L^1 – L^p estimates for radial solutions of the wave equation and application

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Abstract It is well known that, for space dimension $n > 3$, one cannot generally expect L^1 – L^p estimates for the solution of

$$u_{tt} - \Delta u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = g(x),$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. In this paper, we investigate the benefits in the range of $1 \leq p \leq q$ such that L^p – L^q estimates hold under the assumption of radial initial data. In the particular case of odd space dimension, we prove L^1 – L^q estimates for $1 \leq q < \frac{2n}{n-1}$ and apply these estimates to study the global existence of small data solutions to the semilinear wave equation with power nonlinearity $|u|^\sigma$, $\sigma > \sigma_c(n)$, where the critical exponent $\sigma_c(n)$ is the Strauss index.

Keywords Wave equation · L^1 estimates · Asymptotic behavior of solutions · Critical exponent · Global existence of small data solutions

Mathematics Subject Classification 35L05 (primary) · 35B33 · 35B40 · 74G25

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1 Introduction

For $1 \leq p, q \leq \infty$, estimates in the L^p – L^q framework have been well investigated for solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = 0, & x \in \mathbb{R}^n, \\ u_t(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (1)$$

Let $S_t : g \rightarrow u(t, \cdot)$ be the operator that takes the initial data g into the solution $u(t, \cdot)$ at time t , and let $\|S_t\|$ be the operator norm from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. From the estimates in Peral [16] and Strichartz [21], S_t is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if, and only if, the point $(1/p, 1/q)$ belongs to the closed triangle with vertices

$$P_1 = \left(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1} \right), \quad P_2 = \left(\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} - \frac{1}{n-1} \right),$$

and $P_3 = \left(\frac{1}{2} + \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1} \right).$ (2)

In the case that $n = 1$ or 2 , we define $P_2 = (0, 0)$ and $P_3 = (1, 1)$. Moreover, the asymptotic behavior on t of $\|S_t\|$ follows by homogeneity, namely $\|S_t\| = t^{1-n/p+n/q} \|S_1\|$, i.e., there exists a positive constant C such that

$$\|u(t, \cdot)\|_{L^q} \leq C t^{1-n/p+n/q} \|g\|_{L^p} \quad (3)$$

hold uniformly for any $t > 0$.

The Strichartz's line condition [21] and duality arguments are the reasons why (3) is not obtained outside the triangle $P_1 P_2 P_3$. Indeed, Strichartz proved that if $\|S_t\|$ is finite, then we are above the line $P_1 P_2$; i.e., $\frac{n}{q} \geq \frac{n-3}{2} + \frac{1}{p}$. Additionally, if we suppose $\|S_t\|$ finite on the right-hand side of the line $P_1 P_3$, then by duality arguments $\|S_t\|$ would be finite below the Strichartz's line. In particular, when $p = 1$ and $q > \frac{2n}{n-1}$, one cannot generally expect L^1 – L^q estimates. However, the case $q = 1$ remains true if, and only if, $n = 1, 2, 3$ ([15, 16]).

It is interesting to compare the Cauchy problem (1) with the Klein–Gordon equation

$$\begin{cases} v_{tt} - \Delta v + v = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ v(0, x) = 0, & x \in \mathbb{R}^n, \\ v_t(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (4)$$

In [14], the authors proved that the operator $T_t : g \rightarrow v(t, \cdot)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if, and only if, $(1/p, 1/q)$ is on the same closed triangle $P_1 P_2 P_3$. The sharpness of the region is a consequence of Lemma 10 in [14]. More precisely, denoting the operator norm from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ by $\|T_t\|$, the authors proved that $\|T_t\|$ blows-up in the region $\frac{1}{q} < \frac{n}{p} - \frac{n+1}{2}$; i.e., for $(1/p, 1/q)$ on the right-hand side of the line $P_1 P_3$, even when they took radial initial data. By duality, the authors extended the argument to the region below the line $P_1 P_2$.

From [14], the small time behavior of $\|T_t\|$ is identical to the behavior of the homogeneous free wave operator, namely $t^{1-n/p+n/q}$, whereas for large t and $1/p > 1/n + 1/2$, the relation $\|T_t\| > \|S_t\|$ holds. Thus, in this last spectrum, we cannot conclude a blow-up result for the operator S_t for radial initial data from the previous result for $\|T_t\|$.

