact locally conformal calibrated G_2 -structure.

Proposition 4.4 *Let (M, φ) be a connected locally conformal calibrated G_2 -manifold with Lee form θ . Let X be the g_φ -dual vector field of θ , i.e., $\theta(\cdot) = g_\varphi(X, \cdot)$, and define the 2-form $\omega := i_X \varphi$. Then, $\mathcal{L}_X \varphi = 0$ if and only if $\theta(X)\varphi = d_\theta \omega$. Moreover, if $\mathcal{L}_X \varphi = 0$, then $\theta(X) = |X|^2$ is a nonzero constant.*

Proof We have

$$\begin{aligned}\mathcal{L}_X \varphi &= d(i_X \varphi) + i_X d\varphi \\ &= d\omega + i_X(-\theta \wedge \varphi) \\ &= d\omega - \theta(X)\varphi + \theta \wedge \omega.\end{aligned}$$

Therefore, $\mathcal{L}_X \varphi = 0$ if and only if $\theta(X)\varphi = d_\theta \omega$.

If $\mathcal{L}_X \varphi = 0$, from Lemma 4.1, we have that $\theta(X) = |X|^2$ is a nonzero constant, since $\theta(X)\varphi = d_\theta \omega$ and $X = \theta^\#$, where the map $\cdot^\# : \Lambda^1(M) \rightarrow \mathfrak{X}(M)$ is an isomorphism. $\square$

5 Locally conformal calibrated G_2 Lie algebras

In this section, we show that locally conformal calibrated G_2 -structures defined on seven-dimensional Lie algebras are closely related to coupled $SU(3)$ -structures on six-dimensional Lie algebras. This generalizes the result proved in [18] for calibrated G_2 -structures on seven-dimensional Lie algebras from symplectic half-flat Lie algebras. First, we need to recall some definitions and results about $SU(3)$ - and G_2 -structures on Lie algebras.

Let $\mathfrak{g}$ be a seven-dimensional Lie algebra. A G_2 -structure on $\mathfrak{g}$ is a 3-form φ on $\mathfrak{g}$ which can be written as in (1) with respect to some basis $\{e^1, \dots, e^7\}$ of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. φ is said to be a *locally conformal calibrated G_2 -structure* on $\mathfrak{g}$ if

$$d\varphi = -\theta \wedge \varphi,$$

for some closed 1-form θ on $\mathfrak{g}$, where d denotes the Chevalley–Eilenberg differential on $\mathfrak{g}^*$.

We say that a six-dimensional Lie algebra $\mathfrak{h}$ has an $SU(3)$ -structure if there exists a pair (ω, ψ_+) of forms on $\mathfrak{h}$, where ω is a 2-form and ψ_+ a 3-form, which can be expressed as

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some basis $\{e^1, \dots, e^6\}$ of the dual space $\mathfrak{h}^*$. If $\{e^1, \dots, e^6\}$ is such a basis, the dual basis $\{e_1, \dots, e_6\}$ of $\mathfrak{h}$ is called *$SU(3)$ -basis*. An $SU(3)$ -structure (ω, ψ_+) on $\mathfrak{h}$ is said to be a *coupled $SU(3)$ -structure* if

$$\hat{d}\omega = c\psi_+$$

for some nonzero real constant c , where $\hat{d}$ is the Chevalley–Eilenberg differential on $\mathfrak{h}^*$.

If $\mathfrak{h}$ is a six-dimensional Lie algebra and $D \in \text{Der}(\mathfrak{h})$ is a derivation of $\mathfrak{h}$, then the vector space

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$$

is a Lie algebra with the Lie bracket given by

$$[U, V] = [U, V]_{\mathfrak{h}}, \quad [\xi, U] = D(U), \quad (3)$$

for any $U, V \in \mathfrak{h}$.

