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Sharp Moser–Trudinger inequalities on Riemannian manifolds with negative curvature

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Abstract Let M be a complete, simply connected Riemannian manifold with negative curvature. We obtain some Moser–Trudinger inequalities with sharp constants on M.

Keywords Moser–Trudinger inequality · Riemannian manifold · Negative curvature · Sharp constant

Mathematics Subject Classification Primary 46E35 · 58E35

1 Introduction

Moser [14] found the largest positive constant β_0 such that if Ω is an open domain in $\mathbb{R}^n$, $n \geq 2$, with finite n-measure, then there exists a constant C_0 which depends only on n such that if u is smooth and has compact support contained in Ω, then

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$$\int_{\Omega} \exp(\beta |u|^{n/(n-1)}) \mathrm{d}x \leq C_0 |\Omega| \tag{1.1}$$

for any $\beta \leq \beta_0$ when u is normalized so that

$$\int_{\Omega} |\nabla u(x)|^n \mathrm{d}x \leq 1.$$

In fact, Moser showed $\beta_0 = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^n$. This inequality sharpened the result of N. S. Trudinger [18]. In 1988, D. Adams extended such inequality to high-order Sobolev spaces in $\mathbb{R}^n$ via a quite different method. In the case of unbounded domains, Ruf [16] and Li-Ruf [11] obtained the following inequality:

$$\int_{\mathbb{R}^n} \left(\exp(\beta_0 |u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta_0^k |u|^{kn/(n-1)}}{k!} \right) \mathrm{d}x \leq C \tag{1.2}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$ when u is normalized so that

$$\int_{\mathbb{R}^n} (|\nabla u(x)|^n + |u(x)|^n) \mathrm{d}x \leq 1.$$

The constant β_0 in (1.2) is also sharp.

There has also been substantial progress for Moser–Trudinger inequalities on Riemannian manifolds. In the case of compact Riemannian manifolds, the study of Trudinger-Moser inequalities can be traced back to Aubin [3], Cherrier [4,5], and Fontana [6]. In particular, the following Moser–Trudinger inequality is held in n-dimensional compact Riemannian manifold (M, g) (see [6]):

$$\sup_{\int_M u \mathrm{d}v_g = 0, \int_M |\nabla_g u|^n \mathrm{d}v_g \leq 1} \int_M \exp(\beta_0 |u|^{n/(n-1)}) \mathrm{d}v_g < \infty. \tag{1.3}$$

The constant β_0 in (1.3) is also sharp. In the case of complete noncompact Riemannian manifolds, Yang [19] has showed that if the Ricci curvature has a lower bound and the injectivity radius has a positive lower bound, then Trudinger-Moser inequality holds. However, the constant obtained in [19] is not sharp. Furthermore, if M is the hyperbolic space $\mathbb{H}^2$, Mancini and Sandeep [12] (see also [2]) proved the following inequality on $\mathbb{H}^2$:

$$\sup_{u \in C_0^\infty(\mathbb{B}^2), \int_{\mathbb{B}^2} |\nabla u|^2 \mathrm{d}x \leq 1} \int_{\mathbb{B}^2} \frac{e^{4\pi u^2} - 1}{(1 - |x|^2)^2} \mathrm{d}x < \infty, \tag{1.4}$$

where $\mathbb{B}^2$ is the unit ball at origin of $\mathbb{R}^2$. Furthermore, the constants 4π is sharp. Later, inequality (1.4) has been extended by themselves and Tintarev [13] to any dimension.

To our knowledge, much less is known about sharp constants of Moser–Trudinger inequalities on complete noncompact Riemannian manifolds except Euclidean spaces and Hyperbolic spaces. The aim of this paper is to look for the sharp constants of Moser–Trudinger inequalities on a complete, simply connected Riemannian manifold M with negative curvature. In fact, the optimal constants turn out to be the same for every such M as they are in Euclidean space. For simplicity, we also denote by Δ the Laplace-Beltrami operator on M and by ∇ the corresponding gradient. Let Ω be a domain in M. The Sobolev space $W_0^{1,n}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$\left(\int_{\Omega} |\nabla u|^n \mathrm{d}V \right)^{\frac{1}{n}} + \left(\int_{\Omega} |u|^n \mathrm{d}V \right)^{\frac{1}{n}}.$$

One of our main results is the following

Theorem 1.1 *Let M be a complete, simply connected Riemannian manifold of dimension $n \geq 2$ and Ω be a domain in M with $|\Omega| = \int_\Omega dV < \infty$. There exists a positive constant $C_1 = C_1(n, M)$ such that for all $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dV \leq 1$, the following uniform inequality holds*

$$\frac{1}{|\Omega|} \int_\Omega \exp(\beta_0 |u|^{n/(n-1)}) dV \leq C_1. \tag{1.5}$$

Furthermore, the constant β_0 in (1.5) is sharp.

