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Large data existence result for the steady full compressible magnetohydrodynamic flows in three dimensional

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Abstract This paper considers the equations of the steady viscous, compressible, and heat conducting magnetohydrodynamic flows in a bounded three-dimensional domain. By an approximation scheme and a weak convergence method, for any $\gamma > \frac{4}{3}$, the existence of a weak solution to the three-dimensional steady full magnetohydrodynamic equations with large data is obtained. Here, γ describes the heat capacity ratio.

Keywords Steady magnetohydrodynamic flows · Weak solutions · Compressible fluids

Mathematics Subject Classification 76W05 · 35D30 · 76N10

1 Introduction

We consider the full system of partial differential equations for the three-dimensional viscous steady compressible magnetohydrodynamic flows in the Eulerian coordinates [14,15]

$$\mathbf{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\mathbf{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P(\rho,\theta) = (\nabla\times\mathbf{H})\times\mathbf{H} + \mathbf{div}\Psi(\mathbf{u}) + \rho\mathbf{F}, \tag{1.2}$$

$$\mathbf{div}(\mathbf{u}(\Phi' + P)) = \mathbf{div}\left((\mathbf{u}\times\mathbf{H})\times\mathbf{H} + \nu\mathbf{H}\times(\nabla\times\mathbf{H}) + \mathbf{u}\Psi(\mathbf{u}) + \kappa(\theta)\nabla\theta\right), \tag{1.3}$$

$$\nabla\times(\mathbf{u}\times\mathbf{H}) = \nabla\times(\nu\nabla\times\mathbf{H}), \quad \mathbf{divH} = 0. \tag{1.4}$$

where ρ is the density, $\mathbf{u} \in \mathbf{R}^3$ is the velocity, $\mathbf{H} \in \mathbf{R}^3$ is the magnetic field, and θ is the temperature; $\mathbf{F}$ is the external force; Ψ is the viscous stress tensor given by

$$\Psi(\mathbf{u}) = \mu\mathbf{D}(\mathbf{u}) + \lambda\mathbf{divuI},$$

and Φ is the total energy given by

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$$\Phi = \rho(e + \frac{1}{2}|\mathbf{u}|^2) + \frac{1}{2}|\mathbf{H}|^2 \quad \text{and} \quad \Phi' = \rho(e + \frac{1}{2}|\mathbf{u}|^2),$$

with the internal energy $e(\rho,\theta)$, the kinetic energy $\frac{1}{2}\rho|\mathbf{u}|^2$, and the magnetic energy $\frac{1}{2}|\mathbf{H}|^2$. $\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^T$ is the symmetric part of the velocity gradient, $\nabla\mathbf{u}^T$ is the transpose of the matrix $\nabla\times\mathbf{u}$, and $\mathbf{I}$ is the 3×3 identity matrix. The viscosity coefficients λ, μ of the flow satisfy $2\mu + 3\lambda > 0$ and $\mu > 0$; $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, $\kappa > 0$ is the heat conductivity. Equations (1.1), (1.2), (1.3) describe the conservation of mass, momentum, and energy, respectively. It is well known that the electromagnetic fields are governed by Maxwell's equations. In magnetohydrodynamics, the displacement current can be neglected [14,15]. As a consequence, Eq. (1.4) is called the induction equation, and the electric field can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$,

$$\mathbf{E} = \nu\nabla\times\mathbf{H} - \mathbf{u}\times\mathbf{H}.$$

The pressure $P(\rho,\theta)$ is determined through a general constitutive equation:

$$P(\rho,\theta) = p_e(\rho) + \theta p_\theta(\rho) \tag{1.5}$$

with certain functions $p_e, p_\theta \in \mathbf{C}[0,\infty) \cap \mathbf{C}^1(0,\infty)$. The basic principles of classical thermodynamics imply that the internal energy e and pressure P are interrelated through Maxwell's relationship:

$$\partial_\rho e = \frac{1}{\rho^2}(P - \theta\partial_\theta P), \quad \partial_\theta e = \partial_\theta Q = c_\nu(\theta),$$

where $c_\nu(\theta)$ denotes the specific heat and $Q = Q(\theta)$ is a function of θ. Thus, the constitutive relation (1.5) implies that the internal energy e can be decomposed as a sum:

$$e(\rho,\theta) = P_e(\theta) + Q(\theta), \tag{1.6}$$

where

$$P_e(\rho) = \int_1^\rho \frac{p_e(t)}{t^2}\mathrm{d}t, \quad Q(\theta) = \int_0^\theta c_\nu(t)\mathrm{d}t. \tag{1.7}$$

We impose the slip boundary condition

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \tau_k\cdot(\mathbf{T}(P,\mathbf{u})\mathbf{n}) + f\mathbf{u}\cdot\tau_k = 0, \quad \text{at } \partial\Omega,$$

where τ_k $(k = 1, 2)$ are two perpendicular tangent vectors to $\partial\Omega$, $\mathbf{n}$ is the outer normal vector, and $T(P,\mathbf{u}) = -PI + \Psi(\mathbf{u})$ is the stress tensor. The friction coefficient f is nonnegative (if $f = 0$ we assume additionally that Ω is not axially symmetric).

For temperature, we assume that

$$\kappa(\theta)\frac{\partial\theta}{\partial\mathbf{n}} + L(\theta)(\theta - \theta_0) = 0, \text{ at } \partial\Omega, \tag{1.8}$$

where $\theta_0 : \partial\Omega \longrightarrow \mathbf{R}^+$ is a strictly positive sufficiently smooth given function, and there exist $\bar{\theta}, \underline{\theta} \in \mathbf{R}^+$ such that

$$0 < \underline{\theta} \le \theta_0 \le \bar{\theta} < +\infty \quad \text{for almost all } (a.a.)\ x \in \Omega,$$

and

$$L(\theta) = c(1 + \theta^l), \quad l \in \mathbf{R}^+. \tag{1.9}$$

We also add the prescribed mass of the gas

$$\int_{\Omega} \rho \mathrm{d}x = M > 0, \tag{1.10}$$

and the heat conductivity depending on the temperature

$$\kappa(\theta) = \kappa_0(1 + \theta^m), \quad \kappa_0, m > 0. \tag{1.11}$$

The study of steady flows of compressible fluids is also intriguing mathematical questions. Lions [16] proved the existence of weak solutions to the steady compressible Navier–Stokes equation under the assumption that the heat capacity ratio $\gamma > 1$ in two dimensions and $\gamma > \frac{5}{3}$ in three dimensions. Meanwhile, he got rid of the smallness of the data. Roughly speaking, the condition on γ comes from the integrability of the density ρ in $\mathbf{L}^p$. The higher integrability of ρ has, the smaller γ can be allowed. If there is potential, then weak solutions are shown to exist for any $\gamma > \frac{3}{2}$, see [20]. Frehse, Steinhauer, and Weigant [6] established the existence of weak solutions to the Dirichlet problem in three dimensions for any $\gamma > \frac{4}{3}$. Also, the existence of a weak solution to the steady compressible Navier–Stokes equation with periodic or mixed boundary conditions was obtained in the two-dimensional isothermal case ($\gamma = 1$) [7]. Mucha and Pokorný [17] modified the method in [20] to reduce the number of technical tricks; then, they obtained the existence of weak solutions to the steady compressible Navier–Stokes equations in the isentropic regime.

The steady compressible Navier–Stokes–Fourier system is also considered in [18] with the slip boundary condition ($\gamma > 3$ and $m > \frac{3\gamma-1}{3\gamma-7}$). The method from [18] has been extended to $\gamma > \frac{7}{3}$ for both no-slip and slip boundary conditions in [19]. The condition for m remains the same. However, as the integrability of the density and velocity gradient is lower than for the case $\gamma > 3$. Recently, Novotný and Pokorný [22] extended the method in [18,21] and obtained the existence of weak solution and variable entropy solution to Navier–Stokes–Fourier system for $\gamma > \frac{4}{3}$ and $\gamma > \frac{3+\sqrt{41}}{8}$, respectively.

In fact, one of important restriction to the value of γ is due to the priori estimates. In the present paper, inspired by the work of [18,22], we will prove that the steady viscous, compressible, and heat conducting magnetohydrodynamic flows (1.1)–(1.4) have a weak solution for $\gamma > \frac{4}{3}$ by working with suitable priori estimates (see Lemma 2).

If the solution is smooth, multiplying the momentum Eq. (1.2) by $\mathbf{u}$ and the induction equation by $\mathbf{H}$, and summing them together, we obtain

$$\begin{aligned}\mathbf{div}\left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u}\right) + \nabla P \cdot \mathbf{u} &= \mathbf{div}\Psi(\mathbf{u}) \cdot \mathbf{u} + (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} \\ &\quad + \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H} - \nabla \times (\nu \nabla \times \mathbf{H}) \cdot \mathbf{H}.\end{aligned} \tag{1.12}$$

Then using

$$\mathbf{div}(\nu \mathbf{H} \times (\nabla \times \mathbf{H})) = \nu |\nabla \times \mathbf{H}|^2 - \nabla \times (\nu \nabla \times \mathbf{H}) \cdot \mathbf{H},$$

and

$$\mathbf{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H}) = (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H},$$

subtracting (1.12) from (1.3), we get the internal energy equation

$$\mathbf{div}(\rho \mathbf{u} e) + (\mathbf{div} \mathbf{u}) P = \nu |\nabla \times \mathbf{H}|^2 + \Psi(\mathbf{u}) : \nabla \mathbf{u} + \mathbf{div}(\kappa(\theta) \nabla \theta). \tag{1.13}$$

It follows from multiplying the continuity equation (1.1) by $(\rho p_e(\rho))'$ that

$$\mathbf{div}(\rho \mathbf{u} p_e(\rho)) + p_e(\rho) \mathbf{div} \mathbf{u} = 0.$$

Then, subtracting above equation from (1.13), we obtain the thermal energy equation

$$\mathbf{div}(\kappa(\theta)\nabla\theta) - \mathbf{div}(\rho Q(\theta)\mathbf{u}) + \Psi(\mathbf{u}) : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2 - \theta p_\theta(\rho)\mathbf{div}\mathbf{u} = 0, \quad (1.14)$$

where

$$\Psi(\mathbf{u}) : \nabla\mathbf{u} = \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right)^2 + \lambda|\mathbf{div}\mathbf{u}|^2.$$

There have been much work on magnetohydrodynamics by mathematicians because of its physical importance, complexity, and widely application (see [1,14,15,23]). Magnetohydrodynamics (MHD) is a combination of the compressible Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Duvaut and Lions [4], Sermange and Temam [24] obtained some existence and long-time behavior results for incompressible case. For compressible case, Ducomet and Feireisl [3] proved existence of global in time weak solutions to a multi-dimensional nonisentropic MHD system for gaseous stars coupled with the Poisson equation with all the viscosity coefficients and the pressure depends on temperature and density asymptotically, respectively. Hu and Wang [8] studied the global variational weak solution to the three-dimensional full magnetohydrodynamic equations with large data by an approximation scheme and a weak convergence method. In [9], by using the Faedo–Galerkin method and the vanishing viscosity method, they also studied the existence and large-time behavior of global weak solutions for the three-dimensional equations of compressible magnetohydrodynamic isentropic flows (1.1)–(1.3). They [10] showed that the convergence of weak solutions of the compressible MHD system to a weak solution of the viscous incompressible MHD system. Jiang, et all. [11,12] obtained that the convergence toward the strong solution of the ideal incompressible MHD system in the whole space and periodic domain, respectively. Recently, Yan [26] showed the weak-strong uniqueness property for full compressible magnetohydrodynamics flows. After that, he [27,28] obtained that the existence of time-periodic weak solution for compressible magnetic fluids in three-dimensional torus, and the existence of weak solution for the three-dimensional density-dependent generalized incompressible magnetohydrodynamic flows, respectively.

The main difficulty of the study of MHD is the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow of large oscillation. This leads to many fundamental problems for MHD are still open. For example, the global existence of classical solution to the full perfect MHD equations with large data in one-dimensional case is unsolved. But the corresponding problem about Navier–Stokes equation has solved in [13] a long time ago. In present paper, we study one of the fundamental problem about MHD, that is, the existence of weak solution for the equations of the steady viscous, compressible, and heat conducting magnetohydrodynamic flows in a bounded three-dimensional domain. Inspired by the work of [2,5,8,16,18], we will overcome a lack of a priori estimates on MHD and the large oscillation to establish the existence of weak solution for the steady full compressible MHD for any $\gamma > \frac{4}{3}$.

The paper is organized in the following way: In the next section, we provide the precise definition of the notion of weak solution to system (1.1)–(1.2), (1.4), and (1.14) after introducing the appropriate function spaces. The main result of this paper is also stated. In Sect. 3, we introduce the corresponding approximation system and prove the existence of solution about it. In last section, an important quantity: the effective viscous flux is introduced; then, the convergence of the approximation solution is proved.

2 Some notations and main results

Before given the definition of the weak solution to the problem (1.1)–(1.2), (1.4), and (1.14) with the boundary condition (1.8) and (1.9), we state the following notation of relevant Banach spaces of functions defined on a bounded domain $\Omega \subset \mathbf{R}^3$. For any $p \in [1, \infty]$, $\mathbf{L}^p(\Omega)$ denotes the Lebesgue spaces with the norm $\|\cdot\|_{\mathbf{L}^p(\Omega)}$ and , $\mathbf{W}^{a,p}(\Omega)$ denotes the Sobolev spaces with the norm $\|\cdot\|_{\mathbf{W}^{a,p}(\Omega)}$. We do not distinguish between function spaces for scalar and vector valued functions. Generic constants are denoted by C; their values may vary in the same formula or in the same line.