Motivated by recent studies ([5, 8]) on convolution operators acting on radial $L^p(\mathbb{R}^n)$ functions, denoted by $L_{\text{rad}}^p \doteq L_{\text{rad}}^p(\mathbb{R}^n)$, we are interested in the following question: for which values of (p, q) the operator S_t is bounded from L_{rad}^p to L_{rad}^q ? Clearly, if $(1/p, 1/q)$ belongs to the closed triangle with vertices $P_1 P_2 P_3$, then (3) holds. In this paper, we will prove L^1 – L^q estimates for the Cauchy problem (1) for $1 \leq q < \frac{2n}{n-1}$, by considering spherically symmetric data L_{rad}^1 in the odd space dimension (see Theorem 1). In the even space dimension, the pointwise estimates for the kernel are more delicate, and some additional logarithmic singularities appear (see Lemma 3.2 in [1]). As a consequence, for some range of (p, q) , we can still get L^p – L^q estimates, but we no longer expect L_{rad}^1 – L_{rad}^q estimates.

The application of linear estimates is a very useful tool to study global existence of small data solutions for semilinear problems. Much has been devoted to the case of the equation

$$\begin{cases} u_{tt} - \Delta u = |u|^\sigma, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = 0, & x \in \mathbb{R}^n, \\ u_t(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (5)$$

For $1 < \sigma < \sigma_K(n) = \frac{n+1}{n-1}$, in [10] Kato proved the nonexistence of global generalized solutions to (5), for small data with compact support. In [17], the authors proved a result close to the one of Kato for a wide class of nonlinear hyperbolic type inequality, with no assumption on the support of the data.

On the other hand, John [9] showed that $\sigma = 1 + \sqrt{2}$ is the critical exponent for the global existence of classical small data solution when $n = 3$. Classical solution means $u \in C^2([0, \infty) \times \mathbb{R}^n)$. A bit later, Strauss [20] conjectured that the critical exponent $\sigma_c(n)$ is the positive root of

$$(n-1)\sigma^2 - (n+1)\sigma - 2 = 0.$$

Glassey ([6, 7]) verified this conjecture for classical solutions when $n = 2$. For $n > 3$, Sideris [19] proved the nonexistence of global solutions in $C([0, \infty), L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n))$ with arbitrarily small data for $1 < \sigma < \sigma_c(n)$. Later in the supercritical case $\sigma > \sigma_c(n)$, Lindblad and Sogge [11] proved the existence of weak global solution in $L^\infty([0, \infty), L^q d\mu(\mathbb{R}^n))$, with $d\mu$ a weight measure, up to $n \leq 8$ and for all n in the case of radial initial data. In [4], the authors removed the assumption of spherical symmetry. The global existence also breaks down at the critical index $\sigma = \sigma_c(n)$. Schaeffer [18] proved it for $n = 2, 3$ and Yordanov–Zhang [22] for $n \geq 4$.

Our second goal is, assuming spherically symmetric data, particularly, under the assumption $\sigma_c(n) < \sigma < \frac{n+3}{n-1}$, give a simple proof for the global existence of small data solution to (5) in the odd space dimension. The power $\sigma_{\text{conf}} = \frac{n+3}{n-1}$ is well known as “conformal power” and one can obtain optimal results [12] when $\sigma \geq \sigma_{\text{conf}}$.

The paper is organized as follows. In Sect. 2, we present the main results. In Sect. 3, we recall a representation of the solution to (1) in terms of Bessel functions; we also state well known pointwise estimates for his kernel. Section 4 presents the proof of the L^1 – L^q linear estimates. Section 5 deals with the semilinear problem.

2 Main results

Notation 1 Let $f, g : [0, \infty) \rightarrow (0, \infty)$ be two positive functions. We use the notation $f \lesssim g$ if there exists a positive constant C such that $f(t) \leq Cg(t)$, for all $t \geq 0$.

