Annali di Matematica (2016) 195:659–680
DOI 10.1007/s10231-015-0482-2

The Gurov–Reshetnyak inequality on semi-axes

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Received: 21 August 2014 / Accepted: 28 January 2015 / Published online: 19 February 2015

Abstract An elementary method to study functions from the Gurov–Reshetnyak class is proposed, and sharp limiting positive and negative summability exponents for monotone functions from the Gurov–Reshetnyak class on semi-axis are found. Moreover, other properties of functions from the class mentioned are studied.

Keywords Mean oscillation · Gurov–Reshetnyak inequality · Limiting summability exponent

Mathematics Subject Classification Primary 26D10; Secondary 42B25

1 Introduction

Let us consider functions $f : R \mapsto \mathbb{R}^+$ where R is an interval of $\mathbb{R}$. In what follows, R is $\mathbb{R}$ or $\mathbb{R}^+ = [0, \infty)$. In certain cases, the condition $f \geq 0$ can be dropped and this will be mentioned in due course. It is supposed that f is locally summable on R, i.e. it is summable on each bounded interval $I \subset R$.

The mean oscillation of the function f on a bounded interval I is defined by

$$\Omega(f; I) = \frac{1}{|I|} \int_I |f(x) - f_I| \, \mathrm{d}x,$$

where $f_I = \frac{1}{|I|} \int_I f(x) \, \mathrm{d}x$ is the mean value of f on I, and $|\cdot|$ denotes the Lebesgue measure. Note that even for the sign changing functions f, the mean value $f_I = \gamma$ is uniquely defined

The author would like to express his sincere thanks to Victor Didenko for his assistance in preparation of this paper for publication.

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by the condition

$$\int_{I(f\geq\gamma)} (f(x)-\gamma)\,\mathrm{d}x = \int_{I(f\leq\gamma)} (\gamma - f(x))\,\mathrm{d}x,$$

where by $E(P)$ we denote the set of all points $x \in E$ which satisfy the condition $P = P(x)$. It is also easily seen that

$$\Omega(f;I) = \frac{2}{|I|}\int_{I(f\geq f_I)} (f(x)-f_I)\,\mathrm{d}x = \frac{2}{|I|}\int_{I(f\leq f_I)} (f_I - f(x))\,\mathrm{d}x$$

and the condition $f \geq 0$ can be dropped as well.

For any given ε $(0 < \varepsilon \leq 2)$, the Gurov–Reshetnyak class $\mathcal{GR} = \mathcal{GR}(\varepsilon) = \mathcal{GR}_R(\varepsilon)$ is defined as the set of all functions $f \geq 0$ which are locally summable on R and such that the Gurov–Reshetnyak condition

$$\Omega(f;I) \leq \varepsilon\, f_I,$$

is satisfied on all bounded intervals $I \subset R$ (see [2]). Note that any function $f \geq 0$ on every interval I satisfies the inequality $\Omega(f;I) \leq 2f_I$, the class $\mathcal{GR}_R(2)$ is trivial and coincide with the class of all functions locally summable on R. Note that for any $0 < \varepsilon < 2$, the class $\mathcal{GR}_R(\varepsilon)$ is non-trivial (see [7]). For an interval $I \subset R$, the expression $\langle f \rangle_I = \Omega(f;I)/f_I$ is called the relative oscillation of f on the interval I, and we set $\langle f \rangle_{\mathcal{GR}_R} = \sup_{I\subset R} \langle f \rangle_I$.

One of the fundamental properties of functions from the Gurov–Reshetnyak class consists in the possibility to improve their summability exponents. It is precisely this fact that is the basis for numerous applications of the Gurov–Reshetnyak class. Recall that the study of quantitative estimates of the summability exponents usually consists of the following two steps.

Step 1. *Estimates of the distribution functions or, equivalently, the estimates of the equimeasurable rearrangements of functions from the Gurov–Reshetnyak class.* Here we do not present such estimates since it requires appropriate definitions and terms which are aside from the main stream of this work. Moreover, the author is not aware of any sharp estimates for isotropic Gurov–Reshetnyak classes, whereas for functions from anisotropic classes, there are sharp estimates of equimeasurable rearrangements, in particular in the one-dimensional case [4]. Note that the knowledge of such estimates allows one to reduce the study of multivariate functions to the case of monotonic functions of one variable, which drastically simplifies the investigation.

Step 2. *Determination of the summability exponents for monotonic functions satisfying the Gurov–Reshetnyak condition.* The present work proposes a new approach to the computation of these exponents. This approach is based on the study of more delicate properties of functions from the Gurov–Reshetnyak class (see Theorem 3.4 and Remark 3.5). In addition, the proofs of the corresponding results rely on elementary computations given in the Sect. 2. Similar arguments can also be used to study other classes of functions such as the classes of Muckenhoupt, Gehring and so on.

2 On monotone functions satisfying the Gurov–Reshetnyak property

Let us first formulate an auxiliary statement.

Lemma 2.1 (cf. [3]) *If the interval J is contained in the interval I, function f is monotonic on I, and $f_J = f_I$, then*

$$\Omega(f; J) \leq \Omega(f; I).$$

Immediate consequence of this lemma is that for any function f monotonic on $\mathbb{R}^+$, the following identity

$$\langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} = \sup_{\eta>0} \langle f \rangle_{(0,\eta)}$$

holds. If f is monotonic on $R = [0, 1]$, then

$$\langle f \rangle_{\mathcal{GR}_R} = \sup_{0<\eta<1} \max \left\{ \langle f \rangle_{(0,\eta)}, \langle f \rangle_{(1-\eta,1)} \right\}.$$

Let f be monotonic on $\mathbb{R}$ and

$$0 \leq A = \inf_{x\in\mathbb{R}} f(x) < B = \sup_{x\in\mathbb{R}} f(x) \leq \infty.$$

Using Lemma 2.1, one easily obtains that $\sup_{I\subset\mathbb{R}} \Omega(f; I) = (B - A)/2 \leq \infty$. Moreover, $\inf_{I\subset\mathbb{R}} f_I = A$, hence

$$\langle f \rangle_{\mathcal{GR}_{\mathbb{R}}} \leq \frac{\sup_{I\subset\mathbb{R}} \Omega(f; I)}{\inf_{I\subset\mathbb{R}} f_I} = \frac{B - A}{2A}. \tag{2.1}$$

However, it turns out that, this estimate is exaggerated for any $A < B$ (see Remark 2.7 below).

Let χ_E refer to the characteristic function of the set E. We calculate the supremum of the relative oscillations of an elementary function.

Lemma 2.2 *Let* $0 \leq A < B < \infty$, $\alpha \in \mathbb{R}$,

$$g(x) = A\chi_{(-\infty,\alpha)}(x) + B\chi_{[\alpha,\infty)}(x) \quad (x \in \mathbb{R}). \tag{2.2}$$

Then $\langle g \rangle_{\mathcal{GR}_{\mathbb{R}}} = 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$.

Proof If $A = 0$, then for $I = (\alpha - \eta, \alpha + \delta\eta)$ $(\delta, \eta > 0)$, one has

$$g_I = \frac{B\delta}{1+\delta}, \quad \Omega(g; I) = \frac{2}{(1+\delta)\eta}\delta\eta\,(B - g_I) = \frac{2\delta B}{(1+\delta)^2},$$
$$\frac{\Omega(g; I)}{g_I} = \frac{2}{1+\delta} \to 2 \quad (\delta \to 0+).$$

Thus, our lemma is valid for $A = 0$.

Assume now that $A > 0$, $I = (\alpha - \eta, \alpha + \delta\eta)$ $(\delta, \eta > 0)$, and find the expressions

$$g_I = \frac{A + \delta B}{1+\delta}, \quad \Omega(g; I) = \frac{2}{(1+\delta)\eta}\delta\eta\,(B - g_I) = \frac{2\delta}{(1+\delta)^2}(B - A),$$
$$\frac{\Omega(g; I)}{g_I} = 2(B - A)\frac{\delta}{(1+\delta)(A+\delta B)}.$$

If we define the function $\varphi(\delta) = \frac{\delta}{(1+\delta)(A+\delta B)}$, then $\varphi'(\delta) = \frac{A-\delta^2 B}{(1+\delta)^2(A+\delta B)^2}$. Therefore,

$$\max_{\delta>0} \varphi(\delta) = \varphi\left(\frac{\sqrt{A}}{\sqrt{B}}\right) = \frac{1}{\left(\sqrt{B}+\sqrt{A}\right)^2},$$

which gives us

$$\langle g \rangle_{\mathcal{GR}_{\mathbb{R}}} = \sup_{I\subset\mathbb{R}} \frac{\Omega(g; I)}{g_I} = 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}},$$

and the proof is completed. □

Remark 2.3 Let g be as above. The proof of Lemma 2.2 shows that for $A > 0$, the extremal value of the parameter δ is $\sqrt{A}/\sqrt{B}$ regardless of the choice of $\eta > 0$. Therefore, one can think that $I = \left(\alpha - \eta\sqrt{B}, \alpha + \eta\sqrt{A}\right)$ with arbitrary $\eta > 0$ is an optimal interval and get $g_I = \sqrt{AB}$, $\Omega(g; I) = 2\sqrt{AB}\left(\sqrt{B}-\sqrt{A}\right)/\left(\sqrt{B}+\sqrt{A}\right)$.

