On Bott-Chern cohomology of compact complex surfaces

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Abstract We study Bott-Chern cohomology on compact complex non-Kähler surfaces. In particular, we compute such a cohomology for compact complex surfaces in class VII and for compact complex surfaces diffeomorphic to solvmanifolds.

Keywords Compact complex surfaces · Bott-Chern cohomology · Class VII · Solvmanifold

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1 Introduction

For a given complex manifold X , many cohomological invariants can be defined, and many are known for compact complex surfaces.

Among these, one can consider *Bott-Chern and Aeppli cohomologies*. They are defined as follows:

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\operatorname{im} \partial \bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\operatorname{im} \partial + \operatorname{im} \bar{\partial}}.$$

Note that the identity induces natural maps

$$\begin{array}{ccccc} & & H_{BC}^{\bullet,\bullet}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\bar{\partial}}^{\bullet,\bullet}(X) & & H_{dR}^{\bullet}(X; \mathbb{C}) & & H_{\partial}^{\bullet,\bullet}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H_A^{\bullet,\bullet}(X) & & \end{array}$$

where $H_{\bar{\partial}}^{\bullet,\bullet}(X)$ denotes the Dolbeault cohomology and $H_{\partial}^{\bullet,\bullet}(X)$ its conjugate, and the maps are morphisms of (graded or bi-graded) vector spaces. For compact Kähler manifolds, the natural map $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ is an isomorphism.

Assume that X is compact. The Bott-Chern and Aeppli cohomologies are isomorphic to the kernel of suitable 4th-order differential elliptic operators, see [19, Section 2.b, Section 2.c]. In particular, they are finite-dimensional vector spaces. In fact, fixed a Hermitian metric g , its associated $\mathbb{C}$ -linear Hodge- $\ast$ -operator induces the isomorphism

$$H_{BC}^{p,q}(X) \xrightarrow{\sim} H_A^{n-q,n-p}(X),$$

for any $p, q \in \{0, \dots, n\}$, where n denotes the complex dimension of X . In particular, for any $p, q \in \{0, \dots, n\}$, one has

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_{BC}^{q,p}(X) = \dim_{\mathbb{C}} H_A^{n-p,n-q}(X) = \dim_{\mathbb{C}} H_A^{n-q,n-p}(X).$$

For the Dolbeault cohomology, the Frölicher inequality relates the Hodge numbers and the Betti numbers: for any $k \in \{0, \dots, 2n\}$,

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \geq \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

Similarly, for Bott-Chern cohomology, the following inequality *à la* Frölicher has been proven in [3, Theorem A]: for any $k \in \{0, \dots, n\}$,

$$\sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) \geq 2 \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

The equality in the Frölicher inequality characterizes the degeneration of the Frölicher spectral sequence at the first level. This always happens for compact complex surfaces. On the other side, in [3, Theorem B], it is proven that the equality in the inequality *à la* Frölicher for the Bott-Chern cohomology characterizes the validity of the $\partial\bar{\partial}$ -Lemma, namely the property that every ∂ -closed $\bar{\partial}$ -closed d-exact form is $\partial\bar{\partial}$ -exact too, [8]. The validity of the $\partial\bar{\partial}$ -Lemma

implies that the first Betti number is even, which is equivalent to Kählerness for compact complex surfaces. Therefore, the positive integer numbers

$$\Delta^k := \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) - 2b_k \in \mathbb{N},$$

varying $k \in \{1, 2\}$, measure the non-Kählerness of compact complex surfaces X .

Compact complex surfaces are divided into seven classes, according to the Kodaira and Enriques classification, see, e.g., [4]. In this note, we compute Bott-Chern cohomology for some classes of compact complex (non-Kähler) surfaces. In particular, we are interested in studying the relations between Bott-Chern cohomology and de Rham cohomology, looking at the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$. This can be intended as a weak version of the $\partial\bar{\partial}$ -Lemma, compare also [10].

More precisely, we start by proving that the non-Kählerness for compact complex surfaces is encoded only in Δ^2 , namely Δ^1 is always zero. This gives a partial answer to a question by T. C. Dinh to the third author.

Theorem 1.1 *Let X be a compact complex surface. Then:*

- (i) *the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\partial}^{2,1}(X)$ induced by the identity is injective;*
- (ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

For compact complex surfaces in class VII, we show the following result, where we denote $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X)$ for $p, q \in \{0, 1, 2\}$.

Theorem 1.2 *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccc} & & h_{BC}^{0,0} = 1 & & \\ & h_{BC}^{1,0} = 0 & & h_{BC}^{0,1} = 0 & \\ h_{BC}^{2,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & h_{BC}^{0,2} = 0 \\ & h_{BC}^{2,1} = 1 & & h_{BC}^{1,2} = 1 & \\ & & h_{BC}^{2,2} = 1 & & \end{array}$$

According to Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\partial}^{2,1}(X)$ is injective for any compact complex surface. One is then interested in studying the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity, at least for compact complex surfaces diffeomorphic to solvmanifolds. In fact, by definition, the property of satisfying the $\partial\bar{\partial}$ -Lemma [8], is equivalent to the natural map $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ being injective. Note that, for a compact complex manifold of complex dimension n , the injectivity of the map $H_{BC}^{n,n-1}(X) \rightarrow H_{dR}^{2n-1}(X; \mathbb{C})$ implies the $(n-1, n)$ -th weak $\partial\bar{\partial}$ -Lemma in the sense of Fu and Yau [10, Definition 5].

