Global Kähler–Ricci flow on complete non-compact manifolds

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Abstract In this paper, we study the global Kähler–Ricci flow on a complete non-compact Kähler manifold. We prove the following result. Assume that (M, g_0) is a complete non-compact Kähler manifold such that there is a potential function f of the Ricci tensor, i.e.,

$$R_{i\bar{j}}(g_0) = f_{i\bar{j}}.$$

Assume that the quantity $|f|_{C^0} + |\nabla_{g_0} f|_{C^0}$ is finite and the L^2 Sobolev inequality holds true on (M, g_0) . Then the Kähler–Ricci flow with the initial metric g_0 either blows up at finite time or infinite time to Ricci-flat metric or exists globally with Ricci-flat limit at infinite time. Related results are also discussed.

Keywords Global Kähler–Ricci flow · Non-local collapsing · Ricci flat

Mathematics Subject Classification 53C44 · 32Q20 · 58E11

1 Introduction

In the study of Ricci flow, the main problem is to understand the global behavior of it [12]. Since one may meet singularities of Ricci flow in finite time, it is interesting to know the singularity structure of the Ricci flow. Hamilton conjectured that after suitable normalization, the Ricci flow at the blow-up point looks like a Ricci soliton. In this paper, we study the global behavior of the Kähler–Ricci flow on a complete non-compact manifold with bounded curvature and we show in this case Hamilton’s conjecture is right. We remark that the local existence of the flow has been proved by Shi [23] and we use this local flow in this paper.

We prove the following result.

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Theorem 1 Assume that (M^n, g_0) ($n = \dim_{\mathbb{C}} M$) is a complete non-compact Kähler manifold with bounded curvature and there is a potential function f of the Ricci tensor, i.e.,

$$R_{i\bar{j}}(g_0) = f_{i\bar{j}}.$$

Assume the quantity $|f|_{C^0} + |\nabla_{g_0} f|_{C^0}$ is finite. We also assume that the L^2 Sobolev inequality holds true on (M, g_0) . Then the Kähler–Ricci flow

$$\partial_t g_{i\bar{j}} = -R_{i\bar{j}}(g) \quad (1)$$

with the initial metric g_0 either blows up at finite time or infinite time to Ricci-flat metric or exists globally with Ricci-flat limit at infinite time.

Our assumptions are natural. Note that in the statement of our result, we assume L^2 Sobolev inequality, namely there exists a global constant $C > 0$ such that for any $u \in C_0^1(M, g_0)$,

$$C \left(\int_M |u|^{2n/(n-1)} dV \right)^{(n-1)/n} \leq C \int_M (|\nabla_{g_0} u|^2 + u^2) dV,$$

which, plus the bounded curvature assumption, implies the Gross Log–Sobolev inequality and the non-local collapsing for the metric g_0 . Hence the W -functional introduced by Perelman [20] makes sense on the Riemannian manifold (M, g_0) . Along the Ricci flow, as long as the curvature is bounded, all the Riemannian metrics are equivalent each other before the first singularity. This implies that the W -functional is well defined along the Ricci flow before it blows up. In our proof of Theorem 1, we shall use the equivalent inequality for the W -functional (and see [9] for more discussions).

Without the non-local collapsing for the metric g_0 , we have the following result.

Theorem 2 Assume that (M, g_0) is a complete non-compact Kähler manifold with bounded curvature and there is a potential function f of the Ricci tensor, i.e.,

$$R_{i\bar{j}}(g_0) = f_{i\bar{j}}.$$

Assume only $|\nabla_{g_0} f|_{C^0}$ is finite and the initial metric has nonnegative bisectional curvature. There is a global Kähler–Ricci flow with the initial metric g_0 .

To understand the global behavior of the Kähler–Ricci flow established by Shi [23], we need to use some invariants for the various curvature quantities. The key problem is whether the positivity of the Ricci curvature under the Kähler–Ricci flow holds true, which is central for understanding the convergence of the Kähler–Ricci flow on Kähler–Einstein manifolds. It is well known that the Kähler–Ricci flow preserves the positivity of the curvature operator, the positivity of the bisectional curvature [2, 14] and the positivity of the scalar curvature. Phong and Sturm [18] have found that the positivity of the Ricci curvature is preserved by the Kähler–Ricci flow on compact Kähler manifolds of dimension two, under the assumption that the sum of any two eigenvalues of the traceless curvature operator on traceless $(1, 1)$ -forms is nonnegative. Since their argument only uses the maximum principle, one may easily extend their result to a complete non-compact manifold with the same curvature assumptions above. As applications of our Theorem 1, we can prove that in either cases as above, there is a global Kähler–Ricci flow on the Kähler manifold with Ricci-flat metric as its limit at time infinity.

