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On the well-posedness of the exp-Rabelo equation

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Abstract The exp-Rabelo equation describes pseudo-spherical surfaces. It is a nonlinear evolution equation. In this paper, the well-posedness of bounded from above solutions for the initial value problem associated with this equation is studied.

Keywords Existence · Uniqueness · Stability · Entropy solutions · Conservation laws · The exp-Rabelo equation

Mathematics Subject Classification 35G31 · 35L65 · 35L05

1 Introduction

Bäcklund transformations have been useful in the calculation of soliton solutions of certain nonlinear evolution equations of physical significance [9,22,25,26] restricted to one space variable x and a time coordinate t. The classical treatment of the surface transformations, which provide the origin of Bäcklund theory, was developed in [11]. Bäcklund transformations are local geometric transformations, which construct from a given surface of constant Gaussian curvature -1 a two parameter family of such surfaces. To find such transformations, one needs to solve a system of compatible ordinary differential equations [10].

In [15,16], the authors used the notion of differential equation for a function $u(t, x)$ that describes a pseudo-spherical surface, and they derived some Bäcklund transformations for

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nonlinear evolution equations which are the integrability condition $sl(2, R)$-valued linear problems [13,14,18,19,26].

In [20], the authors had derived some Bäcklund transformations for nonlinear evolution equations of the Ablowitz–Kaup–Newell–Segur (AKNS) class. These transformations explicitly express the new solutions in terms of the known solutions of the nonlinear evolution equations and corresponding wave functions which are solutions of the associated AKNS system [2,32].

In [17], the authors used Bäcklund transformations derived in [15,16] in the construction of exact soliton solutions for some nonlinear evolution equations describing pseudo-spherical surfaces which are beyond the AKNS class. In particular, they analyzed the following equation [3]:

$$\partial_x \left(\partial_t u + \alpha g(u)\partial_x u + \beta \partial_x u\right) = \gamma g'(u), \quad \alpha,\ \beta,\ \gamma \in \mathbb{R}, \tag{1.1}$$

where $g(u)$ is any solution of the linear ordinary differential equation

$$g''(u) + \mu g(u) = \theta, \quad \mu,\ \theta \in \mathbb{R}. \tag{1.2}$$

(1.1) include the sine-Gordon, sinh-Gordon and Liouville equations, in correspondence of $\alpha = 0$.

In [24], Rabelo proved that the system of the Eqs. (1.1) and (1.2) describes pseudo-spherical surfaces and possesses a zero-curvature representation with a parameter.

We consider (1.1) and assume that $\alpha \neq 0$.

When

$$\alpha = -1, \qquad \mu = 0, \qquad \theta = 1, \tag{1.3}$$

(1.2) reads

$$g''(u) = 1. \tag{1.4}$$

A solution of (1.4) is

$$g(u) = \frac{u^2}{2}. \tag{1.5}$$

Taking $\beta = 0,\ \gamma > 0$, substituting (1.3), and (1.5) in (1.1), we get

$$\partial_x \left(\partial_t u - \frac{1}{6}\partial_x u^3\right) = \gamma u, \tag{1.6}$$

that was also introduced recently by Schäfer and Wayne [29] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers.

Integrating (1.6) in x, we gain the integro-differential formulation of (1.6) (see [27])

$$\partial_t u - \frac{1}{6}\partial_x u^3 = \gamma \int^x u(t, y)\mathrm{d}y, \tag{1.7}$$

that is equivalent to

$$\partial_t u - \frac{1}{6}\partial_x u^3 = \gamma P, \qquad \partial_x P = u. \tag{1.8}$$

In [4,6,8], the authors investigated the well-posedness in classes of discontinuous functions for (1.7) or (1.8). In particular, they proved that (1.7) or (1.8) admits a unique entropy solution in the sense of the following definition:

Definition 1.1 We say that $u \in L^\infty((0, T) \times \mathbb{R}),\ T > 0$, is an entropy solution of (1.7) or (1.8) if

(i) u is a distributional solution of (1.7) or equivalently of (1.8);
(ii) for every convex function $\eta \in C^2(\mathbb{R})$, the entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0, \qquad q(u) = -\int^u \frac{\xi^2}{2} \eta'(\xi)\, \mathrm{d}\xi, \tag{1.9}$$

holds in the sense of distributions in $(0, \infty) \times \mathbb{R}$.