Definition A vector $(\rho, \mathbf{u}, \theta, \mathbf{H})$ is said to be a weak solution to the problem (1.1)–(1.2), (1.4), and (1.14) of the full compressible steady MHD equations if the following conditions hold:

- The density $\rho \in \mathbf{L}^p(\Omega)$, $p \geq \gamma$, the velocity $\mathbf{u} \in \mathbf{W}^{1,2}(\Omega)$, The temperature $\theta \in \mathbf{W}^{1,2}(\Omega)$, $\theta^m \nabla\theta \in \mathbf{L}^1(\Omega)$, and the magnetic field $\mathbf{H} \in \mathbf{W}_0^{1,2}(\Omega)$ satisfy the Eqs. (1.1)–(1.2), (1.4), and (1.14) in the sense of distributions, and
$$\int_\Omega \rho\mathbf{u} \cdot \nabla\varphi \mathrm{d}x = 0,$$
for any $\varphi \in \mathbf{C}^\infty(\overline{\Omega})$.
- The temperature nonnegative θ function, the velocity function $\mathbf{u}$, and the magnetic field $\mathbf{H}$ satisfying
$$\begin{aligned}\int_\Omega (-\rho\mathbf{u} \otimes \mathbf{u} : \nabla\varphi + 2\mu\mathbf{D}(\mathbf{u}) : \mathbf{D}(\varphi) + \lambda\mathbf{div}\mathbf{u} \cdot \mathbf{div}\varphi - P(\rho, \theta)\mathbf{div}\varphi)\mathrm{d}x \\ + \int_\Omega (\mathbf{H}^T \nabla\varphi\mathbf{H} + \frac{1}{2}\nabla(|\mathbf{H}|^2) \cdot \varphi)\mathrm{d}x + f\int_{\partial\Omega} (\mathbf{u} \odot \tau) \cdot (\varphi \odot \tau)\mathrm{d}\sigma \\ = \int_\Omega \rho\mathbf{F} \cdot \varphi \mathrm{d}x,\end{aligned}$$
$$\begin{aligned}&\int_\Omega (\kappa(\theta)\nabla\theta \cdot \nabla\varphi - \rho Q(\theta)\mathbf{u} \cdot \nabla\varphi)\mathrm{d}x + \int_{\partial\Omega} L(\theta)(\theta - \theta_0)\varphi \mathrm{d}x \\ &= \int_\Omega \left(2\mu|D(\mathbf{u})|^2\Psi + \lambda(\mathbf{div}\mathbf{u})^2\varphi + \nu|\nabla \times \mathbf{H}|^2 \cdot \varphi - \theta p_\theta(\rho)\mathbf{div}\mathbf{u} \cdot \varphi\right) \mathrm{d}x,\end{aligned}$$
$$\int_\Omega (\nu\mathbf{curl}\mathbf{H} \cdot \mathbf{curl}\varphi + (\mathbf{div}\mathbf{u} + \mathbf{u} \cdot \nabla)\mathbf{H} \cdot \varphi - (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \varphi)\, \mathrm{d}x = 0,$$
for any $\varphi \in \mathbf{C}^\infty(\overline{\Omega})$, $\varphi \cdot \mathbf{n} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$ at $\partial\Omega$ in the sense of traces.
The aim of this paper was to establish the following result.

Theorem 1 *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain of class $\mathbf{C}^2$ which is not axially symmetric if $f = 0$, $\mathbf{F} \in \mathbf{L}^\infty(\Omega)$ and $\frac{2\gamma}{3(\gamma-1)} < m = 1 + l < m^+$, where m^+ is given in (3.48). Suppose that the following conditions hold:*

(i) The pressure $P(\rho, \theta)$ is given by (1.5), where $p_e, p_\theta \in \mathbf{C}^1[0, \infty)$ and
$$\begin{aligned}&p_e(0) = 0, \quad p_\theta(0) = 0, \\ &p_e'(\rho) \geq a_1\rho^{\gamma-1}, \quad p_\theta'(\rho) \geq 0,\end{aligned} \tag{2.1}$$

$$p_e(\rho) \leq a_2\rho^{\gamma}, \quad p_\theta(\rho) \leq a_3\rho^{\frac{\gamma}{3}},$$

with some constant $\gamma > \frac{4}{3}$, $a_1 > 0$, $a_2 > 0$ *and* $a_3 > 0$.

(ii) The specific heat $c_\nu(\theta)$ *satisfies*

$$0 < \underline{c_\nu} \leq c_\nu(\theta) \leq \overline{c_\nu} < +\infty, \tag{2.2}$$

where $\underline{c_\nu}$ *and* $\overline{c_\nu}$ *are two positive constants.*

Then, the steady full compressible MHD has a weak solution $(\rho, \mathbf{u}, \theta, \mathbf{H})$ *such that for* $1 \leq p < \infty$

$$\rho \in \mathbf{L}^\infty(\Omega), \quad \mathbf{u} \in \mathbf{W}^{1,p}(\Omega), \quad \theta \in \mathbf{W}^{1,p}(\Omega), \quad \mathbf{H} \in \mathbf{W}^{1,2}(\Omega).$$

Moreover, the temperature $\theta > 0$ *a.e. in* Ω.

Remark 1 The assumption on the specific heat $c_\nu(\theta)$ in (ii) means that it can be controlled by a positive constant. Moreover, by (1.7), we can get that $\underline{c_\nu}\theta \leq Q(\theta) \leq \overline{c_\nu}\theta$. This is used in deriving (3.8) in Lemma 1. In addition, this assumption on $c_\nu(\theta)$ is the same as the work of [8,9].

Finally, note that, in order to simplify the presentation, we will put $a_1 = a_2 = a_3 = 1$.

3 The existence of solution for the approximation system

In this section, we first introduce an auxiliary function $K(\cdot)$ defined by number $k > 0$ as follows

$$K(t) := \begin{cases} 1 & \text{for } t < k-1, \\ \in [0, 1] & \text{for } k-1 \leq t \leq k, \\ 0 & \text{for } t > k. \end{cases}$$

Moreover, we assume that $K'(t) < 1$ for $t \in (k-1, k)$, where $k \in \mathbf{R}^+$. Then, an approximation problem which consists of a system of regularized equations can be showed

$$\mathbf{div}(K(\rho)\rho\mathbf{u}) + \epsilon\rho - \epsilon\triangle\rho = \epsilon h K(\rho), \tag{3.1}$$

$$\frac{1}{2}\mathbf{div}(K(\rho)\rho\mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho, \theta) = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{div}\Psi(\mathbf{u}) + K(\rho)\rho\mathbf{F}, \tag{3.2}$$

$$\begin{aligned} &-\mathbf{div}((1+\theta^m)\frac{(\epsilon+\theta)}{\theta}\nabla\theta) + \mathbf{div}(K(\rho)\rho Q(\theta)\mathbf{u}) + \mathbf{div}\left(\mathbf{u}p_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\right)\theta \\ &\quad -K(\rho)\mathbf{u}\theta p'_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\nabla\rho \\ &\quad = \nu|\nabla \times \mathbf{H}|^2 + \Psi(\mathbf{u}) : \nabla\mathbf{u}, \end{aligned} \tag{3.3}$$

$$\nabla \times (\mathbf{u} \times \mathbf{H}) = \nabla \times (\nu\nabla \times \mathbf{H}), \tag{3.4}$$

where $h = \frac{M}{|\Omega|}$ and

$$P_1(\rho, \theta) = p_e\left(\int_0^\rho K(t)\mathrm{d}t\right) + \theta p_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right) = p_b(\rho) + \theta p_c(\rho). \tag{3.5}$$

To estimate the temperature, the entropy is introduced by

$$s = \ln \theta.$$

Then, the corresponding entropy equation of (3.3) can be reformulated

$$\begin{aligned}
&-\mathbf{div}((1+e^{sm})\frac{(\epsilon+e^{s})}{e^{s}}\nabla s) + \mathbf{div}(K(\rho)\rho\mathbf{u})Q(e^{s})e^{-s} + K(\rho)\rho\mathbf{u}Q'(e^{s})\nabla s \\
&+\mathbf{div}\left(\mathbf{u}p_{\theta}\left(\int_0^{\rho} K(t)\mathrm{d}t\right)\right) - K(\rho)\mathbf{u}p'_{\theta}(\int_0^{\rho} K(t)\mathrm{d}t)\nabla\rho \\
&= (1+e^{sm})(\epsilon+e^{s})|\nabla s|^2 e^{-s} + \nu|\nabla\times\mathbf{H}|^2 e^{-s} + (\Psi:\nabla\mathbf{u})e^{-s}.
\end{aligned} \tag{3.6}$$

We consider the boundary condition at $\partial\Omega$ for the approximation system (3.1)–(3.4):

$$\begin{aligned}
&(1+\theta^m)(\epsilon+\theta)\frac{\partial s}{\partial\mathbf{n}} + L(\theta)(\theta-\theta_0) + \epsilon s = 0, \\
&\tau_k\cdot(\mathbf{T}(P,\mathbf{u})\mathbf{n}) + f\mathbf{u}\cdot\tau_k = 0, \quad k = 1, 2, \\
&\frac{\partial\rho}{\partial\mathbf{n}} = 0, \quad \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{H}|_{\partial\Omega} = 0.
\end{aligned} \tag{3.7}$$

The main tool of solving (3.1)–(3.4) is by the standard Leray–Schauder fixed point theorem. Firstly, the solvability of the continuity equation (3.1) is taken from [20] (also can be found in [17,18]). Denote

$$\mathbf{X}^p = \{\mathbf{u}\in\mathbf{W}^{2,p}(\Omega) : \mathbf{u}\cdot\mathbf{n} = 0 \text{ at } \partial\Omega\}.$$

Lemma 1 *Let $p > 3$. Then, for any fix $\mathbf{u}\in\mathbf{X}^p$, the continuity equation (3.1) with the boundary condition $\frac{\partial\rho}{\partial\mathbf{n}} = 0$ has a solution $\rho\in\mathbf{W}^{2,q}(\Omega)$. Furthermore, for $1 < q < \infty$, the operator*

$$\mathcal{F}:\ \mathbf{X}^p \longrightarrow \mathbf{W}^{2,q}(\Omega)$$

is a well-defined continuous operator from $\mathbf{X}^p$ to $\mathbf{W}^{2,q}(\Omega)$ such that $\mathcal{F}(\mathbf{u}) = \rho$. Moreover,

$$\begin{aligned}
&\|\rho\|_{\mathbf{W}^{1,q}(\Omega)} \le C(k,\epsilon)\left(1+\|\mathbf{u}\|_{\mathbf{L}^q(\Omega)}\right), \quad 1<q<\infty, \\
&\|\rho\|_{\mathbf{W}^{2,q}(\Omega)} \le C(k,\epsilon)\left(1+\|\mathbf{u}\|_{\mathbf{W}^{1,q}(\Omega)}(1+\|\mathbf{u}\|_{\mathbf{L}^3(\Omega)})\right), \quad 1<q<3, \\
&\|\rho\|_{\mathbf{W}^{2,q}(\Omega)} \le C(k,\epsilon)\left(1+\|\mathbf{u}\|^2_{\mathbf{W}^{1,q}(\Omega)}\right), \quad 3\le q<\infty.
\end{aligned} \tag{3.8}$$

Before showing the approximation system (3.2)–(3.4) is solvable, we derive a priori estimates. We recall the following basic theory to the stationary Stokes system (see [20] for details). Consider the following problem:

$$\begin{aligned}
&\mathbf{div}\Theta = p_b(\rho) - \overline{p_b(\rho)}, \quad x\in\Omega, \\
&\Theta = 0, \quad x\in\partial\Omega,
\end{aligned}$$

where $\overline{p_b(\rho)} = \frac{1}{|\Omega|}\int_{\partial\Omega} p_b(\rho)\mathrm{d}x$. Then, above problem possess a solution Θ such that

$$\|\Theta\|_{H_0^1(\Omega)} \le C\|p_b(\rho)\|_{\mathbf{L}^2(\Omega)}. \tag{3.9}$$

Furthermore, using the interpolation inequality and (1.10), we deduce that for any $\delta > 0$,

$$\overline{p_b(\rho)} \le \delta\|p_b(\rho)\|_{\mathbf{L}^2(\Omega)} + C(\delta, M).$$

Lemma 2 *Under the assumptions of Theorem 1, let* $(\rho_\epsilon, \mathbf{u}_\epsilon, \theta_\epsilon, \mathbf{H}_\epsilon)$ *be the solution of approximation system* (3.1)–(3.4) *in* $\mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{1,2}(\Omega)$, *for any* $p < \infty$, *and* $\theta > 0$. *Then,*

$$0 \le \rho \le k, \quad \int_\Omega \rho dx \le M$$

and

$$\begin{aligned}
&\|\mathbf{u}\|_{H^1(\Omega)} + \|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)} + \|P(\rho,\theta)\|_{\mathbf{L}^{2\gamma}(\Omega)} + \|\theta\|_{\mathbf{L}^{3m}(\Omega)} + \|\nabla\theta\|_{\mathbf{L}^{\frac{3m}{m+1}}(\Omega)} \\
&+\|\mathbf{H}\|_{H^1(\Omega)} + \int_{\partial\Omega} (e^s + e^{-s}) d\sigma + \|\nabla s\|_{\mathbf{L}^{2\gamma}(\Omega)} \\
&\le C(\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}, M),
\end{aligned} \tag{3.10}$$

where $C(\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}, M)$ *is independent of* ϵ, k, s, *and* m.