Using some pointwise estimates derived for the kernel of the radial solution u of (1), we obtain the following result:

Theorem 1 *Let $n \geq 3$ be an odd integer. Then, for all $1 \leq q < \frac{2n}{n-1}$, the solution u of (1), with $g \in L^1_{\text{rad}}$, satisfies*

$$\|u(t, \cdot)\|_{L^q} \lesssim t^{1-n+n/q} \|g\|_{L^1}, \quad \text{for all } t > 0. \quad (6)$$

After interpolating with the estimates obtained in Peral [16] and Strichartz [21], we proved the following:

Corollary 1 *Let $n \geq 3$ be an odd integer, and let P_1 and P_2 be as in (2). Moreover, let Δ be a closed triangle with vertices P_2 , $P_4 = (1, \frac{n-1}{2n})$ and $P_5 = (1, 1)$, and L be the line segment with end points P_1 and P_4 , open in P_1 . Then, for all $(1/p, 1/q) \in \Delta - L$, the solution u of (1), with $g \in L^p_{\text{rad}}$, satisfies*

$$\|u(t, \cdot)\|_{L^q} \lesssim t^{1-n/p+n/q} \|g\|_{L^p}, \quad \text{for all } t > 0. \quad (7)$$

Remark 1 If $q > pn/(n-p)$, a singularity appears in (7) as $t \rightarrow 0$. However, for $1-n/p+n/q \geq 0$, (7) holds uniformly for all $t \geq 0$. In fact, using the homogeneity of the free wave equation, it would be enough to prove (6) and (7) for $t = 1$.

Under the assumption $g \in L^p_{\text{rad}} \cap L^2_{\text{rad}}$, with $(p, q) \in \mathcal{A} =]1, 2[\times [2, \frac{2n}{n-1}]$, the singularity in (7) disappears, thanks to $1 - n/2 + n/q \geq 0$ and Remark 1. Hence, for all $(p, q) \in \mathcal{A}$, (7) implies in

$$\|u(t, \cdot)\|_{L^q} \lesssim \begin{cases} \|g\|_{L^2} \leq \|g\|_{L^p \cap L^2}, & t \in [0, 1) \\ (1+t)^{1-n/p+n/q} \|g\|_{L^p \cap L^2}, & t \in [1, \infty). \end{cases} \quad (8)$$

In the next two results, we fix the indices

$$r \doteq \frac{2n}{n-1} \quad \text{and} \quad \sigma p \doteq r, \quad r > \sigma. \quad (9)$$

Remark 2 Let us explain the reason to choose $r = \frac{2n}{n-1}$ (critical point on the L^1 - L^q linear estimates). Following a suggestion of Sideris in [19], in order to avoid the assumption of initial data with compact support on the existence of local solution, it is sufficient to assume the

relation $q = \sigma p$, with (p, q) on the admissible region of validity of L^p – L^q linear estimate. So, the choice $q = r$ and p close to 1, but not $p = 1$, due to fact that we do not have linear estimates at the end point $(1, \frac{2n}{n-1})$, is of the best nature and produce sharp results.

On the base of the Duhamel principle and using the linear estimates (7), we have the following result.

Theorem 2 [Local existence] *Let $n \geq 3$ be an odd integer, $g \in L^2_{\text{rad}}$ and suppose that $1 \leq \sigma < \frac{n+3}{n-1}$. Then, there exists $T > 0$ and a unique weak solution $u \in \mathcal{C}([0, T], L^r(\mathbb{R}^n))$ to (5).*

From the scale argument, smallness of $\|g\|_{L^{\frac{n(\sigma-1)}{\sigma+1}}}$ is a necessary condition to have global existence to (5). Indeed, if u is a solution to (5), then

$$\lambda^h u(\lambda t, \lambda x), \quad \text{with } h \doteq \frac{2}{\sigma-1},$$

is a solution to the equation in (5) for any $\lambda > 0$, with initial velocity $\lambda^{h+1} g(\lambda x)$. We have

$$\lambda^{h+1} \|g(\lambda \cdot)\|_{L^q} = \lambda^{h+1-\frac{n}{p}} \|g\|_{L^q},$$

so that the L^q norm is invariant if, and only if, $h+1 = n/q$. Hence, we find $q = \frac{n(\sigma-1)}{\sigma+1}$.

In the next result, for $\sigma > \sigma_c(n)$, where $\sigma_c(n)$ is the Strauss index, we prove the existence of a weak global solution to (5), but without to introduce mixed norm spaces with weight as done in [11].