Definition 1.1 makes sense because the weak solution of (1.7) lies in L^∞, see [4,6,8]. Here, we consider the case

$$\alpha = 1. \tag{1.10}$$

Taking $\mu = -1,\ \theta = 0$, (1.2) reads

$$g''(u) - g(u) = 0. \tag{1.11}$$

A solution of (1.11) is

$$g(u) = e^u. \tag{1.12}$$

Taking $\beta = 0,\ \gamma = -1$, and substituting (1.10), and (1.12) in (1.1), we get

$$\partial_x \left(\partial_t u + \partial_x e^u \right) = -e^u, \tag{1.13}$$

which is known as the exp-Rabelo equation (see [12,28]), and describes pseudo-spherical surfaces with constant negative curvature.

Our aim is to investigate the well-posedness for the initial value problem in classes of discontinuous functions for (1.13). Therefore, we augment (1.13) with the initial datum

$$u(0, x) = u_0(x), \tag{1.14}$$

on which we assume that

$$\sup u_0 < \infty, \qquad \int_{\mathbb{R}} e^{u_0(x)} \mathrm{d}x < \infty. \tag{1.15}$$

Integrating (1.13) in $(0, x)$, we gain the integro-differential formulation of (1.13) (see [1,28, 31])

$$\begin{cases} \partial_t u + \partial_x e^u = -\int_0^x e^{u(t,y)} \mathrm{d}y, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.16}$$

that is equivalent to

$$\begin{cases} \partial_t u + \partial_x e^u = -P, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = e^u, & t > 0,\ x > 0, \\ P(t, 0) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.17}$$

We assume (1.15) because the unique useful conserved quantity is

$$t \longmapsto \int_{\mathbb{R}} e^{u(t,x)} \mathrm{d}x.$$

Moreover, we prove that the weak solutions of (1.13) may not belong to L^∞, but they are only bounded from above.

Therefore, to have the well-posedness of weak solution, we have to consider the following definition of the entropy:

Definition 1.2 A pair of functions (η, q) is called an entropy–entropy flux pair if $\eta : \mathbb{R} \to \mathbb{R}$ is a C^2 function and $q : \mathbb{R} \to \mathbb{R}$ is defined by

$$q(u) = \int_0^u \eta'(\xi) f'(\xi) \mathrm{d}\xi.$$

An entropy–entropy flux pair $(\eta,\ q)$ is called convex/compactly supported if, in addition, η is convex/compactly supported.

In light of (1.15) and Definition 1.2, we give the following definition of solution:

Definition 1.3 We say that u, such that

$$u \in L^\infty_{loc}((0,\infty)\times\mathbb{R}), \qquad \sup u(t,\cdot) < \infty, \qquad \int_{\mathbb{R}} e^{u(t,x)} dx < \infty, \quad t > 0, \tag{1.18}$$

is an entropy solution of the initial value problem (1.13) and (1.14) if

(i) u is a distributional solution of (1.16) or equivalently of (1.17);
(ii) for every convex function $\eta \in C^2(\mathbb{R})$, the entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u) \int_0^x e^u \mathrm{d}y \le 0, \qquad q(u) = \int^u e^\xi \eta'(\xi)\, \mathrm{d}\xi, \tag{1.19}$$

holds in the sense of distributions in $(0,\infty)\times\mathbb{R}$.

The main result of this paper is the following theorem.