Next we consider the Moser–Trudinger inequalities on the whole space M. The basic idea of the proof is given by Lam and Lu [8,9], and the main result is the following

Theorem 1.2 *Let M be a complete, simply connected Riemannian manifold of dimension $n \geq 2$ and τ be any positive number. There exists a constant $C_2 = C_2(\tau, n, M)$ such that for all $u \in W_0^{1,n}(M)$ with $\int_M (|\nabla u|^n + \tau |u|^n) dV \leq 1$, the following uniform inequality holds*

$$\int_M \left(\exp(\beta_0 |u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta_0^k |u|^{kn/(n-1)}}{k!} \right) dV \leq C_2. \tag{1.6}$$

Furthermore, the constant β_0 in (1.6) is sharp.

2 Notations and preliminaries

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [7,10,17] for more precise information about this subject.

Let M be an n-dimensional complete Riemannian manifold with Riemannian metric $\mathrm{d}s^2$. If $\{x^i\}_{1 \leq i \leq n}$ is a local coordinate system, then we can write

$$\mathrm{d}s^2 = \sum g_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

so that the Laplace-Beltrami operator Δ in this local coordinate system is

$$\Delta = \sum \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Denote by ∇ the corresponding gradient.

Let K be the sectional curvature on M. M is said to be with negative curvature (respectively, with strictly negative curvature) if $K \leq 0$ (respectively, $K \leq c < 0$) along each plane section at each point of M. If M is with negative curvature, then for each $p \in M$, M contains no points conjugate to p. Furthermore, if M is simply connected, then the exponential mapping $\mathrm{Exp}_p : T_p M \to M$ is a diffeomorphism, where $T_p M$ is the tangent space to M at p (see e.g. [7]).

From now on, we let M be a complete, simply connected Riemannian manifold with negative curvature. Let $p \in M$ and denote by $\rho(x) = \mathrm{dist}(x, p)$ for all $x \in M$, where $\mathrm{dist}(\cdot, \cdot)$ denotes the geodesic distance. Then $\rho(x)$ is smooth on $M \setminus \{p\}$ and it satisfies

$$|\nabla \rho(x)| = 1, \quad x \in M \setminus \{p\}.$$

By Gauss's lemma, the radial derivative $\partial_\rho = \frac{\partial}{\partial \rho}$ satisfies

$$|\partial_\rho f| \leq |\nabla f|, \quad f \in C^1(M). \tag{2.1}$$

For any $\delta > 0$, denote by $B_\delta(p) = \{x \in M : \rho(x) < \delta\}$ the geodesic ball in M. We introduce the density function $J_p(\theta, t)$ of the volume form in normal coordinates as follows (see e.g. [7], page 166-167). Choose an orthonormal basis $\{\theta, e_2, \dots, e_n\}$ on T_pM and let $c(t) = \mathrm{Exp}_p t\theta$ be a geodesic. $\{Y_i(t)\}_{2\le i\le n}$ are Jacobi fields satisfying the initial conditions

$$Y_i(0) = 0, \quad Y_i'(0) = e_i, \quad 2 \le i \le n$$

so that the density function can be given by

$$J_p(\theta, t) = t^{-n+1}\sqrt{\det(\langle Y_i(t), Y_j(t)\rangle)}, \quad t > 0.$$

We note that $J_p(\theta, t)$ does not depend on $\{e_2, \dots, e_n\}$ and $J_p(\theta, t) \in C^\infty(T_pM\backslash\{p\})$ by the definition of $J_p(\theta, t)$. Furthermore, if we set $J_p(\theta, 0) \equiv 1$, then $J_p(\theta, t) \in C(T_pM)$ and

$$J_p(\theta, t) = 1 + O(t^2) \ \text{ as } \ t \to 0, \tag{2.2}$$

since $Y_i(t)$ has the asymptotic expansion (see e.g. [7], page 169)

$$Y_i(t) = te_i - \frac{t^3}{6}R(c'(t), e_i)c'(t) + o(t^3),$$

where $R(\cdot, \cdot)$ is the curvature tensor on M.

By the definition of $J_p(\theta, t)$, we have the following formula in polar coordinates on M:

$$\int_M f\mathrm{d}V = \int_0^\infty \int_{\mathbb{S}^{n-1}} f\rho^{n-1}J_p(\theta, \rho)\mathrm{d}\rho\mathrm{d}\sigma, \quad f \in L^1(M),$$

where $\mathrm{d}\sigma$ denotes the canonical measure of the unit sphere of $T_p(M)$.

If M is with constant sectional curvature, then $J_p(\theta, t)$ depends only on t. We denote by $J_b(t)$ the corresponding density function if $K \equiv -b$ for some $b \ge 0$. It is well known that $J_0(t) = 1$ for $t > 0$ since in this case M is isomorphic to the Euclidean space.