Proof The proof of the estimate of ρ is similar with [17]. Hence, we only prove (3.10). Multiplying the approximative momentum equation (3.2) by $\mathbf{u}$, integrating over Ω, we get

$$\begin{aligned}
&\int_\Omega (2\mu D^2(\mathbf{u}) + \lambda \mathbf{div}^2\mathbf{u})\mathrm{d}x + \int_\Omega f|\mathbf{u} \odot \tau|^2 \mathrm{d}x + \int_\Omega \mathbf{u} \cdot \nabla p_b(\rho)\mathrm{d}x \\
&= \int_\Omega (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u}\mathrm{d}x + \int_\Omega K(\rho)\rho\mathbf{u}\mathbf{F}\mathrm{d}x + \int_\Omega \theta p_c(\rho)\mathbf{div}\mathbf{u}\mathrm{d}x.
\end{aligned} \tag{3.11}$$

Denote $P_f(\rho) = \int_0^\rho \frac{p_e'(t)}{t}\mathrm{d}t$. Using (3.1) and (3.5), we have

$$\begin{aligned}
\int_\Omega \mathbf{u} \cdot \nabla p_b(\rho)\mathrm{d}x &= \int_\Omega \mathbf{u}K(\rho)\rho\nabla P_f(\rho)\mathrm{d}x \\
&= -\int_\Omega \nabla(\mathbf{u}K(\rho)\rho)P_f(\rho)\mathrm{d}x \\
&= \epsilon\int_\Omega (\rho - hK(\rho))P_f(\rho)\mathrm{d}x + \epsilon\int_\Omega \nabla\rho\nabla P_f(\rho)\mathrm{d}x \\
&= \epsilon\int_\Omega (\rho - hK(\rho))P_f(\rho)\mathrm{d}x + \epsilon\int_\Omega p_e'(\rho)\nabla\rho\nabla\ln\rho\mathrm{d}x.
\end{aligned} \tag{3.12}$$

Note that

$$-\int_\Omega (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u}\mathrm{d}x = \int_\Omega (\mathbf{H}^T\nabla\mathbf{u}\mathbf{H} + \frac{1}{2}\nabla(|\mathbf{H}|^2) \cdot \mathbf{u})\mathrm{d}x. \tag{3.13}$$

Thus, by (3.11)–(3.13), we derive

$$\begin{aligned}
&\int_\Omega \Psi(\mathbf{u}) : \nabla\mathbf{u}\mathrm{d}x + \int_\Omega f|\mathbf{u} \odot \tau|^2\mathrm{d}x + \epsilon\int_\Omega (\rho - hK(\rho))P_f(\rho)\mathrm{d}x + \epsilon\int_\Omega p_e'(\rho)\nabla\rho\nabla\ln\rho\mathrm{d}x \\
&+ \int_\Omega (\mathbf{H}^T\nabla\mathbf{u}\mathbf{H} + \frac{1}{2}\nabla(|\mathbf{H}|^2) \cdot \mathbf{u})\mathrm{d}x - \int_\Omega \theta p_c(\rho)\mathbf{div}\mathbf{u}\mathrm{d}x \\
&\le C(1 + \int_\Omega |K(\rho)\rho\mathbf{u} \cdot \mathbf{F}|\mathrm{d}x).
\end{aligned} \tag{3.14}$$

Multiplying the approximative equation (3.2) by $\mathbf{H}$, integrating over Ω, and using the boundary condition and $\mathbf{div}\mathbf{H} = 0$, we get

$$\int_\Omega \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H}\mathrm{d}x = \int_\Omega \nabla \times (\nu\nabla \times \mathbf{H}) \cdot \mathbf{H}\mathrm{d}x. \tag{3.15}$$

Direct calculations show that

$$\int_\Omega \nabla\times(\mathbf{u}\times\mathbf{H})\cdot\mathbf{H}\mathrm{d}x=\int_\Omega(\mathbf{H}^T\nabla\mathbf{u}\mathbf{H}+\frac{1}{2}\nabla(|\mathbf{H}|^2)\cdot\mathbf{u})\mathrm{d}x,$$
$$\int_\Omega \nabla\times(\nu\nabla\times\mathbf{H})\cdot\mathbf{H}\mathrm{d}x=\nu\int_\Omega|\nabla\times\mathbf{H}|^2\mathrm{d}x.$$

Thus, by (3.1), it has

$$\nu\int_\Omega|\nabla\times\mathbf{H}|^2\mathrm{d}x=\int_\Omega(\mathbf{H}^T\nabla\mathbf{u}\mathbf{H}+\frac{1}{2}\nabla(|\mathbf{H}|^2)\cdot\mathbf{u})\mathrm{d}x,$$

which together with (3.14) yields

$$\begin{aligned}&\int_\Omega\Psi(\mathbf{u}):\nabla\mathbf{u}\mathrm{d}x+\int_\Omega f|\mathbf{u}\odot\tau|^2\mathrm{d}x+\epsilon\int_\Omega\rho P_f(\rho)\mathrm{d}x+\epsilon\int_\Omega p_e'(\rho)\nabla\rho\nabla\ln\rho\mathrm{d}x\\&\quad+\nu\int_\Omega|\nabla\times\mathbf{H}|^2\mathrm{d}x-\int_\Omega\theta p_c(\rho)\mathbf{divu}\mathrm{d}x\\&\le C(1+\int_\Omega|K(\rho)\rho\mathbf{u}\cdot\mathbf{F}|\mathrm{d}x).\end{aligned}\tag{3.16}$$

Furthermore, by Hölder's inequality, we obtain

$$\begin{aligned}&\|\mathbf{u}\|^2_{H^1(\Omega)}+\nu\|\mathbf{H}\|^2_{H^1(\Omega)}+\epsilon\int_\Omega\rho P_f(\rho)\mathrm{d}x+\epsilon\int_\Omega p_e'(\rho)\nabla\rho\nabla\ln\rho\mathrm{d}x\\&\le C(1+\int_\Omega|K(\rho)\rho\mathbf{u}\cdot\mathbf{F}|\mathrm{d}x+\int_\Omega|\theta p_c(\rho)|^2\mathrm{d}x).\end{aligned}\tag{3.17}$$

In what follows, our target is to estimate the temperature θ, the term $p_b(\rho)$, the velocity $\mathbf{u}$, and the magnetic field $\mathbf{H}$. We notice that the estimation of last three term is related to the temperature θ, so we first estimate it.

Integrating (3.3) and by the first boundary condition of (3.7), we get

$$\begin{aligned}\int_{\partial\Omega}(L(\theta)(\theta-\theta_0)+\epsilon s)\mathrm{d}\sigma&=\int_\Omega(\nu|\nabla\times\mathbf{H}|^2+\Psi:\nabla\mathbf{u})\mathrm{d}x\\&\quad-\int_\Omega(K(\rho)\rho Q(\theta))\mathbf{divu}\mathrm{d}x.\end{aligned}\tag{3.18}$$

Denote s^+ and s^- as the positive and negative parts of the entropy, respectively. Summing up (3.16) and (3.18) yields

$$\begin{aligned}&\int_{\partial\Omega}(L(\theta)\theta+\epsilon s^+)\mathrm{d}\sigma+\epsilon\int_\Omega\rho P_f(\rho)\mathrm{d}x+\epsilon\int_\Omega p_e'(\rho)\nabla\rho\nabla\ln\rho\mathrm{d}x\\&\quad+\int_\Omega(K(\rho)\rho Q(\theta)-\theta p_c(\rho))\mathbf{divu}\mathrm{d}x\\&\le\int_{\partial\Omega}\epsilon s^-\mathrm{d}\sigma+C(1+\int_\Omega|K(\rho)\rho\mathbf{u}\cdot\mathbf{F}|\mathrm{d}x).\end{aligned}\tag{3.19}$$

Integrating the entropy equation (3.6) over Ω, we have

$$\begin{aligned}&\int_{\partial\Omega}\left(\frac{L(\theta)(\theta-\theta_0)}{\theta}+\epsilon se^{-s}\right)\mathrm{d}\sigma-\int_\Omega K(\rho)\mathbf{u}p_\theta'\left(\int_0^\rho K(t)\mathrm{d}t\right)\nabla\rho\mathrm{d}x\\&=\int_\Omega\left(\frac{(1+\theta^m)(\epsilon+\theta)|\nabla s|^2}{\theta}+\nu\frac{|\nabla\times\mathbf{H}|^2}{\theta}+\frac{\Psi:\nabla\mathbf{u}}{\theta}\right)\mathrm{d}x,\end{aligned}\tag{3.20}$$

which implies that

$$\int_\Omega\left(\frac{(1+\theta^m)(\epsilon+\theta)|\nabla s|^2}{\theta}+\nu\frac{|\nabla\times\mathbf{H}|^2}{\theta}+\frac{\Psi:\nabla\mathbf{u}}{\theta}-K(\rho)\mathbf{u}p_\theta'(\int_0^\rho K(t)\mathrm{d}t)\nabla\rho\right)\mathrm{d}x$$
$$+\int_{\partial\Omega}\left(\frac{L(\theta)\theta_0}{\theta}+\epsilon|s^-||e^{s^-}|\right))\mathrm{d}\sigma\le\int_{\partial\Omega}(L(\theta)+\epsilon s^+e^{-s^+})\mathrm{d}\sigma. \tag{3.21}$$

It derives from (3.19) and (3.21) that

$$\int_\Omega(\frac{(1+\theta^m)|\nabla\theta|^2}{\theta^2}+\nu\frac{|\nabla\times\mathbf{H}|^2}{\theta}+\frac{\Psi:\nabla\mathbf{u}}{\theta})\mathrm{d}x+\int_{\partial\Omega}(\frac{L(\theta)\theta_0}{\theta}+\epsilon|s|)\mathrm{d}\sigma$$
$$\le\int_\Omega((\theta p_c(\rho)-K(\rho)\rho Q(\theta)-p_\theta(\int_0^\rho K(t)\mathrm{d}t))\mathbf{divu}\mathrm{d}x$$
$$+C(1+\int_\Omega|K(\rho)\rho\mathbf{u}\cdot\mathbf{F}|\mathrm{d}x). \tag{3.22}$$

Using generalized Hölder inequality, for $\frac{1}{p_1}+\frac{1}{p_1}=\frac{1}{2}$, we derive

$$\left|\int_\Omega\theta p_c(\rho)\mathbf{divu}\mathrm{d}x\right|\le\|\mathbf{u}\|_{H^1(\Omega)}\|p_c(\rho)\|_{\mathbf{L}^{p_1}(\Omega)}\|\theta\|_{\mathbf{L}^{p_2}(\Omega)}, \tag{3.23}$$

$$\left|\int_\Omega K(\rho)\rho Q(\theta)\mathbf{divu}\mathrm{d}x\right|\le\|\mathbf{u}\|_{H^1(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^{p_1}(\Omega)}\|Q(\theta)\|_{\mathbf{L}^{P_2}(\Omega)}, \tag{3.24}$$

$$\left|\int_\Omega p_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\mathbf{divu}\mathrm{d}x\right|\le\|\mathbf{u}\|_{H^1(\Omega)}\|p_\theta(\int_0^\rho K(t)\mathrm{d}t)\|_{\mathbf{L}^2(\Omega)}, \tag{3.25}$$

$$\left|\int_\Omega K(\rho)\rho\mathbf{u}\cdot\mathbf{F}\mathrm{d}x\right|\le\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)}\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}. \tag{3.26}$$

Thus, by (3.22)–(3.26) we obtain

$$\int_\Omega(\frac{(1+\theta^m)|\nabla\theta|^2}{\theta^2}+\nu\frac{|\nabla\times\mathbf{H}|^2}{\theta}+\frac{\Psi:\nabla\mathbf{u}}{\theta})\mathrm{d}x+\int_{\partial\Omega}\left(\frac{L(\theta)\theta_0}{\theta}+\epsilon|s|\right))\mathrm{d}\sigma\le N, \tag{3.27}$$

where

$$N=\|\mathbf{u}\|_{H^1(\Omega)}(\|p_c(\rho)\|_{\mathbf{L}^{p_1}(\Omega)}\|\theta\|_{\mathbf{L}^{p_2}(\Omega)}+\|K(\rho)\rho\|_{\mathbf{L}^{p_1}(\Omega)}\|Q(\theta)\|_{\mathbf{L}^{P_2}(\Omega)}$$
$$+\|p_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\|_{\mathbf{L}^2(\Omega)})+C(1+\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)}\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}). \tag{3.28}$$

It follows from (1.9) and (3.27) that

$$C\left(\int_{\partial\Omega}\theta^{l+1}\mathrm{d}x\right)^{\frac{1}{l+1}}\le\left(\int_{\partial\Omega}L(\theta)\theta\mathrm{d}x\right)^{\frac{1}{l+1}}\le N^{\frac{1}{l+1}},$$
$$C\left(\int_\Omega|\nabla\theta^{\frac{m}{2}}|^2\mathrm{d}x\right)^{\frac{1}{m}}\le\left(\int_\Omega\frac{(1+\theta^m)|\nabla\theta|^2}{\theta^2}\mathrm{d}x\right)^{\frac{1}{m}}\le N^{\frac{1}{m}}.$$

By Poincaré type inequality $\|\mathbf{u}\|_{\mathbf{L}^1(\Omega)}\le C(\Omega)(\|\mathbf{u}\|_{\mathbf{L}^1(\Omega)}+\|\mathbf{u}\|_{H^1(\Omega)})$, we derive

$$\left(\int_\Omega|\theta^{\frac{m}{2}}|^2\mathrm{d}x\right)^{\frac{1}{m}}\le C(\Omega)\left(\left(\int_\Omega|\nabla\theta^{\frac{m}{2}}|^2\mathrm{d}x\right)^{\frac{1}{m}}+\left(\int_{\partial\Omega}\theta^{l+1}\mathrm{d}\sigma\right)^{\frac{1}{l+1}}\right).$$

Then, it derive from $\mathbf{W}^{1,2}(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ that

$$\left(\int_\Omega |\theta^{3m}|^2 \mathrm{d}x\right)^{\frac{1}{3m}} \leq N^{\frac{1}{l+1}} + N^{\frac{1}{m}}. \tag{3.29}$$