We then compute the Bott-Chern cohomology for compact complex surfaces diffeomorphic to solvmanifolds, according to the list given by Hasegawa [11], see Theorem 5.1. More precisely, we prove that the cohomologies can be computed by using just left-invariant forms. Furthermore, for such complex surfaces, we note that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ is injective, see Theorem 5.2.

We note that the above classes do not exhaust the set of compact complex non-Kähler surfaces, the cohomologies of elliptic surfaces being still unknown.

2 Non-Kählerness of compact complex surfaces and Bott-Chern cohomology

We recall that, for a compact complex manifold of complex dimension n , for $k \in \{0, \dots, 2n\}$, we define the “non-Kählerness” degrees, [3, Theorem A],

$$\Delta^k := \sum_{p+q=k} \left(h_{BC}^{p,q} + h_{BC}^{n-q,n-p} \right) - 2b_k \in \mathbb{N},$$

where we use the duality in [19, §2.c] giving $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_A^{n-q,n-p}(X)$. According to [3, Theorem B], $\Delta^k = 0$ for any $k \in \{0, \dots, 2n\}$ if and only if X satisfies the $\partial\bar{\partial}$ -Lemma, namely every ∂ -closed $\bar{\partial}$ -closed d-exact form is $\partial\bar{\partial}$ -exact too. In particular, for a compact complex surface X , the condition $\Delta^1 = \Delta^2 = 0$ is equivalent to X being Kähler, the first Betti number being even [14, 17, 20], see also [15, Corollaire 5.7], and [5, Theorem 11].

We prove that Δ^1 is always zero for any compact complex surface. In particular, a sufficient and necessary condition for compact complex surfaces to be Kähler is $\Delta^2 = 0$.

Theorem 2.1 *Let X be a compact complex surface. Then:*

- (i) *the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is injective;*
- (ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

Proof (i) Let $\alpha \in \wedge^{2,1}X$ be such that $[\alpha] = 0 \in H_{\bar{\partial}}^{2,1}(X)$. Let $\beta \in \wedge^{2,0}X$ be such that $\alpha = \bar{\partial}\beta$. Fix a Hermitian metric g on X , and consider the Hodge decomposition of β with respect to the Dolbeault Laplacian $\bar{\square}$: let $\beta = \beta_h + \bar{\partial}^* \lambda$ where $\beta_h \in \wedge^{2,0}X \cap \ker \bar{\square}$, and $\lambda \in \wedge^{2,1}X$. Therefore, we have

$$\alpha = \bar{\partial}\beta = \bar{\partial}\bar{\partial}^* \lambda = -\bar{\partial} * \underbrace{(\partial * \lambda)}_{\in \wedge^{2,0}X} = -\bar{\partial}(\partial * \lambda) = \partial\bar{\partial}(*\lambda),$$

where we have used that any $(2, 0)$ -form is primitive and hence, by the Weil identity, is self-dual. In particular, α is $\partial\bar{\partial}$ -exact, so it induces a zero class in $H_{BC}^{2,1}(X)$.

(ii) On the one hand, note that

$$\begin{aligned} H_{BC}^{1,0}(X) &= \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0}X}{\operatorname{im} \partial\bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0}X \\ &\subseteq \ker \bar{\partial} \cap \wedge^{1,0}X = \frac{\ker \bar{\partial} \cap \wedge^{1,0}X}{\operatorname{im} \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X). \end{aligned}$$

It follows that

$$\dim_{\mathbb{C}} H_{BC}^{0,1}(X) = \dim_{\mathbb{C}} H_{BC}^{1,0}(X) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) = b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use that the Frölicher spectral sequence degenerates, hence in particular $b_1 = \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X)$.

On the other hand, by part (i), we have

$$\dim_{\mathbb{C}} H_{BC}^{1,2}(X) = \dim_{\mathbb{C}} H_{BC}^{2,1}(X) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{2,1}(X) = \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use the Kodaira and Serre duality $H_{\bar{\partial}}^{2,1}(X) \simeq H^1(X; \Omega_X^2) \simeq H^1(X; \mathcal{O}_X) \simeq H_{\bar{\partial}}^{0,1}(X)$.

By summing up, we get

$$\begin{aligned}\Delta^1 &= \dim_{\mathbb{C}} H_{BC}^{0,1}(X) + \dim_{\mathbb{C}} H_{BC}^{1,0}(X) + \dim_{\mathbb{C}} H_{BC}^{1,2}(X) + \dim_{\mathbb{C}} H_{BC}^{2,1}(X) - 2b_1 \\ &\leq 2 \left(b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) - b_1 \right) = 0,\end{aligned}$$

concluding the proof. $\square$

3 Class VII surfaces

In this section, we compute Bott-Chern cohomology for compact complex surfaces in class VII.

Let X be a compact complex surface. By Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is always injective. Consider now the case when X is in class VII. If X is minimal, we prove that the same holds for cohomology with values in a line bundle. We will also prove that the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ is not injective.

