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Continuity of solutions to singular parabolic equations with coefficients from Kato-type classes

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Abstract We prove local boundedness and continuity of solutions to divergence type quasi-linear singular parabolic equations with measurable coefficients and lower order terms from nonlinear Kato classes.

Keywords Quasi-linear singular parabolic equations · Local boundedness · Continuity

Mathematics Subject Classification 35K65 · 35B65 · 35B45

1 Introduction and main results

In this paper we are concerned with divergence type quasi-linear singular parabolic equation with measurable coefficients and lower order terms. This class of equations has numerous applications and has been attracting attention for several decades (see, e.g. the monographs [7,16,28], survey [8] and reference therein).

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and for any $T > 0$ let Ω_T denote the cylindrical domain $\Omega \times (0, T)$. We consider quasi-linear parabolic differential equation of the form

$$u_t - \operatorname{div} \mathbb{A}(x, t, u, \nabla u) = b(x, t, u, \nabla u), \quad (x, t) \in \Omega_T. \tag{1.1}$$

Throughout the paper we suppose that the functions $\mathbb{A} : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $\mathbb{A}(\cdot, \cdot, u, \xi)$, $b(\cdot, \cdot, u, \xi)$ are Lebesgue measurable for all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, and $\mathbb{A}(x, t, \cdot, \cdot)$, $b(x, t, \cdot, \cdot)$ are continuous for almost all $(x, t) \in \Omega_T$. We also assume that the following structure conditions are satisfied:

Dedicated to the memory of Vitali Liskevich.

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$$\mathbb{A}(x,t,u,\xi)\,\xi \geq \mu_1|\xi|^p - g_1(x)|u|^p - f_1(x),$$
$$|\mathbb{A}(x,t,u,\xi)| \leq \mu_2|\xi|^{p-1} + g_2(x)|u|^{p-1} + f_2(x),$$
$$|b(x,t,u,\xi)| \leq h(x)|\xi|^{p-1} + g_3(x)|u|^{p-1} + f_3(x), \tag{1.2}$$

where $\frac{2n}{n+1} < p < 2$, μ_1, μ_2 are positive constants and $h(x), g_i(x), f_i(x),\ i = 1,2,3$ are nonnegative functions, satisfying conditions which will be specified bellow.

The aim of this paper is to establish basic qualitative properties such as local boundedness of weak solutions and their continuity under minimal possible restrictions on the coefficients in structure conditions (1.2). These properties are indispensable in the qualitative theory of second-order elliptic and parabolic equations. For Eq. (1.1) with $g_i(x), f_i(x),\ i = 1,2,3$ constants the local boundedness and Hölder continuity of solutions was known since mid-1980s (see [7] for the results, references and historical notes), and a resent breakthrough has been made in [9,10], where the Harnack inequality has been proved. Before stating precisely our results we make several remarks related to lower order terms of (1.1) and refer the reader for an extensive survey of the regularity issues to [7–10].

Local boundedness and Hölder continuity of weak solutions to homogeneous linear divergence type second-order elliptic and parabolic equations with measurable coefficients without lower order terms is known since the famous results by De Giorgi [6] and Nash [20], and the Harnack inequality since Moser's celebrated paper [18]. However, in presence of lower order term in the equation weak solutions may have singularities and/or internal zeroes, and the Harnack inequality in general may not be valid, as one can easily realize looking at the equation $-\Delta u + \frac{c}{|x|^2}u = 0$. It was Serrin [21] who generalized Moser's result to the case of quasi-linear equations with lower order terms with conditions expressed in terms of L^q-spaces. Using probabilistic techniques Aizenman and Simon in their famous paper [1] proved the Harnack inequality and continuity of weak solutions to the equation $-\Delta u + Vu = 0$ under the local Kato class condition on the potential V. Moreover, they showed that the Kato-type condition on the potential V is necessary for the validity of the Harnack inequality. Soon after that Chiarenza et al. [5] developed a real variables technique to prove the Harnack inequality for a linear equation of divergence type with measurable coefficients and the potential from the Kato class, thus extending Aizenman, Simon's result. Kurata [14] extended the method of Chiarenza, Fabes and Garofalo and proved the same to the equation $\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + Vu = 0$, with $|b|^2, V$ from the Kato class. Both papers [5] and [14] make a heavy use of Green's functions which makes this approach inapplicable to quasi-linear equations. To treat the quasi-linear case of p-Laplacian with a lower order term Biroli [2] introduced the notion of the nonlinear Kato class and gave the Harnack inequality for positive solutions to $-\Delta_p u + Vu^{p-1} = 0$. This was extended in [25] to the general case of quasi-linear elliptic equations with lower order terms.

For second-order linear parabolic equations with measurable coefficients (without lower order terms) Hölder continuity of solutions was first proved by Nash [20]. Moser [19] proved the validity of the Harnack inequality which was extended to the case of quasi-linear equations with $p = 2$ in the structure conditions and structure coefficients from L^q-classes in [26]. The continuity of weak solutions and the Harnack inequality for second-order linear elliptic equations with lower order coefficients from Kato classes was proved by Zhang [29,30].

The parabolic theory for degenerate quasi-linear equations differs substantially from the "linear" case $p = 2$ which can be already realized looking at the Barenblatt solution to the parabolic p-Laplace equation. Di Benedetto developed an innovative intrinsic scaling method (see [7] and the references to the original papers there; see also a nice exposition in [27] where some recent advances are included) and proved the Hölder continuity of weak solutions to (1.1) for $p \neq 2$ for the case f_1, f_2, f_3, h from L^q-classes, and the Harnack inequality for the parabolic p-Laplace equations. For the case of measurable coefficients in the main part of (1.1) the Harnack inequality was proved in the recent breakthrough paper [9]. The Harnack inequality and continuity of solutions to the porous medium equations and to the degenerate ($p > 2$) parabolic equations with singular lower order terms was proved in [3,4,17]. It is natural to conjecture that the Harnack inequality holds for the singular ($p < 2$) parabolic p-Laplace equation perturbed by lower order terms with coefficients form Kato classes. The difficulty is that seemingly neither De Giorgy nor Moser iteration techniques work in this situation.

In this paper following the strategy of [10] but using a different iteration, namely the Kilpeläinen–Malý technique [13] properly adapted to the parabolic equations [17,23,24], we establish the local boundedness and continuity for solutions of (1.1).

In what follows we use the notion of the Wolf potential of a function $g(x)$, which is defined by

$$W^g_{\alpha,\beta}(x;\rho) := \int_0^\rho \left\{ \frac{1}{r^{n-\alpha\beta}} \int_{B_r(x)} |g(z)|\,\mathrm{d}z \right\}^{\frac{1}{\alpha-1}} \frac{\mathrm{d}r}{r}, \quad \alpha > 1, \quad n > \alpha\beta. \tag{1.3}$$

The corresponding nonlinear Kato-type classes $K_{\alpha,\beta}$ are defined by

$$K_{\alpha,\beta} := \left\{ g \in L^1(\Omega) : \lim_{\rho\to 0} \sup_{x\in\Omega} W^g_{\alpha,\beta}(x;\rho) = 0 \right\}. \tag{1.4}$$

As one can easily see, for $p = 2$, the nonlinear Kato class $K_p := K_{p,1}$ reduces to the standard definition of the Kato class with respect to the Laplacian [1,22].

For the functions in the right-hand sides of (1.2) we assume that

$$g_1, f_1, g_2^{\frac{p}{p-1}}, \quad f_2^{\frac{p}{p-1}} \in K_{p+1,\frac{p}{p+1}}, \quad h^p, g_3, f_3 \in K_p. \tag{1.5}$$

Before formulating the main results let us remind the reader of the definition of a weak solution to Eq. (1.1). We say that the function $u \in V_{\mathrm{loc}}(\Omega_T) := C_{\mathrm{loc}}(0,T; L^2_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a local weak solution of equation (1.1) if for every sub-interval $[t_1,t_2] \subset (0,T]$ the following integral identity is valid

$$\int_\Omega u\varphi\,\mathrm{d}x \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \{-u\,\varphi_t + \mathbb{A}(x,t,u,\nabla u)\nabla\varphi - b(x,t,u,\nabla u)\varphi\}\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{1.6}$$

for any function $\varphi \in W^{1,2}_{\mathrm{loc}}(0,T : L^2(\Omega)) \cap L^p_{\mathrm{loc}}(0,T; \overset{\circ}{W}{}^{1,p}_{\mathrm{loc}}(\Omega))$.

Further on we assume without loss of generality that $\frac{\partial u}{\partial t} \in L^2(\Omega_T)$ since otherwise we can pass to the Steklov averages (see, e.g. [7]).

Remark 1 The parameters $\{n, p, \mu_1, \mu_2\}$ are the data and we say that generic constant $\gamma = \gamma(n,p,\mu_1,\mu_2)$ depends upon the data if it can be quantitatively determined a priori only in terms of the indicated parameters.

In what follows we use the following quantities

$$W_f(\rho) := \sup_{x\in\Omega} W^{f_1+f_2^{\frac{p}{p-1}}}_{p+1,\frac{p}{p+1}}(x;\rho) + \sup_{x\in\Omega} W^{f_3}_{p,1}(x;\rho),$$

$$W_g(\rho) := \sup_{x\in\Omega} W^{g_1+g_2^{\frac{p}{p-1}}}_{p+1,\frac{p}{p+1}}(x;\rho) + \sup_{x\in\Omega} W^{g_3}_{p,1}(x;\rho),$$

$$W_h(\rho) := \sup_{x\in\Omega} W^{h^p}_{p,1}(x;\rho).$$

The first main result of this paper is the local boundedness of solutions.

Let $x \in \Omega$, $0 < s < T$, for any $\rho, \tau > 0$ we define $B_\rho(x) = \{y : |x - y| < \rho\}$, $Q_{\rho,\tau}(x,s) := Q^-_{\rho,\tau}(x,s) \cup Q^+_{\rho,\tau}(x,s)$, where $Q^-_{\rho,\tau}(x,s) := B_\rho(x) \times (s - \tau, s)$, $Q^+_{\rho,\tau}(x,s) := B_\rho(x) \times (s, s+\tau)$.

Theorem 1.1 *Let the conditions (1.2), (1.5) be fulfilled and u be a local weak solution to Eq. (1.1). Then there exists* $\nu_1 \in (0,1)$ *depending only on the data such that the inequality*

$$W_h(32\rho) + W_g(32\rho) \le \nu_1 \tag{1.7}$$

implies that either

$$\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}} \le W_f(32\rho), \tag{1.8}$$

or

$$\operatorname*{ess\,sup}_{Q^-_{\frac{\rho}{2},t-s}(y,t)} |u| \le \gamma(t-s)^{-\frac{n}{\varkappa}} \left(\operatorname*{ess\,sup}_{2s-t<\tau<t} \int_{B_\rho(y)} |u(x,\tau)|\,dx\right)^{\frac{p}{\varkappa}} + \gamma\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}},$$
$$\varkappa = p + n(p-2) > 0, \tag{1.9}$$

for all cylinders $Q^-_{\rho,2(t-s)}(y,t) \subset \Omega_T$.

Having established the local boundedness we proceed with the continuity.

Theorem 1.2 *Let conditions (1.2), (1.5) be fulfilled and u be a bounded local weak solution to Eq. (1.1). Then u is continuous, that is* $u \in C(\Omega_T)$.

For a fixed cylinder $Q^-_{2\rho,(2\rho)^p\theta}(y,s) \subset \Omega_T$ denote by $\mu_\pm$ and ω, nonnegative numbers such that

$$\mu_+ \ge \operatorname*{ess\,sup}_{Q^-_{2\rho,(2\rho)^p\theta}(y,s)} u, \quad \mu_- \le \operatorname*{ess\,inf}_{Q^-_{2\rho,(2\rho)^p\theta}(y,s)} u, \quad \omega \ge \mu_+ - \mu_-.$$

The next is a De Giorgi-type lemma (cf. [10]), and its formulation is almost the same as in [10]. However, due to the different structure conditions the De Giorgi-type iteration cannot be used, so we adapt the Kilpeläinen–Malý iteration [13], combined with ideas from [17,23,24], where the Kilpeläinen–Malý technique was adapted to parabolic equations.

