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Higher-order functional inequalities related to the clamped 1-biharmonic operator

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Abstract We consider the problem of finding the optimal constant for the embedding of the space

$$W^{2,1}_{\Delta,0}(\Omega) := \{u \in W^{1,1}_0(\Omega) \,\big|\, \text{there exists } \{u_k\} \subset C^\infty_c(\Omega) \text{ s.t. } \|\Delta u_k - \Delta u\|_1 \to 0\}$$

into the space $L^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class $C^{1,1}$. This is equivalent to find the first eigenvalue $\Lambda^c_{1,1}(\Omega)$ of the clamped 1-biharmonic operator. In this paper, we identify the correct relaxation of the problem on $BL_0(\Omega)$, the space of functions whose distributional Laplacian is a finite Radon measure, we obtain the associated Euler–Lagrange equation, and we give lower bounds for $\Lambda^c_{1,1}(\Omega)$, investigating the validity of an inequality of Faber–Krahn type.

Keywords Higher order Sobolev embeddding · Minimization problem · Clamped 1-biharmonic operator · Faber-Krahn type inequality

Mathematics Subject Classification 46E35, 35G15, 35P30, 49J52

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $W^{2,1}_{\Delta}(\Omega)$ be defined as

$$W^{2,1}_{\Delta}(\Omega) := \left\{ u \in W^{1,1}_0(\Omega) \mid \Delta u \in L^1(\Omega) \right\}.$$

This function space turns out to be strictly larger than the Sobolev space $W^{2,1}(\Omega) \cap W^{1,1}_0(\Omega)$, in which all second-order derivatives are taken into account. This is in contrast to the case $p > 1$ where one always has $W^{2,p}_{\Delta}(\Omega) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, provided $\partial\Omega$ is sufficiently smooth: The equivalence between the full Sobolev norm and $\|\Delta u\|_p$ can be achieved by standard elliptic regularity theory, see [11, Lemma 9.17]. This difference is highlighted by the corresponding sharp Sobolev embeddings: in particular, in the so-called limiting case $n = 2$ ($p = 1 = \frac{n}{2}$), one has $W^{2,1}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ (see e.g., [1]), while this embedding fails for the larger space $W^{2,1}_{\Delta}(\Omega)$, which embeds only into $L_{exp}(\Omega)$ (see [5], [2]).

In [18], the authors addressed the minimization problem

$$\Lambda_{1,1}(\Omega) = \inf_{u \in W^{2,1}_{\Delta}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1}{\|u\|_1}, \tag{1}$$

which is strictly related to the optimal constant for the embedding of $W^{2,1}_{\Delta}(\Omega)$ into $L^1(\Omega)$. A physical interpretation of $\Lambda_{1,1}(\Omega)$ was given in terms of the L^{∞}-norm of the torsion function, and a Faber–Krahn-type result was proved. Actually, the infimum is not attained in $W^{2,1}_{\Delta}(\Omega)$, but in the broader space

$$BL_0(\Omega) := \{u \in W^{1,1}_0(\Omega) \,\big|\, |\Delta u|(\Omega) < \infty\},$$

consisting of all functions $u \in W^{1,1}_0(\Omega)$ such that Δu is a Radon measure with finite total variation. The minimizers satisfy, in an appropriate sense, the (formal) eigenvalue problem

$$\begin{cases} \Delta_1^2 u = \lambda \dfrac{u}{|u|} & \text{in } \Omega, \\ u = \dfrac{\Delta u}{|\Delta u|} = 0 & \text{on } \partial\Omega, \end{cases}$$

where we denote by $\Delta_1^2 u$ the 1-biharmonic operator $\Delta_1^2 u := \Delta\left(\frac{\Delta u}{|\Delta u|}\right)$, which can be seen as the limiting case, for $p \to 1$, of the p-biharmonic operator $\Delta_p^2 u := \Delta\left(|\Delta u|^{p-2}\Delta u\right)$. The 1-biharmonic operator, hence, can be interpreted as a 'higher-order' case of the well-known 1-Laplacian operator $\Delta_1 u := \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ which has been widely studied in recent years due to its numerous applications.

How does problem (1) change if we replace $W^{2,1}_{\Delta}(\Omega)$ with the subspace of the smooth compactly supported functions, $C^{\infty}_c(\Omega)$? The aim of this paper is to investigate the minimization problem

$$\Lambda^c_{1,1}(\Omega) = \inf_{u \in C^{\infty}_c(\Omega)\setminus\{0\}} \frac{\int_{\Omega} |\Delta u|}{\int_{\Omega} |u|}, \tag{2}$$

which can be seen as the L^1-counterpart of the minimization of the quotient

$$\mu_1(\Omega) = \inf_{u \in W^{2,2}_0(\Omega)\setminus\{0\}} \frac{\int_{\Omega} |\Delta u|^2}{\int_{\Omega} |u|^2}. \tag{3}$$

The value μ_1 coincides with the first eigenvalue of the clamped plate equation; when the domain Ω is a ball, its value can be expressed as the smallest positive root of a functional equality involving Bessel functions.

Investigating the minimization problem (2) is equivalent to determine the value of the quantity

$$\inf_{u \in W^{2,1}_{\Delta,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\int_\Omega |u|}, \tag{4}$$

where $W^{2,1}_{\Delta,0}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$, with respect to $\|\Delta \cdot \|_1$, that is,

$$W^{2,1}_{\Delta,0}(\Omega) := \left\{ u \in W^{1,1}_0(\Omega) \,\middle|\, \text{there exists } \{u_k\} \subset C_c^\infty(\Omega) \text{ s.t. } \|\Delta u_k - \Delta u\|_1 \to 0 \right\},$$

Hence, $\Lambda^c_{1,1}(\Omega)$ is the inverse of the optimal constant for the embedding

$$W^{2,1}_{\Delta,0}(\Omega) \hookrightarrow L^1(\Omega).$$

Embeddings of $W^{2,1}_{\Delta,0}(\Omega)$ into other function spaces were investigated in [5] and more recently in [10]. However, the method used in this paper, which avoids looking for particular minimizing sequences when computing the value of $\Lambda^c_{1,1}(\Omega)$, is new in this context.

Set in the space $W^{2,1}_{\Delta,0}(\Omega)$, the minimization problem (4) resembles the analogous one, (3), in $W^{2,2}_0(\Omega)$. Nevertheless, while the existence of a minimizer for (3) in $W^{2,2}_0(\Omega)$ is easily shown, the quotient (4) does not admit minimizers in $W^{2,1}_{\Delta,0}(\Omega)$, but in the broader space $BL_0(\Omega)$. Of course, one needs to find the relaxation of the functional in $BL_0(\Omega)$, and therefore, the main difficulty is its correct identification. Observe that we cannot relax the functional simply replacing $\|\Delta u\|_1$ by $|\Delta u|(\Omega)$, the total variation of Δu measured in Ω, since this would give the infimum of the quotient on $W^{2,1}_\Delta(\Omega)$, as shown in [18]; therefore, one can think to penalize the numerator by adding the L^1-norm of the normal derivative u_n on $\partial\Omega$, in order to "force" the functions to have zero normal derivative, that is, to consider the new minimization problem

$$\inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{|\Delta u|(\Omega) + \|u_n\|_{L^1(\partial\Omega)}}{\|u\|_1}.$$

This intuitive approach turns out to be correct, but it requires some work in order to give it a precise mathematical meaning. First of all, we have to make sure that functions in $BL_0(\Omega)$ admit a normal derivative on $\partial\Omega$ which belongs to $L^1(\partial\Omega)$ and that an integration by parts formula holds; this question was addressed by Brezis and Ponce, [4, Theorem 1.2], who also showed that the numerator in the previous ratio is nothing but the total variation of Δu calculated in the whole space $\mathbb{R}^n$, that is,

$$|\Delta u|(\mathbb{R}^n) = |\Delta u|(\Omega) + \|u_n\|_{L^1(\partial\Omega)}$$

so that one can equivalently consider one of the following minimization problems

$$\begin{aligned} \Lambda^c_{1,1}(\Omega) &= \inf_{u \in W^{2,1}_{\Delta,0}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1}{\|u\|_1} = \inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1} \\ &= \inf_{u \in W^{2,1}_{\Delta}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1 + \|u_n\|_{L^1(\partial\Omega)}}{\|u\|_1}. \end{aligned} \tag{5}$$