Proposition 3.1 *Let X be a compact complex surface in class VII₀. Let $L \in H^1(X; \mathbb{C}^*) = \text{Pic}^0(X)$. The natural map $H_{BC}^{2,1}(X; L) \rightarrow H_{\bar{\partial}}^{2,1}(X; L)$ induced by the identity is injective.*

Proof Let $\alpha \in \wedge^{2,1} X \otimes L$ be a $\bar{\partial}_L$ -exact $(2, 1)$ -form. We need to prove that α is $\partial_L \bar{\partial}_L$ -exact too. Consider $\alpha = \bar{\partial}_L \vartheta$, where $\vartheta \in \wedge^{2,0} X \otimes L$. In particular, $\partial_L \vartheta = 0$, hence $\bar{\partial}$ defines a class in $H_{\bar{\partial}}^{0,2}(X; L)$. Note that $H_{\bar{\partial}}^{0,2}(X; L) \simeq H^2(X; \mathcal{O}_X(L)) \simeq H^0(X; K_X \otimes L^{-1}) = \{0\}$ for surfaces of class VII₀, [9, Remark 2.21]. It follows that $\bar{\partial} = -\bar{\partial}_L \bar{\eta}$ for some $\eta \in \wedge^{1,0} X \otimes L$. Hence $\alpha = \partial_L \bar{\partial}_L \eta$, that is, α is $\partial_L \bar{\partial}_L$ -exact. $\square$

We now compute the Bott-Chern cohomology of class VII surfaces.

Theorem 3.2 *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccc} & & h_{BC}^{0,0} = 1 & & \\ & h_{BC}^{1,0} = 0 & & h_{BC}^{0,1} = 0 & \\ h_{BC}^{2,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & h_{BC}^{0,2} = 0 \\ & h_{BC}^{2,1} = 1 & & h_{BC}^{1,2} = 1 & \\ & & h_{BC}^{2,2} = 1 & & \end{array}$$

Proof It holds $H_{BC}^{1,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0} X}{\text{im } \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0} X \subseteq \ker \bar{\partial} \cap \wedge^{1,0} X = \frac{\ker \bar{\partial} \cap \wedge^{1,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X) = \{0\}$ hence $h_{BC}^{1,0} = h_{BC}^{0,1} = 0$.

On the other side, by Theorem 1.1, $0 = \Delta^1 = 2 \left(h_{BC}^{1,0} + h_{BC}^{2,1} - b_1 \right) = 2 \left(h_{BC}^{2,1} - 1 \right)$ hence $h_{BC}^{2,1} = h_{BC}^{1,2} = 1$.

Similarly, it holds $H_{BC}^{2,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{2,0} X}{\text{im } \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{2,0} X \subseteq \ker \bar{\partial} \cap \wedge^{2,0} X = \frac{\ker \bar{\partial} \cap \wedge^{2,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{2,0}(X) = \{0\}$ hence $h_{BC}^{2,0} = h_{BC}^{0,2} = 0$.

Note that, from [3, Theorem A], we have $0 \leq \Delta^2 = 2 \left(h_{BC}^{2,0} + h_{BC}^{1,1} + h_{BC}^{0,2} - b_2 \right) = 2 \left(h_{BC}^{1,1} - b_2 \right)$ hence $h_{BC}^{1,1} \geq b_2$. More precisely, from [3, Theorem B] and Theorem 1.1, we have that $h_{BC}^{1,1} = b_2$ if and only if $\Delta^2 = 0$ if and only if X satisfies the $\partial\bar{\partial}$ -Lemma, in fact X is Kähler, which is not the case.

Finally, we prove that $h_{BC}^{1,1} = b_2 + 1$. Consider the following exact sequences from [21, Lemma 2.3]. More precisely, the sequence

$$0 \rightarrow \frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}} \rightarrow H_{BC}^{1,1}(X) \rightarrow \operatorname{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) \rightarrow 0$$

is clearly exact. Furthermore, fix a Gauduchon metric g . Denote by $\omega := g(J \cdot, \cdot)$ the $(1, 1)$ -form associated to g , where J denotes the integrable almost-complex structure. By definition of g being Gauduchon, we have $\partial \bar{\partial} \omega = 0$. The sequence

$$0 \rightarrow \frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}} \xrightarrow{\langle \cdot | \omega \rangle} \mathbb{C}$$

is exact. Indeed, firstly note that for $\eta = \partial \bar{\partial} f \in \operatorname{im} \partial \bar{\partial} \cap \wedge^{1,1} X$, we have

$$\langle \eta | \omega \rangle = \int_X \partial \bar{\partial} f \wedge \overline{* \omega} = \int_X \partial \bar{\partial} f \wedge \omega = \int_X f \partial \bar{\partial} \omega = 0$$

by applying twice the Stokes theorem. Then, we recall the argument in [21, Lemma 2.3(ii)] for proving that the map

$$\langle \cdot | \omega \rangle : \frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}} \rightarrow \mathbb{C}$$

is injective. Take $\alpha = d\beta \in \operatorname{im} d \cap \wedge^{1,1} X \cap \ker \langle \cdot | \omega \rangle$. Then,