Theorem 1.3 *Let the conditions (1.2), (1.5) be fulfilled and u be a bounded local weak solution to Eq. (1.1). Fix* $\xi, a \in (0,1)$*, there exist numbers* $\nu_1 \in (0,1)$ *depending only on the data and* $B \ge 1$, $\nu \in (0,1)$ *depending on* θ, ξ, ω, a, $\operatorname{ess\,sup}_{\Omega_T} |u|$ *and the data such that if*

$$W_h(32\rho) \le \nu_1 \tag{1.10}$$

and

$$|\{(x,t) \in Q^-_{2\rho,(2\rho)^p\theta}(y,s) : u(x,t) \le \mu_- + \xi\omega\}| \le \nu |Q^-_{2\rho,(2\rho)^p\theta}(y,s)|, \tag{1.11}$$

then either

$$\xi\omega \le B(W_f(32\rho) + W_g(32\rho)), \tag{1.12}$$

or

$$u(x,t) \ge \mu_- + a\xi\omega \quad \text{for almost all (a.a.)} \quad (x,t) \in Q^-_{\rho,\rho^p\theta}(y,s). \tag{1.13}$$

Likewise if (1.10) *holds and*

$$|\{(x,t) \in Q^-_{2\rho,(2\rho)^p\theta}(y,s) : u(x,t) \le \mu_+ - \xi\omega\}| \ge \nu |Q^-_{2\rho,(2\rho)^p\theta}(y,s)|, \tag{1.14}$$

then either (1.12) *holds true, or*

$$u(x,t) \le \mu_+ - a\xi\omega \quad \text{for a.a.} \quad (x,t) \in Q^-_{\rho,\rho^p\theta}(y,s). \tag{1.15}$$

Moreover, if $g_1 = g_2 = g_3 = 0$*, the constants* ν, B *can be chosen independent from* $\operatorname{ess\,sup}_{\Omega_T} |u|$.

Next is a De Giorgi-type lemma involving "initial data".

Theorem 1.4 *Let the conditions* (1.2), (1.5) *be fulfilled and* u *be a bounded local weak solution to Eq.* (1.1)*. Fix* $\xi, a \in (0,1)$*, there exist numbers* $\nu_1 \in (0,1)$ *depending only on the data and* $B \ge 1$, $\nu \in (0,1)$ *depending on* $\theta, \xi, \omega, a, \operatorname{ess\,sup}_{\Omega_T} |u|$ *and the data such that if* (1.10) *holds true and if*

$$u(x,s) \ge \mu_- + \xi\omega \quad \text{for a.a.} \quad x \in B_{2\rho}(y), \tag{1.16}$$

and

$$|\{(x,t) \in Q^+_{2\rho,(2\rho)^p\theta}(y,s) : u(x,t) \le \mu_+ - \xi\omega\}| \le \nu |Q^+_{2\rho,(2\rho)^p\theta}(y,s)|, \tag{1.17}$$

then either (1.12) *holds true, or*

$$u(x,t) \ge \mu_- + a\xi\omega \quad \text{for a.a.} \quad (x,t) \in Q^+_{\rho,(2\rho)^p\theta}(y,s). \tag{1.18}$$

Likewise if (1.10) *holds and if*

$$u(x,s) \le \mu_+ - \xi\omega \quad \text{for a.a.} \quad x \in B_{2\rho}(y), \tag{1.19}$$

and

$$|\{(x,t) \in Q^+_{2\rho,(2\rho)^p\theta}(y,s) : u(x,t) \ge \mu_+ - \xi\omega\}| \le \nu |Q^+_{2\rho,(2\rho)^p\theta}(y,s)|, \tag{1.20}$$

then either (1.12) *holds true, or*

$$u(x,t) \le \mu_+ - a\xi\omega \quad \text{for a.a.}\ (x,t) \in Q^+_{\rho,(2\rho)^p\theta}(y,s). \tag{1.21}$$

If $g_1 = g_2 = g_3 = 0$*, the constants* ν, B *can be chosen independent from* $\operatorname{ess\,sup}_{\Omega_T} |u|$.

The following theorem is an expansion of positivity result, analogous in formulation as well as in the proof [10].

Theorem 1.5 *Let the conditions* (1.2), (1.5) *be fulfilled and* u *be a bounded local weak solution to Eq.* (1.1)*. Assume that for some* $(y,s) \in \Omega_T$ *and some* $\rho > 0$

$$|\{x \in B_\rho(y) : u(x,s) \le \mu_- + N\}| \le (1-\alpha)|B_\rho(y)| \tag{1.22}$$

for some $N > 0$ *and some* $\alpha \in (0,1)$*. Then there exist positive constants* $\nu_1 \in (0,1)$ *depending only on the data and* $B \ge 1$, $\sigma, \varepsilon, b \in (0,1)$ *depending on the data and* α*, such that if* (1.10) *holds true, then either*

$$N \le B(W_f(32\rho) + W_g(32\rho)), \tag{1.23}$$

or

$$u(x,t) \ge \mu_- + \sigma N \quad \text{for a.a.} \quad x \in B_{2\rho}(y), \tag{1.24}$$

for all $s + b(1-\varepsilon)N^{2-p}\rho^p \le t \le s + bN^{2-p}\rho^p$.

If on the other hand

$$|\{x \in B_\rho(y) : u(x,s) \geq \mu_+ - N\}| \leq (1-\alpha)|B_\rho(y)|, \tag{1.25}$$

and if (1.10) *holds, then either* (1.23) *holds true, or*

$$u(x,t) \leq \mu_+ - \sigma N \quad \text{for a.a. } \ x \in B_{2\rho}(y), \tag{1.26}$$

for all $s + b(1-\varepsilon)N^{2-p}\rho^p \leq t \leq s + bN^{2-p}\rho^p$.

The rest of the paper contains the proof of the above theorems.

2 Auxiliary material and integral estimates

2.1 Auxiliary properties and local energy estimates

The following lemmas will be used in the sequel. The first one is the well-known De Giorgi–Poincare lemma (see [15]).

Lemma 2.1 *Let* $u \in W^{1,1}(B_\rho(y))$ *for some* $\rho > 0$ *and* $y \in \mathbb{R}^n$. *Let* k *and* l *be real numbers such that* $k < l$. *Then exists a constant* $\gamma > 0$ *depending only on* n*, such that*

$$(l-k)|A_{k,\rho}||B_\rho(y)\backslash A_{l,\rho}| \leq \gamma \rho^{n+1} \int_{A_{l,\rho}\backslash A_{k,\rho}} |\nabla u|\, \mathrm{d}x, \tag{2.1}$$

where $A_{k,\rho} = \{x \in B_\rho(y) : u(x) < k\}$.

The next lemma is an interpolation lemma.

Lemma 2.2 *Let* $\{y_j\}$, $j = 0, 1, 2, \ldots$ *be a sequence of bounded positive numbers satisfying the recursive inequalities*

$$y_j \leq A a^j y_{j+1}^\sigma,$$

where $A, a > 1$ *and* $\sigma \in (0,1)$ *are given constants. Then there exists a constant* $\gamma > 0$ *depending only on* a, σ *such that*

$$y_0 \leq \gamma A^{\frac{1}{1-\sigma}}.$$

In what follows we will frequently use the following lemma.

Lemma 2.3 *Let* $\varphi \in W_0^{1,p}(B_\rho(y))$, $0 \leq f \in L^1_{\mathrm{loc}}$. *Then there exists* $\gamma > 0$ *such that*

$$\int_{B_\rho(y)} f|\varphi|^p \mathrm{d}x \leq \gamma \sup_{x \in B_{2\rho}(y)} W^f_{p,1}(x; 2\rho)^{p-1} \int_{B_\rho(y)} |\nabla \varphi|^p \mathrm{d}x.$$

Proof Let v be the weak solution to

$$-\Delta_p v = f \quad \text{in } B_\rho(y), \ v \in \overset{\circ}{W}{}^{1,p}(B_\rho(y)) = 0.$$

Then by [13] $\sup_{x \in B_\rho(y)} v(x) \leq \gamma \sup_{x \in B_{2\rho(y)}} W^f_{p,1}(x, 2\rho)$. Multiplying the equation $-\Delta_p v = f$ by $\frac{|\varphi|^p}{(v+\varepsilon)^{p-1}}$, $\varepsilon \to 0$, integrating by parts and letting $\varepsilon \to 0$ we obtain

$$\int_{B_\rho(y)} \frac{|\nabla v|^p |\varphi|^p}{v^p} \mathrm{d}x + \int_{B_\rho(y)} \frac{f|\varphi|^p}{v^{p-1}} \mathrm{d}x \leq \gamma \int_{B_\rho(y)} \frac{|\nabla v|^{p-1}|\varphi|^{p-1}}{v^{p-1}} |\nabla \varphi|\, \mathrm{d}x.$$

Using the Young's inequality we get

$$\int_{B_\rho(y)} \frac{f|\varphi|^p}{v^{p-1}}\mathrm{d}x \le \gamma \int_{B_\rho(y)} |\nabla\varphi|^p \mathrm{d}x.$$

Hence the requited inequality follows. □

Lemma 2.4 *Let u be a solution to Eq. (1.1) in Ω_T. Then there exists $\gamma > 0$ depending only on the data, such that for every cylinder $Q^+_{\rho,\rho^p\theta}(y,s) \subset \Omega_T$, and any $k \in \mathbb{R}$ and any smooth $\xi(x,t)$ which is equal to zero for $(x,t) \in \partial B_\rho(y) \times (s, s+\rho^p\theta)$ one has*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s<t<s+\rho^p\theta} \int_{B_\rho(y)} (u-k)^2_\pm \xi^p(x,t)\,\mathrm{d}x + \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} |\nabla(u-k)_\pm|^p \xi^p \mathrm{d}x\,\mathrm{d}\tau \\
&\le \int_{B_\rho(y)} (u-k)^2_\pm \xi^p(x,s)\,\mathrm{d}x + \gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} \left((u-k)^2_\pm|\xi_\tau| + (u-k)^p_\pm|\nabla\xi|^p\right)\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} \left(|u|^p\left(g_1 + g_2^{\frac{p}{p-1}}\right)\chi((u-k)_\pm>0) + (u-k)_\pm|u|^{p-1}g_3\right)\xi^p\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} (u-k)^p_\pm h^p \xi^p \mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} \left(\left(f_1 + f_2^{\frac{p}{p-1}}\right)\chi((u-k)_\pm > 0) + (u-k)_\pm f_3\right)\xi^p \mathrm{d}x\,\mathrm{d}\tau. \qquad (2.2)
\end{aligned}$$

Proof Test (1.6) by $\varphi = (u-k)_\pm\xi^p$ and use conditions (1.2), the Hölder and Young inequalities. □

2.2 Integral estimates of solutions

Fix a positive number $a \ge 1$ depending only on the data, which will be specified later. Set $v = u_+ + aW_f(32\rho)$ and

$$G(v) = \begin{cases} v & \text{for} \quad v \ge 1 \\ v^2 & \text{for} \quad 0 < v \le 1. \end{cases}$$

Lemma 2.5 *Let the conditions of Theorem 1.1 be fulfilled. Then there exists a constant $\gamma > 0$ depending only on the data, such that for any $l, \delta > 0$, $0 < \lambda < \min(1, \frac{2-p}{p-1})$, $k > p$ and any cylinder $Q^-_{r,\theta}(\bar{x},\bar{t}) \subset Q^-_{\rho,2(t-s)}(y,t)$ and any smooth $\xi(x,t) = \xi^{(1)}(x)\xi^{(2)}(t)$, where $\xi^{(1)}(x) \in C_0^\infty(B_r(\bar{x}))$ and $\xi^{(2)}(t)$ is equal to zero for $t \le \bar{t} - \theta$ one has*

$$\begin{aligned}
&\sup_{\bar{t}-\theta<t<\bar{t}} \int_{L(t)} vG\Big(\frac{v-l}{\delta}\Big)\xi^k \mathrm{d}x + \delta^{-2}\iint_L \bigg(\int_l^v \Big(1+\frac{s-l}{\delta}\Big)^{-1-\lambda}\mathrm{d}s \\
&\quad + v\Big(1+\frac{v-l}{\delta}\Big)^{-1-\lambda}\bigg)|\nabla v|^p\xi^k \mathrm{d}x\,\mathrm{d}t \\
&\le \gamma \iint_L v\frac{v-l}{\delta}|\xi_t|\xi^{k-p}\mathrm{d}x\,\mathrm{d}t + \gamma\delta^{p-2}\iint_L v\frac{v-l}{\delta}|\nabla\xi|^p\xi^{k-p}\mathrm{d}x\,\mathrm{d}t \\
&\quad + \gamma\delta^{-2}\iint_L v^{p+1}\Big(1+\frac{v-l}{\delta}\Big)^{-1-\lambda} h_1\xi^k \mathrm{d}x\,\mathrm{d}t \\
&\quad + \gamma\delta^{-2}\iint_L v^p \int_l^v \Big(1+\frac{s-l}{\delta}\Big)^{-1-\lambda}\mathrm{d}s\, h_2\,\xi^k \mathrm{d}x\,\mathrm{d}t, \qquad (2.3)
\end{aligned}$$

where $L = Q^-_{r,\theta}(\bar{x},\bar{t}) \cap \{v > l\}$, $L(t) = L \cap \{\tau = t\}$, $h_1(x) = g_1(x) + g_2^{\frac{p}{p-1}}(x) + a^{-p} W_f^{-p}(32\rho)(f_1(x) + f_2^{\frac{p}{p-1}}(x))$, $h_2(x) = g_1(x) + g_3(x) + h^p(x) + a^{-p} W_f^{-p}(32\rho) f_1(x) + a^{1-p} W_f^{1-p}(32\rho) f_3(x)$.