We remark that one may formulate more convergence results using the invariant set constructed by Wilking [25], who finds almost all invariant curvature conditions of Ricci flows. See also related interesting works [3–7, 11, 15, 16, 24, 26].

The plan of the paper is below. In Sect. 2, we consider the non-local collapsing result obtained by using Perelman's W -functional and the scalar curvature bound found by Chow in our Kähler–Ricci flow case. In Sects. 3 and 4, we prove Theorems 1 and 2. We also consider some consequences of our Theorem 1 in the last section.

2 Non-local collapsing and Perelman's W -functional and uniform scalar curvature bound

We now recall some facts about Perelman's W -functional on a closed Riemannian manifold (M^n, g) . First, Perelman [20] defines the F -functional as

$$F(g, f) = \int_M (R + |\nabla f|^2) dm$$

where $dm = e^{-f} dv_g$ is a fixed measure and R is the scalar curvature of the metric g . This functional is an extension of Hilbert action on the Riemannian metric space on M . Let v and h be the variations of g and f , respectively. Since dm is fixed, we have the relation $h = tr_h v/2$. Then we have

$$\delta F(v, h) = - \int (v, Rc(g) + D^2 f) dm$$

where $Rc(g)$ is the Ricci tensor of g and $D^2 f$ is the Hessian matrix of the function f . For any fixed parameter $\tau > 0$, the W -functional is defined by

$$W(g, f) = W(g, f, \tau) = \int_M [\tau (R + |\nabla f|^2) + f - n] (4\pi\tau)^{-n/2} dm.$$

That is,

$$W(g, f) = (4\pi\tau)^{-n/2} \tau F(g, f) + (4\pi\tau)^{-n/2} \int_M (f - n) dm.$$

The importance of W -functional is that it is a diffeomorphism invariant. Then we have

$$\delta W(v, h) = -\tau \int_M \left(v, Rc(g) + D^2 f - \frac{1}{2\tau} g \right) (4\pi\tau)^{-n/2} dm.$$

Hence, the L^2 gradient flow of $W(g, f)$ is

$$\partial_t g = -2 \left(Rc(g) + D^2 f - \frac{1}{2\tau} g \right)$$

with

$$f_t = -\Delta_g f - R + \frac{n}{2\tau}.$$

Let $\phi(t)$ be the flow generated by the time-dependent vector field $\nabla_g f$. Let $\bar{g}(t) = \phi(t)^* g(t)$ and $\bar{f}(t) = \phi(t)^* f(t)$. Then we have

$$\partial_t \bar{g} = -2 \left(Rc(\bar{g}) - \frac{1}{2\tau} \bar{g} \right)$$

with

$$\bar{f}_t = -\Delta_{\bar{g}} \bar{f} - \bar{R} + |\nabla_{\bar{g}} \bar{f}|^2 + \frac{n}{2\tau}.$$

Let $C(s) = 1 - \frac{s}{\tau}$,

$$t(s) = -\tau \log C(s).$$

and

$$\check{g}(s) = C(s)\bar{g}(t(s)).$$

Then we have

$$\partial_t \check{g} = -2Rc(\check{g})$$

and

$$\check{f}_t = -\Delta_{\check{g}} \check{f} - \check{R} + \left| \nabla_{\check{g}} \check{f} \right|^2 + \frac{n}{2\tau C(s)}.$$

Define

$$\mu(g, \tau) = \inf \left\{ W(g, f, \tau), \int_M dm = 1 \right\}.$$

Then for any fixed $\tau > 0$, along the Ricci flow, $\mu(g(t), \tau)$ is non-decreasing.

We assume that the scalar curvature of the Kähler–Ricci flow is uniformly bounded, which can be proved in our situation (see Sect. 3) as in Theorem 1. In a Kähler manifold (M^n, g) (where n is the complex dimension), we normalize the W -functional as

$$W(g, f, \tau) = (4\pi\tau)^{-n} \int_M [2\tau(R + |\nabla f|^2) + f - 2n] e^{-f} dV$$

with $(4\pi\tau)^{-n} \int_M e^{-f} dV = 1$ and define

$$\mu(g, \tau) = \inf_{f | (4\pi\tau)^{-n} \int_M e^{-f} dV = 1} W(g, f, \tau)$$

Assume now that (M, g) is a complete non-compact Riemannian manifold. Note that for $u = e^{-f/2}$, we have