Theorem 1.1 *Let $T > 0$ be given and assume* (1.15). *The initial value problems* (1.13) *and* (1.14) *possess a unique entropy solution u in the sense of Definition 1.3. Moreover, if u and w are two entropy solutions of* (1.13) *in the sense of Definition 1.3, the following inequality holds*

$$\|u(t,\cdot) - w(t,\cdot)\|_{L^1(-R,R)} \le e^{C(T)t} \|u(0,\cdot) - w(0,\cdot)\|_{L^1(-R-C(T)t,R+C(T)t)}, \tag{1.20}$$

for almost every $0 < t < T$, $R > 0$, and some suitable constant $C(T) > 0$ that depends only on $R,\ T,\ \sup u(0,\cdot),\ \sup w(0,\cdot)$.

The existence argument is based on passing to the limit using the compensated compactness argument of [30] in a vanishing viscosity approximation of (1.17) (see Sect. 2). Moreover, we argue as in [6,8,21] for the uniqueness and stability of the solutions of (1.17).

The paper is organized as follows. In Sect. 2, we prove several a priori estimates on a vanishing viscosity approximation of (1.17). Those play a key role in the proof of our main result, that is given in Sect. 3.

2 Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.17).

Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x e^{u_\varepsilon} = -P_\varepsilon + \varepsilon\partial^2_{xx}\left(e^{u_\varepsilon}\right), & t > 0,\ x \in \mathbb{R}, \\ \partial_x P_\varepsilon = e^{u_\varepsilon}, & t > 0,\ x \in \mathbb{R}, \\ P_\varepsilon(t,0) = 0, & t > 0, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a $C^{\infty}(0,\infty)$ approximation of u_0 such that

$$\begin{aligned}
&u_{0,\varepsilon} \to u_0, \quad \text{a.e. and in } L^p_{loc}(\mathbb{R}),\ 1 \le p < \infty,\\
&\sup u_{\varepsilon,0} \le \sup u_0, \quad \varepsilon > 0,\\
&\int_{\mathbb{R}} e^{u_{\varepsilon,0}(x)}\mathrm{d}x \le \int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x, \quad \varepsilon > 0.
\end{aligned} \tag{2.2}$$

Clearly, (2.1) is equivalent to the integro-differential problem

$$\begin{cases}
\partial_t u_\varepsilon + \partial_x e^{u_\varepsilon} = -\int_0^x e^{u_\varepsilon(t,y)}\mathrm{d}y + \varepsilon \partial_{xx}^2\left(e^{u_\varepsilon}\right), & t>0,\ x\in\mathbb{R},\\
u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x\in\mathbb{R}.
\end{cases} \tag{2.3}$$

The existence of solutions of (2.1) can be obtained by fixing a small number $\delta > 0$ and considering the further approximation of (2.1) (see [5,7])

$$\begin{cases}
\partial_t u_{\varepsilon,\delta} + \partial_x e^{u_{\varepsilon,\delta}} = -P_{\varepsilon,\delta} + \varepsilon \partial_{xx}^2\left(e^{u_{\varepsilon,\delta}}\right), & t>0,\ x\in\mathbb{R},\\
-\delta\partial_{xx}^2 P_{\varepsilon,\delta} + \partial_x P_{\varepsilon,\delta} = e^{u_{\varepsilon,\delta}}, & t>0,\ x\in\mathbb{R},\\
P_{\varepsilon,\delta}(t,0) = 0, & t>0,\\
u_{\varepsilon,\delta}(0,x) = u_{\varepsilon,0}(x), & x\in\mathbb{R},
\end{cases}$$

and then sending $\delta \to 0$.

Observe that, multiplying (2.3) by $e^{u_\varepsilon(t,x)}$, we have

$$\partial_t\left(e^{u_\varepsilon}\right) + \frac{1}{2}\partial_x\left(e^{2u_\varepsilon}\right) = -e^{u_\varepsilon}\int_0^x e^{u_\varepsilon(t,y)}\mathrm{d}y + \varepsilon e^{u_\varepsilon}\partial_{xx}^2\left(e^{u_\varepsilon}\right). \tag{2.4}$$