$$\langle \Lambda \alpha | 1 \rangle = \langle \alpha | \omega \rangle = 0,$$

where Λ is the adjoint operator of $\omega \wedge \cdot$ with respect to $\langle \cdot | \cdot \rangle$. Then, $\Lambda \alpha \in \ker \langle \cdot | 1 \rangle = \operatorname{im} \Lambda \partial \bar{\partial}$, by extending [16, Corollary 7.2.9] by $\mathbb{C}$ -linearity. Take $u \in C^\infty(X; \mathbb{C})$ such that $\Lambda \alpha = \Lambda \partial \bar{\partial} u$. Then, by defining $\alpha' := \alpha - \partial \bar{\partial} u$, we have $[\alpha'] = [\alpha] \in \frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}}$, and $\Lambda \alpha' = 0$, and $\alpha' = d\beta'$ where $\beta' := \beta - \bar{\partial} u$. In particular, α' is primitive. Since α' is primitive and of type $(1, 1)$, then, it is anti-self-dual by the Weil identity. Then,

$$\begin{aligned} \|\alpha'\|^2 &= \langle \alpha' | \alpha' \rangle = \int_X \alpha' \wedge \overline{* \alpha'} = - \int_X \alpha' \wedge \overline{\alpha'} = - \int_X d\beta' \wedge \overline{d\beta'} \\ &= - \int_X d \left(\beta' \wedge \overline{d\beta'} \right) = 0 \end{aligned}$$

and hence $\alpha' = 0$, and therefore $[\alpha] = 0$.

Since the space $\frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}}$ is finite-dimensional, being a subspace of $H_{BC}^{1,1}(X)$, and since the space $\operatorname{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right)$ is finite-dimensional, being a subspace of $H_{dR}^2(X; \mathbb{C})$, we get that

$$\dim_{\mathbb{C}} \frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}} \leq \dim_{\mathbb{C}} \mathbb{C} = 1,$$

and hence

$$\begin{aligned} b_2 &< \dim_{\mathbb{C}} H_{BC}^{1,1}(X) = \dim_{\mathbb{C}} \operatorname{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) \\ &\quad + \dim_{\mathbb{C}} \frac{\operatorname{im} d \cap \wedge^{1,1} X}{\operatorname{im} \partial \bar{\partial}} \leq b_2 + 1. \end{aligned}$$

We get that $\dim_{\mathbb{C}} H_{BC}^{1,1}(X) = b_2 + 1$. □

Finally, we prove that the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ is not injective.

Proposition 3.3 *Let X be a compact complex surface in class VII. Then, the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ induced by the identity is the zero map and not an isomorphism.*

Proof Note that, for class VII surfaces, the pluri-genera are zero. In particular, $H_{\bar{\partial}}^{1,2}(X) \simeq H_{\bar{\partial}}^{1,0}(X) = \{0\}$, by Kodaira and Serre duality. By Theorem 3.2, one has $H_{BC}^{1,2}(X) \neq \{0\}$. $\square$

3.1 Cohomologies of Calabi–Eckmann surface

In this section, as an explicit example, we list the representatives of the cohomologies of a compact complex surface in class VII: namely we consider the Calabi–Eckmann structure on the differentiable manifolds underlying the Hopf surfaces.

Consider the differentiable manifold $X := \mathbb{S}^1 \times \mathbb{S}^3$. As a Lie group, $\mathbb{S}^3 = SU(2)$ has a global left-invariant co-frame $\{e^1, e^2, e^3\}$ such that $de^1 = -2e^2 \wedge e^3$ and $de^2 = 2e^1 \wedge e^3$ and $de^3 = -2e^1 \wedge e^2$. Hence, we consider a global left-invariant co-frame $\{f, e^1, e^2, e^3\}$ on X with structure equations

$$\begin{cases} df = 0 \\ de^1 = -2e^2 \wedge e^3 \\ de^2 = 2e^1 \wedge e^3 \\ de^3 = -2e^1 \wedge e^2 \end{cases}.$$

Consider the left-invariant almost-complex structure defined by the $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + i e^2 \\ \varphi^2 := e^3 + i f \end{cases}.$$

By computing the complex structure equations, we get

$$\begin{cases} \partial\varphi^1 = i\varphi^1 \wedge \varphi^2 \\ \partial\varphi^2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} \bar{\partial}\varphi^1 = i\varphi^1 \wedge \bar{\varphi}^2 \\ \bar{\partial}\varphi^2 = -i\varphi^1 \wedge \bar{\varphi}^1 \end{cases}.$$

We note that the almost-complex structure is in fact integrable.

The manifold X is a compact complex manifold not admitting Kähler metrics. It is bi-holomorphic to the complex manifold $M_{0,1}$ considered by Calabi and Eckmann [6], see [18, Theorem 4.1].

Consider the Hermitian metric g whose associated $(1, 1)$ -form is

$$\omega := \frac{i}{2} \sum_{j=1}^2 \varphi^j \wedge \bar{\varphi}^j.$$

As for the de Rham cohomology, from the Künneth formula we get

$$H_{dR}^{\bullet}(X; \mathbb{C}) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C} \langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}2} \rangle \oplus \mathbb{C} \langle \varphi^{12\bar{1}\bar{2}} \rangle,$$

(where, here and hereafter, we shorten, e.g., $\varphi^{12\bar{1}} := \varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1$).