$$(4\pi\tau)^{-n} \int_M u^2 dV = 1$$

and

$$W(g, f, \tau) = (4\pi\tau)^{-n} \int_M [2\tau(Ru^2 + 4|\nabla u|^2) + u^2 \log u^2 - 2nu^2] dV$$

which will be written as $W(g, u, \tau)$. This latter formulation allows us to define the W -functional for $u \in C_0^\infty(M)$ (that is $e^{-f/2} \in C_0^\infty(M)$ and this is noticed by Perelman, see Remark 3.2 in [20], see also p. 240 of [9]) and this fact is very useful in analyzing the properties of W -functional. Applying the decay estimates of Hamilton [12] and Dai and Ma [10], Li–Yau–Hamilton gradient estimate [9], and Perelman’s Proposition 9.1 [20], we can show that W -functional is also non-decreasing along the Ricci flow on a complete non-compact Riemannian manifold with the initial bounded curvature. Since the argument is well known to experts, we omit the detail.

The important application about W -functional is the no local collapsing result due to Perelman: *using this property and the scalar curvature bound of the Ricci flow on $(0, T)$ with $T < \infty$, we can get the no local collapsing at T* , and the no collapsing data depend only on the initial metric and the global scalar curvature bound. For the full proof of this result, one

may refer to Perelman's paper [20]. See also Sesum–Tian's paper [22] on compact Kähler manifolds. We remark that the argument can be carried out to our complete non-compact case. Since the argument is almost the same, we omit the detail.

We now recall some formulae from Chow's paper [8]. Along the Kähler–Ricci flow (1), we have

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det (g_{k\bar{l}}) = \partial_i \partial_{\bar{j}} f, \quad R = \Delta f,$$

and

$$\partial_t R_{i\bar{j}} = -\nabla_i \nabla_{\bar{j}} \left(g^{k\bar{l}} \partial_t g_{k\bar{l}} \right) = \nabla_i \nabla_{\bar{j}} R,$$

which implies that $\partial_t \partial_{\bar{j}} (\partial_t - \Delta) f = 0$. We can normalize f such that it satisfies $\partial_t f = \Delta f$ with the uniform estimate $t|\nabla f|^2 + f^2 \leq C$, where C is some uniform constant depending only on the initial data f_0 and $|\nabla f|^2 = g^{k\bar{l}} f_k f_{\bar{l}}$.

Recall that the scalar curvature evolution equation is

$$(\partial_t - \Delta)R = |R_{i\bar{j}}|^2 \geq \frac{1}{n} R^2. \quad (2)$$

Applying the maximum principle, we get the lower bound $R \geq -\frac{n}{t}$. We remark that we can use the maximum principle [1] for the quantities below before the first singularity of the Ricci flow since we always use the local solution obtained by Shi [23] with bounded curvature. The following result is from Chow's paper [8].

Theorem 3 *Assume that there is a potential function f on M such that*

$$R_{i\bar{j}}(g_0) = f_{i\bar{j}}$$

with bounded gradient $|\nabla_{g_0} f|_{C^0}$. Then there is a smooth function $f = f(t)$ such that

$$R_{i\bar{j}}(g(t)) = f(t)_{i\bar{j}}$$

and we also have some uniform constant $C_0 > 0$ depending only on the initial metric such that

$$R + |\nabla_{g(t)} f|^2 \leq C_0.$$

The proof of Theorem 4 below is similar to Chow's argument [8], but some new barrier function should be constructed.

Theorem 4 *Assume the same condition for the global Kähler–Ricci flow $g(t)$ with the potential function f being a bound function. Then we also have some uniform constant $C_1 > 0$ depending only on the initial metric and the dimension n such that*

$$R + |\nabla_{g(t)} f|^2 \leq C_1/t.$$

Proof of Theorem 4 By direct computation [8], we have

$$(\partial_t - \Delta) [t (|\nabla f|^2 + R)] \leq -\frac{2t}{n} R^2 + R + 2|\nabla f|^2. \quad (3)$$

Note that

$$\frac{t}{n} R^2 = \frac{t}{n} (R + 2|\nabla f|^2)^2 - \frac{4t|\nabla f|^2}{n} (R + |\nabla f|^2).$$

Then we get from (3) that

$$(\partial_t - \Delta) [t (|\nabla f|^2 + R)] \leq -\frac{t}{n} (R + 2|\nabla f|^2)^2 - \frac{t}{n} R^2 + R + 2|\nabla f|^2 + \frac{4t|\nabla f|^2}{n} (R + |\nabla f|^2). \quad (4)$$

Assume that $R \geq 0$, using $|\nabla f|^2 \leq \frac{C}{t}$, we obtain from (4) that

$$(\partial_t - \Delta) [t (|\nabla f|^2 + R)] \leq -\frac{t}{n} (R + 2|\nabla f|^2)^2 - \frac{t}{n} R^2 + C (R + 2|\nabla f|^2). \quad (5)$$