Finally, we recall a useful fact of $J_p(\theta, t)$ which play an important role in the study of Moser–Trudinger inequalities. If the sectional curvature K on M satisfies $K \le -b$, then (see [7], page 172, line -2, the proof of Bishop-Gunther comparison theorem)

$$\frac{1}{J_p(\theta, t)} \cdot \frac{\partial J_p(\theta, t)}{\partial t} \ge \frac{J_b'(t)}{J_b(t)}, \quad t > 0. \tag{2.3}$$

Therefore, since M is with negative curvature, we have

$$\frac{1}{J_p(\theta, t)} \cdot \frac{\partial J_p(\theta, t)}{\partial t} \ge \frac{J_0'(t)}{J_0(t)} = 0,$$

which means $J_p(\theta, t)$, as a function of t on $[0, +\infty)$, is monotonically increasing.

3 Proof of Theorem 1.1

We firstly show the following pointwise estimates for $f \in C_0^\infty(M)$.

Lemma 3.1 *There holds, for any $f \in C_0^\infty(M)$ and $p \in M$,*

$$|f(p)| \le \frac{1}{\omega_{n-1}}\int_M |\nabla f| \frac{1}{\rho^{n-1}J_p(\theta, \rho)}dV, \tag{3.1}$$

where ω_{n-1} is the surface measure of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Proof Since f has compact support, taking the radial derivative in an arbitrary direction, we have

$$-f(p)=\int_0^\infty \frac{\partial f}{\partial\rho}\mathrm{d}\rho.$$

Integrating both sides over the unit sphere yields

$$-\left(\int_{\mathbb{S}^{n-1}}\mathrm{d}\sigma\right)f(p)=\int_0^\infty\int_{\mathbb{S}^{n-1}}\frac{\partial f}{\partial\rho}\mathrm{d}\rho\mathrm{d}\sigma.$$

Using polar coordinate and (2.1), we have

$$\begin{aligned}|f(p)|&\le\frac{1}{\omega_{n-1}}\int_0^\infty\int_{\mathbb{S}^{n-1}}|\partial_\rho f|\mathrm{d}\rho\mathrm{d}\sigma\\&\le\frac{1}{\omega_{n-1}}\int_0^\infty\int_{\mathbb{S}^{n-1}}|\nabla f|\mathrm{d}\rho\mathrm{d}\sigma\\&=\frac{1}{\omega_{n-1}}\int_M|\nabla f|\frac{1}{\rho^{n-1}J_p(\theta,\rho)}\mathrm{d}V.\end{aligned}$$

This concludes the proof of lemma 3.1.

We now recall the rearrangement of functions on M. Suppose F is a nonnegative function on M. The non-increasing rearrangement of is defined by

$$F^*(t)=\inf\{s>0:\lambda_F(s)\le t\},\tag{3.2}$$

where $\lambda_F(s)=|\{x\in M:F(x)>s\}|$. Here we use the notation $|\Sigma|$ for the measure of a measurable set $\Sigma\subset M$.

Lemma 3.2 *Let* $g=\frac{1}{\rho^{n-1}J_p(\theta,\rho)}$ *be in the Lemma 3.1. Then*

$$g^*(t)\le\left(\frac{nt}{\omega_{n-1}}\right)^{-(n-1)/n},\quad t>0.$$

Proof Define, for any $s>0$,

$$\lambda_g(s)=\int_{\{x\in M:g(x)>s\}}\mathrm{d}V=\int_{\{(\rho,\theta)\in M:\rho^{n-1}J_p(\theta,\rho)<s^{-1}\}}\mathrm{d}V.\tag{3.3}$$

We note that $\rho^{n-1}J_p(\theta,\rho)$, as a function of ρ on $[0,+\infty)$, is strictly decreasing since $J_p(\theta,\rho)$, as a function of ρ on $[0,+\infty)$, is monotonically increasing. Therefore, for every $\theta\in\mathbb{S}^{n-1}$ and $s>0$, the equation $\rho^{n-1}J_p(\theta,\rho)=s^{-1}$ has only one solution in $(0,+\infty)$ and we denote it by $\rho_\theta(s)$. Then $\rho_\theta(s)$ satisfies

$$\rho_\theta(s)^{n-1}J_p(\theta,\rho_\theta(s))=s^{-1}$$

and

$$\lambda_g(s)=\int_{\{(\rho,\theta)\in M:\rho^{n-1}J_p(\theta,\rho)<s^{-1}\}}\mathrm{d}V=\int_{\mathbb{S}^{n-1}}\int_0^{\rho_\theta(s)}\rho^{n-1}J_p(\theta,\rho)d\sigma\mathrm{d}\rho.$$