By [12, Appendix II, Theorem 9.5], one has that a model for the Dolbeault cohomology is given by

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) \simeq \bigwedge \langle x_{2,1}, x_{0,1} \rangle,$$

where $x_{i,j}$ is an element of bi-degree (i, j) . In particular, we recover that the Hodge numbers $\{h_{\bar{\partial}}^{p,q} := \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X)\}_{p,q \in \{0,1,2\}}$ are

$$\begin{array}{ccccc} & & h_{\bar{\partial}}^{0,0} = 1 & & \\ & h_{\bar{\partial}}^{1,0} = 0 & & h_{\bar{\partial}}^{0,1} = 1 & \\ h_{\bar{\partial}}^{2,0} = 0 & & h_{\bar{\partial}}^{1,1} = 0 & & h_{\bar{\partial}}^{0,2} = 0 \\ & h_{\bar{\partial}}^{2,1} = 1 & & h_{\bar{\partial}}^{1,2} = 0 & \\ & & h_{\bar{\partial}}^{2,2} = 1 & & \end{array} .$$

We note that the sub-complex

$$\iota: \bigwedge \langle \varphi^1, \varphi^2, \bar{\varphi}^1, \bar{\varphi}^2 \rangle \hookrightarrow \wedge^{\bullet,\bullet} X$$

is such that $H_{\bar{\partial}}(t)$ is an isomorphism. More precisely, we get

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle [\varphi^{\bar{2}}] \rangle \oplus \mathbb{C} \langle [\varphi^{1\bar{1}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Dolbeault Laplacian of g .

By [2, Theorem 1.3, Proposition 2.2], we have also $H_{BC}(t)$ isomorphism. In particular, we get

$$H_{BC}^{\bullet,\bullet}(X) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle [\varphi^{1\bar{1}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}}] \rangle \oplus \mathbb{C} \langle [\varphi^{1\bar{1}\bar{2}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Bott-Chern Laplacian of g .

By [19, Section 2.c], we have

$$H_A^{\bullet,\bullet}(X) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle [\varphi^2] \rangle \oplus \mathbb{C} \langle [\varphi^{\bar{2}}] \rangle \oplus \mathbb{C} \langle [\varphi^{2\bar{2}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Aeppli Laplacian of g .

Note in particular that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is an isomorphism, and that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity is injective.

4 Complex surfaces diffeomorphic to solvmanifolds

Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$. By [11, Theorem 1], X is (A) either a complex torus, (B) or a hyperelliptic surface, (C) or a Inoue surface of type $\mathcal{S}_M$, (D) or a primary Kodaira surface, (E) or a secondary Kodaira surface, (F) or a Inoue surface of type $\mathcal{S}^{\pm}$, and, as such, it is endowed with a left-invariant complex structure.

In each case, we recall the structure equations of the group G , see [11]. More precisely, take a basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra $\mathfrak{g}$ naturally associated to G . We have the following commutation relations, according to [11]:

(A) differentiable structure underlying a *complex torus*:

$$[e_j, e_k] = 0 \quad \text{for any } j, k \in \{1, 2, 3, 4\};$$

(hereafter, we write only the non-trivial commutators);

(B) differentiable structure underlying a *hyperelliptic surface*:

$$[e_1, e_4] = e_2, \quad [e_2, e_4] = -e_1;$$

(C) differentiable structure underlying a *Inoue surface of type S_M* :

$$[e_1, e_4] = -\alpha e_1 + \beta e_2, \quad [e_2, e_4] = -\beta e_1 - \alpha e_2, \quad [e_3, e_4] = 2\alpha e_3,$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$;

(D) differentiable structure underlying a *primary Kodaira surface*:

$$[e_1, e_2] = -e_3;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$[e_1, e_2] = -e_3, \quad [e_1, e_4] = e_2, \quad [e_2, e_4] = -e_1;$$

(F) differentiable structure underlying a *Inoue surface of type $S^\pm$* :

$$[e_2, e_3] = -e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = e_3.$$

Denote by $\{e^1, e^2, e^3, e^4\}$ the dual basis of $\{e_1, e_2, e_3, e_4\}$. We recall that, for any $\alpha \in \mathfrak{g}^*$, for any $x, y \in \mathfrak{g}$, it holds $d\alpha(x, y) = -\alpha([x, y])$. Hence we get the following structure equations:

(A) differentiable structure underlying a *complex torus*:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = 0 \\ de^4 = 0 \end{cases};$$

(B) differentiable structure underlying a *hyperelliptic surface*:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = 0 \\ de^4 = 0 \end{cases};$$

(C) differentiable structure underlying a *Inoue surface of type S_M* :

$$\begin{cases} de^1 = \alpha e^1 \wedge e^4 + \beta e^2 \wedge e^4 \\ de^2 = -\beta e^1 \wedge e^4 + \alpha e^2 \wedge e^4 \\ de^3 = -2\alpha e^3 \wedge e^4 \\ de^4 = 0 \end{cases};$$

(D) differentiable structure underlying a *primary Kodaira surface*:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = e^1 \wedge e^2 \\ de^4 = 0 \end{cases};$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = e^1 \wedge e^2 \\ de^4 = 0 \end{cases};$$

(F) differentiable structure underlying a *Inoue surface of type $S^\pm$* :

$$\begin{cases} de^1 = e^2 \wedge e^3 \\ de^2 = e^2 \wedge e^4 \\ de^3 = -e^3 \wedge e^4 \\ de^4 = 0 \end{cases}.$$

In cases (A), (B), (C), (D), (E), consider the G -left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and} \quad Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 \quad \text{and} \quad Je_4 := -e_3.$$