Remark 2.4 For the function $h = 1/g$, where g is defined by (2.2), extremal values of the corresponding parameters (for $\alpha = 0$) are $I = \left(-\eta\sqrt{A}, \eta\sqrt{B}\right)$ with arbitrary $\eta > 0$, $h_I = 1/\sqrt{AB}$, $\Omega(h; I) = 2/\sqrt{AB} \times \left(\sqrt{B}-\sqrt{A}\right) / \left(\sqrt{B}+\sqrt{A}\right)$. Thus $\langle h \rangle_{\mathcal{GR}_{\mathbb{R}}} = \langle g \rangle_{\mathcal{GR}_{\mathbb{R}}}$, but the supremum of the relative oscillations of functions g and h is attained on distinct intervals.

Lemma 2.5 *Let function f be defined on $\mathbb{R}$ and*

$$0 \le \limsup_{x\to-\infty} f(x) \le A < B \le \liminf_{x\to\infty} f(x) \le \infty.$$

Then

$$\langle f \rangle_{\mathcal{GR}_{\mathbb{R}}} \ge 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}. \tag{2.3}$$

Note that if $B = \infty$ then the right-hand side of (2.3) *is replaced by* 2.

Proof It suffices to prove this lemma in the case $A = \limsup_{x\to-\infty} f(x)$, $B = \liminf_{x\to\infty} f(x)$, otherwise one can use the obvious inequality

$$\frac{\sqrt{B_1}-\sqrt{A_1}}{\sqrt{B_1}+\sqrt{A_1}} \ge \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}},$$

where $A_1 \le A < B \le B_1$.

Assume first that $0 < A < B < \infty$. Fix a $\xi > 0$ and chose a sufficiently small $\sigma > 0$ such that

$$2\frac{\sqrt{B-\sigma}-\sqrt{A+\sigma\dfrac{A}{B-\sigma}}}{\sqrt{B-\sigma}+\sqrt{A+\sigma\dfrac{A}{B-\sigma}}} > 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} - \frac{\xi}{2}. \tag{2.4}$$

One can also assume that $\sigma < B - \sqrt{AB}$, which will lead to the inequalities

$$B_\sigma \equiv B - \sigma > \sqrt{AB}, \quad A_\sigma \equiv A + \sigma\frac{A}{B-\sigma} < \sqrt{AB}.$$

Fix such a σ and chose an interval $I = (\alpha, \beta)$ with the property that $f(x) \le A_\sigma$ for $x \le \alpha$; $f(x) \ge B_\sigma$ for $x \ge \beta$ and $f_I = \sqrt{AB}$. We also fix an η such that

$$\eta > \frac{4}{\xi}\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}\frac{B-\sigma}{\sqrt{AB}+B-\sigma}(\beta-\alpha) \tag{2.5}$$

and construct an interval

$$I_\sigma \equiv (\alpha_\sigma, \beta_\sigma) \supset \left(\alpha-\eta, \beta+\frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\cdot\eta\right),$$

such that

$$f_{I_\sigma} = \sqrt{AB}. \tag{2.6}$$

Now the equalities

$$A_\sigma B_\sigma = A\,B, \quad \left(B_\sigma - \sqrt{AB}\right)\frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}} = \sqrt{AB} - A_\sigma \tag{2.7}$$

imply

$$f_{I_\sigma\setminus I} = \sqrt{A_\sigma B_\sigma} = \sqrt{AB} = f_I.$$

Set

$$g(y) = A_\sigma\chi_{(-\infty,0)}(y) + B_\sigma\chi_{[0,\infty)}(y) \quad \left(y \in \left(-\eta, \frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\eta\right) \equiv J\right),$$

and the application of Remark 2.3 to the function g on J gives us

$$g_J = \sqrt{A_\sigma B_\sigma}, \quad \frac{\Omega(g;J)}{g_J} = 2\frac{\sqrt{B_\sigma}-\sqrt{A_\sigma}}{\sqrt{B_\sigma}+\sqrt{A_\sigma}}. \tag{2.8}$$

However, since $f(x) \geq B_\sigma$ for $x \geq \beta$, then

$$\frac{1}{\beta_\sigma-\beta}\int_\beta^{\beta_\sigma}\left(f(x)-f_{I_\sigma\setminus I}\right)\,\mathrm{d}x \geq \frac{1}{\frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\eta}\int_0^{\frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\eta}(g(y)-g_J)\,\mathrm{d}y,$$

and the inequality $f(x) \leq A_\sigma$ $(x \leq \alpha)$ implies that

$$\frac{1}{\alpha-\alpha_\sigma}\int_{\alpha_\sigma}^{\alpha}\left(f_{I_\sigma\setminus I}-f(x)\right)\,\mathrm{d}x \geq \frac{1}{\eta}\int_{-\eta}^{0}(g_J-g(y))\,\mathrm{d}y.$$

Let us rewrite these two inequalities in the form

$$\frac{\beta_\sigma-\beta}{\int_\beta^{\beta_\sigma}\left(f(x)-f_{I_\sigma\setminus I}\right)\,\mathrm{d}x} \leq \frac{\frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\eta}{\int_0^{\frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\eta}(g(y)-g_J)\,\mathrm{d}y},$$

$$\frac{\alpha-\alpha_\sigma}{\int_{\alpha_\sigma}^{\alpha}\left(f_{I_\sigma\setminus I}-f(x)\right)\,\mathrm{d}x} \leq \frac{\eta}{\int_{-\eta}^{0}(g_J-g(y))\,\mathrm{d}y}.$$

One can observe that the denominators in the left-hand sides of the fractions are equal to $\frac{1}{2}\int_{I_\sigma\setminus I}\left|f(x)-f_{I_\sigma\setminus I}\right|\,\mathrm{d}x$, whereas the ones in the right-hand sides are $\frac{1}{2}\int_J|g(y)-g_J|\,\mathrm{d}y$. Therefore, summing these two inequalities one obtains

$$\frac{|I_\sigma\setminus I|}{\frac{1}{2}\int_{I_\sigma\setminus I}\left|f(x)-f_{I_\sigma\setminus I}\right|\,\mathrm{d}x} \leq \frac{|J|}{\frac{1}{2}\int_J|g(y)-g_J|\,\mathrm{d}y}.$$

This implies the inequality

$$\Omega\left(f;\, I_\sigma \setminus I\right) \geq \Omega\left(g;\, J\right).$$

However,

$$\begin{aligned}\Omega(f;\, I_\sigma) &= \frac{1}{|I_\sigma|}\int_{I_\sigma}\left|f(x) - f_{I_\sigma}\right|\,\mathrm{d}x \\ &\geq \frac{1}{|I_\sigma|}\int_{I_\sigma\setminus I}\left|f(x) - f_{I_\sigma\setminus I}\right|\,\mathrm{d}x = \left(1 - \frac{|I|}{|I_\sigma|}\right)\Omega\left(f;\, I_\sigma \setminus I\right).\end{aligned}$$

Finally, using relations (2.6), (2.7), (2.8), (2.5), and (2.4), one obtains

$$\frac{\Omega\left(f;\, I_\sigma\right)}{f_{I_\sigma}} \geq \left(1 - \frac{|I|}{|I_\sigma|}\right)\frac{\Omega(g;\, J)}{g_J} = \left(1 - \frac{\beta - \alpha}{\beta_\sigma - \alpha_\sigma}\right)\cdot 2\frac{\sqrt{B_\sigma} - \sqrt{A_\sigma}}{\sqrt{B_\sigma} + \sqrt{A_\sigma}}$$

$$\geq \left(1 - \frac{\beta - \alpha}{\left(\frac{\sqrt{A + \sigma\frac{A}{B-\sigma}}}{\sqrt{B - \sigma}} + 1\right)\eta + \beta - \alpha}\right)\left(2\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}} - \frac{\xi}{2}\right) \geq 2\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}} - \xi.$$

Since ξ is an arbitrary positive number, the proof of our lemma in the case $0 < A < B < \infty$ is completed.

On the other hand, if $A = 0$ or $B = \infty$, then one can use the already proven part of the lemma for arbitrary A_1, B_1 such that $A < A_1 < B_1 < B$ and get

$$\sup_{I\subset\mathbb{R}}\frac{\Omega(f;\, I)}{f_I} \geq 2\frac{\sqrt{B_1} - \sqrt{A_1}}{\sqrt{B_1} + \sqrt{A_1}}.$$

It remains to pas to the limits when A_1 tends to $A + 0$ and B_1 tends to $B - 0$. □

If f is a non-decreasing on $\mathbb{R}$ function, the inequality (2.3) from Lemma 2.5 becomes an equality. Moreover, the following result holds.

Theorem 2.6 *Let function f be monotonic on $\mathbb{R}$, $0 \leq \inf_{x\in\mathbb{R}} f(x) = A < B = \sup_{x\in\mathbb{R}} f(x) \leq \infty$. Then*

$$\langle f \rangle_{\mathcal{GR}_\mathbb{R}} = 2\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}}.$$

Note that for $B = \infty$, the right-hand side is set to be 2.