It is well known that there exists a real representation of the 3×3 complex matrices via

$$\rho : \mathfrak{gl}(3, \mathbb{C}) \longrightarrow \mathfrak{gl}(6, \mathbb{R}).$$

More in detail, if $A \in \mathfrak{gl}(3, \mathbb{C})$, then $\rho(A)$ is the matrix $(B_{ij})_{i,j=1}^3$ with

$$B_{ij} = \begin{pmatrix} \operatorname{Re} A_{ij} & \operatorname{Im} A_{ij} \\ -\operatorname{Im} A_{ij} & \operatorname{Re} A_{ij} \end{pmatrix},$$

where A_{ij} is the (i, j) component of A .

Now, suppose that (ω, ψ_+) is a coupled $\mathrm{SU}(3)$ -structure on a six-dimensional Lie algebra $\mathfrak{h}$, and let D be a derivation of $\mathfrak{h}$ such that $D = \rho(A)$, where $A \in \mathfrak{sl}(3, \mathbb{C})$. Then, the matrix representation of D with respect to an $\mathrm{SU}(3)$ -basis $\{e_1, \dots, e_6\}$ of $\mathfrak{h}$ is

$$D = \left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & a_{11} & -a_{14} & a_{13} & -a_{16} & a_{15} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ -a_{32} & a_{31} & -a_{34} & a_{33} & -a_{36} & a_{35} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & -a_{11} - a_{33} & -a_{12} - a_{34} \\ -a_{52} & a_{51} & -a_{54} & a_{53} & a_{12} + a_{34} & -a_{11} - a_{33} \end{array} \right), \quad (4)$$

where $a_{ij} \in \mathbb{R}$.

Proposition 5.1 *Let (ω, ψ_+) be a coupled $\mathrm{SU}(3)$ -structure on a Lie algebra $\mathfrak{h}$ of dimension 6, and let $D = \rho(A)$, $A \in \mathfrak{sl}(3, \mathbb{C})$, be a derivation of $\mathfrak{h}$ whose matrix representation with respect to an $\mathrm{SU}(3)$ -basis $\{e_1, \dots, e_6\}$ of $\mathfrak{h}$ is as in (4). Then, the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi,$$

with the Lie bracket given by (3), has a locally conformal calibrated G_2 -structure.

Proof We define the G_2 form φ on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ by

$$\varphi = \omega \wedge \eta + \psi_+, \quad (5)$$

where η is the 1-form on $\mathfrak{g}$ such that $\eta(X) = 0$ for all $X \in \mathfrak{h}$ and $\eta(\xi) = 1$. We will see that

$$d\varphi = -c\eta \wedge \varphi,$$

where c is the coupled constant of the coupled $\mathrm{SU}(3)$ -structure on $\mathfrak{h}$.

Suppose that $X, Y, Z, U \in \mathfrak{h}$. Then, it is clear that $(d\omega \wedge \eta)(X, Y, Z, U) = 0$, and

$$d\varphi(X, Y, Z, U) = d\psi_+(X, Y, Z, U) = \hat{d}\psi_+(X, Y, Z, U) = 0,$$

since ψ_+ is $\hat{d}$ -closed and $d\eta = 0$.

Let us consider $X, Y, Z \in \mathfrak{h}$. Using (5), we have

$$\begin{aligned} d\varphi(X, Y, Z, \xi) &= -\varphi([X, Y], Z, \xi) + \varphi([X, Z], Y, \xi) - \varphi([X, \xi], Y, Z) \\ &\quad - \varphi([Y, Z], X, \xi) + \varphi([Y, \xi], X, Z) - \varphi([Z, \xi], X, Y) \\ &= -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &\quad - \psi_+([X, \xi], Y, Z) + \psi_+([Y, \xi], X, Z) - \psi_+([Z, \xi], X, Y) \\ &= d\omega(X, Y, Z) + \psi_+(D(X), Y, Z) + \psi_+(X, D(Y), Z) + \psi_+(X, Y, D(Z)). \end{aligned}$$

Taking into account the expressions of D and ψ_+ in terms of the $\mathrm{SU}(3)$ -basis $\{e_1, \dots, e_6\}$, it is easy to check that

$$\psi_+(D(e_i), e_j, e_k) + \psi_+(e_i, D(e_j), e_k) + \psi_+(e_i, e_j, D(e_k)) = 0,$$

for every triple (e_i, e_j, e_k) of elements of the $\mathrm{SU}(3)$ -basis. Therefore,

$$d\varphi(X, Y, Z, \xi) = d\omega(X, Y, Z) = \hat{d}\omega(X, Y, Z) = c\psi_+(X, Y, Z).$$