Analogously to the eigenvalue problem studied in [18], a minimizer satisfies formally

$$\begin{cases} \Delta\left(\dfrac{\Delta u}{|\Delta u|}\right) = \lambda \dfrac{u}{|u|} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Observe that this boundary value problem seems under-determined, since the boundary condition $u_n = 0$ does not appear; indeed, minimizers in $BL_0(\Omega)$ need not have zero normal derivative on $\partial\Omega$. This feature is actually shared with the eigenvalue problem for the 1-Laplacian operator: in that case, one minimizes the Rayleigh quotient

$$\frac{|Du|(\mathbb{R}^n)}{\|u\|_1} = \frac{|Du|(\Omega) + \|u\|_{L^1(\partial\Omega)}}{\|u\|_1}$$

among all non-trivial functions $u \in BV(\Omega)$. Since minimizers are given by characteristic functions of Cheeger sets, whose boundary always intersect $\partial\Omega$, the Dirichlet boundary condition $u = 0$ is never satisfied on the whole boundary; we refer to [13] for further details on the eigenvalue problem for the 1-Laplacian and to [17] for general properties of Cheeger sets.

Let us now go back to the first eigenvalue of the clamped plate Eq. (3). A long-standing conjecture (see Payne [19]) states that, for any bounded domain Ω, the ratio

$$\frac{\mu_1(\Omega)}{\mu_1(\Omega^\#)}$$

is bounded from below by 1, where $\Omega^\#$ denotes the ball in $\mathbb{R}^n$ having the same volume of Ω. The question was first investigated by Talenti in [22] and then solved in the case $n = 2$ by Nadirashvili [16] and for $n = 3$ by Ashbaugh–Benguria [3], but it is still an open problem for $n \geq 4$. One of the major difficulties is the fact that the associated first eigenfunction may be sign-changing, as is, for example, the case when Ω is a square. Here, we address the analogous question for the first eigenvalue of the 'clamped' 1-biharmonic operator: does a Faber–Krahn-type inequality hold? The answer is affirmative in the case $n = 1$ and $n = 2$, whereas we give a (non-optimal) lower bound for the ratio

$$\frac{\Lambda^c_{1,1}(\Omega)}{\Lambda^c_{1,1}(\Omega^\#)}$$

if $n \geq 3$, and we leave the question as an open problem.

We are now ready to state our main theorems that summarize the results described above. The following theorem provides equivalent formulations for the minimization problem (4), (2), the existence of a minimizer in $BL_0(\Omega)$ and the associated Euler–Lagrange equation. We denote by $D^{1,2}(\mathbb{R}^n)$ the closure of $C^\infty_c(\mathbb{R}^n)$ with respect to the norm $\|u\| := \|\nabla u\|_2$ (see, for instance [14]).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,1}$. Consider the minimization problem*

$$\Lambda^c_{1,1}(\Omega) := \inf_{u \in W^{2,1}_{\Delta,0}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1}{\|u\|_1}. \tag{7}$$

Then,

$$\Lambda^c_{1,1}(\Omega) = \inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1} = \inf_{u \in W^{2,1}_{\Delta}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1 + \|u_n\|_{L^1(\partial\Omega)}}{\|u\|_1}. \tag{8}$$

Moreover,

(i) *the first infimum in* (8) *is attained: There exists* $v \in BL_0(\Omega) \setminus \{0\}$ *such that* $|\Delta v|(\mathbb{R}^n) = \Lambda^c_{1,1}(\Omega)\|v\|_1$*;*
(ii) *the minimizer satisfies formally*

$$\begin{cases} \Delta\left(\dfrac{\Delta u}{|\Delta u|}\right) = \lambda \dfrac{u}{|u|} & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

in the sense that for any measurable sign selection $s \in Sgn(u)$ *there exists* $z \in D^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *such that*

- $\|z\|_\infty = 1$*, supp* $\Delta z \subset \Omega$*,* $\Delta z \in L^n(\Omega)$*;*
- $|\Delta u|(\mathbb{R}^n) = \displaystyle\int_\Omega u \Delta z$*;*
- $\Delta z = \lambda s$ *almost everywhere in* Ω*, with* $\lambda = \dfrac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1}$.

The next theorem provides a lower bound for the ratio $\frac{\Lambda^c_{1,1}(\Omega)}{\Lambda^c_{1,1}(\Omega^\#)}$ and states a Faber–Krahn-type inequality when the minimization problem is restricted to the positive cone of $W^{2,1}_{\Delta,0}(\Omega)$. We recall that, by Theorem 1.3 in [18],

$$\Lambda_{1,1}(\Omega) \geq \Lambda_{1,1}(\Omega^\#) = \frac{2n}{R^2}$$

where $\Omega^\#$ is the ball of radius R having the same measure of $\Omega : |\Omega| = |\Omega^\#| = \omega_n R^n$, ω_n being the measure of the unit ball.

Theorem 1.2 *For any bounded domain* Ω *with boundary of class* $C^{1,1}$*,*

$$\begin{aligned} \Lambda^c_{1,1}(\Omega) &\geq \Lambda^c_{1,1}(\Omega^\#) = 2 \cdot \Lambda_{1,1}(\Omega^\#) \qquad \text{if } n = 1, 2 \\ \Lambda^c_{1,1}(\Omega) &\geq \frac{1}{2^{\frac{n-2}{n}}} \cdot \Lambda^c_{1,1}(\Omega^\#) \qquad \text{if } n \geq 3 \end{aligned} \tag{9}$$

Further, if

$$\mathscr{W}^+(\Omega) := \left\{u \in W^{2,1}_{\Delta,0}(\Omega) \,|\, u \geq 0\right\}$$

denotes the cone of positive functions in $W^{2,1}_{\Delta,0}(\Omega)$*, and*

$$\Lambda^{c,+}_{1,1}(\Omega) = \inf_{u \in \mathscr{W}^+(\Omega)} \frac{\|\Delta u\|_1}{\|u\|_1}$$

then

$$\Lambda^{c,+}_{1,1}(\Omega) \geq \Lambda^{c,+}_{1,1}(\Omega^\#) \tag{10}$$

and equality holds if $\Omega = B_R$*, where* B_R *is the ball of radius* R*, when*

$$\Lambda^{c,+}_{1,1}(B_R) = \frac{4n}{R^2}$$

The paper is organized as follows: after giving the necessary definitions and proving some preliminary results in Sect. 2, we study the existence and the properties of the normal derivative for functions in $BL_0(\Omega)$ (Sect. 3). In the fourth section, we discuss the well-posedness of our minimization problem and its relaxation on $BL_0(\Omega)$, while in Sect. 5 we describe the Euler–Lagrange problem satisfied by the minimizers; Sect. 5 ends with the proof of Theorem 1.1, which summarizes all the arguments introduced above. Section 6 is devoted to the radial case: when the domain is a ball, we solve the minimization problem when restricted to the subspace of radial functions. The last section is dedicated to the discussion on Faber–Krahn-type inequalities for $\Lambda^c_{1,1}$ (Theorem 1.2).

The authors would like to thank Samuel Littig for providing an earlier version of [15].