Multiplying (3.2) by Θ, using (3.9) and (3.16), we conclude after standard estimates that

$$\|p_b(\rho)\|_{\mathbf{L}^2(\Omega)} \leq C\left(1 + \int_\Omega |K(\rho)\rho\mathbf{u}\otimes\mathbf{u}|^2 \mathrm{d}x + \int_\Omega |\theta p_c(\rho)|^2 \mathrm{d}x\right). \tag{3.30}$$

Note that

$$\begin{aligned}\int_\Omega |K(\rho)\rho\mathbf{u}\otimes\mathbf{u}|^2 \mathrm{d}x &\leq C\|\mathbf{u}\|^4_{H^1(\Omega)}\|K(\rho)\rho\|^2_{\mathbf{L}^6(\Omega)}\\ &\leq C\|\mathbf{u}\|^4_{H^1(\Omega)}\|K(\rho)\rho\|^{\frac{2(\gamma-1)}{3(2\gamma-1)}}_{\mathbf{L}^1(\Omega)}\|K(\rho)\rho\|^{\frac{10\gamma}{3(2\gamma-1)}}_{\mathbf{L}^{2\gamma}(\Omega)}\\ &\leq \delta\|p_b(\rho)\|_{\mathbf{L}^2(\Omega)} + C(\delta, M)\|\mathbf{u}\|^{\frac{6(2\gamma-1)}{3\gamma-4}}_{H^1(\Omega)},\end{aligned} \tag{3.31}$$

where we use (getting by the form of $p_b(\rho)$ and (2.1))

$$\|p_b(\rho)\|_{\mathbf{L}^2(\Omega)} \geq C\left(\int_\Omega (K(\rho)\rho)^{2\gamma}\mathrm{d}x + \int_\Omega (\int_0^\rho K(t)\mathrm{d}t)^{2\gamma}\mathrm{d}x\right). \tag{3.32}$$

Thus, by a suitable choice of δ, it deduce from (3.30)–(3.31) that

$$\|p_b(\rho)\|_{\mathbf{L}^2(\Omega)} \leq C\left(1 + \|\mathbf{u}\|^{\frac{6(2\gamma-1)}{3\gamma-4}}_{H^1(\Omega)}\mathrm{d}x + \int_\Omega |\theta p_c(\rho)|^2\mathrm{d}x\right). \tag{3.33}$$

Combining (3.32)–(3.33), we get

$$\begin{aligned}&\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)} + \|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}\\ &\quad\leq C\left(1 + \|\mathbf{u}\|^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}}_{H^1(\Omega)}\mathrm{d}x + \left(\int_\Omega |\theta p_c(\rho)|^2\mathrm{d}x\right)^{\frac{1}{2\gamma}}\right).\end{aligned} \tag{3.34}$$

Using (2.1) and the Interpolation between 1 and 2γ, we have for $m > \frac{2}{3}$

$$\begin{aligned}\left(\int_\Omega |\theta p_c(\rho)|^2\mathrm{d}x\right)^{\frac{1}{2\gamma}} &\leq \|\theta\|^{\frac{1}{\gamma}}_{\mathbf{L}^{3m}(\Omega)}\|p_c(\rho)\|^{\frac{1}{\gamma}}_{\mathbf{L}^{\frac{6m}{3m-2}}(\Omega)}\\ &\leq \|\theta\|^{\frac{1}{\gamma}}_{\mathbf{L}^{3m}(\Omega)}\|\int_0^\rho K(t)\mathrm{d}t\|^{\frac{1}{3}}_{\mathbf{L}^{\frac{6m}{3m-2}}(\Omega)}\\ &\leq \|\theta\|^{\frac{1}{\gamma}}_{\mathbf{L}^{3m}(\Omega)}\|\int_0^\rho K(t)\mathrm{d}t\|^{\frac{4m\gamma-3m+2}{(3m-2)(2\gamma-1)}}_{\mathbf{L}^1(\Omega)}\|\int_0^\rho K(t)\mathrm{d}t\|^{\frac{2\gamma(m-2)}{(3m-2)(2\gamma-1)}}_{\mathbf{L}^{2\gamma}(\Omega)}.\end{aligned} \tag{3.35}$$

This combining with (3.34), by Young inequality, for $m > \frac{4\gamma-6}{4\gamma-3}$, we derive

$$\begin{aligned}&\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)} + \|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}\\ &\quad\leq C\left(1 + \|\mathbf{u}\|^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}}_{H^1(\Omega)} + \|\theta\|^{\frac{(3m-2)(2\gamma-1)}{\gamma(4m\gamma-3m-4\gamma+6)}}_{\mathbf{L}^{3m}(\Omega)}\right).\end{aligned} \tag{3.36}$$

Now we estimate N. By the Interpolation inequality, for $1 < p_1 < 2\gamma$ and $1 < p_2 < 3m$, we derive

$$\begin{aligned}&\|\mathbf{u}\|_{H^1(\Omega)}\|p_c(\rho)\|_{\mathbf{L}^{p_1}(\Omega)}\|\theta\|_{\mathbf{L}^{p_2}(\Omega)}\\&\leq \|\mathbf{u}\|_{H^1(\Omega)}\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^1(\Omega)}^{\frac{2\gamma-p_1}{p_1(2\gamma-1)}}\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{2\gamma(p_1-1)}{p_1(2\gamma-1)}}\|\theta\|_{\mathbf{L}^1(\Omega)}^{\frac{2(3m-p_2)}{p_2(3m-1)}}\|\theta\|_{\mathbf{L}^{3m}(\Omega)}^{\frac{6m(p_2-1)}{p_2(3m-1)}},\end{aligned}\tag{3.37}$$

$$\begin{aligned}&\|\mathbf{u}\|_{H^1(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^{p_1}(\Omega)}\|Q(\theta)\|_{\mathbf{L}^{p_2}(\Omega)}\\&\leq C\|\mathbf{u}\|_{H^1(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^1(\Omega)}^{\frac{2\gamma-p_1}{p_1(2\gamma-1)}}\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{2\gamma(p_1-1)}{p_1(2\gamma-1)}}\|\theta\|_{\mathbf{L}^1(\Omega)}^{\frac{2(3m-p_2)}{p_2(3m-1)}}\|\theta\|_{\mathbf{L}^{3m}(\Omega)}^{\frac{6m(p_2-1)}{p_2(3m-1)}},\end{aligned}\tag{3.38}$$

$$\begin{aligned}&\|\mathbf{u}\|_{H^1(\Omega)}\|p_\theta(\int_0^\rho K(t)\mathrm{d}t)\|_{\mathbf{L}^2(\Omega)}\leq C(\gamma)(\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{\gamma}{\gamma-1}}+\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{3}})\\&\leq C(\gamma)(\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{\gamma}{\gamma-1}}+\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^1(\Omega)}^{\frac{\gamma-3}{3(2\gamma-1)}}\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{5\gamma}{3(2\gamma-1)}}),\end{aligned}\tag{3.39}$$

$$\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)}\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}\leq C\|\mathbf{u}\|_{H^1(\Omega)}\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{\gamma}{3(2\gamma-1)}}.\tag{3.40}$$

Let $l+1=m$ in (3.29). By (2.2), (3.28)–(3.29), and (3.37)–(3.40), using standard Hölder inequality, it derive

$$\begin{aligned}\|\theta\|_{\mathbf{L}^{3m}(\Omega)}&\leq C(1+\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{1}{m}}(\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{2\gamma(p_1-1)}{mp_1(2\gamma-1)}}+\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{2\gamma(p_1-1)}{mp_1(2\gamma-1)}}\\&+\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{\gamma}{3m(2\gamma-1)}})+\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{\gamma}{m(\gamma-1)}}+\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}^{\frac{5\gamma}{3m(2\gamma-1)}}).\end{aligned}\tag{3.41}$$

Inserting (3.41) into (3.36), by Hölder inequality and direct computation, we conclude that

$$\begin{aligned}&\|K(\rho)\rho\|_{\mathbf{L}^{2\gamma}(\Omega)}+\|\int_0^\rho K(t)\mathrm{d}t\|_{\mathbf{L}^{2\gamma}(\Omega)}\\&\qquad\leq C\left(1+\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}}+\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{(3m-2)(2\gamma-1))}{m(4m\gamma-3m-4\gamma+6)(\gamma-1)}}\right),\end{aligned}\tag{3.42}$$

which implies that

$$\|\theta\|_{\mathbf{L}^{3m}(\Omega)}\leq C(1+\|\mathbf{u}\|_{H^1(\Omega)}^{\frac{\gamma}{m(\gamma-1)}}),\tag{3.43}$$

where $m>0$ for $\gamma\in(\frac{4}{3},\gamma^*)$, $m_0^-\leq m\leq m_0^+$ for $\gamma>\gamma^*$ and

$$\begin{aligned}m_0^-&=\frac{3(7\gamma^2-14\gamma+6)-\sqrt{\Delta}}{6(4\gamma-1)(\gamma-1)}<\frac{4\gamma-6}{4\gamma-3}<1,\\m_0^+&=\frac{3(7\gamma^2-14\gamma+6)+\sqrt{\Delta}}{6(4\gamma-1)(\gamma-1)}>1,\\\Delta&=153\gamma^4-876\gamma^3+1632\gamma^2-1224\gamma+324,\\&2<\gamma^*<3\ \text{ is the zero solution of }\ \Delta=0.\end{aligned}\tag{3.44}$$

Here, we use the restriction

$$\frac{(3m-2)(2\gamma-1)}{m(4m\gamma-3m-4\gamma+6)(\gamma-1)}<\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4},\quad\text{for }\gamma>\frac{4}{3}.$$

Furthermore, by (3.26), (3.35), (3.40), and (3.43), we derive

$$\int_\Omega |\theta p_c(\rho)|^2 \mathrm{d}x \le \|\theta\|^2_{\mathbf{L}^{3m}(\Omega)} \|\int_0^\rho K(t)\mathrm{d}t\|^{\frac{4\gamma^2(m-2)}{(3m-2)(2\gamma-1)}}_{\mathbf{L}^{2\gamma}(\Omega)}$$
$$\le C\left(1+\|\mathbf{u}\|^{\frac{2\gamma}{m(\gamma-1)}+\frac{12\gamma(m-2)}{(3\gamma-4)(3m-2)}}_{\mathbf{L}^{2\gamma}(\Omega)}\right), \tag{3.45}$$

$$\int_\Omega |K(\rho)\rho\mathbf{u}\cdot\mathbf{F}|\mathrm{d}x \le C\left(1+\|\mathbf{u}\|^{\frac{3\gamma-3}{3\gamma-4}}_{\mathbf{L}^{2\gamma}(\Omega)}\right). \tag{3.46}$$

Combining (3.45)–(3.46) with (3.17), we obtain

$$\|\mathbf{u}\|^2_{H^1(\Omega)} + \nu\|\mathbf{H}\|^2_{H^1(\Omega)} \le C(\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}, M),$$

where $C(\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}, M)$ is positive constant depending on $\|\mathbf{F}\|_{\mathbf{L}^\infty(\Omega)}$ and M. Here we need

$$\frac{2\gamma}{m(\gamma-1)} + \frac{12\gamma(m-2)}{(3\gamma-4)(3m-2)} < 2. \tag{3.47}$$

In fact for $\gamma \ge 4$, there holds $\frac{2\gamma}{3\gamma-4} < 1$. So for any $m > 0$, it is easy to check

$$\frac{2\gamma}{m(\gamma-1)} + \frac{12\gamma(m-2)}{(3\gamma-4)(3m-2)} < \frac{2\gamma}{m(\gamma-1)} + \frac{6(m-2)}{3m-2} < 2.$$

Next, we prove that (3.47) holds for $\frac{4}{3} < \gamma < 4$. Note that

$$\frac{4}{3m} < \frac{2\gamma}{m(\gamma-1)} < \frac{8}{m}, \quad \text{for } m > 0,$$
$$\frac{12\gamma(m-2)}{(3\gamma-4)(3m-2)} < \frac{48(m-2)}{(3\gamma-4)(3m-2)} < (1-\frac{4}{3m-2})\frac{16}{3\gamma-4}, \quad \text{for } m > 0.$$

Thus, to make (3.47) hold, we need

$$(1-\frac{4}{3m-2})\frac{16}{3\gamma-4} < 1,$$

which gives that

$$0 < m < m^+ = \frac{(4-\gamma)(3\gamma-2)+\sqrt{\Delta}}{6(4-\gamma)(\gamma-1)} > \frac{2\gamma}{3(\gamma-1)} > 1, \tag{3.48}$$

with

$$\Delta = -63\gamma^4 + 372\gamma^3 - 524\gamma^2 + 160\gamma + 64.$$

Note that $m = 1 + l$, $l \in \mathbf{R}^+_0$. Finally, we conclude that the result holds for $1 < m < m^+$. This completes the proof. □

To solve the approximation system (3.2)–(3.4), we need to use the Leary-Schauder fixed point theorem. Define the operator

$$\mathcal{G}: \mathbf{X}^p \times \mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{1,2}(\Omega) \longrightarrow \mathbf{X}^p \times \mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{1,2}(\Omega)$$

such that

$$\mathcal{G}(\mathbf{u}, s, \mathbf{H}) = (\mathbf{v}, z, \mathbf{m}).$$

Here, $(\mathbf{v}, z, \mathbf{m})$ is the solution to the system

$$-\mathbf{div}\Psi(\mathbf{v}) = -\frac{1}{2}\mathbf{div}(K(\rho)\rho\mathbf{u}\otimes\mathbf{u}) - \nabla P(\rho, e^s) + (\nabla\times\mathbf{H})\times\mathbf{H} + K(\rho)\rho\mathbf{F}, \quad (3.49)$$

$$\begin{aligned}-\mathbf{div}((1+e^{ms})(\epsilon+e^s)\nabla z) &= -\mathbf{div}(K(\rho)\rho Q(e^s)\mathbf{u}) - e^s\mathbf{div}(\mathbf{u}p_\theta(\int_0^\rho K(t)\mathrm{d}t)) \\ &\quad + e^s K(\rho)\mathbf{u}p'_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\nabla\rho + \nu|\nabla\times\mathbf{H}|^2 + \Psi(\mathbf{u}):\nabla\mathbf{u},\end{aligned} \quad (3.50)$$

$$\nabla\times(\nu\nabla\times\mathbf{m}) = \nabla\times(\mathbf{u}\times\mathbf{H}), \quad \mathbf{div}\mathbf{m} = 0, \quad (3.51)$$

with the boundary condition

$$\begin{aligned}&(1+e^{ms})(\epsilon+e^s)\nabla z + \epsilon z + L(e^s)(e^s-\theta_0) = 0, \quad x\in\partial\Omega \\ &\mathbf{n}\cdot\Psi(\mathbf{v})\cdot\tau_k + f\mathbf{v}\cdot\tau_k = 0, \quad k=1,2, \\ &\mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{m}|_{\partial\Omega} = 0.\end{aligned} \quad (3.52)$$