Therefore, since $g^*(t)=\inf\{s>0:\lambda_g(s)\le t\}$,

$$t=\lambda_g(g^*(t))=\int_{\mathbb{S}^{n-1}}\int_0^{\rho_\theta(g^*(t))}\rho^{n-1}J_p(\theta,\rho)\mathrm{d}\sigma\mathrm{d}\rho,\tag{3.4}$$

where $\rho_\theta(g^*(t))$ satisfies

$$\rho_\theta(g^*(t))^{n-1} J_p(\theta, \rho_\theta(g^*(t))) = \frac{1}{g^*(t)}. \tag{3.5}$$

For simplicity, we set $\rho_\theta(t) = \rho_\theta(g^*(t))$ in the rest of proof. Then,

$$t = \lambda_g(g^*(t)) = \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(t)} \rho^{n-1} J_p(\theta, \rho) \mathrm{d}\sigma \mathrm{d}\rho$$

and $\rho_\theta(t)$ satisfies

$$\rho_\theta(t)^{n-1} J_p(\theta, \rho_\theta(t)) = \frac{1}{g^*(t)}.$$

Thus, since $J_p(\theta, \rho)$, as a function of ρ on $[0, +\infty)$, is monotonically increasing and $J_p(\theta, \rho) \geq J_p(\theta, 0) = 1$, we have

$$\begin{aligned}
t &= \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(t)} \rho^{n-1} J_p(\theta, \rho) \mathrm{d}\sigma \mathrm{d}\rho \\
&\leq \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(t)} \rho^{n-1} J_p(\theta, \rho_\theta(t)) \mathrm{d}\sigma \mathrm{d}\rho \\
&= \int_{\mathbb{S}^{n-1}} J_p(\theta, \rho_\theta(t)) \left(\int_0^{\rho_\theta(t)} \rho^{n-1} d\rho \right) \mathrm{d}\sigma \\
&= \frac{1}{n} \int_{\mathbb{S}^{n-1}} J_p(\theta, \rho_\theta(t)) \rho_\theta^n(t) \mathrm{d}\sigma \\
&\leq \frac{1}{n} \int_{\mathbb{S}^{n-1}} J_p^{n/(n-1)}(\theta, \rho_\theta(t)) \rho_\theta^n(t) \mathrm{d}\sigma \\
&= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left[J_p(\theta, \rho_\theta(t)) \rho_\theta^{n-1}(t) \right]^{n/(n-1)} \mathrm{d}\sigma \\
&= \frac{1}{n} [g^*(t)]^{-n/(n-1)} \omega_{n-1}.
\end{aligned}$$

The desired result follows.

Define $F^{**}(t) = \frac{1}{t} \int_0^t F^*(t) \mathrm{d}t$, where F^* is defined in (3.2). Before the proof of Theorem 1.1, we need the following lemma from Adams' paper [1].

Lemma 3.3 *Let $a(s, t)$ be a nonnegative measurable function on $(-\infty, +\infty) \times [0, +\infty)$ such that (a.e.)*

$$a(s, t) \leq 1, \quad when \ \ 0 < s < t,$$

$$\sup_{t>0} \left(\int_{-\infty}^0 a(s, t)^{n'} ds + \int_t^\infty a(s, t)^{n'} \mathrm{d}s \right)^{1/n'} = b < \infty,$$

where $n' = \frac{n}{n-1}$. Then there is a constant $c_0 = c_0(n, b)$ such that if for $\phi \geq 0$ with $\int_{-\infty}^\infty \phi(s)^n ds \leq 1$, then

$$\int_0^\infty e^{-F(t)} dt \leq c_0,$$

where

$$F(t) = t - \left(\int_{-\infty}^\infty a(s, t) \phi(s) ds \right)^{n'}.$$

Proof of Theorem 1.1. The proof use ideas from [1] and the main tool is O'Neil's lemma ([15], Lemma 1.5). Let $u \in C_0^\infty(\Omega)$ be such that $\int_\Omega |\nabla u|^n \mathrm{d}V \le 1$. Without loss of generality, we may assume $u \ge 0$. By Lemma 3.1 and O'Neil's lemma, for $t > 0$,

$$u^*(t) \le \frac{1}{\omega_{n-1}}\left(t|\nabla u|^{**}(t)g^{**}(t) + \int_t^\infty |\nabla u|^*(s)g^*(s)\mathrm{d}t\right), \tag{3.6}$$

where $g = \frac{1}{\rho^{n-1}J_p(\theta,\rho)}$. By Lemma 3.2,

$$g^*(t) \le \left(\frac{nt}{\omega_{n-1}}\right)^{-(n-1)/n}, \quad g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)\mathrm{d}s \le n\left(\frac{nt}{\omega_{n-1}}\right)^{-(n-1)/n}. \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$u^*(t) \le \left(\frac{1}{n\omega_{n-1}^{1/(n-1)}}\right)^{(n-1)/n}\left(nt^{-\frac{n-1}{n}}\int_0^t |\nabla u|^*(s)\mathrm{d}s + \int_t^\infty |\nabla u|^*(s)s^{-\frac{n-1}{n}}\mathrm{d}s\right). \tag{3.8}$$