Consider the G -left-invariant $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + ie^2 \\ \varphi^2 := e^3 + ie^4 \end{cases}.$$

In case (F), consider the G -left-invariant almost-complex structure J on X defined by $Je_1 := e_2$ and $Je_2 := -e_1$ and $Je_3 := e_4 - q e_2$ and $Je_4 := -e_3 - q e_1$, where $q \in \mathbb{R}$. Consider the G -left-invariant $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + ie^2 + i q e^4 \\ \varphi^2 := e^3 + ie^4 \end{cases}.$$

With respect to the G -left-invariant coframe $\{\varphi^1, \varphi^2\}$ for the holomorphic tangent bundle $T^{1,0} \Gamma \backslash G$, we have the following structure equations. (As for notation, we shorten, e.g., $\varphi^{1\bar{2}} := \varphi^1 \wedge \bar{\varphi}^2$.)

(A) *torus*:

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = 0 \end{cases}$$

(B) *hyperelliptic surface*:

$$\begin{cases} d\varphi^1 = -\frac{1}{2} \varphi^{12} + \frac{1}{2} \varphi^{1\bar{2}} \\ d\varphi^2 = 0 \end{cases}$$

(C) *Inoue surface S_M* :

$$\begin{cases} d\varphi^1 = \frac{\alpha - i\beta}{2i} \varphi^{12} - \frac{\alpha - i\beta}{2i} \varphi^{1\bar{2}} \\ d\varphi^2 = -i\alpha \varphi^{2\bar{2}} \end{cases}$$

(where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$);

(D) *primary Kodaira surface*:

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = \frac{i}{2} \varphi^{1\bar{1}} \end{cases}$$

(E) *secondary Kodaira surface*:

$$\begin{cases} d\varphi^1 = -\frac{1}{2} \varphi^{12} + \frac{1}{2} \varphi^{1\bar{2}} \\ d\varphi^2 = \frac{i}{2} \varphi^{1\bar{1}} \end{cases}$$

(F) *Inoue surface* $\mathcal{S}^\pm$:

$$\begin{cases} d\varphi^1 = \frac{1}{2i} \varphi^{12} + \frac{1}{2i} \varphi^{2\bar{1}} + \frac{qi}{2} \varphi^{2\bar{2}} \\ d\varphi^2 = \frac{1}{2i} \varphi^{2\bar{2}} \end{cases}.$$

5 Cohomologies of complex surfaces diffeomorphic to solvmanifolds

In this section, we compute the Dolbeault and Bott-Chern cohomologies of the compact complex surfaces diffeomorphic to a solvmanifold.

We prove the following theorem.

Theorem 5.1 *Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$; denote the Lie algebra of G by $\mathfrak{g}$. Then, the inclusion $(\wedge^{\bullet, \bullet} \mathfrak{g}^*, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet, \bullet} X, \partial, \bar{\partial})$ induces an isomorphism both in Dolbeault and in Bott-Chern cohomologies. In particular, the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies and the degrees of non-Kählerness are summarized in Table 5.*

Proof Firstly, we compute the cohomologies of the sub-complex of G -left-invariant forms. The computations are straightforward from the structure equations.

In Tables 1, 2, 3 and 4, we summarize the results of the computations. The sub-complexes of left-invariant forms are depicted in Fig. 1 (each dot represents a generator, vertical arrows depict the $\bar{\partial}$ -operator, horizontal arrows depict the ∂ -operator, and trivial arrows are not shown.) The dimensions are listed in Table 5.

On the one side, recall that the inclusion of left-invariant forms into the space of forms induces an injective map in Dolbeault and Bott-Chern cohomologies, see, e.g., [7, Lemma 9], [1, Lemma 3.6]. On the other side, recall that the Frölicher spectral sequence of a compact complex surface X degenerates at the first level, equivalently, the equalities

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) = \dim_{\mathbb{C}} H_{dR}^1(X; \mathbb{C})$$

and

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{2,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,2}(X) = \dim_{\mathbb{C}} H_{dR}^2(X; \mathbb{C})$$

hold. By comparing the dimensions in Table 5 with the Betti numbers case by case, we find that the left-invariant forms suffice in computing the Dolbeault cohomology for each case. Then, by [1, Theorem 3.7], see also [2, Theorem 1.3, Theorem 1.6], it follows that also the Bott-Chern cohomology is computed using just left-invariant forms. $\square$

We prove the following result.

Table 1 Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

(p, q)	(A) Torus				(B) Hyperelliptic				(C) Inoue S_M			
	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
$(0, 0)$	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1
$(1, 0)$	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^2 \rangle$	1	$\langle \varphi^2 \rangle$	1	(0)	0	(0)	0
$(0, 1)$	$\langle \bar{\varphi}^1, \bar{\varphi}^2 \rangle$	2	$\langle \bar{\varphi}^1, \bar{\varphi}^2 \rangle$	2	$\langle \bar{\varphi}^2 \rangle$	1	$\langle \bar{\varphi}^2 \rangle$	1	$\langle \bar{\varphi}^2 \rangle$	1	(0)	0
$(2, 0)$	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
$(1, 1)$	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	(0)	0	$\langle \varphi^{2\bar{2}} \rangle$	1
$(0, 2)$	$\langle \bar{\varphi}^{1\bar{2}} \rangle$	1	$\langle \bar{\varphi}^{1\bar{2}} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
$(2, 1)$	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
$(1, 2)$	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
$(2, 2)$	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