By the identities

$$\nabla\times(\nabla\times\mathbf{m}) = \nabla\mathbf{div}\mathbf{m} - \triangle\mathbf{m},$$

and

$$\nabla\times(\mathbf{u}\times\mathbf{H}) = (\mathbf{div}\mathbf{H} + \mathbf{H}\cdot\nabla)\mathbf{u} - (\mathbf{div}\mathbf{u} + \mathbf{u}\cdot\nabla)\mathbf{H}.$$

together with the constraint $\mathbf{div}\mathbf{m} = 0$ and $\mathbf{div}\mathbf{H} = 0$, the equation (3.51) can be expressed as

$$\nu\triangle\mathbf{m} = (\mathbf{div}\mathbf{u})\mathbf{H} + (\mathbf{u}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u}. \quad (3.53)$$

We notice that the system (3.49)–(3.50), and (3.53) is strictly elliptic for $\epsilon > 0$. $\mathbf{W}^{1,p}(\Omega)$-space is algebra for $p > 3$, and the boundary term belongs to $\mathbf{W}^{1-\frac{1}{p},p}(\partial\Omega)$, so the right-hand side of the system (3.49)–(3.50), and (3.53) belongs to $\mathbf{L}^p(\Omega)$. Meanwhile, the coefficient in the operator in the left-hand side of (3.50) are of the $\mathbf{C}^{1+\alpha}(\bar{\Omega})$-class. Hence, by the standard elliptic theory, the existence of the solution of the system (3.49)–(3.50), and (3.53) in space $\mathbf{X}^p\times\mathbf{W}^{2,p}(\Omega)\times\mathbf{W}^{1,2}(\Omega)$ can be obtained. $\|\mathbf{v}\|_{\mathbf{W}^{2,p}(\Omega)} + \|z\|_{\mathbf{W}^{2,p}(\Omega)} + \mu\|\mathbf{m}\|_{\mathbf{W}^{1,2}(\Omega)}$ can be controlled by the right-hand side of the system (3.49)–(3.50) and (3.53) under suitable norm. Combining with the right-hand side of the system (3.49)–(3.50), and (3.53) being at most of the first-order derivative of sought functions, the continuous and compactness of the operator $\mathcal{G}$ is obtained.

Hence, we conclude the following result on the continuous and compact of the operator $\mathcal{G}$.

Lemma 3 *Under the assumption of Theorem 1, let $p > 3$. Then, $\mathcal{G}$ is a continuous and compact operator from $\mathbf{X}^p\times\mathbf{W}^{2,p}(\Omega)\times\mathbf{W}^{1,2}(\Omega)$ to $\mathbf{X}^p\times\mathbf{W}^{2,p}(\Omega)\times\mathbf{W}^{1,2}(\Omega)$.*

To apply the Leray–Schauder fixed point theorem, we also need to verify that all the solution satisfying for any $t\in[0,1]$

$$t\mathcal{G}(\mathbf{v}, z, \mathbf{m}) = (\mathbf{v}, z, \mathbf{m}) \quad (3.54)$$

are bounded in $\mathbf{X}^p \times \mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{1,2}(\Omega)$. Note that $\rho = \mathcal{F}(\mathbf{u})$ given by Lemma 1. (3.54) leads us to consider

$$-\,\mathbf{div}\Psi(\mathbf{v}) = -\frac{t}{2}\mathbf{div}(K(\rho)\rho\mathbf{v}\otimes\mathbf{v}) - t\nabla P(\rho, e^z) + t(\nabla\times\mathbf{m})\times\mathbf{m} + tK(\rho)\rho\mathbf{F}, \quad (3.55)$$

$$\begin{aligned}-\,\mathbf{div}((1+e^{mz})(\epsilon+e^z)\nabla z) &= -t\mathbf{div}(K(\rho)\rho Q(e^z)\mathbf{v}) - te^z\mathbf{div}\left(\mathbf{v}p_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\right)\\ &+ te^zK(\rho)\mathbf{v}p'_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\nabla\rho + \nu t|\nabla\times\mathbf{m}|^2 + t\Psi(\mathbf{v}):\nabla\mathbf{v}, \quad (3.56)\end{aligned}$$

$$\nu\triangle\mathbf{m} = t(\mathbf{div}\mathbf{v})\mathbf{m} + t(\mathbf{v}\cdot\nabla)\mathbf{m} - t(\mathbf{m}\cdot\nabla)\mathbf{v}, \quad (3.57)$$

with the boundary condition

$$\begin{aligned}&(1+e^{mz})(\epsilon+e^z)\nabla z + \epsilon z + tL(e^z)(e^z-\theta_0) = 0, \quad x\in\partial\Omega\\ &\mathbf{n}\cdot\Psi(\mathbf{v})\cdot\tau_k + f\mathbf{v}\cdot\tau_k = 0, \quad k=1,2,\\ &\mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{m}|_{\partial\Omega} = 0.\end{aligned} \quad (3.58)$$

The same process as in Lemma 1, the following priori estimates of (3.55)–(3.58) can be obtained.

Lemma 4 *Assume that* $(\mathbf{v}, z, \mathbf{m})$ *be the solution of* (3.55)–(3.57) *with the boundary condition* (3.58). *Then,*

$$\|\mathbf{v}\|_{H^1(\Omega)} + \sqrt{\epsilon}\|\nabla\rho\|_{\mathbf{L}^2(\Omega)} + \|\theta\|_{\mathbf{L}^{3m}(\Omega)} + \|\nabla\theta\|_{\mathbf{L}^{\frac{3m}{m+1}}(\Omega)} + \|\mathbf{m}\|_{H^1(\Omega)} \le C(k),$$

where $\theta = e^z$ *and* $C(k)$ *is independent of* ϵ *and* t.

In the following, we state our main result in this section.

Theorem 2 *Under the assumption of Theorem 1, let* $\epsilon > 0$ *and* $k > 0$. *Then, the approximation system (3.1)–(3.4) has a strong solution* $(\rho, \mathbf{u}, \theta, \mathbf{H})$ *such that*

$$\rho\in\mathbf{W}^{2,p}(\Omega), \quad \mathbf{u}\in\mathbf{W}^{2,p}(\Omega), \quad s\in\mathbf{W}^{2,p}(\Omega), \quad \mathbf{H}\in\mathbf{W}^{1,2}(\Omega), \quad for\ 1\le p<\infty.$$

Moreover, there holds

$$0\le\rho\le k, \quad \int_\Omega \rho dx \le M, \quad for\ x\in\Omega,$$

and

$$\|\mathbf{u}\|_{\mathbf{W}^{1,3m}(\Omega)} + \sqrt{\epsilon}\|\nabla\rho\|_{\mathbf{L}^2(\Omega)} + \|\nabla\theta\|_{\mathbf{L}^{\frac{3m}{m+1}}(\Omega)} + \|\mathbf{H}\|_{H^1(\Omega)} \le C(k),$$

where $\theta = e^s$ *and* $C(k)$ *is independent of* ϵ.

Proof Define

$$\Pi(z) = \int_0^z (1+e^{m\tau})(\epsilon+e^\tau)d\tau. \quad (3.59)$$

Then, we can rewrite the approximation momentum equation(3.56) as

$$
\begin{aligned}
-\triangle\Pi(z) &= -t\,\mathbf{div}(K(\rho)\rho Q(e^z)\mathbf{v}) - te^z\mathbf{div}(\mathbf{v}p_\theta(\int_0^\rho K(t)\mathrm{d}t)) \\
&+te^zK(\rho)\mathbf{v}p'_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\nabla\rho + \nu t|\nabla\times\mathbf{m}|^2 + t\Psi(\mathbf{v}):\nabla\mathbf{v}, \quad x\in\Omega, \qquad (3.60)
\end{aligned}
$$

with the boundary condition

$$
\frac{\partial\Pi(z)}{\partial\mathbf{n}} + \epsilon z + tL(e^z)(e^z-\theta_0) = 0, \quad x\in\partial\Omega.
$$

From Lemma 3.4, we can conclude that

$$
\begin{aligned}
&K(\rho)\rho\mathbf{v} \text{ is bounded in } \mathbf{L}^3(\Omega),\\
&\mathbf{v} \text{ is bounded in } \mathbf{W}^{1,3m}(\Omega),\\
&\mathbf{m} \text{ is bounded in } \mathbf{W}^{1,2}(\Omega),\\
&\rho \text{ is bounded in } \mathbf{W}^{2,3m}(\Omega),
\end{aligned}
$$

for some constant C, which is independent of ϵ.

Multiplying (3.60) by Π and integrating over Ω, we have

$$
\begin{aligned}
\|\nabla\Pi(z)\|_{\mathbf{L}^2(\Omega)} &+ \int_{\partial\Omega}(\epsilon z\Pi + tL(e^z)(e^z-\theta_0))\mathrm{d}\sigma \\
&\le C\|\Pi(z)\|_{\mathbf{L}^6(\Omega)}\|F(\mathbf{v},\theta,\mathbf{m})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)}, \qquad (3.61)
\end{aligned}
$$

where

$$
\begin{aligned}
F(\mathbf{v},\theta,\mathbf{m}) &= -\mathbf{div}(K(\rho)\rho Q(e^z)\mathbf{v}) - e^z\mathbf{div}\left(\mathbf{v}p_\theta\left(\int_0^\rho K(t)\mathrm{d}t\right)\right) \\
&+e^zK(\rho)\mathbf{v}p'_\theta(\int_0^\rho K(t)\mathrm{d}t)\nabla\rho + \nu|\nabla\times\mathbf{m}|^2 + \Psi(\mathbf{v}):\nabla\mathbf{v}.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
\Pi(z) &\sim \epsilon z && \text{for } z\longrightarrow -\infty,\\
\Pi(z) &\sim \epsilon e^{(m+1)z} && \text{for } z\longrightarrow +\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{\partial\Omega}(\epsilon z\Pi + tL(e^z)(e^z-\theta_0))I_{\{\Pi\le 0\}}\mathrm{d}\sigma \ge C_1\epsilon^2\|\Pi I_{\{\Pi\le 0\}}\|^2_{\mathbf{L}^2(\partial\Omega)} - C_2,\\
&\int_{\partial\Omega}(\epsilon z\Pi + tL(e^z)(e^z-\theta_0))I_{\{\Pi\ge 0\}}\mathrm{d}\sigma \ge C_1\epsilon\|\Pi I_{\{\Pi\ge 0\}}\|^2_{\mathbf{L}^1(\partial\Omega)} - C_2.
\end{aligned}
$$

This combines with Lemma 3.4 and (3.61), we derive

$$
\begin{aligned}
&\|\Pi\|_{\mathbf{W}^{1,2}(\Omega)} \le C, \quad \|\theta^{m+1}\|_{\mathbf{L}^6(\Omega)} = \|e^{(m+1)z}\|_{\mathbf{L}^6(\Omega)} \le C,\\
&\|\nabla\theta\|_{\mathbf{L}^2(\Omega)} = \|e^z\nabla z\|_{\mathbf{L}^2(\Omega)} \le C,
\end{aligned}
$$

where C independent of t.