Proof First note that

$$\left(\int_l^v \left(1+\frac{s-l}{\delta}\right)^{-1-\lambda} \mathrm{d}s\right)_+ \asymp \frac{v-l}{1+\frac{v-l}{\delta}}, \tag{2.4}$$

and

$$\int_l^v w\,\mathrm{d}w \int_l^w \left(1+\frac{s-l}{\delta}\right)^{-1-\lambda} \mathrm{d}s \geq \gamma v(v-l) \int_l^{\frac{v+l}{2}} \left(1+\frac{s-l}{\delta}\right)^{-1-\lambda} \mathrm{d}s$$
$$= \gamma \delta v(v-l) \int_0^{\frac{v-l}{2\delta}} (1+s)^{-1-\lambda} \mathrm{d}s \geq \gamma \delta^2 v G\left(\frac{v-l}{\delta}\right). \tag{2.5}$$

Test (1.6) by $\varphi = v\Phi(v)\xi^k$, $\Phi(v) = (\int_l^v (1+\frac{s-l}{\delta})^{-1-\lambda}\mathrm{d}s)_+$, using (1.2) and the Young inequality we have for any $t \in (\bar{t}-\theta, \bar{t})$

$$\int_{L(t)} \int_l^v w\Phi(w)\,\mathrm{d}w \xi^k \mathrm{d}x + \iint_L \left(\Phi(v) + v\left(1+\frac{v-l}{\delta}\right)^{-1-\lambda}\right) |\nabla v|^p \xi^k \mathrm{d}x\,\mathrm{d}t$$
$$\leq \gamma \iint_L \int_l^v w\Phi(w)\,\mathrm{d}w |\xi_t| \xi^{k-1} \mathrm{d}x\,\mathrm{d}t$$
$$+\gamma \iint_L \Phi^p(v)\left(1+\frac{v-l}{\delta}\right)^{(1+\lambda)(p-1)} |\nabla\xi|^p \xi^{k-p} \mathrm{d}x\,\mathrm{d}t$$
$$+\gamma \iint_L v\left(1+\frac{v-l}{\delta}\right)^{-1-\lambda} \left(v^p(g_1+g_2^{\frac{p}{p-1}}) + f_1 + f_2^{\frac{p}{p-1}}\right) \xi^k \mathrm{d}x\,\mathrm{d}t$$
$$+\gamma \iint_L v^p \Phi(v)(g_1+g_3+h^p)\xi^k \mathrm{d}x\,\mathrm{d}t + \gamma \iint_L \Phi(v)\,(f_1+vf_3)\,\xi^k \mathrm{d}x\,\mathrm{d}t.$$

From this, using (2.4), (2.5) due to the choice of λ we obtain the required (2.3) □

Set

$$\psi(v) = \frac{1}{\delta}\left(\int_l^v s^{\frac{1}{p}} \left(1+\frac{s-l}{\delta}\right)^{-\frac{1+\lambda}{p}} \mathrm{d}s\right)_+.$$

Lemma 2.6 *Let the conditions of Lemma 2.5 be fulfilled and* $0 < \lambda < 2-p$. *Then there exists* $\nu_1 \in (0,1)$ *depending only on the data such that the inequality*

$$W_h(32\rho) + W_g(32\rho) + a^{-1} \leq \nu_1 \tag{2.6}$$

implies that

$$\sup_{\bar{t}-\theta<t<\bar{t}} \int_{L(t)} vG\left(\frac{v-l}{\delta}\right)\xi^k \mathrm{d}x + \delta^{p-2} \iint_L |\nabla\psi|^p \xi^k \mathrm{d}x\,\mathrm{d}t \leq \gamma \iint_L v\frac{v-l}{\delta} |\xi_t| \xi^{k-p} \mathrm{d}x\,\mathrm{d}t$$
$$+\gamma\delta^{p-2} \iint_L v\frac{v-l}{\delta} |\nabla\xi|^p \xi^{k-p} \mathrm{d}x\,\mathrm{d}t + \gamma l^{p+1}\delta^{-2}\theta \int_{B_r(\bar{x})} h_1 \mathrm{d}x + \gamma l^p \delta^{-1}\theta \int_{B_r(\bar{x})} h_2 \mathrm{d}x. \tag{2.7}$$

Proof Recall that $\Phi(v) \asymp \frac{v-l}{1+\frac{v-l}{\delta}}$ and apply (2.6) and Lemma 2.3. Then since

$$W^{h_1}_{p,1}(x;2\rho) \le \gamma W^{g_1+g_2^{\frac{p}{p-1}}}_{p+1,\frac{p}{p+1}}(x;2\rho) + \gamma a^{-\frac{p}{p-1}},$$
$$W^{h_2}_{p,1}(x;2\rho) \le \gamma W^{g_1}_{p+1,\frac{p}{p+1}}(x;2\rho) + \gamma W^{g_3+h^p}_{p,1}(x;2\rho) + \gamma (a^{-\frac{p}{p-1}} + a^{-1}),$$

we have

$$\begin{aligned}
&\gamma \iint_L (v-l)^{p+1}\Big(1+\frac{v-l}{\delta}\Big)^{-1-\lambda} h_1 \xi^k \mathrm{d}x\,\mathrm{d}t\\
&\le \gamma \nu_1^{p-1} \iint_L (v-l)\Big(1+\frac{v-l}{\delta}\Big)^{-1-\lambda} |\nabla v|^p \xi^k \mathrm{d}x\,\mathrm{d}t\\
&\quad + \gamma \nu_1^{p-1} \iint_L (v-l)^{p+1}\Big(1+\frac{v-l}{\delta}\Big)^{-1-\lambda} |\nabla \xi|^p \xi^{k-p} \mathrm{d}x\,\mathrm{d}t\\
&\le \gamma \nu_1^{p-1} \iint_L v\Big(1+\frac{v-l}{\delta}\Big)^{-1-\lambda} |\nabla v|^p \xi^k \mathrm{d}x\,\mathrm{d}t\\
&\quad + \gamma \nu_1^{p-1}\delta^p \iint_L v\frac{v-l}{\delta} |\nabla \xi|^p \xi^{k-p} \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.8}$$

Similarly

$$\begin{aligned}
&\gamma \iint_L (v-l)^p \Phi(v) h_2 \xi^k \mathrm{d}x\,\mathrm{d}t \le \gamma \iint_L \frac{(v-l)^{p+1}}{1+\frac{v-l}{\delta}} h_2 \xi^k \mathrm{d}x\,\mathrm{d}t\\
&\le \gamma \nu_1^{p-1} \iint_L \frac{v-l}{1+\frac{v-l}{\delta}} |\nabla v|^p \xi^k \mathrm{d}x\,\mathrm{d}t + \gamma \nu_1^{p-1} \iint_L \frac{(v-l)^{p+1}}{1+\frac{v-l}{\delta}} |\nabla \xi|^p \xi^{k-p} \mathrm{d}x\,\mathrm{d}t\\
&\le \gamma \nu_1^{p-1} \iint_L \Phi(v) |\nabla v|^p \xi^k \mathrm{d}x\,\mathrm{d}t + \gamma \nu_1 \delta^p \iint_L v\frac{v-l}{\delta} |\nabla \xi|^p \xi^{k-p} \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.9}$$

From this, choosing ν_1 sufficiently small, due to Lemma 2.5 we obtain the required (2.7). □

3 $L^1_{\mathrm{loc}} - L^\infty_{\mathrm{loc}}$ estimate: Proof of Theorem 1.1

In what follows we suppose that

$$\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}} \ge W_f(32\rho). \tag{3.1}$$

For $\sigma \in (0,1)$ and $i = 0, 1, 2, \dots$ set $\rho^{(i)} := \frac{\rho}{2}(2-\sigma^i)$, $t^{(i)} := (t-s)(2-\sigma^{ip})$. Fix a point $(\bar{x}, \bar{t}) \in Q^-_{\rho^{(i)},t^{(i)}}(y,t)$, and let $\rho_0 := \rho_0^{(i)} = \dfrac{\rho\sigma^i(1-\sigma)}{2}$, $t_0 := t_0^{(i)} = (t-s)\sigma^{ip}(1-\sigma^p)$,

$$\delta_0 := \max\left\{\Big(\frac{1}{\eta}\frac{1}{t_0^{1+\frac{n}{p}}} \iint_{Q^-_{\rho_0,t_0}(\bar{x},\bar{t})} v^2 \mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p}{p+\varkappa}}, \Big(\frac{t_0}{\rho_0^p}\Big)^{\frac{1}{2-p}}\right\}, \tag{3.2}$$

where v is defined in Sect. 2, and choose a number $m \ge 0$ so that

$$\delta_0^{2-p}\rho_0^p = 2^{mp} t_0. \tag{3.3}$$

For $\alpha = \frac{\varkappa}{3-p}$, $l > 0$ and $j = 0, 1, 2, \dots$ set $r_j := \frac{\rho_0}{2^{j+m}}$, $t_j := \frac{t_0}{2^{\alpha j}}$, $\delta_j(l) := l - l_j$, $\theta_j(l) := \min(t_j, r_j^p \delta_j^{2-p}(l))$, $B_j := B_{r_j}(\bar{x})$, $Q_j(l) := Q^-_{r_j,\theta_j(l)}(\bar{x}, \bar{t})$, $L_j(l) := Q_j(l) \cap \{u > l_j\}$, $L_j(l, t) := L_j \cap \{\tau = t\}$, $\xi_j(x, t) = \xi_j^{(1)}(x)\xi_j^{(2)}(t)$, where $\xi_j^{(1)}(x) \in C_0^\infty(B_j)$, $\xi_j^{(1)}(x) = 1$ in B_{j+1}, $\xi_j^{(2)}(t) = 1$ for $t \ge \bar{t} - 2^{-\alpha}\theta_j(l)$, $\xi_j^{(2)}(t) = 0$ for $t \le \bar{t} - \theta_j(l)$, $0 \le \xi_j(x, t) \le 1$ and $|\nabla \xi_j| \le \gamma r_j^{-1}$, $|\frac{\partial \xi_j}{\partial t}| \le \gamma \theta_j^{-1}(l)$, and set also

$$A_j(l) := \frac{\delta_j^{(2-p)\frac{n}{p}}(l) l^{-1}}{\theta_j^{1+\frac{n}{p}}(l)} \iint_{L_j(l)} v \frac{v - l_j}{\delta_j(l)} \xi_j^{k-p} \mathrm{d}x\, \mathrm{d}t. \tag{3.4}$$

The sequences of positive numbers $\{l_j\}_{j\in\mathbb{N}}$ and $\{\delta_j\}_{j\in\mathbb{N}}$ are defined inductively as follows. Fix a positive number $\eta \in (0, 1)$ depending only on the data, which will be specified later. Put $l_0 = 0$ and $l_1 = \delta_0$, where δ_0 is defined in (3.2). The following inequality is clear

$$A_0(l_1) \le \eta.$$

Suppose we have chosen $l_1, \dots, l_j$ and $\delta_i = \delta_i(l_{i+1}) = l_{i+1} - l_i$, $i = 0, 1, \dots, j-1$ such that $l_i + \frac{1}{2}\delta_{i-1} \le l_{i+1} \le l_i + 2^{\frac{p-\alpha}{2-p}}\delta_{i-1}$, $i = 1, \dots, j-1$,

$$A_i(l_{i+1}) \le \eta, \quad i = 0, 1, \dots, j-1. \tag{3.5}$$

Let us show how to choose l_{j+1} and δ_j. First we show that

$$A_j(\tilde{l}_j) \le \eta, \quad \tilde{l}_j = l_j + 2^{\frac{p-\alpha}{2-p}}\delta_{j-1}. \tag{3.6}$$

Further we show that $\theta_j(\tilde{l}_j) \le 2^{-\alpha}\theta_{j-1}(l_j)$ and so $Q_j(\tilde{l}_j) \subset Q_{j-1}(l_j)$ and $\{\xi_j \ne 0\} \subset \{\xi_{j-1} = 1\}$.