2 Definitions and preliminary results

The aim of this section is to recall definitions and results about the space $BL_0(\Omega)$ proved in [18], to which we refer for further details.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; unless otherwise specified, we will suppose that its boundary is of class Lipschitz. We define $BL_0(\Omega)$ as the space of functions $u \in W_0^{1,1}(\Omega)$ whose Laplacian Δu is representable by a finite measure μ, i.e.,

$$\int_\Omega \nabla u\, \nabla \varphi = \int_\Omega \varphi d\mu \qquad \forall\, \varphi \in C_c^\infty(\Omega; \mathbb{R})$$

Recalling that the total variation of a measure μ is defined as

$$|\mu|(\Omega) = \sup\left\{ \int_\Omega \varphi\, d\mu \,\middle|\, \varphi \in C_c(\Omega),\ \|\varphi\|_\infty \le 1 \right\}$$

we define the *total variation of the Laplacian of* u (in Ω) the quantity

$$\begin{aligned} |\Delta u|(\Omega) &:= \sup\left\{ \int_\Omega \varphi d\Delta u \,\middle|\, \varphi \in C_c(\Omega),\ \|\varphi\|_\infty \le 1 \right\} \\ &= \sup\left\{ \int_\Omega \varphi d\Delta u \,\middle|\, \varphi \in C_c^\infty(\Omega),\ \|\varphi\|_\infty \le 1 \right\} \\ &= \sup\left\{ \int_\Omega \nabla u \nabla \varphi \,\middle|\, \varphi \in C_c^\infty(\Omega),\ \|\varphi\|_\infty \le 1 \right\} \end{aligned}$$

where we used the fact that $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$. Hence,

$$BL_0(\Omega) = \{u \in W_0^{1,1}(\Omega) \,|\, |\Delta u|(\Omega) < \infty\}.$$

For $n = 1$, the space $BL_0(\Omega)$ coincides with the space of functions of bounded Hessian $BH(\Omega)$ introduced in [7], which are functions whose gradient is locally in $BV(\Omega)$. However, if $n \ge 2$ the latter space is strictly contained in $BL_0(\Omega)$, as a consequence of the results in [6, Theorem 3]; indeed, the authors prove the existence of a function $u : [0, 1] \times [0, 1] \to \mathbb{R}$ such that u_{xx} and u_{yy} are Radon measures with finite total variation, but the total variation of u_{xy} is infinite, so that $u \notin BH(\Omega)$. The space $BL_0(\Omega)$ was already introduced by Brezis and Ponce in [4]; using their notation, it holds $BL_0(\Omega) = W_0^{1,1}(\Omega) \cap \mathbb{X}$.

We will also make use of the total variation of Δu in $\mathbb{R}^n$, that is,

$$|\Delta u|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} \nabla u \nabla \varphi \,\middle|\, \varphi \in C_c^\infty(\mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \right\}$$

and of the space

$$BL_0(\mathbb{R}^n) = \left\{u \in W^{1,1}(\mathbb{R}^n) \;\middle|\; |\Delta u|(\mathbb{R}^n) < \infty\right\}.$$

Finally, we define the space

$$W^{2,1}_{\Delta}(\Omega) := \left\{u \in W^{1,1}_0(\Omega) \;\middle|\; \Delta u \in L^1(\Omega)\right\}$$

and

$$W^{2,1}_{\Delta,0}(\Omega) := \left\{u \in W^{1,1}_0(\Omega) \;\middle|\; \text{there exists } \{u_k\} \subset C^\infty_c(\Omega) \text{ s.t. } \|\Delta u_k - \Delta u\|_1 \to 0\right\}$$

Note that $W^{2,1}_{\Delta,0}(\Omega) \subset W^{2,1}_{\Delta}(\Omega) \subset BL_0(\Omega)$; indeed, for any $u \in W^{2,1}_{\Delta}(\Omega)$ the distributional Laplacian is given by $\Delta u\, dx$, but the inclusion is strict.

We recall the following approximation result, whose proof can be found in [18, Theorem 4.1].

Proposition 2.1 *Let $u \in BL_0(\Omega)$. Then, there exists a sequence of functions $u_k \in C^\infty(\Omega) \cap BL_0(\Omega)$ such that $u_k \to u$ in $W^{1,1}_0(\Omega)$, and $|\Delta u_k|(\Omega) \to |\Delta u|(\Omega)$ as $k \to \infty$.*

Note that, indeed, a function of $C^\infty(\Omega) \cap BL_0(\Omega)$ belongs to $W^{2,1}_{\Delta}(\Omega)$.

In order to investigate the minimization problem (8), we are naturally interested to guarantee existence and uniqueness of solution (in a suitable sense) of the equation

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega \\ \quad\; u = 0 & \text{on } \partial\Omega \end{cases}$$

The question is not trivial, as one can see by means of explicit counterexamples. However, if the boundary of Ω is of class $C^{1,1}$, uniqueness of distributional solutions in the space $W^{1,1}_0(\Omega)$ holds. More precisely, the following slightly more general result was proven in [18, Corollary 4.5].

Proposition 2.2 *Let Ω be a bounded, convex domain or a bounded domain whose boundary is of class $C^{1,\alpha}$ with $\alpha \in (0, 1]$. If $u \in W^{1,1}_0(\Omega)$ is harmonic in Ω in distributional sense, then $u \equiv 0$.*

This allows a decomposition of a function $u \in BL_0(\Omega)$ into its "superharmonic" and "subharmonic" part as in Proposition 3.1; hence, from now on, we will suppose that the boundary of Ω is of class $C^{1,1}$.

3 Normal derivative and integration by parts

As remarked in the introduction, in order to identify a proper relaxation of the minimization problem (4) in $BL_0(\Omega)$, we need to guarantee the existence of a normal derivative on the boundary $\partial\Omega$, for functions u in $BL_0(\Omega)$. This question was addressed by Brezis and Ponce, who proved that every function in $BL_0(\Omega)$ admits a normal derivative in $L^1(\partial\Omega; \mathcal{H}^{n-1})$, the symbol $\mathcal{H}^{n-1}$ standing for the $(n-1)$-dimensional Hausdorff measure ([4, Theorem 1.2]). We observe that our space $BL_0(\Omega)$ corresponds to $\mathbb{X} \cap W^{1,1}_0(\Omega)$ in their notation. For the sake of completeness, we wish to give a slightly different proof, in order to state more precisely the regularity assumptions on the boundary, and to show that functions in $W^{2,1}_{\Delta,0}(\Omega)$ are such that their normal derivative is identically zero on the boundary.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,1}$. Let $u \in BL_0(\Omega)$. Then, there exists a unique linear application $T_n : BL_0(\Omega) \to L^1(\partial\Omega; \mathscr{H}^{n-1})$ such that*

$$\int_\Omega \nabla u \nabla \varphi = -\int_\Omega \varphi \, d\Delta u + \int_{\partial\Omega} (T_n u)\, \varphi$$

for all $\varphi \in C^1(\overline{\Omega})$. Further, it holds

$$\int_{\partial\Omega} |T_n u| \leq \int_\Omega |\Delta u| \tag{11}$$

If $u \in BL_0(\Omega) \cap C^1(\overline{\Omega})$, then $T_n u = u_n$ on $\partial\Omega$. Moreover, if $u \in W^{2,1}_{\Delta,0}(\Omega)$, then $T_n u \equiv 0$.