Furthermore, it can derive from (3.60) that for $1\le q\le q^* = \frac{p^*}{3-p^*} > 3$,

$$
\begin{aligned}
&\|\Pi\|_{\mathbf{W}^{2,p^*}(\Omega)} \le C, \quad \text{for } p^* = \min\left\{\frac{3m}{2}, 2\right\},\\
&\|z\|_{\mathbf{L}^\infty(\Omega)} + \|\theta\|_{\mathbf{L}^\infty(\Omega)} \le C, \quad \|\nabla z\|_{\mathbf{L}^q(\Omega)} + \|\nabla\theta\|_{\mathbf{L}^q(\Omega)} \le C.
\end{aligned}
$$

Then, from (3.56) and (3.59), and the imbedding theorem, it has

$$\begin{aligned}&\|\mathbf{v}\|_{\mathbf{W}^{2,q^*}(\Omega)} \le C, \quad \|z\|_{\mathbf{W}^{2,q^*}(\Omega)} + \|\theta\|_{\mathbf{W}^{2,q^*}(\Omega)} \le C,\\ &\|\nabla z\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla\theta\|_{\mathbf{L}^\infty(\Omega)} \le C.\end{aligned}$$

Thus, we conclude for $1 \le p < \infty$,

$$\|\rho\|_{\mathbf{W}^{2,p}(\Omega)} + \|\mathbf{v}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\mathbf{H}\|_{\mathbf{W}^{1,2}(\Omega)} + \|z\|_{\mathbf{W}^{2,p}(\Omega)} + \|\theta\|_{\mathbf{W}^{2,p}(\Omega)} \le C,$$

where C independent of t. This completes the proof. □

4 Effective viscous flux and limit passage

To define the effective viscous flux, we introduce the Helmholtz decomposition

$$\begin{aligned}\mathbf{u} = \mathcal{H}[\mathbf{u}] + \mathcal{H}^\perp[\mathbf{u}],\\ \mathcal{H}^\perp[\mathbf{u}] = \nabla\phi, \quad \mathcal{H}[\mathbf{u}] = \mathbf{curl}\varphi,\end{aligned}$$

where ϕ is given by the solution to the Neumann problem

$$\begin{cases}\triangle\phi = \mathbf{div}\mathbf{u} \quad x \in \Omega,\\ \frac{\partial\phi}{\partial\mathbf{n}} = 0 \quad x \in \partial\Omega, \quad \int_\Omega \phi \mathrm{d}x = 0,\end{cases} \tag{4.1}$$

and φ satisfying the following elliptic problem

$$\begin{cases}\mathbf{curl}\mathcal{H}^\perp[\mathbf{u}] = \nabla\mathbf{u} = \omega, & x \in \Omega,\\ \mathbf{div}\mathcal{H}^\perp[\mathbf{u}] = 0, & x \in \Omega,\\ \mathcal{H}^\perp[\mathbf{u}]\cdot\mathbf{n} = 0 & x \in \partial\Omega.\end{cases} \tag{4.2}$$

By the classical theory for elliptic equations [25,29], we can get for $1 < p < \infty$

$$\begin{aligned}&\|\nabla\mathcal{H}[\mathbf{u}]\|_{\mathbf{L}^p(\Omega)} \le C\|\omega\|_{\mathbf{L}^p(\Omega)}, \quad \|\triangle\mathcal{H}[\mathbf{u}]\|_{\mathbf{L}^p(\Omega)} \le C\|\omega\|_{\mathbf{W}^{1,p}(\Omega)},\\ &\|\nabla\mathcal{H}^\perp[\mathbf{u}]\|_{\mathbf{L}^p(\Omega)} \le C\|\mathbf{div}\mathbf{u}\|_{\mathbf{L}^p(\Omega)}, \quad \|\triangle\mathcal{H}^\perp[\mathbf{u}]\|_{\mathbf{L}^p(\Omega)} \le C\|\mathbf{div}\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}.\end{aligned}$$

From Theorem 2, we can conclude that for $\epsilon \longrightarrow 0^+$

$$\begin{aligned}&\rho_\epsilon \longrightarrow \rho \quad \text{weak} - * \text{ in } \mathbf{L}^\infty(\Omega),\\ &p_b(\rho_\epsilon) \longrightarrow \overline{p_b(\rho_\epsilon)} \quad \text{weak} - * \text{ in } \mathbf{L}^\infty(\Omega),\\ &\mathbf{u}_\epsilon \longrightarrow \mathbf{u} \quad \text{weak in } \mathbf{W}^{1,3m}(\Omega), \qquad \mathbf{u}_\epsilon \longrightarrow \mathbf{u} \quad \text{in } \mathbf{L}^\infty(\Omega),\\ &K(\rho_\epsilon) \longrightarrow \overline{K(\rho)} \quad \text{weak} - * \text{ in } \mathbf{L}^\infty(\Omega),\\ &K(\rho_\epsilon)\rho_\epsilon \longrightarrow \overline{K(\rho)\rho} \quad \text{weak} - * \text{ in } \mathbf{L}^\infty(\Omega),\\ &\int_0^{\rho_\epsilon} K(t)\mathrm{d}t \longrightarrow \overline{\int_0^{\rho} K(t)\mathrm{d}t} \quad \text{weak} - * \text{ in } \mathbf{L}^\infty(\Omega),\\ &\theta_\epsilon \longrightarrow \theta \quad \text{weak in } \mathbf{W}^{1,\frac{3m}{m+1}}(\Omega), \qquad \theta_\epsilon \longrightarrow \theta \quad \text{in } \mathbf{L}^p(\Omega), \quad \text{for } p < 3m,\\ &\mathbf{H}_\epsilon \longrightarrow \mathbf{H} \quad \text{weak in } \mathbf{W}^{1,2}(\Omega),\end{aligned}$$

where we use the notation that a weak limit of a sequence $\mathbf{u}_\epsilon$ is denoted by $\overline{\mathbf{u}}$, as $\epsilon \longrightarrow 0^+$.

Taking the limit in the weak formulation of the approximation system (3.1)–(3.4), we have

$$\mathbf{div}(\overline{K(\rho)\rho}\mathbf{u}) = 0, \tag{4.3}$$

$$\mathbf{div}\left(\Psi(\mathbf{u}) - \overline{p_b(\rho)}I - \theta\overline{p_c(\rho)}I\right) + (\nabla\times\mathbf{H})\times\mathbf{H} + \overline{K(\rho)\rho}\mathbf{F} = 0, \tag{4.4}$$

$$-\mathbf{div}((1+\theta^m)\nabla\theta) + \mathbf{div}(\overline{K(\rho)\rho}Q(\theta)\mathbf{u}) + \theta\overline{p_\theta(\int_0^\rho K(t)\mathrm{d}t)\mathbf{divu}}$$
$$= \nu|\nabla\times\mathbf{H}|^2 + 2\mu\overline{|\mathbf{D}(\mathbf{u})|^2} + \lambda\overline{(\mathbf{divu})^2}, \tag{4.5}$$
$$\nabla\times(\mathbf{u}\times\mathbf{H}) = \nabla\times(\nu\nabla\times\mathbf{H}), \tag{4.6}$$

In the following, we give some priori estimates, which is dependent of k.

Lemma 5 *Under the assumptions of Theorem 1 and 2, it has*

$$\|\rho_\epsilon\|_{\mathbf{L}^\infty(\Omega)} \le k,$$
$$\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,3m}(\Omega)} \le C(1+k^{\frac{\gamma(3m-2)}{3m}}), \tag{4.7}$$
$$\|\mathbf{H}_\epsilon\|_{\mathbf{W}^{1,r}(\Omega)} \le C(1+k^{\frac{2\gamma(3m-2)}{3m}}) \ \ for\ r > 3m. \tag{4.8}$$

Proof The first estimate on ρ_ϵ follows directly from Theorem 2. We only estimate the second one. Rewrite (3.2) as

$$-\mathbf{div}\Psi(\mathbf{u}_\epsilon) = -\frac{1}{2}\mathbf{div}(K(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon) - \nabla P_1(\rho_\epsilon,\theta_\epsilon) + (\nabla\times\mathbf{H}_\epsilon)\times\mathbf{H}_\epsilon$$
$$+K(\rho_\epsilon)\rho_\epsilon\mathbf{F}. \tag{4.9}$$

Then, we have

$$\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,3m}(\Omega)} \le C(\|K(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon\|_{\mathbf{L}^{3m}(\Omega)} + \|P_1(\rho_\epsilon,\theta_\epsilon)\|_{\mathbf{L}^{3m}(\Omega)} + \|(\mathbf{H}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon\|^2_{\mathbf{L}^{3m}(\Omega)}$$
$$+\|\mathbf{H}_\epsilon\|^2_{\mathbf{L}^{3m}(\Omega)} + \|K(\rho_\epsilon)\rho_\epsilon\mathbf{F}\|_{\mathbf{L}^{\frac{3m}{m+1}}(\Omega)}). \tag{4.10}$$

By (2.1), we deduce

$$\|P_1(\rho_\epsilon,\theta_\epsilon)\|_{\mathbf{L}^{3m}(\Omega)} \le a_1\|p_e(\int_0^{\rho_\epsilon}K(t)\mathrm{d}t)\|_{\mathbf{L}^{3m}(\Omega)} + a_3\|\theta p_\theta(\int_0^{\rho_\epsilon}K(t)\mathrm{d}t)\|_{\mathbf{L}^{3m}(\Omega)}$$
$$\le a_1\|p_e(\int_0^{\rho_\epsilon}K(t)\mathrm{d}t)\|^{\frac{2}{3m}}_{\mathbf{L}^2(\Omega)}\|p_e(\int_0^{\rho_\epsilon}K(t)\mathrm{d}t)\|^{\frac{3m-2}{3m}}_{\mathbf{L}^\infty(\Omega)}$$
$$+a_3\|\theta p_\theta(\int_0^{\rho_\epsilon}K(t)\mathrm{d}t)\|^{\frac{2}{3m}}_{\mathbf{L}^2(\Omega)}\|\theta p_\theta(\int_0^{\rho_\epsilon}K(t)\mathrm{d}t)\|^{\frac{3m-2}{3m}}_{\mathbf{L}^\infty(\Omega)}$$
$$\le Ck^{\frac{\gamma(3m-2)}{9m}}(1+k^{\frac{2\gamma(3m-2)}{9m}}).$$

By the interpolation inequality, the Sobolev imbedding theorem, and Young inequality, we derive

$$\|K(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon\|_{\mathbf{L}^{3m}(\Omega)} \le \|\rho_\epsilon\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{u}_\epsilon\|^2_{\mathbf{L}^{6m}(\Omega)} \le \|\rho_\epsilon\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{u}_\epsilon\|^{2\alpha}_{\mathbf{L}^p(\Omega)}\|\mathbf{u}_\epsilon\|^{2(1-\alpha)}_{\mathbf{L}^q(\Omega)}$$
$$\le \|\rho_\epsilon\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{u}_\epsilon\|^{2\alpha}_{\mathbf{W}^{1,2}(\Omega)}\|\mathbf{u}_\epsilon\|^{2(1-\alpha)}_{\mathbf{W}^{1,3m}(\Omega)}$$
$$\le C(1+\frac{1}{2}\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,3m}(\Omega)}),$$

where $2 < 6m\alpha < p < 6m < q$, $\frac{1}{2} < \alpha < 1$ and

$$q = \frac{6mp(1-\alpha)}{p-6m\alpha} > 6m.$$

Hence, using the Sobolev imbedding theorem and (4.10), we have

$$\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,3m}(\Omega)} \le C\left(1+\frac{1}{2}\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,3m}(\Omega)} + k^{\frac{\gamma(3m-2)}{9m}}(1+k^{\frac{2\gamma(3m-2)}{9m}}) + \|\mathbf{H}_\epsilon\|^2_{\mathbf{W}^{1,2}(\Omega)}\right).$$

Note that $\frac{\gamma(3m-2)}{3m} > 1$ for $\gamma > \frac{4}{3}$ and $m > \frac{2\gamma}{3(\gamma-1)}$. Thus, we get

$$\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,3m}(\Omega)} \le C\left(1 + k^{\frac{\gamma(3m-2)}{3m}}\right).$$

Next, by (3.4), the interpolation inequality and Young inequality , we have

$$\begin{aligned}
\nu\|\nabla\times\mathbf{H}\|_{\mathbf{L}^r(\Omega)} = \nu\|\mathbf{H}\|_{\mathbf{W}^{1,r}(\Omega)} &= \|\mathbf{u}\times\mathbf{H}\|_{\mathbf{L}^r(\Omega)} \\
&\le \|\mathbf{u}\|^{\alpha}_{\mathbf{L}^{3m}(\Omega)}\|\mathbf{H}\|^{1-\alpha}_{\mathbf{L}^{\frac{3mr(1-\alpha)}{3m-r\alpha}}(\Omega)} \\
&\le \frac{2}{\nu}\|\mathbf{u}\|^{2\alpha}_{\mathbf{L}^{3m}(\Omega)} + \frac{\nu}{2}\|\mathbf{H}\|^{2(1-\alpha)}_{\mathbf{L}^{\frac{3mr(1-\alpha)}{3m-r\alpha}}(\Omega)},
\end{aligned}$$

where $\frac{1}{2} < \alpha < 1$ and $1 \le 3m < r < \frac{3mr(1-\alpha)}{3m-r\alpha}$. Then, using the Sobolev imbedding theorem

$$\mathbf{W}^{m,p}(\Omega) \hookrightarrow \mathbf{L}^q(\Omega) \ \ for \ \ p \le q < \infty, \ \ m \ge 1.$$

Thus, we have

$$\|\mathbf{H}\|_{\mathbf{W}^{1,r}(\Omega)} \le C(\nu)\|\mathbf{u}_\epsilon\|^2_{\mathbf{W}^{1,3m}(\Omega)} \le C(\nu)\left(1 + k^{\frac{2\gamma(3m-2)}{3m}}\right).$$

This completes the proof. □

We also need to prove that the limit temperature is positive. This proof is similar with [18]. For reader's convenience, we will show it. From Lemma 2, we have

$$\int_{\partial\Omega}(e^{s}_{\epsilon} + e^{-s_\epsilon})\mathrm{d}\sigma + \int_{\Omega}\nabla s_\epsilon \mathrm{d}x \le C,$$

which gives that

$$\int_{\partial\Omega} s_\epsilon^2 \mathrm{d}\sigma + \int_{\Omega}\nabla s_\epsilon \mathrm{d}x \le C.$$

Note that Ω is bounded. Hence, above inequality means we can choose a subsequence $s_\epsilon \longrightarrow s$ in $\mathbf{L}^2(\Omega)$. Recall that $\theta_\epsilon = e^{s_\epsilon}$ and $\theta_\epsilon \longrightarrow \theta$ strongly in $\mathbf{L}^p$, $p < 3m$. Using Vitali's theorem, we have

$$e^{s_\epsilon} \longrightarrow e^{s} \ \ in \ \ \mathbf{L}^p(\Omega), \ \ p < 3m,$$

with

$$\theta = e^{s}, \ \ s \in \mathbf{L}^2(\Omega).$$

Therefore, we conclude the following result:

Lemma 6 *There exists a subsequence* $\{s_\epsilon\}$ *such that*

$$s_\epsilon \longrightarrow s \ \ in \ \ \mathbf{L}^2(\Omega).$$

Moreover, it holds

$$\theta_\epsilon \longrightarrow \theta \ \ in \ \ \mathbf{L}^p(\Omega), \ \ p < 3m,$$

with $\theta > 0$ *a.e. in* Ω.