Using (5) again, we get

$$d\varphi(X, Y, Z, \xi) = -(c\eta \wedge \varphi)(X, Y, Z, \xi),$$

which completes the proof that the 3-form φ given by (5) defines a locally conformal calibrated G_2 -structure on $\mathfrak{g}$. $\square$

As an application of the previous proposition, we describe two examples of nonisomorphic solvable Lie algebras endowed with a locally conformal calibrated G_2 -structure. They are obtained considering two different derivations on the Iwasawa Lie algebra.

Example 5.2 Consider the six-dimensional Iwasawa Lie algebra $\mathfrak{n}$, and let $\{e_1, \dots, e_6\}$ denote an $SU(3)$ -basis for it. With respect to the dual basis $\{e^1, \dots, e^6\}$, the structure equations of $\mathfrak{n}$ are

$$(0, 0, 0, 0, e^{14} + e^{23}, e^{13} - e^{24})$$

and the pair

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

defines a coupled $SU(3)$ -structure on $\mathfrak{n}$ with $\hat{d}\omega = -\psi_+$.

Let D be the derivation of $\mathfrak{n}$ defined as follows:

$$De_1 = -e_3, \quad De_2 = -e_4, \quad De_3 = e_1, \quad De_4 = e_2, \quad De_5 = 0, \quad De_6 = 0.$$

The Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus_D \mathbb{R}e_7$ has the following structure equations with respect to the basis $\{e^1, \dots, e^6, e^7\}$ of $\mathfrak{s}^*$

$$(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0).$$

By Proposition 5.1, the 3-form

$$\varphi = \omega \wedge e^7 + \psi_+$$

defines then a locally conformal calibrated G_2 -structure on $\mathfrak{s}$ with Lee form $\theta = -e^7$.

Let S denote the simply connected solvable Lie group with Lie algebra $\mathfrak{s}$, let N denote the simply connected nilpotent Lie group such that $\text{Lie}(N) = \mathfrak{n}$, and let $e \in N$ denote the identity element. Observe that $S = \mathbb{R} \ltimes_{\mu} N$, where μ is the unique smooth action of $\mathbb{R}$ on N such that $\mu(t)_{*}e = \exp(tD)$, for any $t \in \mathbb{R}$, and where $\exp$ denotes the map $\exp : \text{Der}(\mathfrak{n}) \rightarrow \text{Aut}(\mathfrak{n})$. Hence, S is *almost nilpotent* in the sense of [19].

Now, in order to show a lattice of S , we proceed as follows. The $SU(3)$ -basis $\{e_1, \dots, e_6\}$ we considered is a rational basis for $\mathfrak{n}$, and with respect to this basis, we have

$$\exp(tD) = \begin{bmatrix} \cos(t) & 0 & \sin(t) & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) & 0 & 0 \\ -\sin(t) & 0 & \cos(t) & 0 & 0 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, $\exp(\pi D)$ is an integer matrix. Therefore, $\exp^N(\mathbb{Z}\langle e_1, \dots, e_6 \rangle)$ is a lattice of N preserved by $\mu(\pi)$ and, consequently,

$$\Gamma = \pi \mathbb{Z} \ltimes_{\mu} \exp^N(\mathbb{Z}\langle e_1, \dots, e_6 \rangle)$$

is a lattice in S (see [20]). Thus, the compact quotient $\Gamma \backslash S$ is a compact solvmanifold endowed with an invariant locally conformal calibrated G_2 -structure φ . Moreover, if $X = -e_7$ denotes the g_φ -dual vector field of the Lee form $\theta = -e^7$, then $i_X \varphi = -\omega$, $\mathcal{L}_X \varphi = 0$ and $\varphi = d_\theta(-\omega)$, as we expected from Proposition 4.4.