Proof Assume for definiteness that f is a non-decreasing function. In account on Lemma 2.5, it suffices to show that for every interval I one has

$$\frac{\Omega(f;\, I)}{f_I} \leq 2\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}}.$$

Let I be an arbitrary interval. Chose an $\alpha \in I$ such that $f(x) \geq f_I$ for $x \geq \alpha$ and $f(x) \leq f_I$ for $x \leq \alpha$. Setting $g(x) = A\chi_{(-\infty,\alpha)}(x) + B\chi_{[\alpha,\infty)}(x)$, we find an interval J with the property $g_J = f_I$. Since $f(x) - f_I \leq g(y) - g_J$ for all $x \in I\,(f \geq f_I)$, $y \in J\,(g \geq g_J)$, then

$$\frac{1}{|I\,(f \geq f_I)|}\int_{I(f\geq f_I)} (f(x) - f_I)\,\mathrm{d}x \leq \frac{1}{|J\,(g \geq g_J)|}\int_{J(g\geq g_J)} (g(x) - g_J)\,\mathrm{d}x,$$

i.e.

$$\frac{|I\,(f \geq f_I)|}{\int_{I(f\geq f_I)} (f(x) - f_I)\,\mathrm{d}x} \geq \frac{|J\,(g \geq g_J)|}{\int_{J(g\geq g_J)} (g(x) - g_J)\,\mathrm{d}x}. \tag{2.9}$$

Analogously, the inequality $f_I - f(x) \leq g_J - g(y)$ $(x \in I\,(f < f_I),\ y \in J\,(g < g_J))$ implies that

$$\frac{|I\,(f < f_I)|}{\int_{I(f< f_I)} (f_I - f(x))\,\mathrm{d}x} \geq \frac{|J\,(g < g_J)|}{\int_{J(g< g_J)} (g_J - g(x))\,\mathrm{d}x}. \tag{2.10}$$

Taking into account that the denominators in the left-hand sides of (2.9) and (2.10) are equal to $\frac{1}{2}\int_I |f(x) - f_I|\,\mathrm{d}x$, and that the denominators in the right-hand sides are $\frac{1}{2}\int_J |g(x) - g_J|\,\mathrm{d}x$, and summing (2.9) and (2.10), one obtains

$$\frac{2}{\Omega(f;I)} \geq \frac{2}{\Omega(g;J)},$$

i.e. $\Omega(f;I) \leq \Omega(g;J)$. However, since $f_I = g_J$, then $\langle\, f\,\rangle_I \leq \langle\, g\,\rangle_J \leq \langle\, g\,\rangle_{\mathcal{GR}_{\mathbb{R}}}$, and one can use Lemma 2.2 to finish the proof. □

Remark 2.7 Since $\frac{B-A}{2A} > 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$ for any $0 \leq A < B \leq \infty$, Theorem 2.6 refines estimate (2.1).

Remark 2.8 If f is monotonic on $\mathbb{R}$ and $0 \leq \inf_{x\in\mathbb{R}} f(x) = A < B = \sup_{x\in\mathbb{R}} f(x) \leq \infty$, $h = 1/f$, then by Theorem 2.6 one has

$$\langle\, h\,\rangle_{\mathcal{GR}_{\mathbb{R}}} = 2\frac{\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{B}}}{\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{A}}} = 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} = \langle\, f\,\rangle_{\mathcal{GR}_{\mathbb{R}}}.$$

Lemma 2.5 gives, in fact, an estimate for relative oscillations of a function on the intervals with the ends tending to $-\infty$ and ∞. A local counterpart of such an estimate at an inner point of the domain of definition contains the next version of Lemma 2.5.

Lemma 2.9 *Assume that function f is defined in a neighbourhood Δ of a point α and*

$$0 \leq \limsup_{x\to\alpha-0} f(x) \leq A < B \leq \liminf_{x\to\alpha+0} f(x) \leq \infty.$$

Then

$$\langle\, f\,\rangle_{\mathcal{GR}_{\Delta}} \geq 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}. \tag{2.11}$$

For $B = \infty$ the right-hand side of (2.11) *is replaced by* 2.

Proof Similarly to Lemma 2.5, it suffices to consider the case where $A = \limsup_{x\to\alpha-0} f(x)$ and $B = \liminf_{x\to\alpha+0} f(x)$. The proof is given only for $0 < A < B < \infty$. The case where

$A = 0$ or $B = \infty$ can be considered analogously to the final step in proof of Lemma 2.5. Hence, assume that $0 < A < B < \infty$, fix an arbitrary $\xi > 0$ and chose a positive σ such that

$$2\frac{\sqrt{B-\sigma} - \sqrt{A + \sigma \dfrac{A}{B-\sigma}}}{\sqrt{B-\sigma} + \sqrt{A + \sigma \dfrac{A}{B-\sigma}}} > 2\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}} - \xi.$$

One can also assume that $\sigma < B - \sqrt{AB}$. Then the following inequalities

$$B_\sigma \equiv B - \sigma > \sqrt{AB}, \quad A_\sigma \equiv A + \sigma \frac{A}{B-\sigma} < \sqrt{AB}$$

holds. Note that $A_\sigma B_\sigma = AB$ and consider an interval $I \subset \Delta$ containing the point α such that $f_I = \sqrt{A_\sigma B_\sigma}$, $f(x) \le A_\sigma$ if $x < \alpha$ $(x \in I)$ and $f(x) \ge B_\sigma$ if $x > \alpha$ $(x \in I)$. Set

$$g(x) = A_\sigma \chi_{(-\infty,\alpha)}(x) + B_\sigma \chi_{[\alpha,\infty)}(x) \quad (x \in \mathbb{R}).$$

Now chose an $\eta > 0$ such that the interval

$$J = \left(\alpha - \eta, \alpha + \frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}} \cdot \eta\right)$$

is contained in I and compute g_J,

$$g_J = \frac{1}{\eta\left(\dfrac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}} + 1\right)} \left[\eta A_\sigma + \eta \frac{\sqrt{A_\sigma}}{\sqrt{B_\sigma}}\right] = \sqrt{A_\sigma B_\sigma} = f_I.$$

Applying Remark 2.3 to the function g on the interval J, one obtains

$$\frac{\Omega(g; J)}{g_J} = 2\frac{\sqrt{B_\sigma} - \sqrt{A_\sigma}}{\sqrt{B_\sigma} + \sqrt{A_\sigma}} > 2\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}} - \xi.$$

Now let us show that $\Omega(f; I) \ge \Omega(g; J)$. It is easily seen that

$$(f - f_I)_{I(f \ge f_I)} \ge (g - g_J)_{J(g \ge g_J)}, \quad (f_I - f)_{I(f < f_I)} \ge (g_J - g)_{J(g < g_J)}.$$

These inequalities can be rewritten in the form

$$\frac{|I\,(f \ge f_I)|}{\int_{I(f \ge f_I)} (f(x) - f_I)\, \mathrm{d}x} \le \frac{|J\,(g \ge g_J)|}{\int_{J(g \ge g_J)} (g(x) - g_J)\, \mathrm{d}x},$$
$$\frac{|I\,(f < f_I)|}{\int_{I(f < f_I)} (f_I - f(x))\, \mathrm{d}x} \le \frac{|J\,(g < g_J)|}{\int_{J(g < g_J)} (g_J - g(x))\, \mathrm{d}x}.$$

The denominators of the fractions in the left-hand sides of these inequalities are equal to $\frac{1}{2}\int_I |f(x) - f_I|\, \mathrm{d}x$, and the ones in the right-hand sides are $\frac{1}{2}\int_J |g(x) - g_J|\, \mathrm{d}x$. Summing these inequalities one gets

$$\frac{|I|}{\frac{1}{2}\int_I |f(x) - f_I|\, \mathrm{d}x} \le \frac{|J|}{\frac{1}{2}\int_J |g(x) - g_J|\, \mathrm{d}x},$$

i.e. $\Omega(f; I) \ge \Omega(g; J)$.

Therefore,

$$\frac{\Omega(f;I)}{f_I} \geq \frac{\Omega(g;J)}{g_J} > 2\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} - \xi,$$

and since $\xi > 0$ is arbitrary, the inequality (2.11) is proved. □

For monotonic functions, Lemma 2.9 can be formulated as follows.

Theorem 2.10 *If a function $f \in \mathcal{GR}_R(\varepsilon)$ is monotonic on R, then for every inner point $\alpha \in R$, the inequality*

$$\frac{\max[f(\alpha+0),\ f(\alpha-0)]}{\min[f(\alpha+0),\ f(\alpha-0)]} \leq \left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2 \tag{2.12}$$

holds, and this inequality is sharp.

Proof Indeed, setting $A = \min[f(\alpha+0),\ f(\alpha-0)]$, $B = \max[f(\alpha+0),\ f(\alpha-0)]$ we rewrite inequality (2.11) as

$$2\frac{\sqrt{\frac{B}{A}}-1}{\sqrt{\frac{B}{A}}+1} \leq \langle f \rangle_{\mathcal{GR}_R} \leq \varepsilon,$$

which is equivalent to (2.12), and this inequality is sharp by Lemma 2.2. □

Theorem 2.6 means that on the real line monotone functions satisfy the Gurov–Reshetnyak property if and only if they are trivial, i.e. if they are bounded from zero and bounded. The situation is completely different if the class $\mathcal{GR}_R$ is considered on the interval R bounded above or below. To show this let us consider the relative oscillations of a function in a one-sided neighbourhood of a point. We need to auxiliary statements which, generally speaking, do not require the condition of monotonicity.

Lemma 2.11 *Let f be defined on an interval I, and let E be a measurable subset of I. Set $g(x) = f_E\chi_E(x) + f(x)\chi_{I\setminus E}(x)$ $(x \in I)$. Then*

$$g_I = f_I, \quad \Omega(g;I) \leq \Omega(f;I),$$

and this inequality becomes an equality if and only if the function $f(x) - f_I$ does not change the sign on E.

Proof The equality $g_I = f_I$ is trivial. Further, one has

$$\int_E |g(x) - g_I|\,\mathrm{d}x = |E|\,|f_E - f_I| \leq \int_E |f(x) - f_I|\,\mathrm{d}x,$$
$$\int_{I\setminus E} |g(x) - g_I|\,\mathrm{d}x = \int_{I\setminus E} |f(x) - f_I|\,\mathrm{d}x.$$

Summing these two equations and dividing the result by $|I|$, one obtains the required inequality $\Omega(g;I) \leq \Omega(f;I)$. The case $\Omega(f;I) = \Omega(g;I)$ obvious. □

Lemma 2.12 *Let a function f be defined on interval $I = E_1 \cup E_2$, where E_1 and E_2 are measurable sets such that $E_1 \cap E_2 = \emptyset$. Set $g(x) = f_{E_1}\chi_{E_1}(x) + f_{E_2}\chi_{E_2}(x)$ $(x \in I)$. Then*

$$g_I = f_I, \quad \Omega(g;I) \leq \Omega(f;I),$$

and the last inequality becomes an equality if and only if

$$(f(x)-f_I)\,(f(y)-f_I)\le 0 \quad (x\in E_1,\ y\in E_2)\,.$$

Proof For the proof, one has to apply Lemma 2.11 twice. □

The following lemma presents an estimate for the relative oscillations of a monotone function in one-sided neighbourhoods of a point.