Proof We begin by supposing that $u \in C^\infty(\Omega) \cap W^{2,p}(\Omega)$ for every $p \geq 1$, and $-\Delta u \geq 0$. Then, $\nabla u \in C(\overline{\Omega})$, $u \geq 0$ and $u_n \leq 0$. Therefore,

$$\int_{\partial\Omega} |u_n| = -\int_{\partial\Omega} u_n = -\int_\Omega \operatorname{div}(\nabla u) = \int_\Omega |\Delta u|. \tag{12}$$

Moreover,

$$\int_\Omega \nabla u \nabla \varphi = -\int_\Omega \varphi \, \Delta u + \int_{\partial\Omega} \varphi u_n$$

for all $\varphi \in C^1(\overline{\Omega})$. A similar reasoning applies in the case $-\Delta u \leq 0$. Let now $u \in W^{2,1}_\Delta(\Omega)$ with $f := -\Delta u \geq 0$. Since $\partial\Omega$ is of class $C^{1,1}$, we can approximate u by a sequence $u_k \in C^\infty(\Omega) \cap W^{2,p}(\Omega)$ for every $p \geq 1$ such that $\Delta u_k \in C^\infty_c(\Omega)$ with $-\Delta u_k \geq 0$ in such a way that $u_k \to u$ in $W^{1,1}_0(\Omega)$ and $\Delta u_k \to \Delta u$ in $L^1(\Omega)$. By (12), the sequence $\{T_n u_k\}$ is a Cauchy sequence in $L^1(\partial\Omega; \mathscr{H}^{n-1})$, and therefore, there exists a (unique) function $g \in L^1(\partial\Omega; \mathscr{H}^{n-1})$ such that $T_n u_k \to g$ in $L^1(\partial\Omega; \mathscr{H}^{n-1})$. We will set $T_n u := g$. Moreover, from

$$\int_\Omega \nabla u_k \nabla \varphi = -\int_\Omega \Delta u_k \, \varphi + \int_{\partial\Omega} (T_n u_k)\, \varphi$$

we obtain

$$\int_\Omega \nabla u \nabla \varphi = -\int_\Omega \Delta u \, \varphi + \int_{\partial\Omega} (T_n u)\, \varphi \tag{13}$$

for all $\varphi \in C^1(\overline{\Omega})$. Furthermore, the equality

$$\int_{\partial\Omega} |T_n u| = \int_\Omega |\Delta u|$$

holds true. To show uniqueness of $T_n u$, suppose by contradiction that there exists another $\widetilde{g} \in L^1(\partial\Omega; \mathscr{H}^{n-1})$ such that (13) holds true. Then,

$$\int_{\partial\Omega} (T_n u - \widetilde{g})\, \varphi = 0 \qquad \text{for every} \quad \varphi \in C^1(\overline{\Omega}). \tag{14}$$

By Whitney's extension Theorem [23], for every $\psi \in C^\infty_c(\partial\Omega)$, there exists $\varphi \in C^1(\overline{\Omega})$ such that $\varphi|_{\partial\Omega} = \psi$. Therefore (14) is equivalent to

$$\int_{\partial\Omega} (T_n u - \widetilde{g})\, \psi = 0 \qquad \text{for every} \quad \psi \in C^\infty_c(\partial\Omega)$$

which implies $\widetilde{g} = T_n u$. Now, we consider an arbitrary function $u \in W^{2,1}_\Delta(\Omega)$. Let w and z the solutions of the problems

$$\begin{cases} -\Delta w = f^+ & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta z = f^- & \text{in } \Omega \\ \quad z = 0 & \text{on } \partial\Omega \end{cases}$$

where f^+ and f^- are the positive and negative parts of f, respectively. Then, $u = w + z$, and we can define

$$T_n u := T_n w + T_n z.$$

Relation (13) holds true also for u, and

$$\int_{\partial\Omega} |T_n u| \leq \int_{\partial\Omega} |T_n w| + \int_{\partial\Omega} |T_n z| = \int_\Omega |\Delta w| + \int_\Omega |\Delta z| = \int_\Omega |\Delta u|.$$

The extension of the result for functions in $BL_0(\Omega)$ can be proved as in [4, Proposition 4.2] (their extension argument works also for domains with boundary of class $C^{1,1}$). If u is of class $C^1(\overline{\Omega})$, it is clear that $T_n u$ must coincide with the usual normal derivative. Moreover, if $u \in W^{2,1}_{\Delta,0}(\Omega)$, then it is the limit of a sequence of functions $u_k \in C^\infty_c(\Omega)$. Since $T_n u_k \equiv 0$ on $\partial\Omega$, by continuity we have $T_n u \equiv 0$ on $\partial\Omega$ as well. □

We will now investigate the trivial extension of functions in $BL_0(\Omega)$. Let us denote with

$$\overline{u}(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Observe that if $v \in W^{1,1}_0(\Omega)$, then

$$\begin{aligned} |\Delta \overline{v}|(\mathbb{R}^n) &= \sup\left\{ \int_\Omega v \Delta\varphi \,\middle|\, \varphi \in C^\infty_c(\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_\Omega \nabla v \nabla\varphi \,\middle|\, \varphi \in C^\infty_c(\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_\Omega \nabla v \nabla\varphi \,\middle|\, \varphi \in C^1(\overline{\Omega}),\ \|\varphi\|_\infty \leq 1 \right\} \end{aligned}$$

In the following, when no ambiguity arises, we will denote by u_n the normal derivative of a function $u \in BL_0(\Omega)$.

Proposition 3.2 *Let $u \in BL_0(\Omega)$. Then, $\overline{u} \in BL(\mathbb{R}^n)$, and*

$$|\Delta \overline{u}|(\mathbb{R}^n) = |\Delta u|(\Omega) + \|u_n\|_{L^1(\partial\Omega)}.$$

In particular,

$$|\Delta \overline{u}|(\mathbb{R}^n) \leq 2|\Delta u|(\Omega).$$

Proof We argue as in [9, Section 5.4, Theorem 1]. Since in particular $u \in W^{1,1}_0(\Omega)$, we have that $\overline{u} \in W^{1,1}(\mathbb{R}^n)$. Let $\varphi \in C^1(\overline{\Omega})$ such that $\|\varphi\|_\infty \leq 1$; then,

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla\overline{u}\ \nabla\varphi = \int_\Omega \nabla u\, \nabla\varphi &= -\int_\Omega \varphi\, d\Delta u + \int_{\partial\Omega} \varphi u_n d\mathscr{H}^{n-1} \\ &\leq |\Delta u|(\Omega) + \|u_n\|_{L^1(\partial\Omega)} \end{aligned} \tag{15}$$

Hence, $u \in BL_0(\Omega)$ implies that $\overline{u} \in BL(\mathbb{R}^n)$ with

$$|\Delta \overline{u}|(\mathbb{R}^n) \leq |\Delta u|(\Omega) + \|u_n\|_{L^1(\partial\Omega)}.$$

To obtain the equality, observe that $d\Delta\overline{u} \in M(\overline{\Omega})$, so that, by the Riesz representation Theorem, it admits a unique decomposition

$$\int_{\overline{\Omega}} \varphi d\Delta\overline{u} = \int_{\Omega} \varphi d\mu + \int_{\partial\Omega} \varphi d\nu \qquad \forall \varphi \in C^1(\overline{\Omega})$$

Hence, we obtain that

$$d\mu = d\Delta u \qquad \text{and} \qquad d\nu = u_n d\mathscr{H}^{n-1}$$

and the first part of the claim follows. The second part is a consequence of Proposition 3.1, since

$$\|u_n\|_{L^1(\partial\Omega)} \leq |\Delta u|(\Omega).$$

□

Looking carefully at the proof of Proposition 2.1, one can observe that it is possible to write $|\Delta\overline{u}|(\mathbb{R}^n)$ instead of $|\Delta u|(\Omega)$, obtaining the following approximation result.

Proposition 3.3 *Let $u \in BL_0(\Omega)$. Then, there exists a sequence of functions $u_k \in C^\infty(\Omega) \cap BL_0(\Omega)$ (and hence in $W^{2,1}_\Delta(\Omega)$) such that $u_k \to u$ in $W^{1,1}_0(\Omega)$, and $|\Delta\overline{u}_k|(\mathbb{R}^n) \to |\Delta\overline{u}|(\mathbb{R}^n)$ as $k \to \infty$.*

Remark 3.4 If $u \in W^{2,1}_\Delta(\Omega)$ it is not true, in general, that $\overline{u} \in W^{2,1}_\Delta(\mathbb{R}^n)$. As an example, one can consider $\Omega = B_1$ in $\mathbb{R}^2$ and the function $u(x) = 1 - |x|^2$ defined on B_1. Clearly, $u \in W^{2,1}_\Delta(B_1)$; on the other hand, for any radial $\varphi \in C^\infty_c(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla\overline{u}\, \nabla\varphi\, dx &= 2\pi \int_0^1 -2r\varphi'(r) r dr = -4\pi\varphi(1) + 8\pi \int_0^1 r\varphi(r)\, dr \\ &= -4\pi\varphi(1) + 4\int_{B_1} \varphi\, dx \neq \int_{\mathbb{R}^2} g\varphi\, dx \quad \text{for any } g \in L^1(\mathbb{R}^2)\end{aligned}$$

This proves that $\overline{u} \notin W^{2,1}_\Delta(\mathbb{R}^2)$.