Example 5.3 Let us consider the coupled $SU(3)$ -structure (ω, ψ_+) on the Iwasawa Lie algebra $\mathfrak{n}$ described in the previous example and the derivation $D \in \text{Der}(\mathfrak{n})$ given by

$$De_1 = 2e_3, \quad De_2 = 2e_4, \quad De_3 = e_1, \quad De_4 = e_2, \quad De_5 = 0, \quad De_6 = 0,$$

with respect to the $SU(3)$ -basis $\{e_1, \dots, e_6\}$ of $\mathfrak{n}$. Then, the Lie algebra $\mathfrak{q} = \mathfrak{n} \oplus_D \mathbb{R}e_7$ has the following structure equations with respect to the basis $\{e^1, \dots, e^6, e^7\}$ of $\mathfrak{q}^*$

$$(e^{37}, e^{47}, 2e^{17}, 2e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0).$$

The 3-form

$$\varphi = \omega \wedge e^7 + \psi_+$$

defines a locally conformal calibrated G_2 -structure on $\mathfrak{q}$ with Lee form $\theta = -e^7$ by Proposition 5.1.

In this case, it is easy to check that if $X = -e_7$ denotes the g_φ -dual vector field of θ , then $\mathcal{L}_X \varphi \neq 0$ and, accordingly with Proposition 4.4, $\varphi \neq d_\theta(i_X \varphi)$. However, φ is d_θ -exact. Indeed, $\varphi = d_\theta \gamma$, where

$$\gamma = \frac{5}{7}e^{12} - \frac{3}{7}e^{14} + \frac{3}{7}e^{23} - \frac{1}{7}e^{34} - e^{56}.$$

As in the previous example, we have an almost nilpotent Lie group $Q = \mathbb{R} \ltimes_\mu N$, where Q is the simply connected Lie group with solvable Lie algebra $\mathfrak{q}$, and μ is the unique smooth action of $\mathbb{R}$ on N such that $\mu(t)_{*e} = \exp(tD)$, for any $t \in \mathbb{R}$. With respect to the rational basis $\{X_1, \dots, X_6\}$ of $\mathfrak{n}$ given by $X_1 = -\frac{1}{\sqrt{2}}e_2 + e_4$, $X_2 = -\frac{1}{\sqrt{2}}e_1 + e_3$, $X_3 = \frac{1}{\sqrt{2}}e_1 + e_3$, $X_4 = \frac{1}{\sqrt{2}}e_2 + e_4$, $X_5 = \sqrt{2}e_5$, $X_6 = \sqrt{2}e_6$, the matrix associated with $\exp(\sqrt{2}D)$ is integer. More in detail, we have

$$\exp(\sqrt{2}D) = \text{diag}(-2, -2, 2, 2, 0, 0).$$

Thus, $\exp^N(\mathbb{Z}\langle X_1, \dots, X_6 \rangle)$ is a lattice of N preserved by $\mu(\sqrt{2})$, and as a consequence,

$$\Gamma = \sqrt{2} \mathbb{Z} \ltimes_\mu \exp^N(\mathbb{Z}\langle X_1, \dots, X_6 \rangle)$$

is a lattice in Q . The compact quotient $\Gamma \backslash Q$ is then a compact solvmanifold endowed with an invariant locally conformal calibrated G_2 -structure φ .

6 Characterization as fiber bundles over S^1

Let us now consider a seven-dimensional compact manifold M endowed with a locally conformal calibrated G_2 -structure φ . We will show that if $\mathcal{L}_X \varphi = 0$, where $X = \theta^\#$ is the g_φ -dual of the Lee form θ , then M is fibered over S^1 , and each fiber is endowed with a coupled $SU(3)$ -structure given by the restriction of $(\omega = i_X \varphi, \psi_+ = d\omega)$ to the fiber.

We begin recalling the following result, which we will use later.