Following [1], we set

$$\phi(s) = (|\Omega|e^{-s})^{1/n}|\nabla u|^*(|\Omega|e^{-s}). \tag{3.9}$$

Then

$$\int_0^\infty \phi(s)^n \mathrm{d}s = \int_0^{|\Omega|}(|\nabla u|^*)^n\mathrm{d}s = \int_\Omega |\nabla u|^n\mathrm{d}V \le 1.$$

The auxiliary function $a(s,t)$ is defined to be

$$a(s,t) = \begin{cases} 0, & s < 0; \\ 1, & s < t; \\ ne^{\frac{t-s}{n'}}, & t \le s < \infty, \end{cases} \tag{3.10}$$

where $n' = n/(n-1)$. It is easy to check

$$\sup_{t>0}\left(\int_{-\infty}^0 + \int_t^\infty a(s,t)^{n'}\mathrm{d}s\right)^{1/n'} = n.$$

By Lemma 3.3,

$$\int_0^\infty e^{-F(t)}dt = \int_0^\infty \exp\left[-t + \left(\int_{-\infty}^\infty a(s,t)\phi(s)\mathrm{d}s\right)^{n'}\right]dt < \infty,$$

where

$$\int_{-\infty}^\infty a(s,t)\phi(s)\mathrm{d}s = n|\Omega|^{-1/n'}e^{t/n'}\int_0^{|\Omega|e^{-t}} |\nabla u|^*(s)\mathrm{d}s + \int_{|\Omega|e^{-t}}^{|\Omega|} |\nabla u|^*(s)s^{-1/n'}\mathrm{d}s.$$

On the other hand, by (3.8),

$$\begin{aligned}\int_\Omega \exp(\beta_0|u|^{n/(n-1)})\mathrm{d}V &= \int_0^{|\Omega|}\exp(\beta_0|u^*(t)|^{n/(n-1)})dt \\ &\le \int_0^{|\Omega|} e^{\frac{\beta_0}{n\omega_{n-1}^{1/(n-1)}}\left(nt^{-\frac{n-1}{n}}\int_0^t |\nabla u|^*(s)\mathrm{d}s + \int_t^\infty |\nabla u|^*(s)s^{-\frac{n-1}{n}}\mathrm{d}s\right)^{n/(n-1)}}\mathrm{d}s\end{aligned}$$

$$= \int_0^{|\Omega|} e^{\left(nt^{-\frac{n-1}{n}} \int_0^t |\nabla u|^*(s)\mathrm{d}s + \int_t^\infty |\nabla u|^*(s)s^{-\frac{n-1}{n}}\mathrm{d}s\right)^{n/(n-1)}} \mathrm{d}s.$$

Using the change of variables $t \to |\Omega|e^{-t}$, one can check that

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega \exp(\beta_0|u|^{n/(n-1)})\mathrm{d}V &\leq \frac{1}{|\Omega|}\int_0^{|\Omega|} e^{\left(nt^{-\frac{n-1}{n}} \int_0^t |\nabla u|^*(s)\mathrm{d}s + \int_t^\infty |\nabla u|^*(s)s^{-\frac{n-1}{n}}\mathrm{d}s\right)^{n/(n-1)}} \mathrm{d}s \\ &= \int_0^\infty e^{-F(t)}dt < \infty.\end{aligned}$$

This concludes the proof of the first statement of the theorem.

To prove the second statement, we let $\Omega = B_1 = \{x \in M : \rho(x) < 1\}$. Set, for each $\varepsilon \in (0, 1)$,

$$f_\varepsilon(x) = \begin{cases} (\ln \varepsilon^{-1})^{-1} \ln \rho, & \text{on } B_1 \setminus B_\varepsilon; \\ 1, & \text{on } B_\varepsilon, \end{cases}$$

where $B_\varepsilon = \{x \in M : \rho(x) < \varepsilon\}$. We compute

$$\|\nabla f_\varepsilon\|_n^{n'} = \left(\int_{B\setminus B_\varepsilon} |\nabla f_\varepsilon|^n \mathrm{d}V\right)^{\frac{1}{n-1}} = \frac{1}{\ln \varepsilon^{-1}} \left(\frac{1}{\ln \varepsilon^{-1}} \int_\varepsilon^1 \int_{\mathbb{S}^{n-1}} \frac{J_p(\theta,\rho)}{\rho} d\rho d\sigma\right)^{\frac{1}{n-1}}$$

and

$$|B_\varepsilon| = \int_{B_\varepsilon} \mathrm{d}V = \int_0^\varepsilon \int_{\mathbb{S}^{n-1}} \rho^{n-1} J_p(\theta, \rho) d\rho d\sigma.$$