Remark 3.5 If $u \in W^{2,1}_{\Delta,0}(\Omega)$, then $T_n u \equiv 0$ and $\overline{u} \in W^{2,1}_\Delta(\mathbb{R}^n)$ by Proposition 3.1. In particular,

$$|\Delta\overline{u}|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\Delta\overline{u}| = \int_{\Omega} |\Delta u|.$$

From now on, we will not distinguish between a function and its trivial extension.

4 An approximation result: relaxation on $BL_0(\Omega)$

The aim of this section is to obtain the relaxation of our original minimization problem (4) on $BL_0(\Omega)$, that is, to prove (8)

$$\Lambda^c_{1,1}(\Omega) = \inf_{u \in W^{2,1}_{\Delta,0}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1}{\|u\|_1} = \inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1}.$$

To this end, we will show that each function u in $BL_0(\Omega)$ can be approximated by a sequence $\{u_k\}$ of smooth functions with compact support, in such a way that $u_k \to u$ in $W^{1,1}_0(\Omega)$ and $\|\Delta u_k\|_1 = |\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$. Our results are inspired by those contained in [15]. We will need some preliminary lemmas.

Lemma 4.1 *[15, Lemma 3.4] Let $\Omega \subset \mathbb{R}^n$ be an open, bounded subset with Lipschitz boundary. Then, there exists a $\tau_0 > 0$ and a family of C^∞-diffeomorphisms $\Phi^\tau : \mathbb{R}^n \to \mathbb{R}^n$, $0 \leq \tau \leq \tau_0$, with inverses Ψ^τ, such that:*

- $\Phi^0 = \Psi^0 = Id$;
- $\Phi^\tau \to Id$ *and* $\Psi^\tau \to Id$ *as* $\tau \to 0$ *uniformly on* $\mathbb{R}^n$;
- $D\Phi^\tau(x) \to Id$ *and* $D\Psi^\tau(x) \to Id$ *as* $\tau \to 0$ *uniformly with respect to* x *on* $\mathbb{R}^n$
- $\Phi^\tau(\overline{\Omega}) \subset\subset \Omega$ *for* $0 < \tau \leq \tau_0$.

Moreover, the higher derivatives of Φ^τ and Ψ^τ converge uniformly to zero as $\tau \to 0$.

Lemma 4.2 *Let $u \in BL_0(\Omega)$. Then, there exists a sequence $\{u_k\}$ in $C^\infty(\Omega) \cap W^{2,p}(\Omega)$ for every $p \geq 1$ such that $u_k \to u$ in $W^{1,1}(\Omega)$ and $|\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$.*

Proof Since $u \in BL_0(\Omega)$, by Proposition 3.3, we can say that there exists a sequence $\{u_k\}$ in $C^\infty(\Omega) \cap W^{2,1}_\Delta(\Omega)$ such that $u_k \to u$ in $W^{1,1}_0(\Omega)$ and $|\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$. Therefore, it is enough to consider a function $u \in C^\infty(\Omega) \cap W^{2,1}_\Delta(\Omega)$. Let $f = -\Delta u$, and let $\{f_k\}$ be a sequence in $C^\infty_c(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$. Let u_k be the solution of

$$\begin{cases} -\Delta u_k = f_k & \text{in } \Omega \\ \quad\; u_k = 0 & \text{on } \partial\Omega \end{cases}$$

Since $\partial\Omega$ is of class $C^{1,1}$, one has that $u_k \in C^\infty(\Omega) \cap W^{2,p}(\Omega)$ for every $p \geq 1$. Moreover, $u_k \to u$ in $W^{1,1}(\Omega)$, $|\Delta u_k|(\Omega) \to |\Delta u|(\Omega)$ since $\|f_k - f\|_1 \to 0$, and by Proposition 3.1

$$\int_{\partial\Omega} |T_n u_k| \to \int_{\partial\Omega} |T_n u|$$

which implies $|\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$. □

The following proposition, which is the main result of this section, is an adaptation of [15, Theorem 3.2].

Proposition 4.3 *Let $u \in BL_0(\Omega)$. Then, there exists a sequence $u_k \in C^\infty_c(\Omega)$ such that $u_k \to u$ in $W^{1,1}_0(\Omega)$ and $|\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$.*

Proof We begin by considering a function $u \in BL_0(\Omega)$ with compact support. Let η be a standard mollifier, and consider the functions $u_\varepsilon := u * \eta_\varepsilon$ where $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. Then, $u_\varepsilon \in C^\infty_c(\Omega)$ for $\varepsilon > 0$ small enough, and $u_\varepsilon \to u$ in $W^{1,1}_0(\Omega)$, which implies

$$|\Delta u|(\mathbb{R}^n) \leq \liminf_{\varepsilon \to 0} |\Delta u_\varepsilon|(\mathbb{R}^n).$$

Moreover, for a function $\varphi \in C^\infty_c(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \nabla \varphi = \int_{\mathbb{R}^n} \nabla(u * \eta_\varepsilon)\nabla\varphi = \int_{\mathbb{R}^n} \nabla u \nabla(\eta_\varepsilon * \varphi) \leq |\Delta u|(\mathbb{R}^n)$$

since $|\eta_\varepsilon * \varphi| \leq 1$. Hence,

$$\limsup_{\varepsilon \to 0} |\Delta u_\varepsilon|(\mathbb{R}^n) \leq |\Delta u|(\mathbb{R}^n).$$

Now, we consider an arbitrary function $u \in BL_0(\Omega)$. We notice that, by Lemma 4.2, it is enough to consider functions in $C^\infty(\Omega) \cap W^{2,p}(\Omega)$ for a fixed $p > n$. Then, for Φ^τ a family of diffeomorphisms with inverses Ψ^τ as in the lemma above, one defines

$$u^\tau(x) = u(\Psi^\tau(x)).$$

The functions u^τ have compact support, and satisfy $u^\tau \to u$ in $L^1(\Omega)$ for $\tau \to 0$ (see Step 1 in [15, Theorem 3.2]). Moreover,

$$\nabla u^\tau(x) = \nabla[u(\Psi^\tau(x))] = [D\Psi^\tau(x)]^T \cdot \nabla u(\Psi^\tau(x)).$$

For every $\varepsilon > 0$, we can choose τ so small that $\|[D\Psi^\tau(x)]^T - Id\| \le \varepsilon$ and $|\det D\Phi^\tau(x)| \le 2$. Therefore,

$$\begin{aligned}|\nabla u^\tau(x) - \nabla u(x)| &= |[D\Psi^\tau(x)]^T \cdot \nabla u(\Psi^\tau(x)) - \nabla u(x)|\\ &\le |[D\Psi^\tau(x)]^T \cdot \nabla u(\Psi^\tau(x)) - \nabla u(\Psi^\tau(x))| + |\nabla u(\Psi^\tau(x)) - \nabla u(x)|\\ &\le \|[D\Psi^\tau(x)]^T - Id\||\nabla u(\Psi^\tau(x))| + |\nabla u(\Psi^\tau(x)) - \nabla u(x)|\\ &\le \varepsilon|\nabla u(\Psi^\tau(x))| + |\nabla u(\Psi^\tau(x)) - \nabla u(x)|\end{aligned}$$

which implies

$$\int_\Omega |\nabla u^\tau(x) - \nabla u(x)| \le \varepsilon \int_\Omega |\nabla u(\Psi^\tau(x))| + \int_\Omega |\nabla u(\Psi^\tau(x)) - \nabla u(x)|.$$