Indeed, $\delta_j(\tilde{l}_j) = 2^{\frac{p-\alpha}{2-p}}\delta_{j-1}$, thus $r_j^p \delta_j^{2-p}(\tilde{l}_j) = 2^{-\alpha} r_{j-1}^p \delta_{j-1}^{2-p}$, therefore $\theta_j(\tilde{l}_j) \le 2^{-\alpha}\theta_{j-1}(l_j)$ and

$$\begin{aligned} A_j(\tilde{l}_j) &\le 2^{-\frac{p-\alpha}{2-p}\frac{\varkappa}{p}} \delta_{j-1}^{-\frac{\varkappa}{p}} \theta_j^{-1-\frac{n}{p}}(\tilde{l}_j) l_j^{-1} \iint_{L_j(\tilde{l}_j)} v(v - l_j)\xi_{j-1}^{k-p} \mathrm{d}x\, \mathrm{d}t \\ &\le 2^{-\frac{p-\alpha}{2-p}\frac{\varkappa}{p} + \alpha(1+\frac{n}{p})} \delta_{j-1}^{(2-p)\frac{n}{p}} \theta_{j-1}^{-1-\frac{n}{p}}(l_j) l_{j-1}^{-1} \iint_{L_{j-1}(l_j)} v \frac{v - l_{j-1}}{\delta_{j-i}} \xi_{j-1}^{k-p} \mathrm{d}x\, \mathrm{d}t \\ &= A_{j-1}(l_j) \le \eta. \end{aligned}$$

Thus inequality (3.6) is proved. If $A_j(l_j + \frac{1}{2}\delta_{j-1}) \le \eta$ we set $l_{j+1} = l_j + \frac{1}{2}\delta_{j-1}$. Note that $A_j(l)$ is continuous as a function of l. So if $A_j(l_j + \frac{1}{2}\delta_{j-1}) > \eta$, the equation $A_j(l) = \eta$ has roots. Denote l_{j+1} the largest root $A_j(l_{j+1}) = \eta$ and in both cases we set $\delta_j = \delta_j(l_{j+1}) = l_{j+1} - l_j$. Note that our choices guarantee that $\delta_j \le 2^{\frac{p-\alpha}{2-p}j}\delta_0 = (\frac{t_j}{r_j^p})^{\frac{1}{2-p}}$ and

$$A_j(l_{j+1}) \le \eta. \tag{3.7}$$

Further we set $\theta_j = \theta_j(l_{j+1})$, $Q_j = Q_j(l_{j+1})$, $L_j = L_j(l_{j+1})$ and $L_j(t) = L_j(l_{j+1}, t)$. The following lemma is a key in the Kilpëlainen–Malý technique [13].

Lemma 3.1 *For all $j \ge 2$ there exists $\gamma > 0$ depending only on the data, such that*

$$\delta_j \le \frac{1}{2}\delta_{j-1} + \gamma l_j \left(r_j^{p-n} \int_{B_j} h_1 \mathrm{d}x \right)^{\frac{1}{p}} + \gamma l_j \left(r_j^{p-n} \int_{B_j} h_2 \mathrm{d}x \right)^{\frac{1}{p-1}}. \tag{3.8}$$

Proof Fix $j \geq 2$ and without loss assume that

$$\delta_j > \frac{1}{2}\delta_{j-1}, \tag{3.9}$$

since otherwise (3.8) is evident. This inequality guarantees that $A_j(l_{j+1}) = \eta$. Let us estimate the term in the right-hand side of (3.4) with $l = l_{j+1}$. For this we decompose L_j as $L_j = L_l' \cup L_j''$, $L_j' = \{(x,t) \in L_j : \frac{v(x,t)-l_j}{\delta_j} \leq \varepsilon\}$, $L_j'' = L_j \backslash L_j'$, where $\varepsilon > 0$ depending on the data is small enough to be determined later. Observe that our choices guarantee that $\theta_j = \delta_j^{2-p} r_j^p \leq t_j$ and for $(x,t) \in L_j$ one has

$$\frac{u(x,t) - l_{j-1}}{\delta_{j-1}} = 1 + \frac{u(x,t) - l_j}{\delta_{j-1}} \geq 1.$$

Since $\xi_{j-1} = 1$ on Q_j we obtain

$$\begin{aligned} \delta_j^{p-2} l_{j+1}^{-1} r_j^{-n-p} \iint_{L_j'} v \frac{v - l_j}{\delta_j} \xi_j^{k-p} \mathrm{d}x\, \mathrm{d}t &\leq \varepsilon \delta_j^{p-2} l_j^{-1} r_j^{-p-n} \iint_{L_j'} v \xi_{j-1}^{k-p} \mathrm{d}x\, \mathrm{d}t \\ &\leq \varepsilon\gamma \delta_{j-1}^{p-2} l_j^{-1} r_{j-1}^{-p-n} \iint_{L_{j-1}} v \frac{v - l_{j-1}}{\delta_{j-1}} \xi_{j-1}^{k-p} \mathrm{d}x\, \mathrm{d}t \leq \varepsilon\gamma A_{j-1}(l_j) \leq \varepsilon\gamma\eta. \end{aligned} \tag{3.10}$$

Let

$$\psi_j = \frac{1}{\delta_j} \left(\int_{l_j}^{v} s^{\frac{1}{p}} \left(1 + \frac{s - l_j}{\delta_j}\right)^{-\frac{1+\lambda}{p}} \mathrm{d}s \right)_+ .$$

Using the evident inequality $\gamma^{-1}(\varepsilon)\psi_j^p \leq v(\frac{v-l_j}{\delta_j})^{p-1-\lambda} \leq \gamma(\varepsilon)\psi_j^p$ for $(x,t) \in L_j''$, the Sobolev inequality and Lemma 2.6 with $l = l_{j+1},\ \delta = \delta_j,\ \theta = \theta_j$ and $0 < \lambda < \min(2 - p, \frac{\varkappa}{n})$, we obtain

$$\begin{aligned} &\delta_j^{p-2} l_{j+1}^{-1} r_j^{-n-p} \iint_{L_j''} v \frac{v - l_j}{\delta_j} \xi_j^{k-p} \mathrm{d}x\, \mathrm{d}t \leq \gamma(\varepsilon) \delta_j^{p-2} l_j^{-1} r_j^{-n-p} \iint_{L_j''} \left(\frac{v - l_j}{\delta_j}\right)^{2-p+\lambda} \psi_j^p \xi_j^{k-p} \mathrm{d}x\, \mathrm{d}t \\ &\leq \gamma(\varepsilon) \delta_j^{p-2} l_j^{-1} r_j^{-n-p} \left(\sup_{0<t<T} \int_{L_j''(t)} \left(\frac{v - l_j}{\delta_j}\right)^{(2-p+\lambda)\frac{n}{p}} \xi_j^{\frac{(k-p)n}{n+p}} \mathrm{d}x \right)^{\frac{p}{n}} \iint_{L_j} |\nabla(\psi_j \xi_j^{\frac{(k-p)n}{(n+1)p}})|^p \mathrm{d}x\, \mathrm{d}t \\ &\leq \gamma(\varepsilon) \delta_j^{p-2} l_j^{-1} r_j^{-n-p} \left(\sup_{0<t<T} \int_{L_j''(t)} \frac{v}{l_j} \left(\frac{v - l_j}{\delta_j}\right) \xi_j^{\frac{(k-p)n}{n+p}} \mathrm{d}x \right)^{\frac{p}{n}} \\ &\times \iint_{L_j} \left(|\nabla\psi_j|^p + v \frac{v - l_j}{\delta_j} \xi_j^{-p} |\nabla\xi_j|^p \right) \xi_j^{\frac{(k-p)n}{n+p}} \mathrm{d}x\, \mathrm{d}t \\ &\leq \gamma(\varepsilon) \left\{ \frac{\delta_j^{p-2} l_j^{-1}}{r_j^{n+p}} \iint_{L_j} v \frac{v - l_j}{\delta_j} \xi_j^{\frac{(k-p)n}{n+p} - p} \mathrm{d}x\, \mathrm{d}t \right. \\ &\qquad \left. + l_j^p \delta_j^{-p} r_j^{p-n} \int_{B_j} h_1 \mathrm{d}x + l_j^{p-1} \delta_j^{1-p} r_j^{p-n} \int_{B_j} h_2 \mathrm{d}x \right\}^{1+\frac{p}{n}} . \end{aligned} \tag{3.11}$$

Choosing k such that $\frac{(k-p)n}{n+p} - p \geq 1$, and using (3.9), we obtain

$$\frac{\delta_j^{p-2} l_j^{-1}}{r_j^{n+p}} \iint_{L_j} v \frac{v - l_j}{\delta_j} \xi_j \mathrm{d}x\, \mathrm{d}t \leq \gamma A_{j-1}(l_j) \leq \gamma\eta. \tag{3.12}$$

Combining (3.10)–(3.12) we get

$$\eta \leq \varepsilon\gamma\eta + \gamma(\varepsilon)\left\{\eta + l_j^p \delta_j^{-p} r_j^{p-n} \int_{B_j} h_1 \mathrm{d}x + l_j^{p-1}\delta_j^{1-p} r_j^{p-n} \int_{B_j} h_2 \mathrm{d}x\right\}^{1+\frac{p}{n}}. \tag{3.13}$$

Choose ε such that $\varepsilon\gamma = \frac{1}{4}$, and η such that $\gamma(\varepsilon)\eta^{\frac{p}{n}} = \frac{1}{4}$. Hence (3.13) yields (3.8), which completes the proof of the lemma. □

In order to complete the proof of Theorem 1.1 we sum up (3.8) with respect to j from 2 to $J-1$,

$$l_J \leq \gamma(\delta_1 + l_1) + \gamma l_J W^{h_1}_{p+1,\frac{p}{p+1}}(2r_0) + \gamma l_J W^{h_2}_{p,1}(2r_0). \tag{3.14}$$

Inequality (1.7) implies that $\gamma W^{h_1}_{p+1,\frac{p}{p+1}}(2r_0) + \gamma W^{h_2}_{p,1}(2r_0) \leq \frac{1}{2}$, then by (3.14) we obtain

$$l_J \leq \gamma(\delta_1 + l_1) \leq \gamma\delta_0, \tag{3.15}$$

where δ_0 is defined in (3.2).

Hence the sequence $\{l_j\}_{j\in\mathbb{N}}$ is convergent, and $\delta_j \to 0$ $(j\to\infty)$, and we can pass to the limit $J\to\infty$ in (3.15). Let $l = \lim_{j\to\infty} l_j$, from (3.7) we conclude that

$$\frac{1}{|Q_j|}\iint_{Q_j} v(v-l)_+ \mathrm{d}x\,\mathrm{d}t \leq \gamma l \delta_j \to 0 \quad (j\to\infty). \tag{3.16}$$

Choosing $(\bar{x},\bar{t})$ as a Lebesgue point of the function $v(v-l)_+$ we conclude that $v(\bar{x},\bar{t}) \leq l$ and hence $v(\bar{x},\bar{t})$ is estimated from above by δ_0. Applicability of the Lebesgue differentiation theorem follows from [12, Chapter II,Section 3]. Taking essential supremum over $Q^-_{\rho^{(i)},t^{(i)}}(y,t)$ we get for any $i = 0, 1, 2, \ldots$

$$M_i := \operatorname*{ess\,sup}_{Q^-_{\rho^{(i)},t^{(i)}}(y,t)} v \leq \gamma\sigma^{-i\gamma}\left((t-s)^{-1-\frac{n}{p}} \iint_{Q^-_{\rho^{(i+1)},t^{(i+1)}}(y,t)} v^2 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{p+\varkappa}}$$
$$+\gamma\sigma^{-\gamma}\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{3.17}$$

If for some $i_0 \geq 0$

$$\sigma^{-i_0\gamma}\left((t-s)^{-1-\frac{n}{p}} \iint_{Q^-_{\rho^{(i_0+1)},t^{(i_0+1)}}(y,t)} v^2 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{p+\varkappa}} \leq \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}},$$

then

$$M_0 \leq M_{i_0} \leq \gamma\sigma^{-\gamma}\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}}.$$

otherwise inequality (3.17) implies that

$$M_i \leq \gamma\sigma^{-i\gamma} M_{i+1}^{\frac{p}{p+\varkappa}}\left((t-s)^{-\frac{n}{p}} \sup_{2s-t<\tau<t}\int_{B_\rho(y)} v\,\mathrm{d}x\right)^{\frac{p}{p+\varkappa}}.$$

From this, by Lemma 2.2, and taking into account (3.1), we conclude that

$$M_0 \leq \gamma\left((t-s)^{-\frac{n}{p}} \sup_{2s-t<\tau<t}\int_{B_\rho(y)} u_+\,\mathrm{d}x\right)^{\frac{p}{\varkappa}} + \gamma\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}},$$

this completes the proof of Theorem 1.1.