By the asymptotic expansion of $J_p(\theta, \rho)$ (see (2.2)), it is easy to check

$$\lim_{\varepsilon\to 0+} \|\nabla f_\varepsilon\|_n^{n'} \ln \varepsilon^{-1} = \omega_{n-1}^{1/(n-1)}, \quad \lim_{\varepsilon\to 0+} \frac{\ln |B_\varepsilon|^{-1}}{\ln \varepsilon^{-1}} = n. \tag{3.11}$$

Now assume that

$$\frac{1}{|B|}\int_B \exp\left[\beta\left(\frac{|f_\varepsilon|}{\|\nabla f_\varepsilon\|_n}\right)^{n'}\right]\mathrm{d}V \leq C_1$$

for some $\beta > 0$. Using the fact $f_\varepsilon \equiv 1$ on B_ε, we have

$$\frac{|B_\varepsilon|}{|B|}\exp\left(\beta \frac{1}{\|\nabla f_\varepsilon\|_n^{n'}}\right) \leq C_1,$$

i.e.,

$$\beta \leq \left(\ln C_1 + \ln |B| + \ln |B_\varepsilon|^{-1}\right) \|\nabla f_\varepsilon\|_n^{n'}.$$

Passing the limit $\varepsilon \to 0+$ and using (3.11) yields

$$\beta \leq n\omega_{n-1}^{1/(n-1)}.$$

This concludes the proof of Theorem.

4 Proof of Theorem 1.2

The proof of Theorem 1.2 follows closely Lam and Lu's proof (see [8], section 2 or [9], section 5). Let $u \in C_0^\infty(M)$ be such that $\int_M (|\nabla u|^n + \tau |u|^n) \mathrm{d}V \leq 1$. Without loss of generality, we may assume $u \geq 0$.

Set $A(u) = 2^{-\frac{1}{n(n-1)}} \tau^{\frac{1}{n}} \|u\|_n$ and $\Omega(u) = \{x \in M : u(x) > A(u)\}$, where $\|u\|_n = \sqrt[n]{\int_M |u|^n \mathrm{d}V}$. Then

$$A(u)^n = 2^{-\frac{1}{n-1}} \tau \|u\|_n^n \leq \tau \|u\|_n^n = \tau \int_M |u|^n \mathrm{d}V \leq 1; \tag{4.1}$$

$$\begin{aligned} |\Omega(u)| &= \int_{\Omega(u)} \mathrm{d}V \leq \frac{1}{A(u)^n} \int_{\Omega(u)} |u|^n \mathrm{d}V \\ &\leq \frac{1}{A(u)^n} \int_M |u|^n \mathrm{d}V = 2^{\frac{1}{n-1}} \tau^{-1}. \end{aligned} \tag{4.2}$$

We write

$$\begin{aligned} &\int_M \left(\exp(\beta_0 |u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta_0^k |u|^{kn/(n-1)}}{k!} \right) \mathrm{d}V \\ &= \int_{\Omega(u)} \left(\exp(\beta_0 |u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta_0^k |u|^{kn/(n-1)}}{k!} \right) \mathrm{d}V + \\ &\int_{M \setminus \Omega(u)} \left(\exp(\beta_0 |u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta_0^k |u|^{kn/(n-1)}}{k!} \right) \mathrm{d}V \\ &=: I_1 + I_2. \end{aligned}$$

By (4.1), $M \setminus \Omega(u) = \{x \in M : 0 \leq u(x) \leq A(u)\} \subset \{x \in M : 0 \leq u(x) \leq 1\}$. Therefore,

$$\begin{aligned} I_2 &= \int_{M \setminus \Omega(u)} \left(\exp(\beta_0 |u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta_0^k |u|^{kn/(n-1)}}{k!} \right) \mathrm{d}V \\ &= \int_{M \setminus \Omega(u)} \sum_{k=n-1}^{\infty} \frac{\beta_0^k}{k!} |u|^{kn/(n-1)} \mathrm{d}V \\ &\leq \int_{\{x \in M : 0 \leq u(x) \leq 1\}} \sum_{k=n-1}^{\infty} \frac{\beta_0^k}{k!} |u|^{kn/(n-1)} \mathrm{d}V \\ &\leq \int_{\{x \in M : 0 \leq u(x) \leq 1\}} \sum_{k=n-1}^{\infty} \frac{\beta_0^k}{k!} |u|^n \mathrm{d}V \\ &\leq \left(\sum_{k=n-1}^{\infty} \frac{\beta_0^k}{k!} \right) \int_M |u|^n \mathrm{d}V \\ &\leq \left(\sum_{k=n-1}^{\infty} \frac{\beta_0^k}{k!} \right) \tau^{-1}. \end{aligned} \tag{4.3}$$