Lemma 2.13 *Let I denote the interval $(\alpha,\beta)\subset\mathbb{R}^+$. If f is a monotone function on $\mathbb{R}^+$ and $f(\beta)>0$, then there is an interval $J=(\alpha,\alpha+(1+\delta)(\beta-\alpha))\supset I$, i.e. $\delta\ge 0$ such that*

$$\langle f\rangle_J \ge 2\frac{\left|\sqrt{f_I}-\sqrt{f(\beta)}\right|}{\sqrt{f_I}+\sqrt{f(\beta)}}. \tag{2.13}$$

Proof Without loss of generality, one can assume that $I=(0,1)$, $f_I=1$. Let $\tau=1+1/\sqrt{f(1)}$, $K=(0,\tau)$, $g(x)=\chi_{[0,1]}(x)+f(1)\chi_{(1,\tau]}(x)$ $(x\in K)$, then

$$g_K=\frac{1}{\tau}[1+(\tau-1)f(1)]=\sqrt{f(1)},$$
$$\Omega(g;K)=\frac{2}{\tau}\left|1-\sqrt{f(1)}\right|=2\sqrt{f(1)}\frac{\left|1-\sqrt{f(1)}\right|}{1+\sqrt{f(1)}},$$
$$\frac{\Omega(g;K)}{g_K}=2\frac{\left|1-\sqrt{f(1)}\right|}{1+\sqrt{f(1)}}.$$

Chose an interval J, $I\subset J\subset K$ such that $f_J=\sqrt{f(1)}$, and set $h(y)=\chi_{[0,1]}(y)+f_{J\setminus I}\chi_{J\setminus I}(y)$ $(y\in J)$. Since $f_J=h_J$, Lemma 2.12 implies that $\Omega(f;J)\ge\Omega(h;J)$. However, $\Omega(h;J)=\frac{2}{|J|}\left|1-\sqrt{f(1)}\right|\ge\frac{2}{|K|}\left|1-\sqrt{f(1)}\right|=\Omega(g;K)$. Since $f_J=h_J=g_K$, we finally arrive at the inequality

$$\frac{\Omega(f;J)}{f_J}\ge 2\frac{\left|1-\sqrt{f(1)}\right|}{1+\sqrt{f(1)}}.$$

□

Remark 2.14 Assume that $f(\beta)=0$. Then for any $\xi>0$, there is an interval $J\supset I$ such that $\langle f\rangle_J>2-\xi$. Indeed, if f is a non-increasing function, then for $J=(\alpha,\beta+\delta)$ $(\delta>0)$ one has $f(x)=0$ $(x\in(\beta,\beta+\delta))$,

$$\langle f\rangle_J=\frac{\Omega(f;J)}{f_J}\ge\frac{1}{f_J}\cdot\frac{2}{\delta+\beta-\alpha}\int_\beta^{\beta+\delta}(f_J-f(x))\,\mathrm{d}x=\frac{2\delta}{\delta+\beta-\alpha},$$

and it suffices to take $\delta>(\beta-\alpha)(2-\xi)/\xi$. On the other hand, if f is a non-decreasing function, then setting $\beta_1=\sup\{x:\ f(x)=0\}\ge\beta$, one obtains $f(x)=0$ $(x\in(\alpha,\beta_1))$. Therefore, for $J=(\alpha,\beta_1+\delta)$ $(\delta>0)$, we have

$$\langle f\rangle_J=\frac{\Omega(f;J)}{f_J}\ge\frac{1}{f_J}\cdot\frac{2}{\delta+\beta_1-\alpha}\int_\alpha^{\beta_1}(f_J-f(x))\,\mathrm{d}x=\frac{2\,(\beta_1-\alpha)}{\delta+\beta_1-\alpha},$$

and it suffice to take a δ such that $\delta<(\beta_1-\alpha)\,\xi/(2-\xi)$.

Remark 2.15 In Remark 2.14, the condition of monotonicity of the function f can be dropped. One can only assume that $f(x)=0$ for $x\ge\beta$, or $f(x)=0$ for $x\in(\alpha,\beta)$. This means that functions from the Gurov–Reshetnyak class $\mathcal{GR}_R(\varepsilon)$ $(\varepsilon<2)$ cannot vanish on any interval. Moreover, if f vanishes on a set of a positive measure, then $\langle f\rangle_{\mathcal{GR}_R}=2$.

Indeed, using Lebesgue's density theorem, for an arbitrary $\xi > 0$, chose an interval J such that $|J(f=0)|/|J| > 1-\xi/2$. It follows

$$\langle f \rangle_J = \frac{1}{f_J} \cdot \frac{2}{|J|} \int_{J(f \le f_J)} (f_J - f(x))\, \mathrm{d}x \ge 2\frac{|J(f=0)|}{|J|} > 2-\xi.$$

Setting $(\alpha, \beta) = (0, x)$ in Lemma 2.13, one obtains the following result.

Corollary 2.16 *If a function f is monotone on $\mathbb{R}^+$ and $\langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} \le \varepsilon < 2$, then*

$$\left(\frac{2-\varepsilon}{2+\varepsilon}\right)^2 f(x) \le \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y \le \left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2 f(x) \quad (x>0). \tag{2.14}$$

Proof Indeed, if $(\alpha, \beta) = (0, x)$, the inequality (2.13) can be rewritten as

$$2\frac{\left|\sqrt{\frac{1}{f(x)} \cdot \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y} - 1\right|}{\sqrt{\frac{1}{f(x)} \cdot \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y} + 1} \le \langle f \rangle_J \le \langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} \le \varepsilon,$$

which implies the inequality (2.14). □

Let we denote

$$p_\varepsilon = \frac{(2+\varepsilon)^2}{8\varepsilon}.$$

Then $p_\varepsilon - 1 = \frac{(2-\varepsilon)^2}{8\varepsilon}$.

Theorem 2.17 (cf. [1,8,9]) *Let f be a monotone function on the semi-axis $\mathbb{R}^+$ and $f \in \mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$ for an $\varepsilon < 2$. Then for any $0 < u < t$, the inequality*

$$\left(\frac{t}{u}\right)^{-1/p_\varepsilon} \frac{1}{u}\int_0^u f(x)\,\mathrm{d}x \le \frac{1}{t}\int_0^t f(x)\,\mathrm{d}x \le \left(\frac{t}{u}\right)^{1/(p_\varepsilon-1)} \frac{1}{u}\int_0^u f(x)\,\mathrm{d}x \tag{2.15}$$

holds.

Proof Set $c = ((2-\varepsilon)/(2+\varepsilon))^2$ $(0 < c < 1)$ and rewrite inequality (2.14) in the form

$$c \cdot \frac{1}{x} \le \frac{f(x)}{\int_0^x f(y)\,\mathrm{d}y} \le \frac{1}{c} \cdot \frac{1}{x}.$$

Since

$$\frac{f(x)}{\int_0^x f(y)\,\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}}x\left(\ln \int_0^x f(y)\,\mathrm{d}y\right),$$

then integrating the previous inequality in (u, t) $(0 < u < t)$, one gets

$$c \cdot \ln\frac{t}{u} \le \ln\frac{\int_0^t f(y)\,\mathrm{d}y}{\int_0^u f(y)\,\mathrm{d}y} \le \frac{1}{c} \cdot \ln\frac{t}{u},$$

i.e.

$$\left(\frac{t}{u}\right)^{c-1} \frac{1}{u}\int_0^u f(y)\,\mathrm{d}y \le \frac{1}{t}\int_0^t f(y)\,\mathrm{d}y \le \left(\frac{t}{u}\right)^{1/c-1} \frac{1}{u}\int_0^u f(y)\,\mathrm{d}y.$$

But $c-1 = -1/p_\varepsilon$, $1/c - 1 = 1/(p_\varepsilon - 1)$, which completes the proof. □

Corollary 2.18 (cf. [2,4]) *Assume that* $f \in \mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$ *for an* $0 < \varepsilon < 2$. *If* f *is a non-decreasing function and a number* p *satisfies the inequality* $0 < p < p_\varepsilon$, *then for any* $t > 0$, *the inequality*

$$\left(\frac{1}{t}\int_0^t f^p(x)\,\mathrm{d}x\right)^{1/p} \le \left(1-\frac{p}{p_\varepsilon}\right)^{-1/p}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x$$

holds. If f *is a non-decreasing function and the number* q *satisfy the inequality* $0 < q < p_\varepsilon - 1$, *then for any* $t > 0$, *the inequality*

$$\left(\frac{1}{t}\int_0^t f^{-q}(x)\,\mathrm{d}x\right)^{-1/q} \ge \left(1-\frac{q}{p_\varepsilon - 1}\right)^{1/q}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x$$

holds.

Proof Let us start with the proof of the first inequality. Since function f is non-decreasing, one can rewrite the left inequality from (2.15) in the form

$$f(u) \le \frac{1}{u}\int_0^u f(x)\,\mathrm{d}x \le \left(\frac{u}{t}\right)^{-1/p_\varepsilon}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x,$$

rise it to the power $0 < p < p_\varepsilon$, integrate in u from 0 to t and rise the result to the power $1/p$.