4 A De Giorgi-type lemmas: Proof of Theorems 1.3 and 1.4

In this section we prove a De Giorgi-type lemmas [10]. Here we assume the structure conditions

$$\begin{aligned} &\mathbb{A}(x,t,u,\xi)\,\xi \geq \mu_1|\xi|^p - F_1(x),\\ &|\mathbb{A}(x,t,u,\xi)| \leq \mu_2|\xi|^{p-2} + F_2(x),\\ &|b(x,t,u,\xi)| \leq h(x)|\xi|^{p-1} + F_3(x), \end{aligned} \tag{4.1}$$

where $F_i(x) = f_i(x) + g_i(x)$, $i = 1, 2, 3$, these assumptions follow from (1.2) due to the boundedness of u.

We provide the proof of (1.13), while the proof of (1.15) is completely similar.

Lemma 4.1 *Let u be a weak solution to Eq. (1.1). Set $v = u - \mu_-$, then for any $l, \delta > 0$, $0 < \lambda < \min(1, \frac{2-p}{p-1})$, $k \geq p$ and any cylinder $Q^-_{r,\tau}(\bar{x}, \bar{t}) \subset Q^-_{2\rho,(2\rho)^p\theta}(y, s)$ and any smooth function $\xi(x,t) = \xi^{(1)}(x)\xi^{(2)}(t)$, where $\xi^{(1)}(x) \in C_0^\infty(B_r(\bar{x}))$ and $\xi^{(2)}(t)$ is equal to zero for $t \leq \bar{t} - \tau$ one has*

$$\begin{aligned} &\sup_{\bar{t}-\tau<t<\bar{t}} \int_{L(t)} (v+l)^{1-2p} G\Big(\frac{l-v}{\delta}\Big)\xi^k dx\\ &\quad+\delta^{-2}\iint_L \Big\{\int_v^l \Big(1+\frac{l-s}{\delta}\Big)^{-1-\lambda} ds + (v+l)\Big(1+\frac{l-v}{\delta}\Big)^{-1-\lambda}\Big\}(v+l)^{-2p}|\nabla v|^p\xi^k dx\,dt\\ &\leq \gamma \iint_L (l+v)^{1-2p}\frac{l-v}{\delta}(|\xi_t| + \delta^{p-2}|\nabla\xi|^p)\xi^{k-p} dx\,dt\\ &\quad+\gamma\delta^{-2}\iint_L (v+l)^{-2p}\int_v^l \Big(1+\frac{l-s}{\delta}\Big)^{-1-\lambda} ds (F_1 + (v+l)F_3)\xi^k dx\,dt\\ &\quad+\gamma\delta^{-2}\iint_L (v+l)^{1-2p}(F_1 + F_2^{\frac{p}{p-1}})\xi^k dx\,dt\\ &\quad+\gamma\delta^{-2}\iint_L (l+v)^{-p}\int_v^l \Big(1+\frac{l-s}{\delta}\Big)^{-1-\lambda} ds\, h^p\xi^k dx\,dt. \end{aligned} \tag{4.2}$$

Proof The proof is similar to that of Lemma 2.5 with the choice of the test function $\varphi = (v+l)^{1-2p}(\int_v^l (1+\frac{l-s}{\delta})^{-1-\lambda} ds)_+\xi^k$. □

Set

$$\psi(v) = \frac{1}{\delta}\Big(\int_v^l \Big(1+\frac{l-s}{\delta}\Big)^{-\frac{1+\lambda}{p}} ds\Big)_+.$$

Lemma 4.2 *Let the conditions of Lemma 4.1 be fulfilled and $0 < \lambda < 2 - p$. Then there exists $\nu_1 \in (0, 1)$ depending only on the data such that the inequality*

$$W_h(32\rho) \leq \nu_1 \tag{4.3}$$

implies that

$$\begin{aligned} &\sup_{\bar{t}-\tau<t<\bar{t}} \int_{L(t)} G\Big(\frac{l-v}{\delta}\Big)\xi^k dx + \delta^{p-2}\iint_L |\nabla\psi|^p\xi^k dx\,dt\\ &\leq \gamma \iint_L \frac{l-v}{\delta}(|\xi_t| + \delta^{p-2}|\nabla\xi|^p)\xi^{k-p} dx\,dt \end{aligned}$$

$$+\gamma l^{-1}\delta^{-1}\tau\int_{B_r(\bar{x})}F_1\mathrm{d}x+\gamma\delta^{-1}\tau\int_{B_r(\bar{x})}F_3\mathrm{d}x+\gamma\delta^{-1}\tau\int_{B_r(\bar{x})}\left(F_1+F_2^{\frac{p}{p-1}}\right)\mathrm{d}x$$
$$+\gamma l^{p-1}\delta^{-1}\tau\int_{B_r(\bar{x})}h^p\mathrm{d}x. \tag{4.4}$$

Proof Note that $(\int_v^l(1+\frac{l-s}{\delta})^{-1-\lambda}\mathrm{d}s)_+\asymp\frac{l-v}{1+\frac{l-s}{\delta}}$, and apply Lemma 2.3 and (4.3), we have

$$\iint_L\left(\frac{1}{v+l}-\frac{1}{2l}\right)^p\frac{l-v}{1+\frac{l-v}{\delta}}h^p\xi^k\mathrm{d}x\,\mathrm{d}t\le\gamma\nu_1l^{-p}\iint_L(v+l)^{-p}\frac{l-v}{1+\frac{l-v}{\delta}}|\nabla v|^p\xi^k\mathrm{d}x\,\mathrm{d}t$$
$$+\gamma\nu_1\iint_L(v+l)^{-2p}\frac{l-v}{1+\frac{l-v}{\delta}}|\nabla v|^p\xi^k\mathrm{d}x\,\mathrm{d}t$$
$$+\gamma\nu_1l^{-p}\iint_L\frac{(l-v)^{p+1}}{(v+l)^p(1+\frac{l-v}{\delta})}|\nabla\xi|^p\xi^{k-p}\mathrm{d}x\,\mathrm{d}t$$
$$\le\gamma\nu_1\iint_L(v+l)^{-2p}\int_v^l\left(1+\frac{s-l}{\delta}\right)^{-1-\lambda}\mathrm{d}s|\nabla v|^p\xi^k\mathrm{d}x\,\mathrm{d}t$$
$$+\gamma\nu_1\delta^p\iint_L(v+l)^{1-2p}\frac{v-l}{\delta}|\nabla\xi|^p\xi^{k-p}\mathrm{d}x\,\mathrm{d}t.$$

Therefore, multiplying (4.2) by l^{2p-1} and using the evident inequality $l\le v(x,t)+l\le 2l$ for $(x,t)\in L$, and choosing ν_1 sufficiently small, we get from (4.2) the required (4.4). □

Further on we assume that

$$\xi\omega\ge B\left(W_f(32\rho)+W_g(32\rho)\right). \tag{4.5}$$

Fix a point $(\bar{x},\bar{t})\in Q^-_{\rho,\rho^p\theta}(y,s)$ and for $\alpha=\frac{\varkappa}{3-p}$, $l>0$ and $j=0,1,2,\dots$ set $r_j:=\frac{\rho}{2^j}$, $\delta_j(l):=l_j-l$, $\tau_j(l):=r_j^p\delta_j^{2-p}(l)$, $B_j:=B_{r_j}(\bar{x})$, $Q_j:=Q^-_{r_j,\tau_j(l)}(\bar{x},\bar{t})$, $L_j(l)=Q_j\cap\{v<l_j\}$, $L_j(l,t)=L_j(l)\cap\{\tau=t\}$, $\xi_j(x,t)=\xi_j^{(1)}(x)\xi_j^{(2)}(t)$, where $\xi_j^{(1)}(x)\in C_0^\infty(B_j)$, $\xi_j^{(1)}(x)=1$ in B_{j+1}, $\xi_j^{(2)}(t)=1$ for $t\ge\bar{t}-2^{-\alpha}\tau_j(l)$, $\xi_j^{(2)}(t)=0$ for $t\le\bar{t}-\tau_j(l)$ and $|\nabla\xi_j|\le\gamma r_j^{-1}$, $|\frac{\partial\xi_j}{\partial t}|\le\gamma\tau_j^{-1}(l)$ and set also

$$A_j(l):=\frac{\delta_j^{p-2}(l)}{r_j^{n+p}}\iint_{L_j(l)}\frac{l_j-v}{\delta_j(l)}\xi_j^{k-p}\mathrm{d}x\,\mathrm{d}t. \tag{4.6}$$

Define also the sequences $\{\alpha_j\}$, $\{\beta_j\}$ by

$$\alpha_j:=\int_0^{r_j}\left(r^{p-n}\int_{B_r(\bar{x})}(F_1+F_2^{\frac{p}{p-1}})\,\mathrm{d}x\right)^{\frac{1}{p}}+\int_0^{r_j}\left(r^{p-n}\int_{B_r(\bar{x})}F_3\,\mathrm{d}x\right)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r},$$
$$\beta_j:=c\int_0^{r_j}\left(r^{p-n}\int_{B_r(\bar{x})}h^p\mathrm{d}x\right)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r},\quad j=-1,0,1,\dots$$

where $c>1$ is fixed number, depending only on the known parameters, which will be defined later.

We start wit the choice of the sequences l_j,δ_j. Put $l_0=\xi\omega$, $\delta_0=\min\left(\frac{(1-a)\xi\omega}{4(1+2^{\frac{2-\alpha}{2-p}})},\theta^{\frac{1}{2-p}}\right)$, $l_1=l_0-\delta_0$. From (1.9) it follows that

$$A_0(l_1)\le\gamma\xi\omega\theta^{-1-\frac{1}{2-p}}\rho^{-n-p}(1+((1-a)\xi\omega)^{p-2}\theta)^{\frac{3-p}{2-p}}|\{Q^-_{2\rho,(2\rho)^p\theta}(y,s):u\le\mu_-+\xi\omega\}|$$

$$\leq \gamma \nu \xi \omega \theta^{-\frac{1}{2-p}}(1+((1-a)\xi\omega)^{p-2}\theta)^{\frac{3-p}{2-p}}.$$

Fix a number $\eta \in (0, 1)$ depending on the known data, which will be specified later, and choose ν from the condition

$$\gamma \nu \xi \omega \theta^{-\frac{1}{2-p}}(1+((1-a)\xi\omega)^{p-2}\theta)^{\frac{3-p}{2-p}} \leq \eta,$$

then we obtain that $A_0(l_1) \leq \eta$. If $c\nu_1 \leq \frac{1}{8}$, then obviously we have

$$\frac{l_0}{2}+\frac{B}{4}\alpha_0+l_0\beta_0 \leq l_1 \leq l_0-\frac{1}{4}(\alpha_{-1}-\alpha_0)-l_0(\beta_{-1}-\beta_0).$$

Lemma 4.3 *Suppose we have chosen $l_1, \ldots, l_j$ and $\delta_i = \delta_i(l_{i+1}) = l_i - l_{i+1}$, $i = 0, 1, \ldots, j-1$ such that*

$$\begin{aligned}
&\max\left(\frac{l_i}{2}+\frac{B}{4}\alpha_i+l_i\beta_i, l_i-2^{\frac{p-\alpha}{2-p}}\delta_{i-1}\right)\\
&\quad\leq l_{i+1} \leq l_i-\min\left(\frac{1}{4}(\alpha_{i-1}-\alpha_1)+l_i(\beta_{i-1}-\beta_i), \frac{1}{2}\delta_{i-1}\right),\\
&\quad\quad i=1,\ldots,j-1, \qquad (4.7)\\
&l_{i+1} \geq \frac{B}{2}\alpha_i+2l_i\beta_i, \quad i=1,\ldots,j-1,\\
&\quad\quad A_i(l_{i+1}) \leq \eta, \quad i=1,\ldots,j-1, \qquad (4.8)
\end{aligned}$$

then

$$A_j(\bar{l}) \leq \eta, \quad \bar{l}=\max\left(\frac{l_j}{2}+\frac{B}{4}\alpha_j+l_j\beta_j, l_j-2^{\frac{p-\alpha}{2-p}}\delta_{j-1}\right). \qquad (4.9)$$