Proposition 6.1 ([21]) *Let V be a seven-dimensional real vector space and $\varphi \in \Lambda^3(V^*)$ a stable 3-form which induces the inner product g_φ on V . Moreover, let $\mathbf{n} \in V$ be a unit vector with $g_\varphi(\mathbf{n}, \mathbf{n}) = 1$, and let $W := \langle \mathbf{n} \rangle^\perp$ denote the g_φ -orthogonal complement of $\mathbb{R}\mathbf{n}$. Then, the pair (ω, ψ_+) defined by*

$$\omega = (i_{\mathbf{n}}\varphi)|_W, \quad \psi_+ = \varphi|_W$$

is a pair of compatible, normalized, stable forms. The inner product h induced by this pair on W satisfies $h = g_\varphi|_W$, and the stabilizer is $\mathrm{SU}(3)$.

The next two lemmas will be useful to prove one part of the main theorem of this section.

Lemma 6.2 *Let (M, g) be a Riemannian manifold and consider two differential forms $\theta \in \Lambda^1(M)$, $\omega \in \Lambda^2(M)$. Then,*

$$|\theta \wedge \omega|^2 = 3|\theta|^2|\omega|^2 - 6|u|^2,$$

where $|\cdot|$ is the pointwise norm induced by g , and $u \in \Lambda^1(M)$ is defined locally as $u = u_i dx^i$, $u_i = g^{rk}\theta_r\omega_{ki}$. From this follows

$$|\theta \wedge \omega|^2 \leq 3|\theta|^2|\omega|^2.$$

Moreover, with respect to the L^2 norm $\|\cdot\|$ induced by the L^2 inner product on k -forms $\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta = \int_M g(\alpha, \beta) * 1$, we have

$$\|\theta \wedge \omega\|^2 \leq 3 \int_M |\theta|^2 |\omega|^2 * 1.$$

Proof Let us recall some definitions in order to clarify the conventions we use. Given a k -covariant tensor η , we define the antisymmetrization of η as

$$\mathrm{Alt}(\eta) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} |\sigma| \eta^\sigma,$$

where $|\sigma|$ is the sign of the permutation σ , and given any k vectors $X_{i_1}, \dots, X_{i_k} \in \mathfrak{X}(M)$, we have $\eta^\sigma(X_{i_1}, \dots, X_{i_k}) = \eta(X_{i_{\sigma(1)}}, \dots, X_{i_{\sigma(k)}})$. Given the differential forms $\alpha \in \Lambda^r(M)$, $\beta \in \Lambda^s(M)$, we define the wedge product $\wedge : \Lambda^r(M) \times \Lambda^s(M) \rightarrow \Lambda^{r+s}(M)$ as

$$\alpha \wedge \beta = \frac{(r+s)!}{r!s!} \mathrm{Alt}(\alpha \otimes \beta).$$

In local coordinates, we then have

$$(\theta \wedge \omega)_{ijk} = \theta_i \omega_{jk} - \theta_j \omega_{ik} + \theta_k \omega_{ij}.$$

We can now start with our computations:

$$\begin{aligned} |\theta \wedge \omega|^2 &= (\theta \wedge \omega)_{ijk} g^{ia} g^{jb} g^{kc} (\theta \wedge \omega)_{abc} \\ &= 3\theta_i \omega_{jk} g^{ia} g^{jb} g^{kc} \theta_a \omega_{bc} - 6\theta_i \omega_{jk} g^{ia} g^{jb} g^{kc} \theta_b \omega_{ac} \\ &= 3(\theta_i g^{ia} \theta_a)(\omega_{jk} g^{jb} g^{kc} \omega_{bc}) - 6(g^{ia} \theta_i \omega_{ac}) g^{ck} (g^{bj} \theta_b \omega_{jk}) \\ &= 3|\theta|^2 |\omega|^2 - 6u_c g^{ck} u_k \\ &= 3|\theta|^2 |\omega|^2 - 6|u|^2. \end{aligned}$$