Analogously, using the right inequality from (2.15) and monotonicity of f, one obtains

$$f(u) \ge \frac{1}{u}\int_0^u f(x)\,\mathrm{d}x \ge \left(\frac{u}{t}\right)^{1/(p_\varepsilon - 1)}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x.$$

Further, one has to rise this inequality to the power $-q$ $(0 < q < p_\varepsilon - 1)$ and integrate in u from 0 to t and rise the result to the power $-1/q$. This leads to the second claim of corollary. □

Corollary 2.19 *Assume that a monotone function* $f \in \mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$ *for an* $0 < \varepsilon < 2$. *Then the functions* f^{p_ε} *and* $f^{-(p_\varepsilon - 1)}$ *are not summable on the interval* $[1, \infty)$.

Proof If f is non-increasing function, then $f^{-(p_\varepsilon-1)}$ is not summable on $[1, \infty)$. Set $u = 1$ in the left inequality (2.15) and apply the right inequality (2.14), for $t \ge 1$ one obtains

$$f(t) \ge \left(\frac{2-\varepsilon}{2+\varepsilon}\right)^2 \frac{1}{t}\int_0^t f(x)\,\mathrm{d}x \ge \left(\frac{2-\varepsilon}{2+\varepsilon}\right)^2 t^{-1/p_\varepsilon}\int_0^1 f(x)\,\mathrm{d}x.$$

This implies that f^{p_ε} is not summable on the interval $[1, \infty)$.

Analogously, if f is a non-decreasing function, then f^{p_ε} is not summable on $[1, \infty)$, the right inequality (2.15) and left inequality (2.14) for $t \ge 1$ imply

$$f(t) \le \left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2 \frac{1}{t}\int_0^t f(x)\,\mathrm{d}x \le \left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2 t^{1/(p_\varepsilon-1)}\int_0^1 f(x)\,\mathrm{d}x.$$

This shows that the function $f^{-(p_\varepsilon-1)}$ is not summable on $[1, \infty)$. □

Remark 2.20 Inequalities (2.13) from Lemma 2.13 and (2.14) from Corollary 2.16 are sharp. It is easily seen for the function $g(x) = A\chi_{[0,1)}(x) + B\chi_{[1,\infty)}(x)$, if one applies Lemma 2.2 with $\alpha = 1$.

Remark 2.21 Theorem 2.17 does not show that the function f is locally summable with $p = p_\varepsilon$ and $q = -(p_\varepsilon - 1)$. However, as it shown below (see Theorem 3.12 and Remark 3.14), that conditions $p < p_\varepsilon$ and $q < p_\varepsilon - 1$ for exponents of local summability of the function f in Corollary 2.18 can be improved. Similarly, in Corollary 2.19, the corresponding exponents can be chosen larger than p_ε and $p_\varepsilon - 1$.

In conclusion of this section, let us give a small refinement of Lemma 2.13.

Lemma 2.22 *Let f be a monotone function on the interval $\mathbb{R}^+$ and let $I = (\alpha, \beta) \subset \mathbb{R}^+$. Then there is a $\delta > 0$ such that for the interval $J = (\alpha, \alpha + (1+\delta)(\beta - \alpha)) \supset I$ the inequality*

$$\langle f \rangle_J \geq 2 \frac{\left|\sqrt{f_I} - \sqrt{f_{J\setminus I}}\right|}{\sqrt{f_I} + \sqrt{f_{J\setminus I}}}$$

holds.

Proof Let us first show that one can chose a δ such that

$$\int_I f(x)\,\mathrm{d}x = \delta \int_\beta^{\beta+\delta(\beta-\alpha)} f(x)\,\mathrm{d}x. \tag{2.16}$$

Indeed, consider the right-hand side of this equation as a function of $\varphi(\delta)$. It is continuous, $\lim_{\delta\to\infty} \varphi(\delta) = \infty$ and $\varphi(0) = 0$. Therefore, there is a $\delta > 0$ such that the equation (2.16) holds. Fix such a δ and introduce $J = (\alpha, \alpha + (1+\delta)(\beta-\alpha)) \supset I$, $B = f_I$, $A = f_{J\setminus I}$, $g(x) = B\chi_I(x) + A\chi_{J\setminus I}(x)$. Then $|J \setminus I| = \delta|I|$, $B/A = \delta^2$,

$$g_J = \frac{1}{(1+\delta)|I|}(B|I| + A\delta|I|) = \frac{B + \delta A}{1+\delta} = f_J,$$

$$\Omega(g; J) = \frac{2}{(1+\delta)|I|}\,|B - g_J|\,|I| = \frac{2\delta}{(1+\delta)^2}|B - A|,$$

$$\frac{\Omega(g; J)}{g_J} = \frac{2\delta}{1+\delta}\,\frac{|B-A|}{B+\delta A} = 2\frac{\left|\sqrt{B} - \sqrt{A}\right|}{\sqrt{B} + \sqrt{A}}.$$

It remains to use the relation $f_J = g_J$ and the inequality $\Omega(f; J) \geq \Omega(g; J)$, which follows from Lemma 2.12. □

Remark 2.23 In notation of Lemma 2.22 one has $\left|f_I - f_{J\setminus I}\right| \geq |f_I - f(\beta)|$. It means that Lemma 2.22 is more accurate in comparison to Lemma 2.13.

3 A weak Gurov–Reshetnyak condition on the semi-axis

As was mentioned before, functions f monotone on the interval $\mathbb{R}^+$ satisfy the relation $\langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} = \sup_{\eta>0} \langle f \rangle_{(0,\eta)}$. For locally summable functions f, let $\langle f \rangle_{\mathcal{GR}^w_{\mathbb{R}^+}} = \sup_{\eta>0} \langle f \rangle_{(0,\eta)}$, and we denote $\mathcal{GR}^w_{\mathbb{R}^+}(\varepsilon) = \left\{ f : \langle f \rangle_{\mathcal{GR}^w_{\mathbb{R}^+}} \leq \varepsilon \right\}$. It is clear that $\langle f \rangle_{\mathcal{GR}^w_{\mathbb{R}^+}} \leq \langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}}$. If f is a monotone function, the last inequality becomes the equality, $\mathcal{GR}^w_{\mathbb{R}^+}(\varepsilon) \supset \mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$ and this is a proper inclusion.

Lemma 3.1 *If function f is defined on an interval I, then for any measurable subset $E \subset I$, the inequality*

$$1 - \frac{1}{2}\frac{|I|}{|E|}\langle f \rangle_I \le \frac{f_E}{f_I} \le 1 + \frac{1}{2}\frac{|I|}{|E|}\langle f \rangle_I \tag{3.1}$$

holds.

Proof One has

$$\begin{aligned} |E|\,|f_E - f_I| &= \left| \int_E (f(x) - f_I)\,\mathrm{d}x \right| \\ &= \left| \int_{E(f \ge f_I)} (f(x) - f_I)\,\mathrm{d}x - \int_{E(f < f_I)} (f_I - f(x))\,\mathrm{d}x \right| \\ &\le \max\left\{ \int_{E(f \ge f_I)} |f(x) - f_I|\,\mathrm{d}x, \int_{E(f < f_I)} |f_I - f(x)|\,\mathrm{d}x \right\} \\ &\le \max\left\{ \int_{I(f \ge f_I)} |f(x) - f_I|\,\mathrm{d}x, \int_{I(f < f_I)} |f_I - f(x)|\,\mathrm{d}x \right\}. \end{aligned}$$

Since in the right-hand side any expression in the curl brackets is equal to $\frac{1}{2}|I|\Omega(f; I)$, then

$$|E|\,|f_E - f_I| \le \frac{1}{2}|I|\Omega(f; I).$$

This implies that

$$\left| \frac{f_E}{f_I} - 1 \right| \le \frac{1}{2}\frac{|I|}{|E|}\langle f \rangle_I .$$

Example 3.2 Let us show that both inequalities in (3.1) are sharp. Indeed, if functions f_1 and f_2 are defined by $f_1(x) = x^{-1/p}$ and $f_2(x) = x^{1/(p-1)}$ $(x \in \mathbb{R}^+,\ p > 1)$, then one can easily show that $\langle f_1 \rangle_{(0,t)} = \langle f_2 \rangle_{(0,t)} = 2(p-1)^{p-1}p^{-p}$ for any $t > 0$. Fix an $\varepsilon \in (0, 2)$ and chose $p > 1$ such that $2(p-1)^{p-1}p^{-p} = \varepsilon$. Then one can observe that for the function f_2 and the set $E = \left(0, ((p-1)/p)^{p-1}\,t\right) \subset (0, t) = I$, the left inequality in (3.1) becomes an equality for any $t > 0$. On the other hand, if $E = \left(0, ((p-1)/p)^{p}\,t\right) \subset (0, t) = I$, then for the function f_1, the right inequality in (3.1) becomes an equality for any $t > 0$.