Introducing the notation

$$v_\varepsilon(t,x) = e^{u_\varepsilon(t,x)} > 0, \tag{2.5}$$

(2.4) reads

$$\partial_t v_\varepsilon + \frac{1}{2}\partial_x v_\varepsilon^2 = -v_\varepsilon\int_0^x v_\varepsilon(t,y)\mathrm{d}y + \varepsilon v_\varepsilon \partial_{xx}^2 v_\varepsilon. \tag{2.6}$$

It follows from (2.5) and $u_\varepsilon(t,\pm\infty) = -\infty$ that

$$v_\varepsilon(t,\pm\infty) = 0. \tag{2.7}$$

Moreover, from (2.2) and (2.5), we get

$$\left\|v_{\varepsilon,0}\right\|_{L^\infty(\mathbb{R})} \le e^{\sup u_0}, \quad \int_{\mathbb{R}} v_{\varepsilon,0}(x)\mathrm{d}x \le \int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x, \quad \varepsilon > 0. \tag{2.8}$$

Let us prove some a priori estimates on v_ε, and, hence on u_ε.

Lemma 2.1 *Let $T > 0$ be given and assume (2.2). We have that*

$$\|v_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})} \le e^{\sup u_0}, \quad \varepsilon > 0. \tag{2.9}$$

In particular, we get

$$\sup u_\varepsilon(t,\cdot) \le \sup u_0, \quad t > 0. \tag{2.10}$$

Proof We begin by observing that, from (2.5) and (2.6), we have

$$\partial_t v_\varepsilon + \partial_x\left(\frac{v_\varepsilon^2}{2}\right) - \varepsilon\partial_{xx}^2\left(\frac{v_\varepsilon^2}{2}\right) \le 0. \tag{2.11}$$

Therefore, a supersolution of (2.6) satisfies the following ordinary differential equation

$$\frac{dz}{dt} = 0, \quad z(0) = e^{\sup u_0},$$

that is

$$z(t) = e^{\sup u_0}. \tag{2.12}$$

It follows from the comparison principle for parabolic equation and (2.5) that

$$0 < v_\varepsilon(t, x) \le e^{\sup u_0}, \tag{2.13}$$

which gives (2.9).

Finally, (2.10) follows from (2.5) and (2.13). □

Lemma 2.2 *Let $\alpha \ge 0$ and $T > 0$ be given and assume (2.2). For each $t > 0$, we have*

$$\begin{aligned}&\int_{\mathbb{R}} v_\varepsilon^{\alpha+1} + \varepsilon(\alpha+1)^2 \int_0^t\int_{\mathbb{R}} v_\varepsilon^\alpha (\partial_x v_\varepsilon)^2 ds dx \\ &\quad + (\alpha+1)\int_0^t\int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\left(\int_0^x v_\varepsilon dy\right) ds dx \le \left(e^{\sup u_0}\right)^\alpha \int_{\mathbb{R}} e^{u_0(x)} dx.\end{aligned} \tag{2.14}$$

In particular, we get

$$\begin{aligned}\int_{\mathbb{R}} e^{(\alpha+1)u_\varepsilon(t,x)} dx &\le \left(e^{\sup u_0}\right)^\alpha \int_{\mathbb{R}} e^{u_0(x)} dx, \\ \varepsilon(\alpha+1)^2 \int_0^t\int_{\mathbb{R}} e^{\alpha u_\varepsilon(t,x)} \left(\partial_x e^{u_\varepsilon(t,x)}\right)^2 ds dx &\le \left(e^{\sup u_0}\right)^\alpha \int_{\mathbb{R}} e^{u_0(x)} dx, \\ (\alpha+1)\int_0^t\int_{\mathbb{R}} e^{(\alpha+1)u_\varepsilon(t,x)}\left(\int_0^x e^{u_\varepsilon(t,x)} dy\right) ds dx &\le \left(e^{\sup u_0}\right)^\alpha \int_{\mathbb{R}} e^{u_0(x)} dx.\end{aligned} \tag{2.15}$$