Table 2 Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 2

(p, q)	(D) Primary Kodaira				(E) Secondary Kodaira				(F) Inoue $S_{\pm}$			
	$H_{\partial}^{p,q}$	$\dim_{\mathbb{C}} H_{\partial}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\partial}^{p,q}$	$\dim_{\mathbb{C}} H_{\partial}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\partial}^{p,q}$	$\dim_{\mathbb{C}} H_{\partial}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0,0)	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
(1,0)	$\langle \varphi^1 \rangle$	1	$\langle \varphi^1 \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(0,1)	$\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^{\bar{1}} \rangle$	1	$\langle \varphi^{\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle \varphi^{\bar{2}} \rangle$	1	$\langle 0 \rangle$	0
(2,0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(1,1)	$\langle \varphi^{1\bar{2}}, \varphi^{2\bar{1}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}} \rangle$	3	$\langle 0 \rangle$	0	$\langle \varphi^{1\bar{1}} \rangle$	1	$\langle 0 \rangle$	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0,2)	$\langle \varphi^{1\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(2,1)	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{2}\bar{1}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}} \rangle$	1
(1,2)	$\langle \varphi^{2\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle 0 \rangle$	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
(2,2)	$\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle$	1

Table 3 de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

k	(A) Torus		(B) Hyperelliptic		(C) Inoue S_M	
	H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^2, \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	4	$\langle \varphi^2, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}}, \varphi^{\bar{1}\bar{2}} \rangle$	6	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}}, \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	4	$\langle \varphi^{12\bar{1}}, \varphi^{1\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

Table 4 de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

k	(D) Primary Kodaira	(E) Secondary Kodaira		(F) Inoue $S^\pm$	
	H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	H^k_{dR}
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$
1	$\langle \varphi^1, \varphi^{\bar{1}}, \varphi^2 - \varphi^{\bar{2}} \rangle$	3	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$
2	$\langle \varphi^{12}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{\bar{1}\bar{2}} \rangle$	4	$\langle 0 \rangle$	0	$\langle 0 \rangle$
3	$\langle \varphi^{12\bar{2}}, \varphi^{2\bar{1}\bar{2}}, \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	3	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}} - q \varphi^{12\bar{2}} - \varphi^{1\bar{1}\bar{2}} + q \varphi^{2\bar{1}\bar{2}} \rangle$
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$

Fig. 1 Double-complexes of left-invariant forms over 4-dimensional solvmanifolds. **a** torus. **b** Hyperelliptic surface. **c** Inoue surface S_M . **d** Primary Kodaira surface. **e** Secondary Kodaira surface. **f** Inoue surface $S^\pm$

Theorem 5.2 *Let X be a compact complex surface diffeomorphic to a solvmanifold. Then, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is an isomorphism, and the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity is injective.*

Proof By the general result in Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is injective. In fact, it is an isomorphism as follows from the computations summarized in Tables 1 and 2. As for the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$, it is a straightforward computation from Tables 1, 2, 3 and 4.

As an example, we offer an explicit calculation of the injectivity of the map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ for the Inoue surfaces of type 0, see [13], see also [22]. We will change a little bit the notation. Recall the construction of Inoue surfaces: let $M \in \mathrm{SL}(3; \mathbb{Z})$ be a unimodular matrix having a real eigenvalue $\lambda > 1$ and two complex eigenvalues $\mu \neq \bar{\mu}$. Take a real eigenvector $(\alpha_1, \alpha_2, \alpha_3)$ and an eigenvector $(\beta_1, \beta_2, \beta_3)$ of M . Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$; on the product $\mathbb{H} \times \mathbb{C}$ consider the following transformations defined as

$$\begin{aligned} f_0(z, w) &:= (\lambda z, \mu w) \\ f_j(z, w) &:= (z + \alpha_j, w + \beta_j) \quad \text{for } j \in \{1, 2, 3\}. \end{aligned}$$

Denote by Γ_M the group generated by $f_0, \dots, f_3$; then, Γ_M acts in a properly discontinuous way and without fixed points on $\mathbb{H} \times \mathbb{C}$, and $S_M := \mathbb{H} \times \mathbb{C} / \Gamma_M$ is an Inoue surface of type 0, as in case (C) in [11]. Denoting by $z = x + iy$ and $w = u + iv$, consider the following differential forms on $\mathbb{H} \times \mathbb{C}$:

Table 5 Summary of the dimensions of de Rham, Dolbeault, and Bott-Chern cohomologies and of the degree of non-Kählerness for compact complex surfaces diffeomorphic to solvmanifolds.