In view of these two classical examples of functions satisfying the Gurov–Reshetnyak conditions on the semi-axis $\mathbb{R}^+$, it is convenient to define the class $\mathcal{GR}^w_{\mathbb{R}^+}(\varepsilon)$ $(0 < \varepsilon < 2)$ as $\mathcal{GR}^w_{\mathbb{R}^+}\left(2(p-1)^{p-1}p^{-p}\right)$ $(p > 1)$. For a fixed $p > 1$, we will use the notation

$$\varepsilon'_p = 2\frac{(p-1)^{p-1}}{p^p},$$

whereas for a $0 < \varepsilon < 2$, by $p'_\varepsilon > 1$ we denote the root of the equation

$$2\frac{(p-1)^{p-1}}{p^p} = \varepsilon.$$

One can easily see that this equation is solvable and has a unique solution. Thus for the functions $f_1(x) = x^{-1/p}$ and $f_2(x) = x^{1/(p-1)}$ in Example 3.2, one has

$$\langle f_1 \rangle_{\mathcal{GR}^w_{\mathbb{R}^+}} = \langle f_2 \rangle_{\mathcal{GR}^w_{\mathbb{R}^+}} = \varepsilon'_p \quad (p > 1).$$

In other words, the functions $F_1(x) = x^{-1/p'_\varepsilon}$ and $F_2(x) = x^{1/(p'_\varepsilon - 1)}$ satisfy the relation

$$\langle F_1 \rangle_{\mathcal{GR}^w_{\mathbb{R}+}} = \langle F_2 \rangle_{\mathcal{GR}^w_{\mathbb{R}+}} = \varepsilon \quad (0 < \varepsilon < 2).$$

Theorem 3.3 *If $f \in \mathcal{GR}^w_{\mathbb{R}+}\left(\varepsilon'_p\right)$ for a given $p > 1$, then*

$$\frac{p-1}{p} f_{(0,(1-1/p)^p t)} \leq f_{(0,t)} \leq \frac{p}{p-1} f_{(0,(1-1/p)^{p-1} t)} \quad (t > 0). \tag{3.2}$$

Proof Let us apply Lemma 3.1 for $I = (0, t)$.

In order to prove the left inequality in (3.2), introduce $\tau = |E|/|I| \in (0, 1)$ and multiply the right inequality in (3.1) by $\tau^{1/p}$. Then

$$\tau^{1/p} \frac{f_{(0,\tau t)}}{f_{(0,t)}} \leq \tau^{1/p} \left(1 + \frac{1}{\tau} \frac{(p-1)^{p-1}}{p^p}\right).$$

Let $\varphi(\tau)$ denote the right-hand side of this inequality. In order to determine the point of the minimum $\tau_0 \in (0, 1)$ of the function $\varphi(\tau)$, we compute the derivative $\varphi'(\tau)$. Thus one has

$$\varphi'(\tau) = \frac{\tau^{-2+1/p}}{p} \left[\tau - \left(1 - \frac{1}{p}\right)^p\right].$$

It shows that the minimum of φ is attained at the point $\tau_0 = (1 - 1/p)^p$, and

$$\min_{0<\tau<1} \varphi(\tau) = \varphi(\tau_0) = 1.$$

Consequently,

$$\frac{f_{(0,\tau_0 t)}}{f_{(0,t)}} \leq \tau_0^{-1/p},$$

which completes the proof of the left inequality.

In order to show the right inequality in (3.2), we again will use the notation $\tau = |E|/|I| \in (0, 1)$ and multiply the left inequality in (3.1) by $\tau^{-1/(p-1)}$. Thus

$$\tau^{-1/(p-1)} \left(1 - \frac{1}{\tau} \frac{(p-1)^{p-1}}{p^p}\right) \leq \tau^{-1/(p-1)} \frac{f_{(0,\tau t)}}{f_{(0,t)}}.$$

Let $\psi(\tau)$ denote the left-hand side of this inequality and let $\tau_1 \in (0, 1)$ denote the point of maximum for the function ψ. To find this point, we compute the derivative $\psi'(\tau)$,

$$\psi'(\tau) = \frac{\tau^{-2-1/(p-1)}}{p-1} \left[-\tau + \left(\frac{p-1}{p}\right)^{p-1}\right].$$

Thus $\tau_1 = (1 - 1/p)^{p-1}$ and

$$\max_{0<\tau<1} \psi(\tau) = \psi(\tau_1) = 1,$$

and, consequently,

$$\frac{f_{(0,\tau_1 t)}}{f_{(0,t)}} \geq \tau_1^{1/(p-1)}.$$

This completes the proof of theorem. □

For the function $f_1(x) = x^{-1/p}$, the left inequality in (3.2) becomes an equality, whereas the function $f_2(x) = x^{1/(p-1)}$ turns the right inequality (3.2) into an equality.

Theorem 3.4 *If $f \in \mathcal{GR}^w_{\mathbb{R}^+}\left(\varepsilon'_p\right)$ for a $p > 1$, then*

$$\left(\frac{p-1}{p}\right)^p \left(\frac{\zeta}{\eta}\right)^{-1/(p-1)} \frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x \le \frac{1}{\eta}\int_0^\eta f(x)\,\mathrm{d}x$$
$$\le \left(\frac{p}{p-1}\right)^{p-1} \left(\frac{\zeta}{\eta}\right)^{1/p} \frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x \quad (0 < \eta \le \zeta). \tag{3.3}$$

Proof Set $a = (p/(p-1))^p > 1$ and rewrite the left inequality from (3.2) in the form

$$\frac{a}{t}\int_0^{t/a} f(x)\,\mathrm{d}x \le a^{1/p}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x.$$

Introducing the function $\Phi(t) = t^{1/p}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x$, one obtains

$$\Phi\left(\frac{t}{a}\right) = t^{1/p}a^{-1/p}\frac{a}{t}\int_0^{t/a} f(x)\,\mathrm{d}x \le t^{1/p}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x = \Phi(t),$$

i.e.

$$\Phi\left(\frac{t}{a}\right) \le \Phi(t). \tag{3.4}$$

Consider $0 < \eta \le \zeta$. It follows from (3.4) that

$$\Phi(\eta) \le \max_{\frac{\zeta}{a}\le t\le \zeta} \Phi(t) \le \left(\frac{\zeta}{a}\right)^{-1+1/p}\int_0^\zeta f(x)\,\mathrm{d}x = a^{1-1/p}\zeta^{1/p}\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x,$$

but this is the right inequality in (3.3).

Analogously, denoting $b = (p/(p-1))^{p-1} > 1$, we rewrite the right inequality in (3.2) in the form

$$\frac{b}{t}\int_0^{t/b} f(x)\,\mathrm{d}x \ge b^{-1/(p-1)}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x.$$

Introducing the function $\Psi(t) = t^{-1/(p-1)}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x$, one obtains

$$\Psi\left(\frac{t}{b}\right) = t^{-1/(p-1)}b^{1/(p-1)}\frac{b}{t}\int_0^{t/b} f(x)\,\mathrm{d}x \ge t^{-1/(p-1)}\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x = \Psi(t),$$

i.e.

$$\Psi\left(\frac{t}{b}\right) \ge \Psi(t).$$

This inequality implies that for arbitrary $0 < \eta \le \zeta$, one gets

$$\Psi(\eta) \ge \min_{\zeta\le t\le b\zeta} \Psi(t) \ge (b\zeta)^{-1-1/(p-1)}\int_0^\zeta f(x)\,\mathrm{d}x$$
$$= b^{-1-1/(p-1)}\zeta^{-1/(p-1)}\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x,$$

which is the left inequality in (3.3), and Theorem 3.4 is proved. □

Remark 3.5 Theorem 3.4 means that the inequality (3.3) is valid for all $0 < \eta \le \zeta$. Discreet analogues of (3.3), i.e. the situation where $\eta = \zeta/a$ and $a > 1$ is fixed, are known (see inequalities (5.67) and (5.95) of [4]).

Remark 3.6 For the function $f_1(x) = x^{-1/p}$, the equation

$$\frac{1}{\eta}\int_0^\eta f_1(x)\,\mathrm{d}x = \left(\frac{\zeta}{\eta}\right)^{1/p} \frac{1}{\zeta}\int_0^\zeta f_1(x)\,\mathrm{d}x \quad (0 < \eta \le \zeta)$$

holds. This means that in the right-hand side of (3.3), the exponent $1/p$ cannot be reduced, even if the factor $(p/(p-1))^{p-1}$ will be replaced by an arbitrary large one independent of ζ and η. The author is not aware which minimal factor depending on p can be chosen in the right-hand side of (3.3) instead of $(p/(p-1))^{p-1}$. However, since $\sup_{p>1}(p/(p-1))^{p-1} = \mathrm{e}$, this factor can be replaced by e and cannot be replaced by the one smaller than 1. The next example shows that for $p = 2$, this minimal factor is greater than 1.

Example 3.7 Consider the function $f(x) = \chi_{[0,1]}(x) + \frac{9}{25}\chi_{(1,\infty)}(x)$ $(x \in \mathbb{R}^+)$. By Lemma 2.2, one has $\langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} = 1/2$, i.e. $p = 2$. If $\zeta > \eta = 1$ is arbitrary, then

$$\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x = \frac{9\zeta + 16}{25\zeta}, \quad \frac{\int_0^1 f(x)\,\mathrm{d}x}{\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x} \cdot \frac{1}{\sqrt{\zeta}} = \frac{25\sqrt{\zeta}}{9\zeta + 16},$$

so the maximum value of the right-hand side is achieved at $\zeta = 16/9$ and it is equal to $25/24 > 1$.

Remark 3.8 For the function $f_2(x) = x^{1/(p-1)}$, the equation

$$\left(\frac{\zeta}{\eta}\right)^{-1/(p-1)} \frac{1}{\zeta}\int_0^\zeta f_2(x)\,\mathrm{d}x = \frac{1}{\eta}\int_0^\eta f_2(x)\,\mathrm{d}x \quad (0 < \eta \le \zeta)$$

holds. This means that in the left-hand side of (3.3), the exponent $-1/(p-1)$ cannot be raised, even if the factor $(1 - 1/p)^p$ will be replaced by a positive small one independent of ζ and η. The author does not know which maximal factor depending on p can be taken in the left-hand side of (3.3) instead of $(1 - 1/p)^p$. The next example shows that for $p = 2$, this maximal factor is smaller than 1.