Note that the term $R + 2|\nabla f|^2$ is bounded by C_0 , and if $R \geq 0$ and $\frac{t}{n} (R + 2|\nabla f|^2) \geq 2CC_0$, we then have

$$(\partial_t - \Delta) [t (|\nabla f|^2 + R)] \leq 0.$$

As in [23], any $T > 0$, we construct a distance like positive function $h(x)$ such that for any $t \in [0, T]$, $\Delta_{g(t)} h \leq 1$, and $h(x) \rightarrow \infty$ as $d(x, \bar{x}) \rightarrow \infty$, where $\bar{x}$ is a fixed point in M . Define the domain

$$\Omega_T = \left\{ (x, t) \in M \times (0, T], \quad R \geq 0, \quad \frac{t}{n} (R + 2|\nabla f|^2) \geq 2CC_0 \right\}.$$

Note that in Ω_T , for $\epsilon > 0$,

$$\begin{aligned} & (\partial_t - \Delta) [t (|\nabla f|^2 + R) + 2\epsilon t - \epsilon h(x)] \\ &= (\partial_t - \Delta) [t (|\nabla f|^2 + R)] - 2\epsilon + \epsilon \Delta_{g(t)} h(x) \leq -\epsilon < 0. \end{aligned}$$

Since the region U_ϵ for the function $t (|\nabla f|^2 + R) - 2\epsilon t - \epsilon h(x) \geq 2CC_0$ is compact and using the maximum principle in U_ϵ , we know that

$$t (|\nabla f|^2 + R) \leq 2CC_0 + 2\epsilon t + \epsilon h(x), \quad (x, t) \in U_\epsilon \cap \Omega_T.$$

Note that $\Omega_T = \bigcup_{\epsilon > 0} U_\epsilon$ and $U_\epsilon \subset U_{\epsilon'}$ for $\epsilon > \epsilon'$. Then for any $(x_0, t_0) \in \Omega_T$, we have some $\epsilon_0 > 0$ such that $(x_0, t_0) \in U_{\epsilon}$ for any $0 < \epsilon < \epsilon_0$ and

$$t (|\nabla f|^2 + R) (x_0, t_0) \leq 2CC_0 + 2\epsilon t_0 + \epsilon h(x_0).$$

Letting $\epsilon \rightarrow 0$, we have

$$t (|\nabla f|^2 + R) (x_0, t_0) \leq 2CC_0.$$

That is to say, in Ω_T , we have $t (|\nabla f|^2 + R) \leq 2CC_0$. Therefore, for any $T > 0$, we always have $|R| \leq 2 \max(CC_0, n)/t$ in $M \times (0, T)$. This completes the proof of Theorem 4. $\square$

3 Global flow and finite time blow up is Ricci flat

In this section, we shall prove Theorem 2. Using our recent result about extension result of Ricci flow [13, 21], which extends the beautiful result of Sesum on compact manifold, we know that if the Kähler–Ricci flow blows up at first finite time T , then the Ricci curvature must blow up at time T . Using the scalar curvature bound of the Kähler–Ricci flow in previous section, we can show the Ricci flatness of the blow-up limit of the flow.

Theorem 5 *Assume the conditions as in Theorem 1. If the Kahler–Ricci flow exists only in finite time, then its blow-up limit at the maximal time exists and the limit is Ricci flat.*

We remark that the blow-up limit or just limit in the results above and below is only for some subsequence, not for full blow-up sequence. The process to prove Theorem 5 is below. First, we note that by using Perelman’s non-local collapsing result at any finite time, we can apply Hamilton’s Cheeger–Gromov convergence theorem [9] to take the blow-up limit $g_\infty(t)$. Then we use the uniform scalar curvature bound of the original Ricci flow to obtain $R(g_\infty) = 0$, which implies the Ricci flatness of the limit metric g_∞ .

Proof for Theorem 5 Let $T < \infty$ be the first blow-up time of the Ricci flow $(M, g(t))$. Choose some constants $C > 0$, the finite time $t_j \rightarrow T-$ and take points x_j such that

$$\sup_{M \times [t_j - C^{-1}K_j^{-1}, t_j]} |Rm(g(t))| \leq CK_j$$

where $K_j = |Rm(g(t_j))|(x_j) \rightarrow \infty$. Define

$$g_j(t) := K_j g \left(t_j + K_j^{-1} t \right).$$

Clearly, we have $|Rm(g_j(t))| \leq C$ and $|R(g_j(t))| \leq C/K_j^{-1} \rightarrow 0$. Using Perelman’s non-local collapsing result, we know the injectivity radius bound condition along the flow (1). Then we can get the limit Ricci flow $g_\infty(t)$ of $g_j(t)$ with $|Rm(g_\infty(t))| \leq C$ and $R(g_\infty(t)) = 0$ for $t \in (-\infty, \omega)$. Using the scalar curvature evolution equation (2) we get that $Rc(g_\infty(t)) = 0$. This completes the proof of Theorem 5.