Observe that

$$\int_\Omega |\nabla u(\Psi^\tau(x))|\, dx = \int_\Omega |\nabla u(y)|\, |\det D\Phi^\tau(y)| dy \le 2\|\nabla u\|_1,$$

while it can be shown as in Step 1 of [15, Theorem 3.2] that

$$\int_\Omega |\nabla u(\Psi^\tau(x)) - \nabla u(x)| \to 0$$

as $\tau \to 0$. Hence, $u^\tau \to u$ in $W_0^{1,1}(\Omega)$. It follows that

$$|\Delta u|(\mathbb{R}^n) \le \liminf_{\tau\to 0} |\Delta u^\tau|(\mathbb{R}^n).$$

It remains to prove that

$$\limsup_{\tau\to 0} |\Delta u^\tau|(\mathbb{R}^n) \le |\Delta u|(\mathbb{R}^n).$$

For every $y \in \partial\Omega$, let $\nu(y)$ be the normal vector, and let us denote by $[J_\Gamma(\Phi)](y)$ the tangential Jacobian at y (see [12, Definition 5.4.2] and [12, Proposition 5.4.3]), defined as

$$[J_\Gamma(\Phi)](y) := |[D\Psi^\tau(y)]^T \nu(y)| \cdot \det D\Phi^\tau(y).$$

It holds

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u^\tau(x)\nabla\varphi(x)\,dx = \int_{\mathbb{R}^n} \nabla[u(\Psi^\tau(x))]\nabla\varphi(x)\,dx \\
&= -\int_{\Omega^\tau} \Delta[u(\Psi^\tau(x))]\varphi(x)\,dx + \int_{\partial\Omega^\tau} \varphi\nabla[u(\Psi^\tau(x))]\cdot\nu(x)\,d\mathscr{H}^{n-1}(x) \\
&\le \int_{\Omega^\tau} \left|\Delta[u(\Psi^\tau(x))]\right|dx + \int_{\partial\Omega^\tau} \left|\nabla[u(\Psi^\tau(x))]\right| d\mathscr{H}^{n-1}(x) \\
&\le \int_{\Omega^\tau} \left|\sum_i \frac{\partial\Psi^\tau}{\partial x_i}(x)\cdot D^2u(\Psi^\tau(x))\cdot\frac{\partial\Psi^\tau}{\partial x_i}(x) + \sum_j \frac{\partial u}{\partial x_j}(\Psi^\tau(x))\Delta\Psi_j^\tau(x)\right| dx \\
&\quad + \int_{\partial\Omega^\tau} |[D\Psi^\tau(x)]^T\cdot\nabla u(\Psi^\tau(x))|\,d\mathscr{H}^{n-1}(x) \\
&\overset{x=\Phi^\tau(y)}{=} \int_\Omega \left|\sum_i \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot\frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\right. \\
&\quad \left. + \sum_j \frac{\partial u}{\partial x_j}(y)\Delta\Psi_j^\tau(\Phi^\tau(y))\right| |\det D\Phi^\tau(y)|\,dy \\
&\quad + \int_{\partial\Omega} |[D\Psi^\tau(\Phi^\tau(y))]^T\cdot\nabla u(y)|\,[J_\Gamma(\Phi)](y)\,d\mathscr{H}^{n-1}(y) \\
&\le \int_\Omega \left|\sum_i \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot\frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\right| |\det D\Phi^\tau(y)|\,dy \\
&\quad + \int_\Omega \sum_j \left|\frac{\partial u}{\partial x_j}(y)\Delta\Psi_j^\tau(\Phi^\tau(y))\right| |\det D\Phi^\tau(y)|\,dy \\
&\quad + \int_{\partial\Omega} |[D\Psi^\tau(\Phi^\tau(y))]^T\cdot\nabla u(y)|\,[J_\Gamma(\Phi)](y)\,d\mathscr{H}^{n-1}(y) \\
&=: I_1^\tau + I_2^\tau + I_3^\tau
\end{aligned}
$$

For the change of variable in the boundary integral, see [12, Proposition 5.4.3]. Since $D\Phi^\tau \to Id$ uniformly, one has that $|\det D\Phi^\tau(y)| \to 1$ and $[J_\Gamma(\Phi)](y) \to 1$ uniformly as $\tau \to 0$. Moreover,

$$
\begin{aligned}
&\int_\Omega \left|\sum_i \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot\frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y)) - e_i\cdot D^2u(y)\cdot e_i\right| |\det D\Phi^\tau(y)|\,dy \\
&= \int_\Omega \left|\sum_i \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot\frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y)) - \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot e_i\right. \\
&\quad \left. + \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot e_i - e_i\cdot D^2u\cdot e_i\right| |\det D\Phi^\tau(y)|\,dy \\
&= \int_\Omega \left|\sum_i \frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y))\cdot D^2u(y)\cdot\left[\frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y)) - e_i\right]\right. \\
&\quad \left. + \left[\frac{\partial\Psi^\tau}{\partial x_i}(\Phi^\tau(y)) - e_i\right]\cdot D^2u(y)\cdot e_i\right| |\det D\Phi^\tau(y)|\,dy
\end{aligned}
$$

$$\leq \sum_i \int_\Omega \left| \frac{\partial \Psi^\tau}{\partial x_i}(\Phi^\tau(y)) \cdot D^2u(y) \cdot \left[\frac{\partial \Psi^\tau}{\partial x_i}(\Phi^\tau(y)) - e_i \right] \right| |\det D\Phi^\tau(y)|\, dy$$
$$+ \sum_i \int_\Omega \left| \left[\frac{\partial \Psi^\tau}{\partial x_i}(\Phi^\tau(y)) - e_i \right] \cdot D^2u(y) \cdot e_i \right| |\det D\Phi^\tau(y)|\, dy$$

Since $\frac{\partial \Psi^\tau}{\partial x_i}$ converges uniformly to e_i as $\tau \to 0$, and $D^2u \in L^1(\Omega)$, we have that

$$\int_\Omega \left| \sum_i \frac{\partial \Psi^\tau}{\partial x_i}(\Phi^\tau(y)) \cdot D^2u(y) \cdot \frac{\partial \Psi^\tau}{\partial x_i}(\Phi^\tau(y)) \right| |\det D\Phi^\tau(y)|\, dy \to \int_\Omega |\Delta u|.$$

Similarly, the I_2^τ converges to zero, while I_3^τ satisfies

$$I_3^\tau \to \int_{\partial\Omega} |u_n|$$

so that finally

$$\limsup_{\tau \to 0} |\Delta u^\tau|(\mathbb{R}^n) \leq |\Delta u|(\mathbb{R}^n)$$

which is the claim. □

5 The Euler–Lagrange problem for the clamped 1-biharmonic operator

In this section, we will derive an Euler–Lagrange equation for the minimization problem (5). Due to the homogeneity of the problem, $\Lambda_{1,1}^c(\Omega)$ can also be defined as

$$\Lambda_{1,1}^c(\Omega) = \min\{|\Delta u|(\mathbb{R}^n) \mid \|u\|_1 = 1\}.$$

Since the functionals involved are not differentiable, we will make use of some results from convex analysis. We will need a suitably modified version of [18, Proposition 5.2]. Let us define the extension of $|\Delta u|(\mathbb{R}^n)$ to the space $L^{n'}(\Omega)$ for $n' = \frac{n}{n-1}$ (observe that $BL_0(\Omega) \subset W_0^{1,1}(\Omega) \subset L^{\frac{n}{n-1}}(\Omega)$)

$$E(u) := \begin{cases} |\Delta u|(\mathbb{R}^n) & \text{if} \quad u \in BL_0(\Omega), \\ +\infty & \text{if} \quad u \in L^{n'}(\Omega) \setminus BL_0(\Omega). \end{cases}$$

As recalled in the Introduction, we will denote by $D^{1,2}(\mathbb{R}^n)$ the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\| := \|\nabla u\|_2$ (see, for instance, [14].)