Example 3.9 Consider the function $f(x) = \chi_{[0,1]}(x) + \frac{25}{9}\chi_{(1,\infty)}(x)$ $(x \in \mathbb{R}^+)$. By Lemma 2.2 one has $\langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} = 1/2$, i.e. $p = 2$. If $\zeta > \eta = 1$, then

$$\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x = \frac{25\zeta - 16}{9\zeta}, \quad \frac{\int_0^1 f(x)\,\mathrm{d}x}{\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x} \cdot \zeta = \frac{9\zeta^2}{25\zeta - 16},$$

so the minimal value of the right-hand side is achieved at $\zeta = 32/25$ and is equal to $576/625 < 1$.

Remark 3.10 Lemma 3.1 and Theorems 3.3, 3.4 do not assume the monotonicity of the function f but only that $f \in \mathcal{GR}^w_{\mathbb{R}^+}(\varepsilon)$.

As was mentioned in Corollary 2.19, non-increasing functions f belonging to $\mathcal{GR}^w_{\mathbb{R}^+}(\varepsilon)$ are not summable with power $p_\varepsilon = (2+\varepsilon)^2/(8\varepsilon) > 1$ on $\mathbb{R}^+$. Therefore, they are also not summable with power 1. The next lemma shows that the functions $f \in \mathcal{GR}^w_{\mathbb{R}^+}(\varepsilon)$ $(\varepsilon < 2)$ are not summable on $\mathbb{R}^+$ even if they are not monotonic.

Lemma 3.11 *If f is a function summable on $\mathbb{R}^+$, then $\langle f \rangle_{\mathcal{GR}^w_{\mathbb{R}+}} = 2$.*

Proof One can assume that $\int_{\mathbb{R}+} f(x)\,\mathrm{d}x = 1$. For a given $\xi > 0$, chose an interval $I = (0, \Delta)$ such that $\int_I f(x)\,\mathrm{d}x > 1 - \xi$. Set $\rho = \xi/\Delta$. Then

$$\int_I f(x)\,\mathrm{d}x - \int_{I(f>\rho)} f(x)\,\mathrm{d}x \le \rho\Delta = \xi,$$

i.e.

$$\int_{I(f>\rho)} f(x)\,\mathrm{d}x \ge \int_I f(x)\,\mathrm{d}x - \xi \ge 1 - 2\xi.$$

Further, let us chose $D \ge \Delta$ such that $\frac{1}{D}\int_0^D f(x)\,\mathrm{d}x \le \rho$ and let $J = (0, D) \supset I$. Then

$$\begin{aligned}\frac{\Omega(f; J)}{f_J} &= \frac{\frac{2}{D}\int_{J(f>f_J)} (f(x) - f_J)\,\mathrm{d}x}{\frac{1}{D}\int_J f(x)\,\mathrm{d}x} \ge 2\int_{J(f>f_J)} (f(x) - f_J)\,\mathrm{d}x \\ &\ge 2\int_{I(f>f_J)} (f(x) - f_J)\,\mathrm{d}x \ge 2\int_{I(f>\rho)} (f(x) - f_J)\,\mathrm{d}x \\ &\ge 2\int_{I(f>\rho)} (f(x) - \rho)\,\mathrm{d}x \ge 2\int_{I(f>\rho)} f(x)\,\mathrm{d}x - \rho\Delta \ge 2(1-2\xi) - \xi = 2 - 5\xi.\end{aligned}$$

Since $\xi > 0$ is an arbitrary positive number, it completes the proof. □

Theorem 3.4 allows us to find exact limiting exponents of summability for monotone functions f on the semi-axis $\mathbb{R}^+$. These exponents are given by the following theorem.

Theorem 3.12 *Let $0 < \varepsilon < 2$ and let $f \in \mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$.*

(1) *If f is a non-increasing function, then*

- (a) *the function f^p is locally summable for any $p < p'_\varepsilon$ and it is not necessarily locally summable for $p \ge p'_\varepsilon$;*
- (b) *for any $p \le p'_\varepsilon$, the function f^p is not summable on $[1, \infty)$ and can be summable on $[1, \infty)$ for $p > p'_\varepsilon$.*

(2) *If f is a non-decreasing function, then*

- (a) *the function f^{-q} is locally summable for any $q < p'_\varepsilon - 1$ and it is not necessarily summable for $q \ge p'_\varepsilon - 1$;*
- (b) *for any $q \le p'_\varepsilon - 1$, the function f^{-q} is non-summable on $[1, \infty)$ and can be summable on $[1, \infty)$ for $q > p'_\varepsilon - 1$.*

Proof Let us recall that monotone functions f on the semi-axis $\mathbb{R}^+$ satisfy the equality $\langle f \rangle_{\mathcal{GR}_{\mathbb{R}^+}} = \langle f \rangle_{\mathcal{GR}^w_{\mathbb{R}+}}$. If $p, q \le 0$, then all statements of the above theorem are obvious, so we will consider the case where $p, q > 0$.

In order to show 1(a), let us fix a $\zeta > 0$ and use the monotonicity of f and the right inequality from (3.3). Then for $0 < \eta \le \zeta$, one obtains

$$f(\eta) \le \frac{1}{\eta}\int_0^\eta f(x)\,\mathrm{d}x \le \left(\frac{p'_\varepsilon}{p'_\varepsilon - 1}\right)^{p'_\varepsilon - 1} \left(\frac{\zeta}{\eta}\right)^{1/p'_\varepsilon} \frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x.$$

Raising the last inequality to power $0 < p < p'_\varepsilon$ and integrating in η from 0 to ζ, we get

$$\left(\frac{1}{\zeta}\int_0^\zeta f^p(\eta)\,\mathrm{d}\eta\right)^{1/p} \le \left(\frac{p'_\varepsilon}{p'_\varepsilon-1}\right)^{p'_\varepsilon-1}\left(\frac{p'_\varepsilon}{p'_\varepsilon-p}\right)^{1/p}\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x < \infty.$$

Further, the function $f_1(x) = x^{-1/p'_\varepsilon}$ from Example 3.2 belongs to $\mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$ and it is locally unsummable with any exponent $p \ge p'_\varepsilon$. It proves 1(a), and the same example shows the validity of the second part of 1(b). Finally, setting $\eta = 1$ in the right inequality (3.3) and using the right inequality from (2.14) for $\zeta \ge 1$, one obtains

$$\begin{aligned} f(\zeta) &\ge \left(\frac{2-\varepsilon}{2+\varepsilon}\right)^2\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x \\ &\ge \zeta^{-1/p'_\varepsilon}\left(\frac{2-\varepsilon}{2+\varepsilon}\right)^2\left(\frac{p'_\varepsilon-1}{p'_\varepsilon}\right)^{p'_\varepsilon-1}\int_0^1 f(x)\,\mathrm{d}x > 0. \end{aligned}$$

It implies that for $0 < p \le p'_\varepsilon$, the function f^p is unsummable on $[1,\infty)$, which finishes the proof of statement 1 of Theorem 3.12.

The proof of the second statement of Theorem 3.12 is analogous. To prove 2(a), fix an arbitrary $\zeta > 0$ and using the monotonicity of f and the left inequality from (3.3), for $0 < \eta \le \zeta$, one obtains

$$f(\eta) \ge \frac{1}{\eta}\int_0^\eta f(x)\,\mathrm{d}x \ge \left(\frac{p'_\varepsilon-1}{p'_\varepsilon}\right)^{p'_\varepsilon}\left(\frac{\zeta}{\eta}\right)^{-1/(p'_\varepsilon-1)}\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x.$$

If one raises it to the power $-q$ $(0 < q < p'_\varepsilon - 1)$ and integrate the result in η from 0 to ζ, then

$$\begin{aligned} &\left(\frac{1}{\zeta}\int_0^\zeta f^{-q}(\eta)\,\mathrm{d}\eta\right)^{1/q} \\ &\quad\le \left(\frac{p'_\varepsilon}{p'_\varepsilon-1}\right)^{p'_\varepsilon}\left(\frac{p'_\varepsilon-1}{p'_\varepsilon-q-1}\right)^{1/q}\left(\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x\right)^{-1} < \infty. \end{aligned}$$

Further, the function $f_2(x) = x^{1/(p'_\varepsilon-1)}$ from Example 3.2 belongs to the class $\mathcal{GR}_{\mathbb{R}^+}(\varepsilon)$, but it is not locally summable with either power $-q$ if $q \ge p'_\varepsilon - 1$. This proves statement 2(a), and the same example can be used to show the second part of assertion 2(b). It remains to prove the first part of 2(b). Setting $\eta = 1$ in the left inequality (3.3) and taking into account the left inequality from (2.14) for $\zeta \ge 1$, we get

$$\begin{aligned} f(\zeta) &\le \left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2\frac{1}{\zeta}\int_0^\zeta f(x)\,\mathrm{d}x \\ &\le \zeta^{1/(p'_\varepsilon-1)}\left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2\left(\frac{p'_\varepsilon}{p'_\varepsilon-1}\right)^{p'_\varepsilon}\int_0^1 f(x)\,\mathrm{d}x < \infty. \end{aligned}$$

This implies that for $0 < q \le p'_\varepsilon - 1$, the function f^{-q} is not summable on the interval $[1,\infty)$. □

Remark 3.13 The limiting summability exponents p'_ε and $p'_\varepsilon - 1$ calculated, respectively, in parts 1(a) and 2(a) of Theorem 3.12 have been presented earlier in [4, Theorems 5.34 and 5.48] (see also [5] and [6]). On the other hand, the results of parts 1(b) and 2(b) of Theorem 3.12 are new.