We now prove Theorem 2: assume that we have a local Kähler–Ricci flow with initial metric g_0 with bounded curvature and nonnegative bisectional curvature. Using the result of Bando–Mok, we know that the nonnegativity bisectional curvature is preserved along the flow. Hence the Ricci curvature is nonnegative along the Ricci flow. Using the scalar curvature bound and nonnegativity Ricci curvature condition, we get via the use of the elementary relation $R_{i\bar{j}} \leq Rg_{i\bar{j}}$ the uniform bound for the Ricci curvature. Then using Theorem 1.4 [13], we know that the Kähler–Ricci flow exists globally with bounded curvature.

4 Global Kahler–Ricci flow and its limit at $t = \infty$

In this section, we prove Theorem 1. Note that there is no local collapsing for the Ricci flow at any finite time according to the result of perelman mentioned in section two. If the curvature of the flow blows up at infinite time, we then can take the similar blow-up limit as in the finite time case [13] and show that its blow-up limit is Ricci flat. Otherwise, the curvature of the flow is uniformly bounded and by Theorem 4, we can simply take the limit, which is scalar flat at first. Using the evolution equation (2) again, we know that the limit is also Ricci flat. This completes the proof of Theorem 1.

We give two applications of Theorem 1. One is for the metric g_0 with nonnegative bisectional curvature, which implies the nonnegativity of the Ricci curvature. Note that the nonnegativity of the bisectional curvature property is preserved by Kahler–Ricci flow as proved by Bando–Mok as mentioned in the introduction. Note that in this case, the scalar curvature dominates other curvatures. Since we have global bound of the scalar curvature of the Ricci flow, we must have the global Kähler–Ricci flow with bounded curvature. Hence

by Theorem 1, the flow is global one and has its limit at $t = \infty$ the Ricci-flat metric. Then we have the following result, which may also be derived from the arguments or results in [17, 19].

Theorem 6 *Assume that (M, g) is a complete non-compact Kähler manifold with bounded curvature such that there is a potential function f of the Ricci tensor, i.e.,*

$$R_{i\bar{j}}(g_0) = f_{i\bar{j}}.$$

Assume the quantity $|f|_{C^0} + |\nabla_{g_0} f|_{C^0}$ is finite. Suppose the initial metric has non-local collapsing and the L^2 Sobolev inequality holds true on (M, g_0) . Assume also that the initial metric has nonnegative bisectional curvature. Then the Kähler–Ricci flow with the initial metric g_0 exists globally with Ricci-flat limit at infinite time.

The other is for complex dimension two, where we assume that (M, g) is a complete non-compact Kähler surface with the bounded curvature, positive Ricci curvature and positive partial curvature operator S in the case handled by Phong and Sturm [18]. We make this precise and define the partial curvature operator

$$S_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{1}{n} \left(S_{i\bar{j}} \delta_{k\bar{l}} + \delta_{i\bar{j}} S_{k\bar{l}} \right) + \frac{1}{n^2} R \delta_{i\bar{j}} \delta_{k\bar{l}}$$

acting on the space of traceless $(1, 1)$ forms on M . Here, $R_{i\bar{j}k\bar{l}}$ is the curvature tensor of g , $S_{i\bar{j}} = R_{i\bar{j}} - \frac{1}{n} R g_{i\bar{j}}$.

Then we have

Theorem 7 *Assume that (M, g) is a complete non-compact Kähler surface with bounded curvature such that there is a potential function f of the Ricci tensor, i.e.,*

$$R_{i\bar{j}}(g_0) = f_{i\bar{j}}.$$

Suppose the quantity $|f|_{C^0} + |\nabla_{g_0} f|_{C^0}$ is finite. Assume that the initial metric has non-local collapsing and the L^2 Sobolev inequality holds true on (M, g_0) . Assume also that the initial metric has nonnegative non-trivial Ricci curvature and the nonnegative sum of the two lowest eigenvalues of the operator S acting on the space of traceless $(1, 1)$ forms on M . Then the Kähler–Ricci flow with the initial metric g_0 exists globally with Ricci-flat limit at infinite time.

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