Theorem 3 [Global existence] *Let $n \geq 3$ be an odd integer and r, p given by (9). If $\sigma_c(n) < \sigma < \frac{n+3}{n-1}$, then there exists $\epsilon > 0$ such that for any initial data*

$$g \in L^p_{\text{rad}} \cap L^2_{\text{rad}}, \quad \text{with } \|g\|_{L^p \cap L^2} \leq \epsilon,$$

there exists a unique weak global solution to (5)

$$u \in C([0, \infty), L^r(\mathbb{R}^n)).$$

Moreover, u satisfies

$$\|u(t, \cdot)\|_{L^r} \lesssim (1+t)^{1-n/p+n/r} \|g\|_{L^p \cap L^2}. \quad (10)$$

Remark 3 The assumption $g \in L^p_{\text{rad}} \cap L^2_{\text{rad}}$ implies $g \in L^{\frac{n(\sigma-1)}{\sigma+1}}$, thanks to

$$p < \frac{n(\sigma-1)}{\sigma+1} < 2 \Leftrightarrow \sigma_c(n) < \sigma < \frac{n+2}{n-2}.$$

We complete this section with a remark about the nonexistence of global solution to (5).

Remark 4 For $n \geq 3$, Sideris [19] obtained the existence of local solution $u \in \mathcal{C}([0, T], L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n))$ to (5), with initial data $g \in C_0^\infty(\mathbb{R}^n)$. Moreover, under the assumption $1 \leq \sigma < \sigma_c(n)$ and $\int g > 0$, he proved that T is necessarily finite. Indeed, one can apply the arguments of the proof of Theorem 2 in [19] for all local solution $u \in \mathcal{C}([0, T], L^q(\mathbb{R}^n))$, with $q > \sigma_c(n)$. Therefore, the condition $\sigma > \sigma_c(n)$ in Theorem 3 cannot be relaxed.

3 Kernel estimates in odd dimension

We simply denote a radial function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ (i.e., $g(x_1) = g(x_2)$ for $|x_1| = |x_2|$) with abuse of notation, as $g(x) = g(r)$ for $r = |x|$, and L_{rad}^p is the set of radial functions belonging to $L^p(\mathbb{R}^n)$ doted with standard norm

$$\|g\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |g(x)|^p dx \right)^{1/p} = |S^{n-1}| \left(\int_0^\infty |g(r)|^p r^{n-1} dr \right)^{1/p},$$

where $|S^{n-1}|$ is the area of the unit sphere S^{n-1} .

Consider the Cauchy problem in polar coordinates associated with (1); i.e. $u(t, x) = u(t, r)$ radial, for $r = |x| > 0$ and $t > 0$ given by

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t, r) - \frac{\partial^2}{\partial r^2} u(t, r) - \frac{n-1}{r} \frac{\partial}{\partial r} u(t, r) = 0, \\ u(0, r) = 0, \quad \frac{\partial}{\partial t} u(0, r) = g(r). \end{cases} \quad (11)$$

The solution of this problem can be written as $u(t, r) = \int_0^\infty K(t, r, y) g(y) dy$, where the kernel is given by

$$K(t, r, y) = \sqrt{\pi/2} x^{-\alpha} y^{\alpha+1} \int_0^{+\infty} J_\alpha(xz) J_\alpha(yz) J_{1/2}(tz) (tz)^{1/2} dz$$

where J_α is the Bessel function of the first type with order α (Lemma 1.5 in [1]). Using a result from Macdonald [13] about integrals involving the product of three Bessel functions we have

$$K(t, r, y) = \begin{cases} \pi^{-1} \cos\left(\frac{\pi}{2}(n-2)\right) r^{-\frac{n-1}{2}} y^{\frac{n-1}{2}} \mathbb{Q}_{\frac{n-3}{2}}\left(\frac{t^2 - y^2 - r^2}{2ry}\right), & \text{if } y < t - r \\ \frac{1}{2} r^{-\frac{n-1}{2}} y^{\frac{n-1}{2}} \mathbb{P}_{\frac{n-3}{2}}\left(\frac{y^2 + r^2 - t^2}{2ry}\right), & \text{if } |r - t| < y < r + t \\ 0, & \text{if } y < r - t \text{ or } y > r + t \end{cases}$$

where for $\beta = (n-3)/2$, $\mathbb{P}_\beta(z)$ and $\mathbb{Q}_\beta(z)$ are the Legendre functions of first and second type, respectively. In dimension three, we recover the D'Alembert's formula

$$u(t, r) = \frac{1}{2r} \int_{|r-t|}^{r+t} yg(y) dy = \frac{1}{2r} \int_{|r-t|-r}^t (y+r)g(y+r) dy.$$