Consider the following problem which is set by the properties of the slip boundary condition:

$$
\begin{aligned}
-\mu\triangle\omega_\epsilon = {} & \mathbf{curl}\left((\nabla\times\mathbf{H})\times\mathbf{H}+K(\rho_\epsilon)\rho\mathbf{F}-\frac{\epsilon}{2}hK(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon+\frac{\epsilon}{2}\rho_\epsilon\mathbf{u}_\epsilon\right)\\
& -\mathbf{curl}\left(\frac{\epsilon}{2}\triangle\rho_\epsilon\mathbf{u}_\epsilon\right), \quad x\in\Omega,
\end{aligned}
$$

with the boundary condition

$$
\begin{aligned}
\omega_\epsilon\cdot\tau_1 &= -(2\eta_2-f/\mu)\mathbf{u}_\epsilon\cdot\tau_2 \;\; x\in\partial\Omega,\\
\omega_\epsilon\cdot\tau_2 &= -(2\eta_1-f/\mu)\mathbf{u}_\epsilon\cdot\tau_1 \;\; x\in\partial\Omega.
\end{aligned}
$$

It deduce from the structure of ω_ϵ that

$$
\omega_\epsilon = A_1+A_2+A_3, \tag{4.11}
$$

which satisfying

$$
\begin{aligned}
-\mu\triangle A_1 &= 0, \quad x\in\Omega,\\
A_1\cdot\tau_1 &= -(2\eta_2-f/\mu)\mathbf{u}_\epsilon\cdot\tau_2 \;\; x\in\partial\Omega,\\
A_1\cdot\tau_2 &= -(2\eta_1-f/\mu)\mathbf{u}_\epsilon\cdot\tau_1 \;\; x\in\partial\Omega,\\
\mathbf{div}A_1 &= 0 \;\; x\in\partial\Omega,\\
-\mu\triangle A_2 &= \mathbf{curl}((\nabla\times\mathbf{H})\times\mathbf{H}+K(\rho_\epsilon)\rho\mathbf{F}\\
&\quad -\frac{\epsilon}{2}hK(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon+\frac{\epsilon}{2}\rho_\epsilon\mathbf{u}_\epsilon), \quad x\in\Omega,\\
A_2\cdot\tau_1 &= 0 \;\; x\in\partial\Omega,\\
A_2\cdot\tau_2 &= 0 \;\; x\in\partial\Omega,\\
\mathbf{div}A_2 &= 0 \;\; x\in\partial\Omega,
\end{aligned} \tag{4.12}
$$

and

$$
\begin{aligned}
&\mu\triangle A_3 = \mathbf{curl}\left(\tfrac{\epsilon}{2}\triangle\rho_\epsilon\mathbf{u}_\epsilon\right), && x\in\Omega,\\
&A_2\cdot\tau_1 = 0 && x\in\partial\Omega,\\
&A_2\cdot\tau_2 = 0 && x\in\partial\Omega,\\
&\mathbf{div}A_3 = 0 && x\in\partial\Omega.
\end{aligned} \tag{4.13}
$$

To solve the first elliptic equation about A_1, we transform it to the form

$$
\begin{aligned}
-\mu\triangle(A_1-\beta) &= \mu\triangle\beta, && x\in\Omega,\\
(A_1-\beta)\cdot\tau_1 &= 0 && x\in\partial\Omega,\\
(A_1-\beta)\cdot\tau_2 &= 0 && x\in\partial\Omega,\\
\mathbf{div}(A_1-\beta) &= 0 && x\in\partial\Omega,
\end{aligned}
$$

where β satisfying the following stokes problem

$$
\begin{aligned}
-\mu\triangle\beta+\nabla p_0 &= 0, && x\in\Omega,\\
\mathbf{div}\beta &= 0 && x\in\Omega,\\
\beta\cdot\tau_1 &= -(2\eta_2-f/\mu)\mathbf{u}_\epsilon\cdot\tau_2 && x\in\partial\Omega,\\
\beta\cdot\tau_2 &= -(2\eta_1-f/\mu)\mathbf{u}_\epsilon\cdot\tau_1 && x\in\partial\Omega,\\
\beta\cdot\mathbf{n} &= 0 && x\in\partial\Omega.
\end{aligned}
$$

Note that $\mathbf{u}_\epsilon \in \mathbf{W}^{1-\frac{1}{3m},3m}(\partial\Omega)$. So we have $\beta \in \mathbf{W}^{1,3m}(\Omega)$ with $\|\beta\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,p}(\Omega)}$, for $1 < p \leq 3m$. Furthermore, as done in [18,25,29], we can get

$$\|A_1\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\mathbf{u}_\epsilon\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(1+k^{\frac{\gamma(3m-2)}{3m}}) \text{ for } 1 < p \leq 3m.$$

Lemma 7 *The vorticity* ω_ϵ *written in* (4.11)*, it holds*

$$\|A_1\|_{\mathbf{W}^{1,p}(\Omega)} + \|A_2\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\left(1+k^{\frac{2\gamma(3m-2)}{3m}}\right) \quad for \ \ 2 \leq p \leq 3m,$$

$$\|A_3\|_{\mathbf{L}^q(\Omega)} \leq C(k)\epsilon^{\frac{1}{2}} \quad for \ \ 1 \leq q \leq 2.$$

Proof It derive from (4.8) in Lemma 4.2 and the interpolation inequality that

$$\begin{aligned}\|(\nabla\times\mathbf{H})\times\mathbf{H}\|_{\mathbf{L}^p(\Omega)} &\leq \|\mathbf{H}\|^{\alpha}_{\mathbf{L}^2(\Omega)}\|\mathbf{H}\|^{1-\alpha}_{\mathbf{W}^{1,q}(\Omega)}\\ &\leq \|\mathbf{H}\|^{\alpha}_{\mathbf{W}^{1,2}(\Omega)}\|\mathbf{H}\|^{1-\alpha}_{\mathbf{W}^{1,q}(\Omega)} \leq C\left(1+k^{\frac{2\gamma(3m-2)}{3m}}\right),\end{aligned}$$

where $1 < p \leq 3m < q$, $0 < \alpha < 1$ and $q = \frac{2p(1-\alpha)}{2-\alpha p}$.

Hence, by above estimate, (4.12) and Lemma 4.2, we have for $1 < p \leq 3m$

$$\begin{aligned}\|A_2\|_{\mathbf{W}^{1,p}(\Omega)} &\leq C\|\mathbf{curl}\left((\nabla\times\mathbf{H})\times\mathbf{H}+K(\rho_\epsilon)\rho\mathbf{F}-\frac{\epsilon}{2}hK(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon+\frac{\epsilon}{2}\rho_\epsilon\mathbf{u}_\epsilon\right)\|_{\mathbf{W}^{-1,p}(\Omega)}\\ &\leq C(\|(\nabla\times\mathbf{H})\times\mathbf{H}\|_{\mathbf{L}^p(\Omega)}+\|K(\rho_\epsilon)\rho\mathbf{F}\|_{\mathbf{L}^p(\Omega)}+\|hK(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon\|_{\mathbf{L}^p(\Omega)}\\ &\quad +\|\rho_\epsilon\mathbf{u}_\epsilon\|_{\mathbf{L}^p(\Omega)})\\ &\leq C\left(1+k^{\frac{2\gamma(3m-2)}{3m}}\right).\end{aligned}$$

Finally, we estimate the term A_3. Note that by Lemma 3.4

$$\sqrt{\epsilon}\|\rho_\epsilon\|_{\mathbf{L}^2\Omega} \leq C(k).$$

Thus, for any test function $\chi \in \mathbf{W}^{1,p}(\Omega)$, we derive

$$\|A_3\|_{\mathbf{L}^p\Omega} \leq C\epsilon\|\triangle\rho_\epsilon\mathbf{u}_\epsilon\|_{\mathbf{W}^{-1,p}(\Omega)} \leq C\epsilon\sup_{\chi}\left|\int_\Omega \triangle\rho_\epsilon\mathbf{u}_\epsilon\chi\,\mathrm{d}x\right|.$$

Combining above two estimates and (4.7) in Lemma 4.1, we obtain

$$\|A_3\|_{\mathbf{L}^p\Omega} \leq C\epsilon\left(\|\nabla\rho_\epsilon\|_{\mathbf{L}^2(\Omega)}\|\mathbf{u}_\epsilon\|_{\mathbf{L}^\infty(\Omega)}+\|\nabla\rho_\epsilon\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}_\epsilon\|_{\mathbf{L}^{3m}(\Omega)}\right) \leq C(k)\epsilon^{\frac{1}{2}}.$$

This completes the proof. □

To introduce the effective viscous flux $\mathbf{G}$, we first use the Helmholtz decomposition to the approximation momentum, and get

$$\nabla\mathbf{G}_\epsilon = \mu\triangle\mathcal{H}[\mathbf{v}]+(\nabla\times\mathbf{H})\times\mathbf{H}+K(\rho_\epsilon)\rho\mathbf{F}-\frac{\epsilon}{2}hK(\rho_\epsilon)\rho_\epsilon\mathbf{u}_\epsilon+\frac{\epsilon}{2}\rho_\epsilon\mathbf{u}_\epsilon-\frac{\epsilon}{2}\triangle\rho_\epsilon\mathbf{u}_\epsilon,$$

where

$$\mathbf{G}_\epsilon = -(2\mu+\lambda)\mathbf{div}\mathbf{u}_\epsilon + P(\rho_\epsilon,\theta_\epsilon).$$

$\mathbf{G}$ as the limit version of $\mathbf{G}_\epsilon$ is defined as

$$\mathbf{G} = -(2\mu+\lambda)\mathbf{div}\mathbf{u} + \overline{P(\rho,\theta)}.$$

The quantities G is in the theory of compressible Navier–Stokes equations known as the effective viscous fluxes. Due to the special structure of the steady full compressible magnetohydrodynamic system, we can derive the bound of G.

Lemma 8 *Let* $\epsilon \longrightarrow 0^+$. *Then we have*

$$\mathbf{G}_\epsilon \longrightarrow \mathbf{G} \;\; strongly \; in \; \mathbf{L}^2(\Omega).$$

Moreover,

$$\|\mathbf{G}\|_{\mathbf{L}^\infty(\Omega)} \leq C\left(1 + k^{\frac{2\gamma(3m-2)}{3m}}\right).$$

Proof Decompose the function $\mathbf{G}_\epsilon$ as

$$\mathbf{G}_\epsilon = G_{1\epsilon} + G_{2\epsilon},$$

with

$$\int_\Omega G_{2\epsilon} \mathrm{d}x = 0 \quad and \quad \nabla G_{2\epsilon} = -\frac{\epsilon}{2}\triangle \rho_\epsilon \mathbf{u}_\epsilon - \mu \mathbf{curl} A_2.$$

Then, using Lemma 4.3, we have for $1 \leq p < 2$

$$\begin{aligned}\|G_{2\epsilon}\|_{\mathbf{L}^p(\Omega)} &\leq C(\epsilon\|\triangle\rho_\epsilon \mathbf{u}_\epsilon\|_{\mathbf{W}^{-1,p}(\Omega)} + \mu\|\mathbf{curl} A_2\|_{\mathbf{W}^{-1,p}(\Omega)})\\ &\leq C(k)\epsilon^{\frac{1}{2}}.\end{aligned}$$

Recalling calculations from Lemma 4.3, we have

$$\|G_{1\epsilon}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\left(1 + k^{\frac{2\gamma(3m-2)}{3m}}\right).$$

Thus, there exist a subsequence (denoted by itself) such that

$$\begin{aligned}&G_{1\epsilon} \longrightarrow G_1 \;\; in \; \mathbf{L}^\infty(\Omega),\\ &G_{2\epsilon} \longrightarrow 0 \;\; in \; \mathbf{L}^2(\Omega).\\ &\mathbf{G}_\epsilon = G_{1\epsilon} + G_{2\epsilon} \longrightarrow G_1 \;\; in \; \mathbf{L}^p(\Omega), \quad 1 \leq p \leq 2.\end{aligned}$$

Furthermore, we obtain

$$\|\mathbf{G}\|_{\mathbf{L}^\infty(\Omega)} \leq C(p)\|\mathbf{G}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(p)\sup_{\epsilon>0}\|\mathbf{G}_\epsilon\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(p)\left(1 + k^{\frac{2\gamma(3m-2)}{3m}}\right).$$

This completes the proof. □

By the small modification of Theorem 3 in [18], we can prove the pointwise convergence of the density.

Lemma 9 *There exist a sufficiently large number* $k_0 > 0$ *such that for* $k > k_0$,

$$\frac{k-3}{k}(k-3)^\gamma - \|\mathbf{G}\|_{\mathbf{L}^\infty(\Omega)} \geq 1$$

and for a subsequence $\epsilon \longrightarrow 0^+$ *it holds*

$$\lim_{\epsilon \longrightarrow 0^+} |\{x \in \Omega : \rho_\epsilon(x) > k-3\}| = 0.$$

Moreover, $\overline{K(\rho)\rho} = \rho$ *a.e. in* Ω.

Now we will adopt the technique in [17,18] to show the pointwise convergence of the density. By Lemma 4.5, we can omit $K(\rho)$ in the limit system (4.3)–(4.6).