Now we will show I_1 is also bounded by a constant $C_3(\tau, n, M)$. Set

$$v(x) = u(x) - A(u), \quad x \in \Omega(u).$$

Then $v \in W_0^{1,n}(\Omega)$ and

$$|u|^{n'} = (v + A(u))^{n'} \le |v|^{n'} + n' 2^{n'-1}(|v|^{n'-1} A(u) + A(u)^{n'}), \tag{4.4}$$

where we used the following elementary inequality

$$(a+b)^q \le a^q + q 2^{q-1}(a^{q-1} b + b^q), \quad q \ge 1, \ a, b \ge 0.$$

By Young's inequality,

$$|v|^{n'-1} A(u) = |v|^{n'-1} A(u) \cdot 1 \le \frac{|v|^{n'} A(u)^n}{n} + \frac{1}{n'}. \tag{4.5}$$

Combing (4.4) and (4.5) yields

$$\begin{aligned} |u|^{n'} &\le |v|^{n'} + \frac{n' 2^{n'-1} A(u)^n}{n} |v|^{n'} + 2^{n'-1} + n' 2^{n'-1} A(u)^{n'} \\ &= \left(1 + \frac{2^{n'-1} |A(u)|^n}{n-1}\right) |v|^{n'} + C_4, \end{aligned} \tag{4.6}$$

where

$$C_4 = 2^{n'-1} + n' 2^{n'-1} A(u)^{n'}. \tag{4.7}$$

Set

$$w = \left(1 + \frac{2^{n'-1} |A(u)|^n}{n-1}\right)^{\frac{n-1}{n}} v.$$

Since $v \in W_0^{1,n}(\Omega)$, so does w. Moreover, by (4.6),

$$|u|^{n'} \le |w|^{n'} + C_4. \tag{4.8}$$

We compute

$$\begin{aligned} \int_{\Omega(u)} |\nabla w|^n \mathrm{d}V &= \left(1 + \frac{2^{n'-1} |A(u)|^n}{n-1}\right)^{n-1} \int_{\Omega(u)} |\nabla v|^n \mathrm{d}V \\ &= \left(1 + \frac{2^{n'-1} |A(u)|^n}{n-1}\right)^{n-1} \int_{\Omega(u)} |\nabla u|^n \mathrm{d}V \\ &\le \left(1 + \frac{2^{n'-1} |A(u)|^n}{n-1}\right)^{n-1} \int_M |\nabla u|^n \mathrm{d}V \\ &\le \left(1 + \frac{2^{n'-1} |A(u)|^n}{n-1}\right)^{n-1} \left(1 - \tau \int_M |u|^n \mathrm{d}V\right). \end{aligned} \tag{4.9}$$

Therefore,

$$
\begin{aligned}
\left(\int_{\Omega(u)} |\nabla w|^n \mathrm{d}V\right)^{\frac{1}{n-1}} &\le \left(1+\frac{2^{n'-1}|A(u)|^n}{n-1}\right)\left(1-\tau\int_M |u|^n \mathrm{d}V\right)^{\frac{1}{n-1}}\\
&= \left(1+\frac{2^{n'-1}}{n-1}2^{-\frac{1}{n-1}}\tau\|u\|_n^n\right)\left(1-\tau\int_M |u|^n \mathrm{d}V\right)^{\frac{1}{n-1}}\\
&= \left(1+\frac{\tau}{n-1}\int_M |u|^n \mathrm{d}V\right)\left(1-\tau\int_M |u|^n \mathrm{d}V\right)^{\frac{1}{n-1}}\\
&\le \left(1+\frac{\tau}{n-1}\int_M |u|^n \mathrm{d}V\right)\left(1-\frac{\tau}{n-1}\int_M |u|^n \mathrm{d}V\right)\\
&\le 1. \qquad (4.10)
\end{aligned}
$$

To get the second inequality in (4.10), we use the following elementary inequality

$$
(1-a)^q \le 1-qa, \quad 0\le a\le 1, \quad 0<q\le 1.
$$

By Theorem 1.1, there exists a constant $C_5 = C_5(n, M)$ such that

$$
\frac{1}{|\Omega(u)|}\int_{\Omega(u)} \exp(\beta_0|w|^{n/(n-1)})\mathrm{d}V \le C_5. \qquad (4.11)
$$

We have, by (4.8), (4.11) and (4.2),

$$
\begin{aligned}
I_1 &= \int_{\Omega(u)}\left(\exp(\beta_0|u|^{n/(n-1)}) - \sum_{k=0}^{n-2}\frac{\beta_0^k|u|^{kn/(n-1)}}{k!}\right)\mathrm{d}V\\
&\le \int_{\Omega(u)} \exp(\beta_0|u|^{n/(n-1)})\mathrm{d}V = \int_{\Omega(u)} \exp(\beta_0|u|^{n'})\mathrm{d}V\\
&\le e^{C_4}\int_{\Omega(u)} \exp(\beta_0|w|^{n'})\mathrm{d}V\\
&\le e^{C_4}C_5|\Omega(u)|\\
&\le e^{C_4}C_5 2^{\frac{1}{n-1}}\tau^{-1} \qquad (4.12)
\end{aligned}
$$

This concludes the proof of the first statement of the theorem.