Remark 3.14 Theorem 3.12 is more accurate than Corollaries 2.18 and 2.19 (see also Remark 2.21). Indeed, recalling the notation $p_\varepsilon = (2+\varepsilon)^2/(8\varepsilon)$, $0 < \varepsilon < 2$, it suffices to show that $p'_\varepsilon > p_\varepsilon$. Let us formulate this inequality as a separate result.

Lemma 3.15 *For any* $0 < \varepsilon < 2$, *the inequality* $p'_\varepsilon > p_\varepsilon$ *holds.*

Proof Fix $p > 1$ and note that $\varepsilon_p = 2\left(\sqrt{p} - \sqrt{p-1}\right)^2$ is the solution of the equation $(2+\varepsilon)^2/(8\varepsilon) = p$. Then, the required inequality $p'_\varepsilon > p_\varepsilon$ is equivalent to the following one $\varepsilon'_p > \varepsilon_p$, i.e.

$$\frac{(p-1)^{p-1}}{p^p} > \left(\sqrt{p} - \sqrt{p-1}\right)^2 \quad (p > 1). \tag{3.5}$$

To prove (3.5), we set $p = r^2/\left(r^2-1\right)$ $(r > 1)$ and using equivalent transformation, we arrive at the inequality

$$\frac{\ln(1+r)}{r} < \frac{\ln\left(1+\frac{1}{r}\right)}{\frac{1}{r}} \quad (r > 1),$$

which turns into an equality for $r = 1$. Moreover, since $\frac{\ln(1+r)}{r}$ is a strictly decreasing function on $(0, \infty)$, the last inequality is obviously true. □

4 Conclusion

Let $f \in \mathcal{GR}_{\mathbb{R}}(\varepsilon)$ for an $0 < \varepsilon < 2$. It is easily seen that the inequality (3.3) can be written in the form

$$\left(\frac{p'_\varepsilon - 1}{p'_\varepsilon}\right)^{p'_\varepsilon} \left(\frac{\zeta}{\eta}\right)^{-1/(p'_\varepsilon - 1)} \frac{1}{\zeta}\int_\alpha^{\alpha+\zeta} f(x)\,\mathrm{d}x \le \frac{1}{\eta}\int_\alpha^{\alpha+\eta} f(x)\,\mathrm{d}x$$
$$\le \left(\frac{p'_\varepsilon}{p'_\varepsilon - 1}\right)^{p'_\varepsilon - 1} \left(\frac{\zeta}{\eta}\right)^{1/p'_\varepsilon} \frac{1}{\zeta}\int_\alpha^{\alpha+\zeta} f(x)\,\mathrm{d}x \quad (0 < \eta \le \zeta), \tag{4.1}$$

$$\left(\frac{p'_\varepsilon - 1}{p'_\varepsilon}\right)^{p'_\varepsilon} \left(\frac{\zeta}{\eta}\right)^{-1/(p'_\varepsilon - 1)} \frac{1}{\zeta}\int_{\alpha-\zeta}^{\alpha} f(x)\,\mathrm{d}x \le \frac{1}{\eta}\int_{\alpha-\eta}^{\alpha} f(x)\,\mathrm{d}x$$
$$\le \left(\frac{p'_\varepsilon}{p'_\varepsilon - 1}\right)^{p'_\varepsilon - 1} \left(\frac{\zeta}{\eta}\right)^{1/p'_\varepsilon} \frac{1}{\zeta}\int_{\alpha-\zeta}^{\alpha} f(x)\,\mathrm{d}x \quad (0 < \eta \le \zeta), \tag{4.2}$$

with an arbitrary $\alpha \in \mathbb{R}$. However, it remains an open problem whether the exponents $-1/\left(p'_\varepsilon - 1\right)$ and $1/p'_\varepsilon$ in the inequalities (4.1) and (4.2) are sharp, since in this case the author does not have any example of functions similar to f_1 and f_2 from Remarks 3.6 and 3.8. It would be only natural to try to replace those examples by the functions $f_3(x) = |x|^{-1/p'_\varepsilon}$ and $f_4(x) = |x|^{1/(p'_\varepsilon - 1)}$ $(x \in \mathbb{R})$, respectively. Unfortunately, the values of the terms $\varepsilon'' = \langle f_3 \rangle_{\mathcal{GR}_{\mathbb{R}}}$ and $\varepsilon''' = \langle f_4 \rangle_{\mathcal{GR}_{\mathbb{R}}}$ are not known to the author. Nevertheless, one still can show that for every $0 < \varepsilon < 2$, each of them is greater than ε.

Lemma 4.1 *For any* $0 < \varepsilon < 2$, *the inequality* $\varepsilon'' > \varepsilon$ *holds.*

Proof It suffices to show that for any $p > 1$, the function $f(x) = |x|^{-1/p}$ $(x \in \mathbb{R})$ satisfies the inequality

$$\langle f \rangle_{\mathcal{GR}_{\mathbb{R}}} > 2\frac{(p-1)^{p-1}}{p^p}.$$

For $\eta \geq 0$, let $I = (-\eta, 1)$. Compute

$$f_I = \frac{p}{p-1}\frac{1}{1+\eta}\left(1+\eta^{(p-1)/p}\right)$$

and chose an $x_\eta \in (0,1)$ such that $x_\eta^{-1/p} = f_I$. Assume that η is small enough, so that $\eta^{-1/p} \geq f_I$. Then

$$\begin{aligned}
\Omega(f;I) &= \frac{2}{1+\eta}\int_{x_\eta}^{1}\left(f_I - x^{-1/p}\right)\,\mathrm{d}x \\
&= \frac{2}{1+\eta}\left[f_I\left(1+\frac{1}{p-1}x_\eta\right) - \frac{p}{p-1}\right], \\
\langle f \rangle_I &= \frac{2}{1+\eta}\left[\frac{(p-1)^{p-1}}{p^p}\left(\frac{1+\eta}{1+\eta^{(p-1)/p}}\right)^p + \frac{\eta^{(p-1)/p}-\eta}{1+\eta^{(p-1)/p}}\right],
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f \rangle_I \\
&\quad = \frac{p-1}{p}\frac{\eta^{-1/p}}{\left(1+\eta^{(p-1)/p}\right)^2}\left[1-\left(\frac{p-1}{p}\right)^{p-1}\left(\frac{1+\eta}{1+\eta^{(p-1)/p}}\right)^{p-1}\right] \\
&\qquad - \frac{1}{(1+\eta)^2}\left[1-\left(\frac{p-1}{p}\right)^{p}\left(\frac{1+\eta}{1+\eta^{(p-1)/p}}\right)^{p}\right].
\end{aligned} \tag{4.3}$$

The expression in the first square brackets in the right-hand side of (4.3) has a positive limit as $\eta \to 0+$. Therefore, the function in the right-hand side of the first line of (4.3) tends to $+\infty$. On the other hand, the term in the second line of (4.3) is bounded. Consequently,

$$\lim_{\eta \to 0+}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f \rangle_I = +\infty.$$

and it follows that for sufficiently small $\eta > 0$, the inequality

$$\langle f \rangle_I > \langle f \rangle_{(0,1)} = 2\frac{(p-1)^{p-1}}{p^p}$$

holds, which completes the proof. □

Lemma 4.2 *For any* $0 < \varepsilon < 2$*, the inequality* $\varepsilon''' > \varepsilon$ *holds.*

Proof Similarly to the proof of Lemma 4.1, it suffices to show that for any $p > 1$, the function $f(x) = |x|^{1/(p-1)}$ $(x \in \mathbb{R})$ satisfies the inequality

$$\langle f \rangle_{\mathcal{GR}_{\mathbb{R}}} > 2\frac{(p-1)^{p-1}}{p^p}.$$

Let $I = (-\eta, 1)$ $(\eta \geq 0)$. Compute

$$f_I = \frac{p-1}{p}\frac{1}{1+\eta}\left(1+\eta^{p/(p-1)}\right)$$

and chose an $x_\eta \in (0,1)$ such that $x_\eta^{1/(p-1)} = f_I$. Additionally, assume that η sufficiently small so that $\eta^{1/(p-1)} \le f_I$. Then

$$\Omega(f;I) = \frac{2}{1+\eta}\int_{x_\eta}^{1}\left(x^{1/(p-1)} - f_I\right)\mathrm{d}x = \frac{2}{1+\eta}\left[f_I\left(\frac{1}{p}x_\eta - 1\right) + \frac{p-1}{p}\right],$$

$$\langle f\rangle_I = \frac{2}{1+\eta}\left[\frac{(p-1)^{p-1}}{p^p}\left(\frac{1+\eta^{p/(p-1)}}{1+\eta}\right)^{p-1} + \frac{\eta-\eta^{p/(p-1)}}{1+\eta^{p/(p-1)}}\right],$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f\rangle_I = \frac{1}{(1+\eta)^2}\left[1-\left(\frac{p-1}{p}\right)^{p-1}\left(\frac{1+\eta^{p/(p-1)}}{1+\eta}\right)^{p-1}\right]$$
$$-\frac{p}{p-1}\frac{\eta^{1/(p-1)}}{\left(1+\eta^{p/(p-1)}\right)^2}\left[1-\left(\frac{p-1}{p}\right)^{p}\left(\frac{1+\eta^{p/(p-1)}}{1+\eta}\right)^{p}\right].$$

It is easily seen that $\frac{1}{2}\lim_{\eta\to 0+}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f\rangle_I = 1-(1-1/p)^{p-1} > 0$. Therefore, if $\eta > 0$ is sufficiently small, the inequality

$$\langle f\rangle_I > \langle f\rangle_{(0,1)} = 2\frac{(p-1)^{p-1}}{p^p}$$

holds. Lemma 4.2 is proved. □

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