Proof Multiplying (2.6) by v_ε^α, we have

$$v_\varepsilon^\alpha \partial_t v_\varepsilon + v_\varepsilon^{\alpha+1}\partial_x v_\varepsilon = -v_\varepsilon^{\alpha+1}\int_0^x v_\varepsilon dy + \varepsilon v_\varepsilon^{\alpha+1}\partial_{xx}^2 v_\varepsilon.$$

It follows from (2.5), (2.6) and an integration on $\mathbb{R}$ that

$$\begin{aligned}\frac{1}{\alpha+1}\frac{d}{dt}\int_{\mathbb{R}} v_\varepsilon^{\alpha+1} &= \int_{\mathbb{R}} v_\varepsilon^\alpha \partial_t v_\varepsilon \\ &= \varepsilon\int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\partial_{xx}^2 v_\varepsilon dx - \int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\partial_x v_\varepsilon - \int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\left(\int_0^x v_\varepsilon dy\right) dx \\ &= -\varepsilon(\alpha+1)\int_{\mathbb{R}} v_\varepsilon^\alpha(\partial_x v_\varepsilon)^2 dx - \int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\left(\int_0^x v_\varepsilon dy\right) dx,\end{aligned}$$

that is,

$$\frac{d}{dt}\int_{\mathbb{R}} v_\varepsilon^{\alpha+1} + \varepsilon(\alpha+1)^2\int_{\mathbb{R}} v_\varepsilon^\alpha(\partial_x v_\varepsilon)^2 dx + (\alpha+1)\int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\left(\int_0^x v_\varepsilon dy\right) dx = 0. \tag{2.16}$$

An integration on $(0, t)$ gives

$$\begin{aligned}&\int_{\mathbb{R}} v_\varepsilon^{\alpha+1} dx + \varepsilon(\alpha+1)^2\int_0^t\int_{\mathbb{R}} v_\varepsilon^\alpha(\partial_x v_\varepsilon)^2 ds dx \\ &\quad + (\alpha+1)\int_0^t\int_{\mathbb{R}} v_\varepsilon^{\alpha+1}\left(\int_0^x v_\varepsilon dy\right) dx = \int_{\mathbb{R}} v_{\varepsilon,0}^{\alpha+1} dx.\end{aligned} \tag{2.17}$$

From (2.5) and (2.8),

$$\int_{\mathbb{R}} v_{\varepsilon,0}^{\alpha+1}\mathrm{d}x \le \left\| v_{\varepsilon,0}\right\|^{\alpha}_{L^{\infty}((0,\infty)\times\mathbb{R})} \int_{\mathbb{R}} v_{\varepsilon,0}\mathrm{d}x \le \left(e^{\sup u_0}\right)^{\alpha} \int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x. \tag{2.18}$$

Therefore, (2.17) and (2.18) give (2.14).

Finally, (2.15) follows from (2.6) and (2.14) □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We begin with the following result

Lemma 3.1 *Fix $T > 0$. There exists a subsequence $\{v_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{v_\varepsilon\}_{\varepsilon>0}$ and a limit function $v \in L^{\infty}((0,\infty)\times\mathbb{R})$ such that*

$$v_{\varepsilon_k} \to v \text{ a.e. and in } L^{p}_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty. \tag{3.1}$$

In particular, we have

$$u_{\varepsilon_k} \to \log v = u \text{ a.e. and in } L^{p}_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty \tag{3.2}$$

Proof Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q'(v) = v\eta'(v)$. By multiplying (2.6) with $\eta'(v_\varepsilon)$ and using the chain rule, we get

$$\begin{aligned}\partial_t\eta(v_\varepsilon) + \partial_x q(v_\varepsilon) = & \underbrace{\varepsilon\partial_x\left(\eta'(v_\varepsilon)v_\varepsilon\partial_x v_\varepsilon\right)}_{=:\mathcal{L}_{1,\varepsilon}} \underbrace{-\varepsilon\eta''(v_\varepsilon)v_\varepsilon\left(\partial_x u_\varepsilon\right)^2}_{=:\mathcal{L}_{2,\varepsilon}} \\ & \underbrace{-\varepsilon\eta'(v_\varepsilon)\left(\partial_x v_\varepsilon\right)^2}_{=:\mathcal{L}_{3,\varepsilon}} \underbrace{-\eta'(v_\varepsilon)v_\varepsilon\int_0^x v_\varepsilon\mathrm{d}y}_{=:\mathcal{L}_{4,\varepsilon}},\end{aligned}$$