□

For manifolds endowed with a G_2 -structure, we can prove the following

Lemma 6.3 *Let M be a manifold endowed with a G_2 -structure φ . Consider a vector field $X \in \mathfrak{X}(M)$ and define the 2-form $\omega := i_X \varphi$. Then,*

$$|\omega|^2 = 3|X|^2.$$

Proof Using the identity $\varphi \wedge (i_X \varphi) = 2 * (i_X \varphi)$, which reads $\varphi \wedge \omega = 2 * \omega$ in our case, we have

$$\begin{aligned} |\omega|^2 * 1 &= \omega \wedge * \omega \\ &= \frac{1}{2} \omega \wedge \varphi \wedge \omega \\ &= \frac{1}{2} (i_X \varphi) \wedge (i_X \varphi) \wedge \varphi \\ &= 3|X|^2 * 1. \end{aligned}$$

□

We can now prove the main result of this section.

Theorem 6.4 *Let M be a connected, compact, seven-dimensional manifold endowed with a locally conformal calibrated G_2 -structure φ such that $\mathcal{L}_X \varphi = 0$, where X is the g_φ -dual vector field of the Lee form θ . Then,*

- (1) *M is fibered over S^1 , and each fiber is endowed with a coupled $SU(3)$ -structure given by the restriction of $(\omega = i_X \varphi, \psi_+ = d\omega)$ to the fiber.*
- (2) *M has a locally conformal calibrated G_2 -structure $\hat{\varphi}$ such that $d\hat{\varphi} = -\hat{\theta} \wedge \hat{\varphi}$, where $\hat{\theta}$ is a 1-form with integral periods.*

Proof (1) First of all, observe that since the closed 1-form θ is nowhere vanishing, we can consider the foliation $\mathcal{F}_\theta$ generated by the integrable distribution $\ker(\theta)$. We prove now that the pair $(\omega = i_X \varphi, \psi_+ = d\omega)$ defines a coupled $SU(3)$ -structure when restricted to each leaf of this foliation; to do this, we consider the tangent space to each point of the leaves of the foliation and apply Proposition 6.1.

Under our hypothesis, we have a positive 3-form φ such that $d\varphi = -\theta \wedge \varphi$, $X = \theta^\#$ and $\theta(X)\varphi = d\omega + \theta \wedge \omega$, where $\omega = i_X \varphi$ and $\theta(X)$ is a nonzero constant (see Proposition 4.4). Let L be a leaf of the foliation $\mathcal{F}_\theta$; then, for any $p \in L$, we have

$$T_p L = \{Y_p \in T_p M \mid \theta_p(Y_p) = 0\} \subset T_p M,$$

and since $\theta(\cdot) = g_\varphi(X, \cdot)$, we also have

$$\ker(\theta) = \langle X \rangle^\perp.$$

Thus, $T_p L = \langle X_p \rangle^\perp$ is a six-dimensional subspace of $T_p M$. Moreover, we can suppose that X is normalized since it is nowhere zero, then $\theta(X) = |X|^2 = 1$, and by Proposition 6.1, we have that the pair $((i_{X_p} \varphi)|_{T_p L}, \varphi|_{T_p L})$ defines an $SU(3)$ -structure on $T_p L$. Now,

$$(i_{X_p} \varphi)|_{T_p L} = \omega_p|_{T_p L},$$

and for any choice of tangent vectors $Q_p, Y_p, Z_p \in T_p L$, we have

$$\begin{aligned} \varphi_p(Q_p, Y_p, Z_p) &= (d\omega_p + \theta_p \wedge \omega_p)(Q_p, Y_p, Z_p) \\ &= (d\omega_p)(Q_p, Y_p, Z_p), \end{aligned}$$

since θ_p evaluated on any vector of $T_p L$ is zero. We then obtain that

$$\varphi|_{T_p L} = d\omega|_{T_p L}.$$

Summarizing, the pair $(\omega, d\omega)$ defines a coupled $SU(3)$ -structure when restricted to each leaf L of the foliation.