In particular, when β is an integer greater than -1 , i.e. n is odd, there exists substantial reduction in the kernel $K(t, r, y)$ (for more details, see [13])

$$K(t, r, y) = \begin{cases} \frac{1}{2} r^{-\frac{n-1}{2}} y^{\frac{n-1}{2}} \mathbb{P}_{\frac{n-3}{2}}\left(\frac{y^2 + r^2 - t^2}{2ry}\right), & \text{if } |r - t| < y < r + t \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

Moreover, $\mathbb{P}_{(n-3)/2}\left(\frac{y^2 + r^2 - t^2}{2ry}\right)$ is uniformly bounded for $|r - t| < y < r + t$. When n is even, the behavior of the kernel is radically modified. Now, the kernel is no longer supported in the region $|r - t| < y < r + t$, and some additional logarithmic singularities appear on the kernel $K(t, r, y)$. We refer the Lemma 3.2 in [1] to apprise these comments.

4 Proof of Theorem 1

The explicit solution $u(t, x)$ of the Cauchy problem (1) is given by

$$u(t, x) = \int_0^\infty K(t, |x|, y)g(y)dy,$$

where $K(t, |x|, y)$ satisfies (12). To estimate $\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)}^q$ we may write

$$\int_{\mathbb{R}^n} |u(x, t)|^q dx \leq G_1(t) + G_2(t)$$

with

$$G_1(t) = \int_{|x| \leq t} \left| |x|^{-\frac{n-1}{2}} \int_{t-2|x|}^t (y + |x|)^{\frac{n-1}{2}} g(y + |x|) dy \right|^q dx$$

and

$$G_2(t) = \int_{|x| \geq t} \left| |x|^{-\frac{n-1}{2}} \int_{-t}^t (y + |x|)^{\frac{n-1}{2}} g(y + |x|) dy \right|^q dx.$$

To deal with G_2 for $q = 1$, we use that $(y + |x|)/|x|$ is bounded and on the basis of Minkowski integral inequality, we have $\|u(t, \cdot)\|_{L^q} \lesssim t \|g\|_{L^1}$. If $1 < q$, using the Hölder inequality (with $\gamma' = q$ and $\gamma = q/(q - 1)$), for $|x| \geq t$ we have

$$\begin{aligned} \int_{-t}^t (y + |x|)^{\frac{n-1}{2}} g(y + |x|) dy &= \int_{-t}^t (y + |x|)^{\frac{n-1}{2} - \frac{n-1}{\gamma} + \frac{n-1}{\gamma}} |g(y + |x|)|^{1/\gamma + 1/\gamma'} dy \\ &\leq \left(\int_{-t}^t (y + |x|)^{n-1} |g(y + |x|)| dy \right)^{1/\gamma} \left(\int_{-t}^t (y + |x|)^{q(n-1)(1/2 - 1/\gamma)} |g(y + |x|)| dy \right)^{1/q}. \end{aligned}$$

Since $g \in L^1_{\text{rad}}(\mathbb{R}^n)$ implies that $s^{n-1}g(s) \in L^1(\mathbb{R})$, then

$$G_2(t) \lesssim \|g\|_{L^1}^{q-1} \int_{|x| \geq t} \frac{1}{|x|^{q\frac{n-1}{2}}} \left(\int_{-t}^t (y + |x|)^{(n-1)\frac{2-q}{2}} |g(y + |x|)| dy \right) dx.$$

Now, we consider two cases: If $q \leq 2$, $|y| \leq t \leq |x|$ implies that $((y + |x|)/|x|)^{(n-1)\frac{2-q}{2}} \leq 2^{(n-1)\frac{2-q}{2}}$, and that

$$G_2(t) \lesssim \|g\|_{L^1}^{q-1} \int_{|x| \geq t} \frac{1}{|x|^{(n-1)(q-1)}} \left(\int_{-t}^t |g(y + |x|)| dy \right) dx.$$

Using Fubini theorem

$$G_2(t) \lesssim t^{-(n-1)(q-1)} \|g\|_{L^1}^{q-1} \int_{-t}^t \int_{|x| \geq t} |g(y + |x|)| dx dy \lesssim t^{(1-n+n/q)q} \|g\|_{L^1}^q.$$