where $\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon}, \mathcal{L}_{3,\varepsilon}, \mathcal{L}_{4,\varepsilon}$ are distributions. Let us show that

$$\mathcal{L}_{1,\varepsilon} \to 0 \text{ in } H^{-1}((0,T)\times\mathbb{R}),\ T > 0.$$

By Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\begin{aligned}\left\|\varepsilon\eta'(v_\varepsilon)v_\varepsilon\partial_x v_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} &\le \varepsilon^2\left\|\eta'\right\|^2_{L^{\infty}(I)}\left\|v_\varepsilon\right\|_{L^{\infty}((0,\infty)\times\mathbb{R})}\int_0^T \left\|\partial_x v_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\ &\le \varepsilon\left\|\eta'\right\|^2_{L^{\infty}(T)} e^{\sup u_0}\int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x \to 0,\end{aligned}$$

where

$$I = \left(0, e^{\sup u_0}\right).$$

We claim that

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0,T)\times\mathbb{R}),\ T > 0.$$

Again by Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\left\|\varepsilon\eta''(v_\varepsilon)v_\varepsilon(\partial_x v_\varepsilon)^2\right\|_{L^1((0,T)\times\mathbb{R})} \le \left\|\eta''\right\|_{L^\infty(I)} \|v_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})}\,\varepsilon\int_0^T \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s$$
$$\le \left\|\eta''\right\|_{L^\infty(I)} e^{\sup u_0}\int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x.$$

We have that

$$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0,T)\times\mathbb{R}),\, T>0.$$

Again by Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\left\|\varepsilon\eta'(v_\varepsilon)(\partial_x v_\varepsilon)^2\right\|_{L^1((0,T\times\mathbb{R})} \le \left\|\eta'\right\|_{L^\infty(I)}\,\varepsilon\int_0^T \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s$$
$$\le \left\|\eta'\right\|_{L^\infty(I)}\int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x.$$

We claim that

$$\{\mathcal{L}_{4,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0,T)\times\mathbb{R}),\, T>0.$$

Again by Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\left\|\eta'(v_\varepsilon)v_\varepsilon\int_0^x v_\varepsilon dy\right\|_{L^1((0,T)\times\mathbb{R})} \le \left\|\eta'\right\|_{L^\infty(I)}\int_0^T\!\!\int_{\mathbb{R}} v_\varepsilon\left(\int_0^x v_\varepsilon \mathrm{d}y\right)\mathrm{d}s\mathrm{d}x$$
$$\le \left\|\eta'\right\|_{L^\infty(I)}\int_{\mathbb{R}} e^{u_0(x)}\mathrm{d}x.$$

Therefore, Murat's lemma [23] implies that

$$\{\partial_t\eta(v_\varepsilon)+\partial_x q(v_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}((0,T)\times\mathbb{R}). \tag{3.3}$$

The L^∞ bound stated in Lemma 2.1, (3.3), and the Tartar's compensated compactness method [30] give the existence of a subsequence $\{v_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $v\in L^\infty((0,\infty)\times\mathbb{R})$, such that (3.1) holds.