Let us now observe that M is fibered over S^1 , since it is compact, and there is a globally defined closed 1-form θ which is nowhere vanishing. Using the same argument as in [11], we can approximate the 1-form θ by $q\hat{\theta}$ for some integer q and some 1-form $\hat{\theta}$ with integral periods. Indeed, using the fact there is a bijection between $H^1(M, \mathbb{Z})$ and the set of homotopy classes of maps from M into the circle, Tischler showed that there exists a smooth map $\pi : M \rightarrow S^1$ such that $\hat{\theta} = \pi^*(dt)$, where dt is the length form on S^1 . The form $q\hat{\theta}$ (and therefore $\hat{\theta}$) has no zeroes since it is close to θ . Thus, $\pi^*(dt)$ has no zeros, and then π is a submersion. Since M is compact, $\pi : M \rightarrow S^1$ is a fibration. The fibers are defined by the equation $\hat{\theta} = 0 = q\hat{\theta}$, which is close to the equation $\theta = 0$. Thus, the tangent spaces to the fibers are close to the tangent spaces to the leaves.

We can now show that the restriction of ω and $d\omega$ to the fibers of π defines a coupled $SU(3)$ -structure. Let p be a point in a fiber F of π , and let L be the leaf of the foliation $\mathcal{F}_\theta$ such that $p \in L$. Using normal coordinates in M with respect to the Riemannian metric g_φ , we get a diffeomorphism Φ from a neighborhood D_F of p in the fiber F to a neighborhood D_L of p in the leaf L such that $\Phi(p) = p$ and Φ_{*p} is arbitrarily close to the identity. Since $(\omega, d\omega)$ defines a coupled $SU(3)$ -structure when restricted to the leaf L , there exists an orthonormal basis $\{e^1, \dots, e^6\}$ of T_p^*L such that

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245}.$$

Considering the basis $\{f^1 = \Phi^*e^1, \dots, f^6 = \Phi^*e^6\}$ of T_p^*F , we have that $(\Phi^*(\omega), \Phi^*(d\omega))$ defines a coupled $SU(3)$ -structure on T_pF .

(2) Note that from Lemma 6.3 and using the fact that X is normalized, we obtain

$$|\omega|^2 = 3|X|^2 = 3.$$

Define the 3-form $\hat{\varphi} := d\omega + q\hat{\theta} \wedge \omega$, it is a positive 3-form. Indeed, using Lemma 6.2 and the previous observation, we have

$$\begin{aligned} |\hat{\varphi} - \varphi|^2 &= |(q\hat{\theta} - \theta) \wedge \omega|^2 \\ &\leq 3|q\hat{\theta} - \theta|^2|\omega|^2 \\ &= 9|q\hat{\theta} - \theta|^2. \end{aligned}$$

Then,

$$\|\hat{\varphi} - \varphi\|^2 = \int_M |\hat{\varphi} - \varphi|^2 * 1 \leq 9 \int_M |q\hat{\theta} - \theta|^2 * 1 = 9 \|q\hat{\theta} - \theta\|^2.$$

Observe that $\|\hat{\varphi} - \varphi\|$ is arbitrary small since $\|q\hat{\theta} - \theta\|$ is arbitrarily small; therefore, the 3-form $\hat{\varphi}$ is positive since it lies in an arbitrary small neighborhood of the positive 3-form φ , and being a positive 3-form is an open condition. Therefore, since $d\hat{\varphi} = -q\hat{\theta} \wedge \hat{\varphi}$, the 3-form $\hat{\varphi}$ defines a new locally conformal calibrated G_2 -structure with associated Lee form $q\hat{\theta}$, which is a 1-form with integral periods. $\square$

Remark 6.5 By the previous theorem, we have that $M = N_\nu$ is the mapping torus of a six-dimensional manifold, but the diffeomorphism ν in general does not preserve the coupled $SU(3)$ -structure on the fiber.

Note that the compact locally conformal calibrated G_2 -manifold $M = \Gamma \backslash G$ obtained in Example 5.2 satisfies the assumptions of previous theorem. Indeed, it admits a locally conformal calibrated G_2 -structure φ such that $\mathcal{L}_X \varphi = 0$, where X is the g_φ -dual vector field of the Lee form θ . Therefore, M is fibered over S^1 , and each fiber is endowed with a coupled $SU(3)$ -structure given by the restriction of $(-\omega, d(-\omega))$ to the fiber.

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