On the other hand, for $2 < q < \frac{2n}{n-1}$, when we use $(y + |x|)^{(n-1)\frac{2-q}{2}} \leq (y + t)^{(n-1)\frac{2-q}{2}}$ and the Fubini theorem, we have

$$\begin{aligned} G_2(t)^{1/q} &\lesssim t^{-(n-1)/2} \|g\|_{L^1}^{1-1/q} \left(\int_{-t}^t (y + t)^{(n-1)\frac{2-q}{2}} \int_{|x| \geq t} |g(y + |x|)| dx dy \right)^{1/q} \\ &\lesssim t^{1-n+n/q} \|g\|_{L^1}. \end{aligned}$$

To deal with G_1 , we split the integral in two parts, namely $|x| \leq t/2$ and $t/2 \leq |x| \leq t$. On the ring, we follow as we did for $G_2(t)$. On the other hand,

$$\int_{t-2|x|}^t (y + |x|)^{\frac{n-1}{2}} g(y + |x|) dy \leq \frac{1}{(t - |x|)^{(n-1)/2}} \int_{t-2|x|}^t (y + |x|)^{n-1} |g(y + |x|)| dy.$$

Then, for $q < \frac{2n}{n-1}$

$$G_1(t)^{1/q} \lesssim t^{-(n-1)/2} \|g\|_{L^1} \left(\int_{|x| \leq t/2} |x|^{-q(n-1)/2} dx \right)^{1/q} \lesssim t^{1-n+n/q} \|g\|_{L^1}.$$

Summing up, for all $1 \leq q < \frac{2n}{n-1}$, we proved that the solution u of the Cauchy problem (1) satisfies

$$\|u(t, \cdot)\|_{L^q} \lesssim t^{1-n+n/q} \|g\|_{L^1}, \quad \text{for all } t > 0.$$

□

Remark 5 Recently in [5], Garrigós and Seeger characterized radial multiplier operators acting on L_{rad}^p for $1 < p < 2n/(n+1)$ that may be applied for the wave operator S_t . Let $\widehat{S}_1(\xi) = m(\xi) \widehat{f}(\xi)$ be the Fourier representation of the free wave equation operator S_t for $t = 1$, where $m(\xi) = |\xi|^{-1} \sin |\xi|$. In view of Theorem 1.2 in [5], for n odd, we have that the kernel $K \doteq \mathcal{F}^{-1}[m(\cdot)]$ belongs to the homogenous Besov space $\dot{B}_{r,\infty}^p(\mathbb{R}^n)$ for $r \doteq n\left(\frac{1}{p} - 1\right)$, i.e., for ϕ a smooth function compactly supported in $\mathbb{R}_+$ with integral equal 1 and $\Phi_s \doteq \mathcal{F}^{-1}[\phi(s|\cdot|)]$ we have

$$\sup_{s>0} s^{n\left(\frac{1}{p}-1\right)} \|\Phi_{1/s} * K\|_{L^p(\mathbb{R}^n)} < \infty,$$

for $\frac{1}{2} + \frac{1}{2n} < \frac{1}{p} < 1$.

5 Application to the semilinear problem: Proof of Theorems 2 and 3

Let us consider the Cauchy problem for the semilinear wave equation (5). According to the Duhamel principle, the solution have to satisfy

$$u(t, x) = K_1(t, x) *_{(x)} g(x) + \int_0^t K_1(t-s, x) *_{(x)} |u(s, x)|^\sigma ds,$$

where

$$\widehat{K}_1(t, |\xi|) = \frac{\sin t|\xi|}{|\xi|}. \quad (13)$$

Proof (Theorem 2) With $r = \frac{2n}{n-1}$, we define the Banach space

$$X \doteq \left\{ u \in \mathcal{C}([0, T], L^r(\mathbb{R}^n)) : \|u\|_X \doteq \sup_{t \in [0, T]} (\|u(t, \cdot)\|_{L^r}) < \infty \right\}.$$

For any $u \in X$, we put

$$Pu \doteq u^0 + Fu,$$

where $u^0(t, x) = K_1(t, x) *_{(x)} g(x)$ and $Fu(t, x) = \int_0^t K_1(t-s, x) *_{(x)} |u(s, x)|^q ds$.

Thanks to $g \in L^2$ and Gagliardo–Nirenberg’s inequality, we have

$$\begin{aligned} \|u^0(t + \Delta t, \cdot) - u^0(t, \cdot)\|_{L^r} &\lesssim \left\| \nabla \left(u^0(t + \Delta t, \cdot) - u^0(t, \cdot) \right) \right\|_{L^2}^\theta \\ &\quad \times \|u^0(t + \Delta t, \cdot) - u^0(t, \cdot)\|_{L^2}^{(1-\theta)}, \end{aligned}$$

with $\theta = n(r - 2)/2r$, and the right-hand side goes to zero as $\Delta t \rightarrow 0$ (see [19], p. 381). Therefore $u^0 \in C([0, \infty), L^r(\mathbb{R}^n))$.