Lemma 10 *Let* $0 < p < \infty$. *Then,*

$$\int_\Omega \overline{P(\rho,\theta)\rho} dx \leq \int_\Omega G\rho dx, \qquad \int_\Omega \overline{P(\rho,\theta)}\rho dx = \int_\Omega G\rho dx.$$

Furthermore, there holds

$$\overline{P(\rho,\theta)\rho} dx = \overline{P(\rho,\theta)}\rho dx,$$

and up to a subsequence $\epsilon \longrightarrow 0^+$,

$$\rho_\epsilon \longrightarrow \rho \ \ strongly\ in\ \mathbf{L}^p(\Omega).$$

Proof By (3.8), we can set $\iota = \|\rho\|_{\mathbf{L}^\infty(\Omega)}+1$. Choosing a test function $\ln(\rho+\delta)$ for $\delta \in (0,1)$. Then, we have

$$\int_\Omega \epsilon \triangle\rho_\epsilon(\ln\iota - \ln(\rho_\epsilon+\delta))\mathrm{d}x = \epsilon\int_\Omega \frac{|\nabla\rho_\epsilon|^2}{\rho_n+\delta}\mathrm{d}x \geq 0.$$

It follows from the approximation continuity equation and above equality that

$$\int_\Omega (\mathbf{div}(\rho_\epsilon\mathbf{u}_\epsilon) + \epsilon\rho_\epsilon - \epsilon h)(\ln\iota - \ln(\rho_\epsilon+\delta))\mathrm{d}x \geq 0,$$

which implies that

$$\int_\Omega \left(\frac{\rho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\rho_\epsilon}{\rho_\epsilon+\delta} + \epsilon\rho_\epsilon\ln\frac{\iota}{\rho_\epsilon+\delta}\right)\mathrm{d}x - \epsilon h\int_\Omega \ln\frac{\iota}{\rho_\epsilon+\delta} \geq 0.$$

Let $\delta \longrightarrow 0^+$, it has

$$\int_\Omega \mathbf{u}_\epsilon\cdot\nabla\rho_\epsilon\mathrm{d}x \geq -\epsilon\int_\Omega \rho_\epsilon\ln\frac{\iota}{\rho_\epsilon}\mathrm{d}x,$$

which implies that for sufficient small $\epsilon > 0$,

$$-\int_\Omega \rho_\epsilon\mathbf{divu}_\epsilon\mathrm{d}x \geq o(\epsilon).$$

From the definition $\mathbf{G}_\epsilon$, we derive

$$\int_\Omega P(\rho_\epsilon,\theta_\epsilon)\rho_\epsilon\mathrm{d}x \leq \int_\Omega \rho_\epsilon\mathbf{G}_\epsilon\mathrm{d}x + o(\epsilon).$$

Taking $\epsilon \longrightarrow 0$, we get for $\gamma > 3$

$$\int_\Omega \overline{P(\rho,\theta)\rho}\mathrm{d}x \leq \int_\Omega \mathbf{G}\rho\mathrm{d}x. \tag{4.14}$$

Now, we return to the limit continuity equation $\mathbf{div}(\rho\mathbf{u}) = 0$, i.e.,

$$\int_\Omega \rho\mathbf{u}\cdot\nabla\varphi\mathrm{d}x = 0, \quad \forall\varphi\in\mathbf{C}^\infty(\overline{\Omega}). \tag{4.15}$$

As done in [17], using Friedrich's Lemma, we have

$$\int_\Omega (\rho\mathbf{divu} + \mathbf{u}\cdot\nabla\rho)\mathrm{d}x = 0.$$

Choosing the test function $\varphi = \ln \frac{\delta}{\rho_\epsilon + \delta}$ with $\delta > 0$. Then, by (4.15), it has

$$\int_\Omega \rho \mathbf{u} \cdot \nabla \ln \frac{\delta}{\rho_\epsilon + \delta} \mathrm{d}x = \int_\Omega \rho \frac{\mathbf{u} \cdot \nabla \rho_\epsilon}{\rho_\epsilon + \delta} \mathrm{d}x.$$

Taking $\epsilon \longrightarrow 0$ and $\delta \longrightarrow 0^+$, we have

$$\int_\Omega \overline{P(\rho, \theta)} \rho \mathrm{d}x = \int_\Omega G\rho \mathrm{d}x.$$

It follows from the properties of weak limits that

$$\overline{P(\rho, \theta)\rho} \geq \overline{P(\rho, \theta)}\rho \ \ a.e. \ \text{ in } \Omega.$$

But (4.14) shows that

$$\int_\Omega (\overline{P(\rho, \theta)\rho} - \overline{P(\rho, \theta)}\rho) \leq 0 \mathrm{d}x.$$

So

$$\overline{P(\rho, \theta)\rho} = \overline{P(\rho, \theta)}\rho \ \ a.e. \ \text{ in } \Omega,$$

i.e.,

$$\overline{\rho p_e(\rho)} + \theta \overline{\rho p_\theta(\rho)} = \rho \overline{p_e(\rho)} + \theta \rho \overline{p_\theta(\rho)}.$$

However, $\overline{\rho p_e(\rho)} \geq \rho \overline{p_e(\rho)}$, $\theta \overline{\rho p_\theta(\rho)} \geq \theta \rho \overline{p_\theta(\rho)}$ and the temperature $\theta > 0$. Hence,

$$\overline{\rho p_e(\rho)} = \rho \overline{p_e(\rho)} \ \text{ and } \ \overline{\rho p_\theta(\rho)} = \rho \overline{p_\theta(\rho)} \ \ a.e. \ \text{ in } \Omega.$$

Note that $p_e(t) \geq 0$ for $t \geq 0$. We conclude that for a suitably taken subsequence

$$\lim_{\epsilon \longrightarrow 0} \| p_e(\rho)\rho_\epsilon - p_e(\rho)\rho \|^2_{\mathbf{L}^2(\Omega)} = \int_\Omega p_e^2(\rho)(\overline{\rho} - \rho)^2 \mathrm{d}x \leq \| \overline{\rho p_e(\rho)} - \rho \overline{p_e(\rho)} \|^2_{\mathbf{L}^1(\Omega)} = 0,$$

which implies that

$$p_e(\rho)\rho_\epsilon \longrightarrow p_e(\rho)\rho \ \text{ strongly in } \ \mathbf{L}^2(\Omega).$$

Thus, by the pointwise boundedness of ρ, ρ_ϵ and $p_e(t) \geq 0$ for $t \geq 0$, we can get

$$\rho_\epsilon \longrightarrow \rho \ \ strongly \ in \ \mathbf{L}^p(\Omega) \ \ for \ \ 0 < p < \infty.$$

This completes the proof. □

In what follows, we study the limit of the energy equation and the induction equation. We recall that

$$\begin{aligned}
\rho_\epsilon &\longrightarrow \rho \ \text{ in } \mathbf{L}^p(\Omega) \text{ for } p < \infty, \\
\mathbf{u}_\epsilon &\longrightarrow \mathbf{u} \ \text{ in } \mathbf{W}^{1,p}(\Omega) \text{ for } p < 3m, \\
\mathbf{H}_\epsilon &\longrightarrow \mathbf{H} \ \text{ in } \mathbf{W}^{1,2}(\Omega), \\
\theta_\epsilon &\longrightarrow \theta \ \text{ in } \mathbf{L}^p(\Omega) \text{ for } p < 3m, \\
\theta_\epsilon &\longrightarrow \theta \ \text{ weakly in } \mathbf{W}^{1, \frac{3m}{m+1}}(\Omega).
\end{aligned} \tag{4.16}$$

Since the strong convergence of the density and temperature, and Lemma 10, then

$$p_e(\rho_\epsilon) \longrightarrow p_e(\rho) \ \ strongly \ in \ \mathbf{L}^2(\Omega),$$

$$p_\theta(\rho_\epsilon) \longrightarrow p_\theta(\rho) \;\; strongly \; in \; \mathbf{L}^2(\Omega),$$
$$P(\rho_\epsilon, \theta_\epsilon) \longrightarrow P(\rho, \theta) \;\; strongly \;\; in \; \mathbf{L}^2(\Omega).$$

The strong convergence of the effective flux G_ϵ in Lemma 4.4 implies that

$$\mathbf{divu}_\epsilon \longrightarrow \mathbf{divu} \;\; strongly \; in \; \mathbf{L}^2(\Omega).$$

Moreover, due to the Helmholtz decomposition, we have

$$\mathbf{curlu}_\epsilon \longrightarrow \mathbf{curlu} \;\; strongly \; in \; \mathbf{L}^2(\Omega).$$

It follows from the regularity of (4.1)–(4.2) that

$$\Psi(\mathbf{u}_\epsilon) : \nabla \mathbf{u}_\epsilon \longrightarrow \Psi(\mathbf{u}) : \nabla \mathbf{u} \;\; strongly \; in \; \mathbf{L}^1(\Omega).$$

For a smooth function φ, we consider the weak form of (3.3)–(3.4) as

$$\begin{aligned}
&\int_\Omega ((1+\theta_\epsilon^m)\frac{(\epsilon+\theta_\epsilon)}{\theta_\epsilon}\nabla\theta_\epsilon \cdot \nabla\varphi)\mathrm{d}x + \int_{\partial\Omega} L(\theta_\epsilon)(\theta_\epsilon - \theta_0)\varphi\sigma + \epsilon\int_{\partial\Omega} \ln\theta_\epsilon \cdot \varphi \mathrm{d}\sigma \\
&\quad - \int_\Omega \left(K(\rho_\epsilon)\rho_\epsilon Q(\theta_\epsilon)\mathbf{u}_\epsilon \cdot \nabla\varphi + \mathbf{u}_\epsilon p_\theta (\int_0^{\rho_\epsilon} K(t)\mathrm{d}t)\cdot\nabla(\theta_\epsilon\varphi)\right)\mathrm{d}x \\
&\quad + \int_\Omega p_\theta(\int_0^{\rho_\epsilon} K(t)\mathrm{d}t)\nabla(\mathbf{u}_\epsilon\theta_\epsilon\varphi)\mathrm{d}x \\
&\quad = \int_\Omega \left(\nu|\nabla\times\mathbf{H}_\epsilon|^2\cdot\varphi + (\Psi(\mathbf{u}_\epsilon):\nabla\mathbf{u}_\epsilon)\cdot\varphi\right)\mathrm{d}x,
\end{aligned}$$

and

$$\int_\Omega (\nu\mathbf{curlH}_\epsilon \cdot \mathbf{curl}\varphi + (\mathbf{divu}_\epsilon + \mathbf{u}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon\cdot\varphi - (\mathbf{H}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon\cdot\varphi)\,\mathrm{d}x = 0.$$

Then, by (4.16) and Lemma 4.2, we deduce that

$$\begin{aligned}
&(1+\theta_\epsilon^m)\frac{(\epsilon+\theta_\epsilon)}{\theta_\epsilon}\nabla\theta_\epsilon \longrightarrow (1+\theta^m)\nabla\theta \;\; in \; \mathbf{L}^1(\Omega),\\
&\theta_\epsilon \longrightarrow \theta \;\; strongly \; in \; \mathbf{L}^m(\partial\Omega), \;\; x\in\partial\Omega,\\
&\ln\theta_\epsilon \longrightarrow \ln\theta \;\; strongly \; in \; \mathbf{L}^2(\partial\Omega), \;\; x\in\partial\Omega,\\
&\mathbf{curlH}_\epsilon \longrightarrow \mathbf{curlH} \;\; strongly \; in \; \mathbf{L}^2(\Omega),\\
&(\mathbf{divu}_\epsilon)\mathbf{H}_\epsilon \longrightarrow (\mathbf{divu})\mathbf{H} \;\; strongly \; in \; \mathbf{L}^2(\Omega),\\
&(\mathbf{u}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon \longrightarrow (\mathbf{u}\cdot\nabla)\mathbf{H} \;\; strongly \; in \; \mathbf{L}^2(\Omega),\\
&(\mathbf{H}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon \longrightarrow (\mathbf{H}\cdot\nabla)\mathbf{u} \;\; strongly \; in \; \mathbf{L}^2(\Omega).
\end{aligned}$$

Let $\epsilon \longrightarrow 0^+$. We have

$$\begin{aligned}
&-\int_\Omega (\rho Q(\theta)\mathbf{u}\cdot\nabla\varphi + \mathbf{u}p_\theta(\rho)\cdot\nabla(\theta\varphi))\,\mathrm{d}x + \int_\Omega p_\theta(\rho)\nabla(\mathbf{u}\theta\varphi)\mathrm{d}x \\
&\qquad = \int_\Omega (p_\theta(\rho)\theta\varphi\nabla\mathbf{u} - \rho Q(\theta)\mathbf{u}\cdot\nabla\varphi)\,\mathrm{d}x.
\end{aligned}$$

Hence, let $\epsilon \longrightarrow 0^+$, we obtain

$$\int_{\Omega}(1+\theta^m)\nabla\theta\cdot\nabla\varphi\mathrm{d}x+\int_{\partial\Omega}L(\theta)(\theta-\theta_0)\varphi\mathrm{d}\sigma+\int_{\Omega}(p_\theta(\rho)\theta\varphi\nabla\mathbf{u}-\rho Q(\theta)\mathbf{u}\cdot\nabla\varphi)\,\mathrm{d}x$$
$$=\int_{\Omega}\left(\nu|\nabla\times\mathbf{H}|^2\cdot\varphi+(\Psi(\mathbf{u}):\nabla\mathbf{u})\cdot\varphi\right)\mathrm{d}x, \tag{4.17}$$

and

$$\int_{\Omega}(\nu\mathbf{curlH}\cdot\mathbf{curl}\varphi+(\mathbf{divu}+\mathbf{u}\cdot\nabla)\mathbf{H}\cdot\varphi-(\mathbf{H}\cdot\nabla)\mathbf{u}\cdot\varphi)\,\mathrm{d}x=0.$$

Finally, recall the definition of Π in (3.59), we introduce $\Pi(\theta)=\int_0^\theta(1+t^m)\mathrm{d}t$. Then, it follows from (4.17) that $\theta\in\mathbf{L}^\infty(\Omega)$ and $\mathbf{u}\in\mathbf{W}^{1,p}(\Omega)$ for $0<p<\infty$. Using the energy equation again, we obtain that $\theta\in\mathbf{W}^{1,p}(\Omega)$ for $0<p<\infty$.

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