Proposition 5.1 *Let $u \in BL_0(\Omega)$, and denote by $\partial E(u)$ the subdifferential of E at u. Then, $u^* \in \partial E(u)$ if and only if there exists $z \in D^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that:*

- $\|z\|_\infty \leq 1$;
- $u^* = \Delta z \in L^n(\mathbb{R}^n)$, *supp* $\Delta z \subset \Omega$;
- $E(u) = \int_\Omega u\, \Delta z$.

Moreover, if $u \neq 0$, then $\|z\|_\infty = 1$.

Proof Let us define

$$M^* := \{u^* \in L^n(\Omega) \mid u^* = \Delta z \text{ for some } z \in D^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),\ \|z\|_\infty \leq 1\}.$$

The functions in M^* will be trivially extended outside Ω. We want to show that M^* is closed. To this end, take a sequence $\{u_k^*\}$ in M^* such that $u_k^* \to u^*$ in $L^n(\Omega)$; hence, there exists a sequence $\{z_k\}$ in $D^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the property that $\|z_k\|_\infty \leq 1$ for every k and that $u_k^* = \Delta z_k$ in distributional sense and also weakly, which means

$$\int_\Omega u_k^* \varphi = \int_{\mathbb{R}^n} z_k \, \Delta\varphi = -\int_{\mathbb{R}^n} \nabla z_k \nabla\varphi \text{ for every } \varphi \in C_c^\infty(\mathbb{R}^n)$$

or equivalently

$$-\int_{\mathbb{R}^n} \nabla z_k \nabla\varphi = \int_\Omega u_k^* \varphi \text{ for every } \varphi \in D^{1,2}(\mathbb{R}^n).$$

The sequence $\{z_k\}$ is bounded also in $D^{1,2}(\mathbb{R}^n)$, since $\{u_k^*\}$ is uniformly bounded in $L^n(\Omega)$; this follows by testing the equation with $-z_k$ in order to obtain

$$\int_{\mathbb{R}^n} |\nabla z_k|^2 = \int_\Omega u_k^* z_k \leq \|z_k\|_\infty \|u_k^*\|_1 \leq \|u_k^*\|_1 \leq |\Omega|^{\frac{n-1}{n}} \|u_k^*\|_n.$$

So there exists a function $z \in D^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that, after passing to a subsequence,

$$z_k \rightharpoonup z \quad \text{in } D^{1,2}(\mathbb{R}^n),$$
$$z_k \rightharpoonup^* z \quad \text{in } L^\infty(\mathbb{R}^n),$$

which implies

$$\|z\|_\infty \leq \liminf_{k\to\infty} \|z_k\|_\infty \leq 1$$

and

$$-\int_{\mathbb{R}^n} \nabla z \nabla\varphi = \int_\Omega u^* \varphi \text{ for every } \varphi \in D^{1,2}(\mathbb{R}^n).$$

which means that $u^* = \Delta z$ weakly. Hence, $u^* \in M^*$.

Let $I_{M^*} : L^n(\Omega) \to \mathbb{R}$ be the function defined as

$$I_{M^*}(u^*) = \begin{cases} 0 & \text{if } u^* \in M^* \\ +\infty & \text{otherwise} \end{cases}$$

The conjugate function to I_{M^*} is given by

$$I_{M^*}^*(u) = \sup_{u^* \in L^{p'}(\Omega)} \left\{ \int_\Omega u^* u - I_{M^*}(u^*) \right\} = \sup_{u^* \in M^*} \int_\Omega u^* u.$$

Now take $u \in BL_0(\Omega)$ and $u^* \in M^*$; then, there exists a sequence $\{u_k\}$ in $C_c^\infty(\Omega)$ such that $u_k \to u$ in $W_0^{1,1}(\Omega)$ and $|\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$ as $k \to \infty$. Without loss of generality, due to the embedding of $BL_0(\Omega)$ into $W_0^{1,r}(\Omega)$ for all $r \in [1, n')$ and the fact that $|\Delta u_k|(\mathbb{R}^n) = |\Delta u_k|(\Omega)$, we can suppose that $u_k \to u$ in $L^{n'}(\Omega)$. We have

$$\begin{aligned} \int_\Omega u^* u &= \int_\Omega \Delta z \, u = \lim_{k\to\infty} \int_\Omega \Delta z \, u_k = \lim_{k\to\infty} \int_\Omega z \, \Delta u_k \\ &\leq \|z\|_\infty \lim_{k\to\infty} \int_\Omega |\Delta u_k| \leq \lim_{k\to\infty} \int_\Omega |\Delta u_k| = \lim_{k\to\infty} |\Delta u_k|(\mathbb{R}^n) = |\Delta u|(\mathbb{R}^n). \end{aligned} \quad (16)$$

Hence,

$$I_{M^*}^*(u) = \sup_{u^* \in M^*} \int_\Omega u^* u \leq E(u).$$

Now, we have

$$
\begin{aligned}
E(u) &= \sup\left\{\int_\Omega u\,\Delta\varphi \mid \varphi\in C_c^\infty(\mathbb{R}^n),\ \|\varphi\|_\infty\le 1\right\}\\
&\le \sup\left\{\int_\Omega u\,\Delta z \mid z\in D^{1,2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n),\ \Delta z\in L^n(\Omega),\ \|z\|_\infty\le 1\right\}\\
&= \sup\left\{\int_\Omega u^*\,u \mid u^*\in M^*\right\} = I^*_{M^*}(u).
\end{aligned}
$$

Since the above inequality is true also for $u\in L^{n'}(\Omega)\setminus BL_0(\Omega)$, we obtain

$$
I^*_{M^*}(u) = E(u)
$$

for every $u\in L^{n'}(\Omega)$. M^* is closed and convex and thus $I^*_{M^*}$ is convex and lower semicontinuous, which implies (see [8, Chapter 1, Propositions 3.1 and 5.1])

$$
I_{M^*} = (I^*_{M^*})^* = E^*.
$$

By [8, Chapter 1, Proposition 5.1] one obtains that $u^*\in\partial E(u)$ if and only if

$$
\int_\Omega u\,u^* = E(u) + E^*(u^*) = E(u) + I_{M^*}(u^*),
$$

which implies that $u^*\in\partial E(u)$ if and only if $u^*\in M^*$ and $E(u)=\int_\Omega u^*\,u$, which is the claim. Moreover, if $u\ne 0$, then $E(u)\ne 0$ by Corollary 2.2 and hence $\|z\|_\infty = 1$ from Eq. (16). □

Let us define $G: L^{n'}(\Omega)\to\mathbb{R}$, $n'=\frac{n}{n-1}$, as

$$
G(u) := \int_\Omega |u|.
$$

For $u\in L^{n'}(\Omega)$, one has that $u^*\in\partial G(u)$ if and only if

$$
u^*\in Sgn(u)
$$

(see [13, Proposition 4.23]). We recall that $v\in Sgn(u)$ if and only if:

- $v(x)=1$ if $u(x)>0$;
- $v(x)=-1$ if $u(x)<0$;
- $v(x)\in[-1,1]$ if $u(x)=0$.

We are now ready to characterize the first eigenfunctions of the clamped 1-biharmonic operator.

Proposition 5.2 *Let $u\in BL_0(\Omega)$ be a minimizer of E constrained to the set $\{u\in L^{n'}(\Omega)\mid G(u)=1\}$. Then, for every measurable selection $s\in Sgn(u)$, there exists $z\in D^{1,2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ such that:*

(1) $\|z\|_\infty = 1$;
(2) $\Delta z\in L^n(\mathbb{R}^n)$;
(3) $E(u)=|\Delta u|(\mathbb{R}^n)=\int_\Omega u\,\Delta z$;
(4) $\Delta z=\lambda s$ *almost everywhere in* Ω, *with* $\lambda=E(u)$.