To prove the second statement, we employ the following Moser function sequence:

$$
g_\varepsilon(x) = \frac{1}{\omega_n^{1/n}} \times \begin{cases} (\ln\varepsilon^{-1})^{(n-1)/n}, & \text{on } B_{\varepsilon\delta};\\ (\ln\varepsilon^{-1})^{-1/n}\ln(\delta/\rho), & \text{on } B_\delta\setminus B_{\varepsilon\delta};\\ 0, & \text{on } M\setminus B_\delta, \end{cases}
$$

where $\delta > 0$ and $\varepsilon \in (0, 1)$. We compute

$$
\begin{aligned}
\int_M |g_\varepsilon|^n \mathrm{d}V &= \frac{(\ln\varepsilon^{-1})^{n-1}}{\omega_{n-1}}\int_0^{\varepsilon\delta}\int_{\mathbb{S}^{n-1}}\rho^{n-1}J_p(\theta,\rho)d\rho d\sigma +\\
&\quad \frac{1}{\omega_{n-1}\ln\varepsilon^{-1}}\int_{\varepsilon\delta}^{\delta}\int_{\mathbb{S}^{n-1}}\ln^n(\delta/\rho)\rho^{n-1}J_p(\theta,\rho)d\rho d\sigma;\\
\int_M |\nabla g_\varepsilon|^n \mathrm{d}V &= \frac{\ln\varepsilon^{-1}}{\omega_{n-1}}\int_{\varepsilon\delta}^{\delta}\int_{\mathbb{S}^{n-1}}\frac{J_p(\theta,\rho)}{\rho}d\rho d\sigma.
\end{aligned}
$$

By the asymptotic expansion of $J_p(\theta, \rho)$ (see (2.2)), we have

$$
\int_M |g_\varepsilon|^n \mathrm{d}V = O(\varepsilon^n(\ln\varepsilon^{-1})^{n-1}) + O\left(\frac{1}{\ln\varepsilon^{-1}}\right);
$$

$$\int_M |\nabla g_\varepsilon|^n \mathrm{d}V = 1 + O(\varepsilon^2).$$

Thus

$$\|g_\varepsilon\|_{W_0^{1,n}(M)} = 1 + O\left(\frac{1}{\ln \varepsilon^{-1}}\right).$$

Let $\widetilde{g}_\varepsilon = g_\varepsilon / \|g_\varepsilon\|_{W_0^{1,n}(M)}$. It follows that, for $\beta > \beta_0 = n\omega_{n-1}^{1/(n-1)}$,

$$\begin{aligned}
&\int_M \left(\exp(\beta|\widetilde{g}_\varepsilon|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta^k |\widetilde{g}_\varepsilon|^{kn/(n-1)}}{k!}\right)\mathrm{d}V \\
&\geq \int_{B_{\varepsilon\delta}} \left(\exp(\beta|\widetilde{g}_\varepsilon|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta^k |\widetilde{g}_\varepsilon|^{kn/(n-1)}}{k!}\right)\mathrm{d}V \\
&= \left[\left(\frac{1}{\varepsilon}\right)^{\frac{\beta}{\omega_{n-1}^{1/(n-1)}}} e^{O(1)} + O((\ln \varepsilon^{-1})^{n-2})\right] \int_0^{\varepsilon\delta} \int_{\mathbb{S}^{n-1}} \rho^{n-1} J_p(\theta, \rho) d\rho d\sigma \\
&= \left[\left(\frac{1}{\varepsilon}\right)^{\frac{\beta}{\omega_{n-1}^{1/(n-1)}}} e^{O(1)} + O((\ln \varepsilon^{-1})^{n-2})\right] \omega_{n-1}\varepsilon^n \delta^n (1 + O(\varepsilon^2)) \to +\infty
\end{aligned}$$

as $\varepsilon \to 0+$. This shows

$$\sup_{u \in W_0^{1,n}(M)} \int_M \left(\exp(\beta|u|^{n/(n-1)}) - \sum_{k=0}^{n-2} \frac{\beta^k |u|^{kn/(n-1)}}{k!}\right)\mathrm{d}V = +\infty$$

if $\beta > \beta_0$. The proof of Theorem 1.2 is thereby completed.

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