(3.2) follows from (2.5) and (3.1). □

Proof of Theorem 1.1 We begin by proving that u, defined in (3.2), is an entropy solution of (1.16) or (1.17) in the sense of Definition 1.3. Let $\phi\in C^\infty(\mathbb{R}^2)$ be a positive text function with a support, and let us consider a compactly supported entropy–entropy flux pair (η, q). We have to prove

$$\int_0^\infty\!\!\int_{\mathbb{R}}(\eta(u)\partial_t\phi+q(u)\partial_x\phi)\mathrm{d}t\mathrm{d}x-\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u)\left(\int_0^x e^u\mathrm{d}y\right)\mathrm{d}t\mathrm{d}x$$
$$+\int_{\mathbb{R}}\eta(u_0(x))\phi(0,x)\mathrm{d}x \ge 0. \tag{3.4}$$

Multiplying (2.1) by $\eta'(u_\varepsilon)$, we have

$$\partial_t\eta(u_{\varepsilon_k})+\partial_x q(u_{\varepsilon_k})+\eta'(u_{\varepsilon_k})\int_0^x e^{u_{\varepsilon_k}}\mathrm{d}dy = \varepsilon_k\eta'(u_{\varepsilon_k})\partial^2_{xx}\left(e^{u_{\varepsilon_k}}\right).$$

Since

$$\begin{aligned}\varepsilon_k\eta'(u_{\varepsilon_k})\partial_{xx}^2\left(e^{u_{\varepsilon_k}}\right) =&\partial_x\left(\varepsilon_k\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\right)-\varepsilon_k\eta''(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\partial_x u_{\varepsilon_k}\\ =&\partial_x\left(\varepsilon_k\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\right)-\varepsilon_k\eta''(u_{\varepsilon_k})e^{u_{\varepsilon_k}}(\partial_x u_{\varepsilon_k})^2,\end{aligned}$$

we have

$$\begin{aligned}&\partial_t\eta(u_{\varepsilon_k})+\partial_x q(u_{\varepsilon_k})+\eta'(u_{\varepsilon_k})\int_0^x e^{u_{\varepsilon_k}}\mathrm{d}y\\ &=\partial_x\left(\varepsilon_k\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\right)-\varepsilon_k\eta''(u_{\varepsilon_k})e^{u_{\varepsilon_k}}(\partial_x u_{\varepsilon_k})^2\\ &\le\partial_x\left(\varepsilon_k\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\right).\end{aligned}\tag{3.5}$$

Multiplying (3.5) by ϕ, an integration on $(0,\infty)\times\mathbb{R}$ gives

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}(\eta(u_{\varepsilon_k})\partial_t\phi+q(u_{\varepsilon_k})\partial_x\phi)\mathrm{d}t\mathrm{d}x-\int_0^\infty\int_{\mathbb{R}}\eta'(u_{\varepsilon_k})\left(\int_0^x e^{u_{\varepsilon_k}}\mathrm{d}y\right)\mathrm{d}t\mathrm{d}x\\ &\quad+\int_{\mathbb{R}}\eta(u_{\varepsilon_k,0}(x))\phi(0,x)\mathrm{d}x\\ &\quad+\varepsilon_k\int_0^\infty\int_{\mathbb{R}}\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\partial_x\phi\mathrm{d}t\mathrm{d}x\ge 0.\end{aligned}\tag{3.6}$$

Let us show that

$$\varepsilon_k\int_0^\infty\int_{\mathbb{R}}\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\partial_x\phi\mathrm{d}t\mathrm{d}x\to 0.\tag{3.7}$$

Fix $T>0$. From (2.14) in correspondence of $\alpha=0$, and the Hölder inequality,

$$\begin{aligned}&\varepsilon_k\int_0^\infty\int_{\mathbb{R}}\eta'(u_{\varepsilon_k})\partial_x\left(e^{u_{\varepsilon_k}}\right)\partial_x\phi\mathrm{d}t\mathrm{d}x\\ &\le\varepsilon_k\int_0^\infty\int_{\mathbb{R}}|\eta'(u_{\varepsilon_k})|\left|\partial_x\left(e^{u_{\varepsilon_k}}\right)\right||\partial_x\phi|\mathrm{d}t\mathrm{d}x\\ &\le\varepsilon_k\left\|\eta'\right\|_{L^\infty((0,\infty)\times\mathbb{R})}\left\|\partial_x\left(e^{u_{\varepsilon_k}}\right)\right\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\ &\le\varepsilon_k\left\|\eta'\right\|_{L^\infty((0,\infty)\times\mathbb{R})}\left\|\partial_x\left(e^{u_{\varepsilon_k}}\right)\right\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\ &\le\sqrt{\varepsilon_k}\left\|\eta'\right\|_{L^\infty((0,\infty)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\left(\int_{\mathbb{R}}e^{u_0(x)}\mathrm{d}x\right)^{\frac{1}{2}}\to 0,\end{aligned}$$

that is (3.7).