Proof Let us decompose $L_j(\bar{l}_j)$ as $L_j(\bar{l}) = L'_j(\bar{l}) \cup L''_j(\bar{l})$, $L'_j(\bar{l}) = \{\frac{l_j-v}{\delta_j(\bar{l})} < \varepsilon\}$, $L''_j(\bar{l}) = L_j(\bar{l})\backslash L'_j(\bar{l})$, where $\varepsilon > 0$ depending on the data is small enough to be determined later. Note that

$$\begin{aligned}
&\frac{l_j}{2}-\frac{B}{4}\alpha_j-l_j\beta_j \geq \frac{l_{j-1}}{4}+\frac{B}{8}\alpha_{j-1}-\frac{B}{4}\alpha_j+\frac{1}{2}l_{j-1}\beta_{j-1}-l_j\beta_j\\
&\quad=\frac{\delta_{j-1}}{4}+\frac{l_j}{4}+\frac{B}{8}\alpha_{j-1}-\frac{B}{4}\alpha_j+\frac{1}{2}l_{j-1}\beta_{j-1}-l_i\beta_j\\
&\quad\geq \frac{\delta_{j-1}}{4}+\frac{B}{4}(\alpha_{j-1}-\alpha_j)+l_j(\beta_{j-1}-\beta_j),
\end{aligned}$$

therefore we have

$$\min\left(\frac{\delta_{j-1}}{4}+\frac{B}{4}(\alpha_{j-1}-\alpha_j)+l_j(\beta_{j-1}-\beta_j), 2^{\frac{p-\alpha}{2-p}}\delta_{j-1}\right) \leq \delta_j(\bar{l}) \leq 2^{\frac{p-\alpha}{2-p}}\delta_{j-1}. \quad (4.10)$$

By (4.10) $r_j^p\delta_j^{2-p}(\bar{l}) \leq 2^{-\alpha}r_{j-1}^p\delta_{j-1}^{2-p}$ and hence $\xi_{j-1}(x,t) \equiv 1$ for $(x,t) \in Q_j(\bar{l})$, so if $\delta_j(\bar{l}) = 2^{\frac{p-\alpha}{2-p}}\delta_j$, then by (4.8)

$$\begin{aligned}
A_j(\bar{l}) &\leq 2^{-\frac{p-\alpha}{2-p}(3-p)}\delta_{j-1}^{p-3}r_j^{-n-p}\iint_{L_j(\bar{l})}(l_j-v)\xi_{j-1}^{k-p}\mathrm{d}x\,\mathrm{d}t\\
&\leq 2^{-\frac{p-\alpha}{2-p}(3-p)+n+p}\delta_{j-1}^{p-2}r_{j-1}^{-n-p}\iint_{L_{j-1}(l_j)}\frac{l_{j-1}-v}{\delta_{j-1}}\xi_{j-1}^{k-p}\mathrm{d}x\,\mathrm{d}t=A_{j-1}(l_j) \leq \eta.
\end{aligned} \qquad (4.11)$$

If $\delta_j(\bar{l}) \geq \frac{\delta_{j-1}}{4} + \frac{B}{4}(\alpha_{j-1} - \alpha_j) + l_j(\beta_{j-1} - \beta_j)$, then

$$\delta_j^{p-2}(\bar{l}) r_j^{-n-p} \iint_{L'_j(\bar{l})} \frac{l_j - v}{\delta_j(\bar{l})} \xi_j^{k-p} \mathrm{d}x\, \mathrm{d}t \leq \gamma \varepsilon \delta_{j-1}^{p-2} r_{j-1}^{-n-p} \iint_{L_j(\bar{l})} \xi_{j-1}^{k-p} \mathrm{d}x\, \mathrm{d}t$$
$$\leq \gamma \varepsilon \delta_{j-1}^{p-2} r_{j-1}^{-n-p} \iint_{L_{j-1}(l_j)} \frac{l_{j-1} - v}{\delta_{j-1}} \xi_{j-1}^{k-p} \mathrm{d}x\, \mathrm{d}t \leq \gamma \varepsilon A_{j-1}(l_j) \leq \gamma \varepsilon \eta. \tag{4.12}$$

Define

$$\psi_j = \frac{1}{\delta_j(\bar{l})} \left(\int_v^{l_j} \left(1 + \frac{l_j - s}{\delta_j(\bar{l})}\right)^{-\frac{1+\lambda}{p}} \mathrm{d}s \right)_+,$$

using the evident inequalities $\gamma^{-1}(\varepsilon)\psi_j^{\rho(\lambda)} \leq \frac{l_j - v}{\delta_j(\bar{l})} \leq \gamma(\varepsilon)\psi_j^{\rho(\lambda)}$ for $(x,t) \in L''_j$, $\rho(\lambda) = \frac{p}{p-1-\lambda}$, the Sobolev inequality, and Lemma 4.2 with $l = l_j,\ \delta = \delta_j(\bar{l}),\ \tau = \tau_j(\bar{l}), 0 < \lambda < \min\{1, \frac{2-p}{p-1}, \frac{\varkappa}{n}\}$, and k such that $\frac{(k-p)n}{n+p} - p \geq 1$, similar to (3.11), we obtain

$$\begin{aligned}
&\frac{\delta_j^{p-2}(\bar{l})}{r_j^{n+p}} \iint_{L''_j(\bar{l})} \frac{l_j - v}{\delta_j(\bar{l})} \xi_j^{k-p} \mathrm{d}x\, \mathrm{d}t \\
&\leq \gamma(\varepsilon) \Bigg\{ \frac{\delta_j^{p-2}(\bar{l})}{r_j^{n+p}} \iint_{L_j(\bar{l})} \frac{l_j - v}{\delta_j(\bar{l})} \mathrm{d}x\, \mathrm{d}t + l_j^{-1} \delta_j^{1-p}(\bar{l}) r_j^{p-n} \int_{B_j} F_1 \mathrm{d}x \\
&\quad + \delta_j^{-p}(\bar{l}) r_j^{p-n} \int_{B_j} (F_1 + F_2^{\frac{p}{p-1}})\, \mathrm{d}x + \delta_j^{1-p}(\bar{l}) r_j^{p-n} \int_{B_j} F_3 \mathrm{d}x \\
&\quad + \delta_j^{1-p}(\bar{l}) l_j^{p-1} r_j^{p-n} \int_{B_j} h^p \mathrm{d}x \Bigg\}^{1+\frac{p}{n}}.
\end{aligned} \tag{4.13}$$

Using the inequality $\delta_j(\bar{l}) \geq \frac{\delta_{j-1}}{4}$, similar to (4.12). we obtain

$$\frac{\delta_j^{p-2}(\bar{l})}{r_j^{n+p}} \iint_{L_j(\bar{l})} \frac{l_j - v}{\delta_j(\bar{l})} \mathrm{d}x\, \mathrm{d}t \leq \gamma \eta.$$

Furthermore, (4.10) implies that $\delta_j(\bar{l}) \geq l_j(\beta_{j-1} - \beta_j) + \frac{B}{4}(\alpha_{j-1} - \alpha_j)$, therefore we have

$$\begin{aligned}
&l_j^{-1} \delta_j^{1-p}(\bar{l}) r_j^{p-n} \int_{B_j} F_1 \mathrm{d}x + \delta_j^{-p}(\bar{l}) r_j^{p-n} \int_{B_j} (F_1 + F_2^{\frac{p}{p-1}})\, \mathrm{d}x \\
&\quad + \delta_j^{1-p}(\bar{l}) r_j^{p-n} \int_{B_j} F_3 \mathrm{d}x + \delta_j^{1-p}(\bar{l}) l_j^{p-1} r_j^{p-n} \int_{B_j} h^p \mathrm{d}x \leq \gamma(B^{-p} + B^{1-p} + c^{1-p}).
\end{aligned} \tag{4.14}$$

Combining estimates (4.12)–(4.14) we have

$$A_j(\bar{l}) \leq \gamma \varepsilon \eta + \gamma(\varepsilon)\{\eta + B^{-p} + B^{1-p} + c^{1-p}\}^{1+\frac{p}{n}}. \tag{4.15}$$

First choose ε from the condition $\gamma\varepsilon \leq \frac{1}{4}$, next fix η by $\gamma(\varepsilon)\eta^{\frac{p}{n}} = \frac{1}{4}$ and choosing B and c large enough so that $B^{-p} + B^{1-p} + c^{1-p} \leq \eta$, we conclude from (4.15) that $A_j(\bar{l}) \leq \eta$, which completes the proof of Lemma 4.3. □

Note that $A_j(l)$ is continuous as a function of l. So if $A_j(l_j - \min(\frac{1}{4}(\alpha_{j-1}-\alpha_j) + l_j(\beta_{j-1}-\beta_j), \frac{1}{2}\delta_{j-1})) > \eta$ the equation $A_j(l) = \eta$ has roots. Denote l_{j+1} the largest root $A_j(l_{j+1}) = \eta$. If $A_j(l_j - \min(\frac{1}{4}(\alpha_{j-1}-\alpha_j) + l_j(\beta_{j-1}-\beta_j), \frac{1}{2}\delta_{j-1})) \le \eta$, we set $l_{j+1} = l_j - \min(\frac{1}{4}(\alpha_{j-1}-\alpha_j) + l_j(\beta_{j-1}-\beta_j), \frac{1}{2}\delta_{j-1})$ and in both cases we set $\delta_j = \delta_j(l_{j+1}) = l_j - l_{j+1}$. Note that our choice guarantee that $\delta_j \le 2^{\frac{p-\alpha}{2-p}}\delta_{j-1}$ and $A_j(l_{j+1}) \le \eta$ for $j = 1, 2, \ldots$. Similar to Lemma 3.1 we prove the following lemma.

Lemma 4.4 *For all $j \ge 2$ there exists $\gamma > 0$ depending only on the data, such that*

$$\delta_j \le \frac{1}{2}\delta_{j-1} + \gamma l_j \left(r_j^{p-n}\int_{B_j} h^p \mathrm{d}x\right)^{\frac{1}{p-1}} + \gamma\left(r_j^{p-n}\int_{B_j}(F_1 + F_2^{\frac{p}{p-1}})\,\mathrm{d}x\right)^{\frac{1}{p}}$$
$$+\gamma\left(r_j^{p-n}\int_{B_j} F_3 \mathrm{d}x\right)^{\frac{1}{p-1}}. \tag{4.16}$$

Summing up inequality (4.16) with respect to $j = 2, \ldots, J-1$, we obtain

$$l_2 - l_J \le \delta_1 + \gamma l_0 W_h(32\rho) + \gamma W_f(32\rho) + \gamma W_g(32\rho),$$

or the same

$$\begin{aligned} l_0 &\le l_J + 2\delta_0 + \delta_1 + \gamma l_0 W_h(32\rho) + \gamma W_f(32\rho) + \gamma W_g(32\rho) \\ &\le l_J + 2(1 + 2^{\frac{2-\alpha}{2-p}})\delta_0 + \gamma l_0 W_h(32\rho) + \gamma W_f(32\rho) + \gamma W_g(32\rho). \end{aligned} \tag{4.17}$$

Let $l = \lim_{j\to\infty} l_j$, passing to the limit in (4.17) as $J \to \infty$ and choosing $(\bar{x}, \bar{t})$ as a Lebesgue point of the function $(l - v)_+$, using the definition of δ_0, we conclude that

$$\frac{1+a}{2}\xi\omega \le v(\bar{x}, \bar{t}) + \gamma\xi\omega W_h(32\rho) + \gamma W_f(32\rho) + \gamma W_g(32\rho).$$

Fix $\nu_1 \in (0, 1)$ and B large enough so that $\nu_1\gamma + B^{-1}\gamma \le \frac{1-a}{2}$, we obtain

$$u(\bar{x}, \bar{t}) - \mu_- = v(\bar{x}, \bar{t}) \ge a\xi\omega. \tag{4.18}$$

Since $(\bar{x}, \bar{t})$ is an arbitrary point in $Q^-_{\rho,\rho^p\theta}(y, s)$, from (4.18) the required (1.13) follows, which proves Theorem 1.3.

The proof of Theorem 1.4 is similar to that of Theorem 1.3. Moreover, by taking $l \le \xi\omega$ and a cutoff function $\xi = \xi^{(1)}(x)$ independent of t, the integral involving ξ_t in the right-hand side of (4.4) vanishes. We may now repeat the same arguments as in previous proof for $(l_j - v)_+$ and $A_j(l)$ over the cylinders $Q_j := B_j \times (s, s + (2\rho)^p\theta)$, $A_j(l) := \frac{\delta_j^{p-2}(l)}{r_j^{n+p}} \iint_{L_j(l)} \frac{l_j - v}{\delta_j(l)} \xi_j^{k-p} \mathrm{d}x\,\mathrm{d}t$, $\delta_j(l) = l_j - l,\ L_j(l) = Q_j \cap \{v < l_j\}$.