$\mathbf{k}(\mathbf{p}, \mathbf{q})$	(A) Torus			(B) Hyperell.			(C) Inoue S_M			(D) Prim. Kod.			(E) Sec. Kod.			(F) Inoue $S^\pm$		
	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$
(0,0)	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1
(1,0)	2	2	4	0	1	1	2	0	0	1	1	3	0	0	1	0	0	1
(0,1)	2	2	1	1	1	1	1	0	1	2	1	1	1	0	1	0	1	0
(2,0)	1	1	6	0	0	0	2	0	0	2	1	4	2	0	0	2	0	0
(1,1)	4	4	2	2	2	2	0	1	2	3	3	0	0	1	0	1	0	1
(0,2)	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0
(2,1)	2	2	4	0	1	1	2	0	1	2	2	3	0	1	1	0	1	1
(1,2)	2	2	1	1	1	1	0	1	1	2	2	0	0	1	0	1	0	1
(2,2)	1	1	1	0	1	1	1	0	1	1	1	1	0	1	1	0	1	1

$$e^1 := \frac{1}{y} dx, \quad e^2 := \frac{1}{y} dy, \quad e^3 := \sqrt{y} du, \quad e^4 := \sqrt{y} dv.$$

(Note that e^1 and e^2 , and $e^3 \wedge e^4$ are Γ_M -invariant, and consequently they induce global differential forms on S_M .) We obtain

$$de^1 = e^1 \wedge e^2, \quad de^2 = 0, \quad de^3 = \frac{1}{2} e^2 \wedge e^3, \quad de^4 = \frac{1}{2} e^2 \wedge e^4.$$

Consider the natural complex structure on S_M induced by $\mathbb{H} \times \mathbb{C}$. Locally, we have

$$J e^1 = -e^2 \quad \text{and} \quad J e^2 = e^1 \quad \text{and} \quad J e^3 = -e^4 \quad \text{and} \quad J e^4 = e^3.$$

Considering the Γ_M -invariant (2, 1)-Bott-Chern cohomology of S_M , we obtain that

$$H_{BC}^{2,1}(S_M) = \mathbb{C} \langle [e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4] \rangle.$$

Clearly $\bar{\partial} (e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4) = 0$ and $e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4 = e^1 \wedge e^3 \wedge e^4 + i d(e^3 \wedge e^4)$, therefore, the de Rham cohomology class $[e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4] = [e^1 \wedge e^3 \wedge e^4] \in H_{dR}^3(S_M)$ is nonzero. $\square$

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References

1. Angella, D.: The cohomologies of the Iwasawa manifold and of its small deformations. *J. Geom. Anal.* **23**(3), 1355–1378 (2013)
2. Angella, D., Kasuya H.: Bott-Chern cohomology of solvmanifolds. [arXiv:1212.5708v3](https://arxiv.org/abs/1212.5708v3) [math.DG]
3. Angella, D., Tomassini, A.: On the $\partial\bar{\partial}$ -Lemma and Bott-Chern cohomology. *Invent. Math.* **192**(1), 71–81 (2013)
4. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: *Compact Complex Surfaces*, Second Edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3*. Springer, Berlin (2004)
5. Buchdahl, N.: On compact Kähler surfaces. *Ann. Inst. Fourier (Grenoble)* **49**(1), vii, xi, 287–302 (1999)
6. Calabi, E., Eckmann, B.: A class of compact, complex manifolds which are not algebraic. *Ann. Math.* **58**(2), 494–500 (1953)
7. Console, S., Fino, A.: Dolbeault cohomology of compact nilmanifolds. *Transform. Groups* **6**(2), 111–124 (2001)
8. Deligne, P., Griffiths, Ph, Morgan, J., Sullivan, D.P.: Real homotopy theory of Kähler manifolds. *Invent. Math.* **29**(3), 245–274 (1975)
9. Dloussky, G.: On surfaces of class VII_0^+ with numerically anticanonical divisor. *Am. J. Math.* **128**(3), 639–670 (2006)
10. Fu, J., Yau, S.-T.: A note on small deformations of balanced manifolds. *C. R. Math. Acad. Sci. Paris* **349**(13–14), 793–796 (2011)
11. Hasegawa, K.: Complex and Kähler structures on compact solvmanifolds, conference on symplectic topology. *J. Symplectic Geom.* **3**(4), 749–767 (2005)
12. Hirzebruch, F.: *Topological methods in algebraic geometry*, translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the (1978) edition. *Classics in Mathematics*. Springer, Berlin (1995)
13. Inoue, M.: On surfaces of class VII_0 . *Invent. Math.* **24**(4), 269–310 (1974)
14. Kodaira, K.: On the structure of compact complex analytic surfaces. I. *Am. J. Math.* **86**, 751–798 (1964)
15. Lamari, A.: Courants kählériens et surfaces compactes. *Ann. Inst. Fourier (Grenoble)* **49**(1), vii, x, 263–285 (1999)
16. Lübke, M., Teleman, A.: *The Kobayashi-Hitchin Correspondence*. World Scientific, River Edge (1995)
17. Miyaoka, Y.: Kähler metrics on elliptic surfaces. *Proc. Jpn. Acad.* **50**(8), 533–536 (1974)

18. Parton, M.: Explicit parallelizations on products of spheres and Calabi–Eckmann structures. *Rend. Istit. Mat. Univ. Trieste* **35**(1–2), 61–67 (2003)
19. Schweitzer, M.: Autour de la cohomologie de Bott-Chern. [arXiv:0709.3528v1](https://arxiv.org/abs/0709.3528v1)
20. Siu, Y.T.: Every K3 surface is Kähler. *Invent. Math.* **73**(1), 139–150 (1983)
21. Teleman, A.: The pseudo-effective cone of a non-Kählerian surface and applications. *Math. Ann.* **335**(4), 965–989 (2006)
22. Tricerri, F.: Some examples of locally conformal Kähler manifolds. *Rend. Semin. Mat. Univ. Politec. Torino* **40**(1), 81–92 (1982)