Using Minkowski integral inequality and (7) with $q = r$ and $\sigma p = r$ we get

$$\begin{aligned} \|Fu(t, \cdot)\|_{L^r} &\leq \int_0^t \|K_1(t - s, \cdot) *_{(x)} |u(s, \cdot)|^\sigma\|_{L^r} \, ds \\ &\lesssim \int_0^t (t - s)^{1-n/p+n/r} \| |u(s, \cdot)|^\sigma \|_{L^p} \, ds \\ &\lesssim \|u\|_X^\sigma \int_0^t (t - s)^{1-n/p+n/r} \, ds \lesssim t^{2-n/p+n/r} \|u\|_X^\sigma, \end{aligned}$$

for all $1 \leq \sigma < \frac{n+3}{n-1}$. Therefore P maps X into itself and, if T is sufficiently small, the existence of a unique local solution follows by contraction argument (will be explained with more details in the proof of the next Theorem). $\square$

Proof (Theorem 3) With $r = \frac{2n}{n-1}$ we define the space

$$X \doteq \{u \in C([0, \infty), L^r(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

equipped with norm

$$\|u\|_X \doteq \sup_{t \in [0, \infty)} \left((1+t)^{n/p-1-n/r} \|u(t, \cdot)\|_{L^r} \right) < \infty \},$$

where $r = p\sigma$. For any $u \in X$, we define

$$Pu \doteq K_1(t, x) *_{(x)} g(x) + Fu,$$

where $Fu(t, x) = \int_0^t K_1(t - s, x) *_{(x)} |u(s, x)|^\sigma \, ds$. Using (8), we have

$$\|K_1(t, x) *_{(x)} g(x)\|_X \lesssim \|g\|_{L^p \cap L^2}. \quad (14)$$

Using Minkowski integral inequality and estimate (7) with $q = r$ and $\sigma p = r$ we have

$$\begin{aligned} \|Fu(t, \cdot)\|_{L^r} &\leq \int_0^{[t-1]^+} \|K_1(t - s, \cdot) *_{(x)} |u(s, \cdot)|^\sigma\|_{L^r} \, ds \\ &\quad + \int_{[t-1]^+}^t \|K_1(t - s, \cdot) *_{(x)} |u(s, \cdot)|^\sigma\|_{L^r} \, ds \\ &\lesssim \int_0^{[t-1]^+} (1+t-s)^{1-n/p+n/r} \| |u(s, \cdot)|^\sigma \|_{L^p} \, ds + \int_{[t-1]^+}^t (t-s)^{1-n/p+n/r} \| |u(s, \cdot)|^\sigma \|_{L^p} \, ds, \end{aligned}$$

where $[x]^+ = x$ for $x \geq 0$ and zero otherwise. In the first integral, we avoid the singularity coming from (7) by using that $t - s \geq 1$.

To deal with the first integral, by definition of X

$$\| |u(s, \cdot)|^\sigma \|_{L^p} = \|u(s, \cdot)\|_{L^r}^\sigma \lesssim (1+s)^{(1-n/p+n/r)\sigma} \|u\|_X^\sigma.$$

The hypothesis $\sigma > \sigma_c(n)$ is equivalent to $(n/p - 1 - n/r)\sigma > 1$, so we can apply Proposition 5 in [2] to

$$\|u\|_X^\sigma \int_0^t (1+t-s)^{1-n/p+n/r} (1+s)^{(1-n/p+n/r)\sigma} ds \lesssim (1+t)^{1-n/p+n/r} \|u\|_X^\sigma.$$

To deal with the second integral, we use that for $s \in ([t-1]^+, t)$

$$\begin{aligned} \|u(s, \cdot)\|_{L^r}^\sigma &= (1+s)^{(1-n/p+n/r)\sigma} (1+s)^{(n/p-1-n/r)\sigma} \|u(s, \cdot)\|_{L^r}^\sigma \\ &\lesssim (1+t)^{(1-n/p+n/r)\sigma} \sup_{s \in [0, \infty)} ((1+s)^{(n/p-1-n/r)\sigma} \|u(s, \cdot)\|_{L^r}^\sigma) \\ &\lesssim (1+t)^{(1-n/p+n/r)\sigma} \|u\|_X^\sigma. \end{aligned}$$