5 The expansion of positivity: Proof of Theorem 1.5

In the proof we closely follow [10].

Lemma 5.1 *Assume that for some $(y, s) \in \Omega_T$ and some $\rho > 0$*

$$|\{x \in B_\rho(y) : u(x, s) \le \mu_- + N\}| \le (1-\alpha)|B_\rho(y)| \tag{5.1}$$

for some $N > 0$ *and some* $0 < \alpha < 1$. *There exist* $\nu_1, \varepsilon_0, b \in (0, 1)$, $B \geq 1$ *depending only on the data and* α, *such that the inequality*

$$W_h(32\rho) \leq \nu_1 \tag{5.2}$$

implies that either

$$N \leq B\left(W_f(32\rho) + \gamma W_g(32\rho)\right) \tag{5.3}$$

or

$$|\{x \in B_\rho(y) : u(x,t) \leq \mu_- + \varepsilon_0 N\}| \leq \left(1 - \frac{\alpha^2}{4}\right)|B_\rho(y)| \tag{5.4}$$

for all $t \in (s, s + bN^{2-p}\rho^p)$.

Proof For $k > 0$ and $t > s$ set $A_{k,\rho}(t) = \{x \in B_\rho(y) : u(x,t) \leq k + \mu_-\}$. Write down the estimate (2.2) for the function $(N + \mu_- u)_+$ over the cylinder $Q^+_{\rho,\rho^p\theta}(y,s)$, where $\theta > 0$ is to be chosen. The cutoff function ξ is taken independent on t, nonnegative, and such that $\xi = 1$ on $B_{\rho(1-\sigma)}(y)$, $|\nabla\xi| \leq \frac{\gamma}{\sigma\rho}$, where $\sigma \in (0,1)$ is to be chosen. Lemma 2.4 yields

$$\begin{aligned}
&\int_{B_{\rho(1-\sigma)}(y)\times\{t\}} (N + \mu_- u)_+^2 \mathrm{d}x + \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} |\nabla(N + \mu_- u)_+|^p \xi^p \mathrm{d}x\,\mathrm{d}\tau \\
&\leq \int_{B_\rho(y)\times\{s\}} (N + \mu_- u)_+^2 \mathrm{d}x + \gamma(\sigma\rho)^{-p} \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} (N + \mu_- u)_+^p \mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} (F_1 + F_2^{\frac{p}{p-1}} + (N + \mu_- u)_+ F_3)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} (N + \mu_- u)_+^p h^p \xi^p \mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{5.5}$$

The last integral in the right-hand side of (5.5) we estimate using Lemma 2.3

$$\begin{aligned}
&\gamma \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} (N + \mu_- u)_+^p h^p \xi^p \mathrm{d}x\,\mathrm{d}\tau \leq \gamma\nu_1^{p-1} \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} |\nabla(N + \mu_- u)_+|^p \xi^p \mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma\nu_1^{p-1}(\sigma\rho)^{-p} \iint_{Q^+_{\rho,\rho^p\theta}(y,s)} (N + \mu_- u)_+^p \mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Choosing ν_1 sufficiently small, and using (5.1), (5.3) from (5.5) we conclude that

$$\begin{aligned}
&\int_{B_{\rho(1-\sigma)}(y)\times\{t\}} (N + \mu_- u)_+^2 \mathrm{d}x \leq N^2 \Big\{ 1 - \alpha + \gamma\theta\sigma^{-p} N^{p-2} \\
&\quad + \gamma\theta N^{-2}\rho^{p-n} \int_{B_\rho(y)} (F_1 + F_2^{\frac{p}{p-1}})\,\mathrm{d}x + \gamma\theta N^{-1}\rho^{p-n} \int_{B_\rho(y)} F_3 \mathrm{d}x \Big\} |B_\rho(y)| \\
&\leq N^2\{1 - \alpha + \gamma\theta\sigma^{-p}N^{p-2}\}|B_\rho(y)|.
\end{aligned} \tag{5.6}$$

The left-hand side of (5.6) is estimated below by

$$\int_{B_{\rho(1-\sigma)}(y)\times\{t\}} (N + \mu_- u)_+^2 \mathrm{d}x \geq N^2(1 - \varepsilon_0)^2 |A_{\varepsilon_0 N,\rho(1-\sigma)}(t)|,$$

where $\varepsilon_0 \in (0,1)$ is to be chosen. Next we have

$$|A_{\varepsilon_0 N,\rho}(t)| \leq |A_{\varepsilon_0 N,\rho(1-\sigma)}(t)| + \sigma n |B_\rho(y)|$$

$$\leq \{\sigma n + (1-\varepsilon_0)^{-2}(1-\alpha+\gamma\theta\sigma^{-p}N^{p-2})\}|B_\rho(y)|.$$

Choose $\sigma = \frac{\alpha^2}{4n}$, $\varepsilon_0 = 1-\sqrt{\frac{1-\alpha}{1-\alpha^2}}$, $b = \frac{\sigma^p\gamma^{-1}(1-\varepsilon_0)^2}{4}\alpha^2$ and $\theta = bN^{2-p}$, then the last inequality implies the required (5.4). □

Let the cylinder $Q^+_{16\rho,bN^{2-p}\rho^p}(y,s)$ be contained in Ω_T. In the same way as in [10] we consider the function

$$w(z,\tau) = e^{\frac{\tau}{2-p}}N^{-1}(u(y+z\rho, s+bN^{2-p}\rho^p(1-e^{-\tau}))-\mu_-).$$

Inequality (5.4) translates into w as

$$|\{z\in B_8(0) : w(z,\tau)\leq \varepsilon_0 e^{\frac{\tau_0}{2-p}}\}| \leq \Big(1-\frac{\alpha^2}{4\cdot 8^n}\Big)|B_8(0)|, \tag{5.7}$$

for all $\tau\in(\tau_0,+\infty)$, where $\tau_0>0$ to be chosen.

Since $w\geq 0$, formal differentiation, which can be justified in a standard way, gives

$$w_\tau = \frac{w}{2-p} + \operatorname{div}\tilde{\mathbb{A}}(z,\tau,w,\nabla w) + \tilde{b}(z,\tau,w,\nabla w),$$

where $\tilde{\mathbb{A}}(z,\tau,w,\nabla w)$, $\tilde{b}(z,\tau,w,\nabla w)$ satisfy the inequality

$$\begin{aligned}
&\tilde{\mathbb{A}}(z,\tau,w,\nabla w)\,\nabla w \geq b\mu_1|\nabla w|^p - \tilde{F}_1(x,\tau),\\
&|\tilde{\mathbb{A}}(z,\tau,w,\nabla w)| \leq b\mu_2|\nabla w|^{p-1} + \tilde{F}_2(z,\tau),\\
&|b(z,\tau,w,\nabla w)| \leq b\tilde{h}|\nabla w|^{p-1} + \tilde{F}_3(z,\tau),
\end{aligned} \tag{5.8}$$

where μ_1,μ_2 are the constants in the structure conditions (1.2), b is a number claimed by Lemma 5.1 and

$$\tilde{F}_1(z,\tau) = bN^{-p}\rho^p e^{\frac{p\tau}{2-p}}F_1(y+z\rho), \quad \tilde{F}_2(z,\tau) = bN^{1-p}\rho^{p-1}e^{\frac{p-1}{2-p}\tau}F_2(y+z\rho),$$
$$\tilde{F}_3(z,\tau) = bN^{1-p}\rho^p e^{\frac{p-1}{2-p}\tau}F_3(y+z\rho), \quad \tilde{h}(z,\tau) = \rho h(y+z\rho).$$

Set $k_0 = \varepsilon_0 e^{\frac{\tau_0}{2-p}}$, and $k_s = \frac{k_0}{2^s}$, $s = 0,1,\ldots,s_*$, where s_* to be chosen later. Then (5.7) yields

$$|\{z\in B_8(0) : w(z,\tau)\leq k_s\}| \leq \Big(1-\frac{\alpha^2}{4\cdot 8^n}\Big)|B_8(0)| \tag{5.9}$$

for all $\tau\in(\tau_0,+\infty)$ and for all $0\leq s\leq s_*$.

Let $Q_{\tau_0} = B_8(0)\times(\tau_0+k_0^{2-p}, \tau_0+2k_0^{2-p})$ and $Q'_{\tau_0} = B_8(0)\times(\tau_0,\tau_0+2k_0^{2-p})$ and a nonnegative cutoff function in Q'_{τ_0}, $\xi(z,\tau) = \xi_1(z),\xi_2(\tau)$, where $\xi_1\in C_0^\infty(B_{16}(0))$, $\xi_1 = 1$ in $B_8(0)$, $\xi_2 = 1$ for $\tau\geq\tau_0+k_0^{2-p}$, $\xi_2 = 0$ for $\tau\leq\tau_0$ and $|\nabla\xi_1|\leq\frac{\gamma}{8}$, $|\frac{\partial\xi_2}{\partial\tau}|\leq\gamma k_0^{p-2}$.

Using Lemma 2.4 we have

$$\begin{aligned}
b\iint_{Q'_{\tau_0}} |\nabla(k_s-w)|_+^p\xi^p\,\mathrm{d}z\,\mathrm{d}\tau &\leq \gamma\iint_{Q'_{\tau_0}}((k_s-w)_+^2|\xi_\tau| + b(k_s-w)_+^p|\nabla\xi|^p)\,\mathrm{d}z\,\mathrm{d}\tau\\
&+\gamma\iint_{Q'_{\tau_0}}(\tilde{F}_1+\tilde{F}_2^{\frac{p}{p-1}}+(k_s-w)_+\tilde{F}_3)\,\mathrm{d}z\,\mathrm{d}\tau + \gamma b\iint_{Q'_{\tau_0}}(k_s-w)_+^p\tilde{h}^p\xi^p\,\mathrm{d}z\,\mathrm{d}\tau.
\end{aligned}$$

Taking into account the expressions of $\tilde{h}$, $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$ and k_0, we estimate

$$\gamma\iint_{Q'_{\tau_0}}(k_s-w)_+^p\tilde{h}^p\xi^p\,\mathrm{d}z\,\mathrm{d}\tau \leq \gamma\nu_1^{p-1}\iint_{Q'_{\tau_0}}|\nabla(k_s-w)_+|^p\xi^p\,\mathrm{d}z\,\mathrm{d}\tau$$

$$+\gamma\nu_1^{p-1}\iint_{Q'_{\tau_0}}(k_s-w)_+^p|\nabla\xi|^p \mathrm{d}z\,\mathrm{d}\tau, \tag{5.10}$$

and

$$\iint_{Q'_{\tau_0}}(\tilde F_1+\tilde F_2^{\frac{p}{p-1}}+(k_s-w)_+\tilde F_3)\,\mathrm{d}z\,\mathrm{d}\tau$$
$$\le\gamma b\varepsilon_0^{-p}k_0^2 e^{\frac{2p}{2-p}k_0^{2-p}}\left\{N^{-p}\rho^{p-n}\int_{B_{16\rho}(y)}(F_1+F_2^{\frac{p}{p-1}})\,\mathrm{d}x+N^{1-p}\rho^{p-n}\int_{B_{16\rho}(y)}F_3\mathrm{d}x\right\}. \tag{5.11}$$

From this, choosing $\nu_1$sufficiently small, we obtain

$$\iint_{Q_{\tau_0}}|\nabla(k_s-w)_+|^p\mathrm{d}x\,\mathrm{d}\tau\le\gamma k_s^p|Q_{\tau_0}|\left\{1+\varepsilon_0^{-p}2^{s_*p}e^{\frac{2p}{2-p}k_0^{2-p}}N^{1-p}\right.$$
$$\left.\times\left(W_f(32\rho)+W_g(32\rho)\right)^{p-1}(1+N^{-1}W_f(32\rho)+N^{-1}W_g(32\rho))\right\}.$$

Suppose for the moment that s_* and k_0 have been chosen, and set

$$\gamma_*=\gamma(s_*,\varepsilon_0,\tau_0)=\varepsilon_0^{-p}2^{s_*p}e^{\frac{2p}{2-p}k_0^{2-p}}.$$

Therefore either $N\le\gamma_*\left(W_f(32\rho)+W_g(32\rho)\right)$ or the previous inequality yields

$$\iint_{Q_{\tau_0}}|\nabla(k_s-w)_+|^p\mathrm{d}x\,\mathrm{d}\tau\le\gamma k_s^p|Q_{\tau_0}|$$

with constant γ depending only on the data, $\operatorname{ess\,sup}_{\Omega_T}|u|$ and b.