Therefore, (3.6) follows from (2.2), (3.2), (3.6) and (3.7).

We claim that prove that $u(t,x)$ is unique and (1.20) holds. We consider two entropy solutions $u(t,x),\ w(t,x)$ be of (1.16) or (1.17) such that

$$\begin{aligned}&\sup u(t,\cdot)\le\sup u_0,\quad \sup w(t,\cdot)\le\sup w_0,\quad t>0,\\ &\int_{\mathbb{R}}e^{u_0(x)}\mathrm{d}x<\infty,\quad\int_{\mathbb{R}}e^{w_0(x)}\mathrm{d}x<\infty,\end{aligned}\tag{3.8}$$

Due to (3.8), we have

$$\left|e^u-e^w\right|\le C_0|u-w|,\tag{3.9}$$

where

$$C_0=e^{\sup u_0}+e^{\sup w_0}.$$

Arguing as in [6, Theorem3.1], we can prove that

$$\partial_t(|u-w|) + \partial_x\left((e^u - e^w)\,\mathrm{sign}\,(u-w)\right) + \mathrm{sign}\,(u-w)\int_0^x \left(e^u - e^w\right) \mathrm{d}y \le 0$$

holds in sense of distributions in $(0, \infty) \times \mathbb{R}$, and

$$\begin{aligned}
&\|u(t,\cdot) - w(t,\cdot)\|_{I(t)} \\
&\le \|u_0 - w_0\|_{I(0)} - \int_0^t\int_{I(s)} \mathrm{sign}\,(u-w)\left(\int_0^x \left(e^u - e^w\right)\mathrm{d}y\right)\mathrm{d}s\mathrm{d}x, \quad 0 < t < T, \quad (3.10)
\end{aligned}$$

where

$$I(s) = [-R - C_0(t-s), R + C_0(t-s)].$$

Due to (3.9),

$$\begin{aligned}
&-\int_0^t\int_{I(s)} \mathrm{sign}\,(u-v)\left(\int_0^x \left(e^u - e^w\right) dy\right)\mathrm{d}s\mathrm{d}x \\
&\le \int_0^t\int_{I(s)} \left|\left(\int_0^x \left|e^u - e^w\right| \mathrm{d}y\right)\right| \mathrm{d}s\mathrm{d}x \\
&\le C_0 \int_0^t\int_{I(s)} \left(\left|\int_{I(s)} |u-v|\mathrm{d}y\right|\right)\mathrm{d}s\mathrm{d}x \\
&= C_0 \int_0^t |I(s)|\, \|u(s,\cdot) - v(s,\cdot)\|_{L^1(I(s))}\, \mathrm{d}s. \qquad (3.11)
\end{aligned}$$

Moreover,

$$|I(s)| = 2R + 2C_0(t-s) \le 2R + 2C_0 t \le 2R + 2C_0 T. \qquad (3.12)$$

We consider the following continuous function:

$$G(t) = \|u(t,\cdot) - v(t,\cdot)\|_{L^1(I(t))}, \quad t \ge 0. \qquad (3.13)$$

It follows from (3.10), (3.11), (3.12) and (3.13) that

$$G(t) \le G(0) + C(T)\int_0^t G(s)\mathrm{d}s,$$

where $C(T) = 2R + 2C_0T$. The Gronwall inequality and (3.13) give

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(-R,R)} \le e^{C(T)t}\, \|u_0 - v_0\|_{L^1(-R-C_0t, R+C_0t)},$$

that is (1.20). □

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