Since $\alpha \doteq 1 - n/p + n/r > -1$ thanks to $\sigma < \frac{n+3}{n-1}$, we may estimate

$$\int_{[t-1]^+}^t (t-s)^\alpha ds \leq \int_0^1 r^\alpha dr = \frac{1}{1+\alpha},$$

and

$$\int_{[t-1]^+}^t (t-s)^{1-n+n/q} \| |u(s, \cdot)|^q \|_{L^1} ds \leq (1+t)^{(1-n/p+n/r)\sigma} \|u\|_X^\sigma.$$

Hence, it follows

$$\|Pu\|_X \lesssim \|g\|_{L^p \cap L^2} + C\|u\|_X^\sigma. \quad (15)$$

This leads to $Pu \in L^\infty([0, \infty), L^r(\mathbb{R}^n))$. Moreover, Young's inequality implies

$$\| |u|^\sigma - |v|^\sigma \| \lesssim |u-v|(|u| + |v|)^{\sigma-1},$$

and using Hölder's inequality we conclude

$$\|Pu - Pv\|_X \lesssim \|u - v\|_X (\|u\|_X^{\sigma-1} + \|v\|_X^{\sigma-1}), \quad (16)$$

for any $u, v \in X$ (see, for instance, [3]). Now, we recall that in Theorem 2 we already proved that $u^{(0)}(t, x) = P(u^{(-1)}) = K_1(t, x) *_{(x)} g(x) \in X$. Moreover, similarly to (16), we prove that $Pu \in C([0, \infty), L^r(\mathbb{R}^n))$, i.e., P maps X into itself. Then, estimates (15)–(16) lead to the existence of a unique solution to $u = Pu$. In fact, taking the recurrence sequence $u^{(-1)} = 0$, $u^{(j)} = P(u^{(j-1)})$ for $j = 0, 1, 2, \dots$, we apply (15) with small $\|g\|_{L^p \cap L^2} = \epsilon$, and we inductively see that

$$\|u^{(j)}\|_X \leq C_1 \epsilon, \quad (17)$$

where $C_1 = 2C$ for any $\epsilon \in [0, \epsilon_0]$ with sufficiently small $\epsilon_0 = \epsilon_0(C_1)$.

After checking the uniform estimate (17) we use (16) to find

$$\|u^{(j+1)} - u^{(j)}\|_X \leq C\epsilon^{p-1}, \quad \|u^{(j+1)} - u^{(j)}\|_X \leq 2^{-1} \|u^{(j)} - u^{(j-1)}\|_X, \quad (18)$$

for sufficiently small $\epsilon \leq \epsilon_0$. From (18), we inductively obtain $\|u^{(j)} - u^{(j-1)}\|_X \leq C2^{-j}$ so that $\{u^{(j)}\}$ is a Cauchy sequence in X . For any $m \in \mathbb{N}^*$, let

$$\begin{aligned} X(m) &= \{u \in C([0, m], L^r(\mathbb{R}^n)) : \|u\|_{X(m)} < \infty\}, \\ \|u\|_{X(m)} &\doteq \max_{t \in [0, m]} \left((1+t)^{n/p-1-n/r} \|u(t, \cdot)\|_{L^r} \right). \end{aligned}$$

It is clear that $u^{(j,m)}$, where $u^{(j,m)} \doteq u^{(j)}(t, \cdot)|_{t \in [0, m]}$, is a Cauchy sequence in the Banach space $(X(m), \|\cdot\|_{X(m)})$. Hence, it converges to a unique $u^{(\dagger, m)} \in X(m)$. It is evident that

for any $t \geq 0$, we may consistently define $u(t, \cdot) = u^{(\dagger, m)}(t, \cdot)$, for any $m \geq t$. This gives $Pu(t, \cdot) = u(t, \cdot)$ in L^r ; that is, u is the unique global solution in X to (5). Moreover, $\|u^{(\dagger, m)}\|_{X(m)} \leq C$ for some $C > 0$, uniformly with respect to m . Therefore, $u \in X$ and $\|u\|_X \leq C$. The decay estimates (10) immediately follows from $\|u\|_X < \infty$. $\square$

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