Set $A_s(\tau)=\{B_8(0):w(\cdot,\tau)<k_s\}$, and $A_s=\{Q_{\tau_0}:w<k_s\}$. By Lemma 2.1, (5.9) and (5.11) we have

$$|A_{s+1}|\le\gamma|Q_{\tau_0}|^{\frac{1}{p}}|A_s\backslash A_{s+1}|^{\frac{p-1}{p}}.$$

Taking the $\frac{p}{p-1}$ power and summing up the last inequality with respect to $s=0,\dots,s_*-1$, we conclude that

$$s_*|A_{s_*}|^{\frac{p}{p-1}}\le\gamma|Q_{\tau_0}|^{\frac{p}{p-1}}.$$

Choosing s_* from the condition $\gamma s_*^{-\frac{p-1}{p}}=\nu$, we obtain

$$|\{(z,\tau)\in Q_{\tau_0}:w(z,\tau)\le k_{s_*}\}|\le\nu|Q_{\tau_0}|.$$

Without loss of generality we assume that $2^{s_*(2-p)}$ is an integer, and subdivide cylinder Q_{τ_0} into $2^{s_*(2-p)}$ cylinders, each of length $k_{s_*}^{2-p}$, by setting

$$Q_i=B_8(0)\times(\tau_0+k_0^{2-p}+ik_{s_*}^{2-p},\tau_0+k_0^{2-p}$$
$$+(i+1)k_{s_*}^{2-p}),\quad\text{for}\quad i=0,1,\dots,2^{s_*(2-p)}-1.$$

For at least one of these, say Q_{i_0} there must hold

$$|\{(z,\tau)\in Q_{i_0}:w(z,\tau)\le k_{s_*}\}|\le\nu|Q_{i_0}|. \tag{5.12}$$

Apply Theorem 1.3 to w over Q_{i_0} with $\mu_- = 0,\ \xi\omega = k_{s_*},\ a = \frac{1}{2},\ \theta = k_{s_*}^{2-p}$. It gives

$$w(z, \tau_0 + k_0^{2-p} + (i_0+1)k_{s_*}^{2-p}) \geq \frac{k_{s_*}}{2}, \quad \text{a.a. in } B_4(0), \tag{5.13}$$

and hence, there exists a time level $\tau_1 \in (\tau_0 + k_0^{2-p}, \tau_0 + 2k_0^{2-p})$, such that

$$w(z, \tau_1) \geq \sigma_0 e^{\frac{\tau_0}{2-p}}, \quad \sigma_0 = \varepsilon_0 2^{-(s_*+1)}. \tag{5.14}$$

In terms of the original coordinates (5.14) implies

$$u(x, t_1) \geq \mu_- + \sigma_0 N e^{-\frac{\tau_1-\tau_0}{2-p}} = \mu_- + N_0 \quad \text{in } B_{4\rho}(y), \tag{5.15}$$

$t_1 = s + bN^{2-p}\rho^p(1 - e^{-\tau_1})$.

Apply Lemma 2.4 with $k = \mu_- + N_0,\ \theta = \nu_0 N_0^{2-p}$, and $\xi := \xi(x) \in C_0^\infty(B_{4\rho}(y))$, $\xi(x) = 1$ in $B_{3\rho}(y)$, $|\nabla\xi| \leq \gamma\rho^{-1}$, where $\nu_0 \in (0,1)$ to be chosen.

$$\begin{aligned}
&\int_{B_{3\rho}(y)\times\{t\}} (N_0 + \mu_- u)_+^2 \mathrm{d}x + \iint_{Q^+_{4\rho,(4\rho)^p\theta}(y,s)} |\nabla(N_0 + \mu_- u)_+|^p \xi^p \mathrm{d}x\, \mathrm{d}t \\
&\leq \gamma\rho^{-p} \iint_{Q^+_{4\rho,(4\rho)^p\theta}(y,s)} (N_0 + \mu_- u)_+^p \mathrm{d}x\, \mathrm{d}t \\
&\quad + \gamma \iint_{Q^+_{4\rho,(4\rho)^p\theta}(y,s)} (F_1 + F_2^{\frac{p}{p-1}} + (N_0 + \mu_- u)F_3)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \gamma \iint_{Q^+_{4\rho,(4\rho)^p\theta}(y,s)} (N_0 + \mu_- u)_+^p h^p \xi^p \mathrm{d}x\, \mathrm{d}t.
\end{aligned} \tag{5.16}$$

The last two terms in the right-hand side of (5.16) we estimate similar to (5.10), (5.11), hence (5.16) yields

$$\begin{aligned}
&\int_{B_{3\rho}(y)\times\{t\}} (N_0 + \mu_- u)_+^2 \mathrm{d}x \leq \gamma\nu_0 N_0^2 \left(1 + N_0^{1-p}\left(W_f(32\rho) + W_g(32\rho)\right)^{p-1}\right. \\
&\quad \left. \times \left(1 + N_0^{-1}\left(W_f(32\rho) + W_g(32\rho)\right)\right)\right).
\end{aligned}$$

Therefore either $N \leq \gamma(\sigma, \tau_0, \tau) W_f(32\rho)$ or the previous inequality yields

$$|\{B_{3\rho}(y) : u(\cdot, t) \leq \mu_- + \frac{3}{4}N_0\}| \leq \gamma\nu_0 |B_{3\rho}(y)|, \tag{5.17}$$

for all $t \in (t_1, t_1 + \nu_0 N_0^{2-p}(4\rho)^p)$.

Using Theorem 1.4 over $Q^+_{3\rho,(4\rho)^p\nu_0 N_0^{2-p}}(y, t_1)$ with $\xi\omega = \frac{3}{4}N_0,\ a = \frac{2}{3},\ \nu = \gamma\nu_0$, we get

$$u(x,t) \geq \mu_- + \frac{N_0}{2} = \mu_- + \frac{N\sigma_0 e^{-\frac{\tau_1-\tau_0}{2}}}{2} \geq \mu_- + \frac{N\sigma_0}{2} e^{-\frac{2k_0^{2-p}}{2-p}}, \tag{5.18}$$

in $B_{2\rho}(y)$ and for all times $t_1 \leq t \leq t_1 + \nu_0 N_0^{2-p}(4\rho)^p$.

Now we define τ_0 so that

$$t_1 + \nu_0 N_0^{2-p}(4\rho)^p = s + bN^{2-p}\rho^p,$$

which implies

$$e^{\tau_0} = \frac{b}{4^p \nu_0 \sigma_0^{2-p}}.$$

From the previous $t_1 \le s + (1-\varepsilon) b N^{2-p} \rho^p$, where $\varepsilon = e^{-\tau_0 - 2\varepsilon_0^{2-p} e^{\tau_0}}$. Choose B so large that $B \ge \max(\gamma_*, \gamma(\sigma_0, \tau_0, \tau_1))$, we get the required (1.24). This proves Theorem 1.5.

6 Continuity of solutions: Proof of Theorem 1.2

Fix a point $(x_0, t_0) \in \Omega_T$, let the cylinder $Q^-_{R, R^p M^{2-p}}(x_0, t_0)$ be contained in Ω_T, $M = \operatorname{ess\,sup}_{\Omega_T} |u|$ and R is so small, that $W_h(R) \le \nu_1$, where $\nu_1 \in (0,1)$ is defined in Theorem 1.5.

Let $0 < R_1 < \frac{R}{32}$ and set

$$M_0 = \operatorname*{ess\,sup}_{Q^-_{R, R^p M^{2-p}}(x_0, t_0)} u, \quad m_0 = \operatorname*{ess\,inf}_{Q^-_{R, R^p M^{2-p}}(x_0, t_0)} u, \quad \omega_0 = M_0 - m_0.$$

Lemma 6.1 *There exist constants $B \ge 1$ and $b, \delta \in (0,1)$ that can be quantitatively determined only in terms of the data, such that, if $\omega_0 \ge B W_f(32 R_1)$, setting $\rho_0 = R_1$ and for $j = 0, 1, 2, \dots$*

$$\omega_{j+1} := \delta \omega_j, \quad \theta_j := b \omega_j^{2-p}, \quad \rho_j := 2^{-j} R_1, \quad Q_j := Q^-_{\rho_j, \rho_j^p \theta_j}(x_0, t_0), \tag{6.1}$$

either

$$\operatorname*{ess\,osc}_{Q_j} u \le 2 B W_f(R_1) \quad \textit{or} \quad \operatorname*{ess\,osc}_{Q_j} u \le \omega_j. \tag{6.2}$$

Proof We assume that statement (6.2) holds for j and prove for $j+1$. Set $\mu_j^+ = \operatorname{ess\,sup}_{Q_j} u$, $\mu_j^- = \operatorname{ess\,inf}_{Q_j} u$ and $\bar t_j = t_0 - A b \rho_j^p \omega_j^{2-p}$, $A = \frac{2^{2(p-1)}}{2^p - \delta^{2-p}}$. At least one of the two inequalities

$$|\{B_{\rho_j}(x_0) : u(\cdot, \bar t_j) \le \mu_j^- + \frac{\omega_j}{2}\}| \le \frac{1}{2} |B_{\rho_j}(x_0)|,$$
$$|\{B_{\rho_j}(x_0) : u(\cdot, \bar t_j) \ge \mu_j^+ - \frac{\omega_j}{2}\}| \le \frac{1}{2} |B_{\rho_j}(x_0)|$$

must hold. Assuming the first holds true, apply Theorem 1.5 with $\alpha = \frac{1}{2}$, $N = \frac{\omega_j}{2}$, either

$$\frac{\omega_j}{2} \le B \left(W_f(32\rho_j) + W_g(32\rho_j) \right) \le B \left(W_f(32 R_1) + W_g(32 R_1) \right) \tag{6.3}$$

or

$$u(x, \bar t_{j+1}) \ge \mu_j^- + \frac{\sigma \omega_j}{2} \quad \text{for a.a.} \quad x \in B_{\rho_{j+1}}(x_0),$$

and therefore

$$\operatorname*{ess\,inf}_{Q_{j+1}} u \ge \mu_j^- + \frac{\sigma \omega_j}{2}. \tag{6.4}$$

Choose $\delta = 1 - \frac{\sigma}{2}$, then if (6.3) occurs, $\operatorname{ess\,osc}_{Q_{j+1}} u \le 2B \left(W_f(32 R_1) + W_g(32 R_1) \right)$, if (6.4) occurs, then $\operatorname{ess\,osc}_{Q_{j+1}} u \le \delta \omega_j = \omega_{j+1}$. This proves Lemma 6.1. □

From the construction of Lemma 6.1 it follows that

$$\operatorname*{ess\,osc}_{Q_j} u \leq 2B\left(W_f(32R_1) + W_g(32R_1)\right) + \omega_j,$$

by iteration

$$\operatorname*{ess\,osc}_{Q_j} u \leq 2B\left(W_f(32R_1) + W_g(32R_1)\right) + \delta^j\omega_0. \tag{6.5}$$

Let now $0 < r < R$ be fixed, set $R_1 = r^\mu R^{1-\mu}$, $\mu \in (0,1)$, there exists a nonnegative integer j such that $R_1 2^{-j-1} \leq r \leq R_1 2^{-j}$, this implies

$$\operatorname*{ess\,osc}_{Q_j} u \leq \frac{1}{\delta}\left(\frac{r}{R_1}\right)^{\log_2 \frac{1}{\delta}} \omega_0 + 2B\left(W_f(32R_1) + W_g(32R_1)\right),$$

that is

$$\operatorname*{ess\,osc}_{Q_j} u \leq \frac{1}{\delta}\left(\frac{r}{R}\right)^{\alpha} \omega_0 + 2B\left(W_f(32r^\mu R^{1-\mu}) + W_g(32r^\mu R^{1-\mu})\right), \quad \alpha = (1-\mu)\log_2 \frac{1}{\delta}.$$

To conclude the proof, we observe that since $\delta^j \geq (\frac{r}{R_1})^{\log_2 \frac{1}{\delta}} \geq (\frac{r}{R})^{\alpha}$, the cylinder $Q_{r,r^p\theta_0}(x_0,t_0)$, $\theta_0 = b(\frac{r}{R})^{\alpha(2-p)}\omega_0^{2-p}$ is included in Q_j, and therefore

$$\operatorname*{ess\,osc}_{Q_{r,r^p\theta_0}(x_0,t_0)} u \leq \frac{1}{\delta}\left(\frac{r}{R}\right)^{\alpha} \omega_0 + 2B\left(W_f(32r^\mu R^{1-\mu}) + W_g(32r^\mu R^{1-\mu})\right). \tag{6.6}$$

This completes the proof of the continuity of solution to Eq. (1.1).

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