Proof From [13, Proposition 6.4] (setting $\widetilde{u} = -u$), it follows that for every $g^* \in \partial G(u)$, there exists a $e^* \in \partial E(u)$ and a $\lambda \in \mathbb{R}$ such that $g^* = \lambda\, e^*$, which plays the role of a Lagrange multiplier rule in this non-smooth setting. Multiplying both sides of the equality by u and integrating on Ω, one obtains that $\lambda = E(u)$. The claim follows easily if one remembers how the subdifferentials of E and G are characterized. □

Proposition 5.2 gives only a necessary condition for u to be a first eigenfunction; therefore, since u has support in Ω, one could state the result with a function $z \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfying the same conditions. The Euler–Lagrange equation formally reads

$$\begin{cases} \Delta\left(\dfrac{\Delta u}{|\Delta u|}\right) = \lambda \dfrac{u}{|u|} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The function s in Proposition 5.2 should be considered as a substitute for the possibly undetermined expression $\frac{u}{|u|}$, while z plays the role of $\frac{\Delta u}{|\Delta u|}$. If compared with the problem studied in [18], we remark that the natural boundary condition $\frac{\Delta u}{|\Delta u|} = 0$ has disappeared; indeed, the function z now belongs to $W^{1,2}(\Omega)$ and not necessarily to $W_0^{1,2}(\Omega)$. However, although we consider this problem as a "clamped" eigenvalue problem, the boundary condition $u_n = 0$ is in general not satisfied, as it will be made clear in the following sections. This feature actually appears also in the eigenvalue problem for the 1-Laplacian operator with Dirichlet boundary conditions: first eigenfunctions are given by characteristic functions of Cheeger sets and they never satisfy the condition $u = 0$ on the whole boundary.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let us consider the minimization problem (7). A direct consequence of Proposition 4.3, combined with Remark 3.5, is the first part of Eq. (8),

$$\Lambda^c_{1,1}(\Omega) = \inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1}$$

On the other hand, by Proposition 3.3

$$\inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1} = \inf_{u \in W^{2,1}_\Delta(\Omega)\setminus\{0\}} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1}$$

which is equal, in turns, to

$$\inf_{u \in W^{2,1}_\Delta(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_1 + \|u_n\|_{L^1(\partial\Omega)}}{\|u\|_1}.$$

by Proposition 3.2. The proof of (8) is then complete.

The proof of assertion (i) follows the same lines as in [18, Proposition 5.1]. Let $\{u_k\}$ be a minimizing sequence in $W^{2,1}_{\Delta,0}(\Omega) \subset BL_0(\Omega)$ such that $\|u_k\|_1 = 1$. By Remark 3.5, there exists a $M > 0$ such that $|\Delta u_k|(\Omega) = |\Delta u_k|(\mathbb{R}^n) = \|\Delta u_k\|_1 \le M$, so that, by [21, Theorem 9.1] (see also [18, Theorem 4.2]) the sequence is uniformly bounded in $W_0^{1,r}(\Omega)$ for a fixed $r \in \left(1, \frac{n}{n-1}\right)$. Hence, there exists a function $u \in W_0^{1,r}(\Omega)$ (and hence in $W_0^{1,1}(\Omega)$) such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $W_0^{1,r}(\Omega)$ and $u_k \to u$ strongly in $L^1(\Omega)$; this implies in particular that $\|u_k\|_1 = 1$. As in [18, Remark 2.1], the total variation $|\Delta u|(\mathbb{R}^n)$ is lower semicontinuous with respect to the L^1-convergence, so that

$$|\Delta u|(\mathbb{R}^n) \leq \liminf_{k\to\infty} |\Delta u_k|(\mathbb{R}^n) = \liminf_{k\to\infty} \|\Delta u_k\|_1.$$

This implies that actually $u \in BL_0(\Omega)$ and, further, that the first minimization problem in (i) is attained:

$$\Lambda^c_{1,1}(\Omega) = \inf_{v\in W^{2,1}_{\Delta,0}(\Omega)\setminus\{0\}} \frac{\|\Delta v\|_1}{\|v\|_1} = \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1}.$$

Assertion (ii) is a consequence of Proposition 5.2. □

6 The radial case

The aim of this Section is to discuss the case of radial domains: One may wonder if the minimizers are radial functions, and a first step in this direction is to investigate the minimization problem restricted to the class of radial functions. The result we obtain will be applied in the next section to study Faber–Krahn-type inequalities.

Let $B_R \subset \mathbb{R}^n$ be a ball of radius R, and let us define

$$\mathscr{W}_{rad}(B_R) := \left\{ u \in W^{2,1}_{\Delta,0}(B_R) \,\middle|\, u \text{ radially symmetric} \right\}.$$

We have the following approximation result, which states that radial functions may be approximated by means of radial smooth functions.

Proposition 6.1 *Let $B_R \subset \mathbb{R}^n$ be a ball of radius R, and let $u \in \mathscr{W}_{rad}(B_R)$. Then, there exists a sequence of radially symmetric functions $u_k \in C^\infty_c(\Omega)$ such that $u_k \to u$ in $W^{1,1}_0(\Omega)$ and $|\Delta u_k|(\mathbb{R}^n) \to |\Delta u|(\mathbb{R}^n)$.*

Proof The proof is the same as in Proposition 4.3, taking into account the fact that a possible family of diffeomorphisms in Lemma 4.1 is given for $\tau \in [0, 1]$ by

$$\Phi^\tau(x) = x_0 + \left(1 - \frac{\tau}{2}\right)(x - x_0),$$

where x_0 is the center of the ball. □

We can now approach the study of minimization problems on $\mathscr{W}_{rad}(B_R)$: thanks to Proposition 6.1, we can reduce to radial smooth functions and then apply techniques in the spirit of [5], [2].

Proposition 6.2 *Let $B_R \subset \mathbb{R}^n$ be a ball of radius R. Then,*

$$\inf_{u\in\mathscr{W}_{rad}(B_R)} \frac{\|\Delta u\|_1}{\|u\|_1} = \frac{4n}{R^2}.$$

Proof For the sake of simplicity, we will suppose that B_R is centered in the origin. By the approximation result of Proposition 6.1, we can restrict ourselves to functions $u \in C^\infty_c(\Omega)$ which are radially symmetric. In the following, we will set $r := |x|$. If $n = 1$, $B_R = (-R, R)$, and

$$\|\Delta u\|_1 = 2\int_0^R |u''(r)|dr, \qquad \|u\|_1 = 2\int_0^R |u(r)|dr$$

since $u \in C_c^\infty(-R, R)$ and it is even. Then, for any $0 < t < R$,

$$u'(t) = \frac{1}{2}\int_0^t u''(s)\,ds + \frac{1}{2}\int_t^R -u''(s)ds \Longrightarrow |u'(t)| \le \frac{1}{2}\int_0^R |u''(s)|\,ds = \frac{\|\Delta u\|_1}{4}$$

so that

$$|u(r)| = \left|\int_r^R -u'(t)dt\right| \le \frac{\|\Delta u\|_1}{4}\int_r^R dt = (R-r)\frac{\|\Delta u\|_1}{4}$$

hence

$$\|u\|_1 \le 2\cdot\frac{\|\Delta u\|_1}{4}\int_0^R (R-r)dr = \frac{R^2}{4}\|\Delta u\|_1$$

so that the case $n = 1$ is complete. We will now consider the case $n \ge 2$. From [5, Propositions 14 and 16], we have

$$|u(r)| \le \frac{\|\Delta u\|_1}{4\pi}\log\left(\frac{R}{r}\right)$$

if $n = 2$, and

$$|u(r)| \le \frac{\|\Delta u\|_1}{2n(n-2)\omega_n R^{n-2}}\left(\frac{R^{n-2}}{r^{n-2}} - 1\right)$$

if $n \ge 3$. Remembering that

$$\|u\|_1 = n\omega_n\int_0^R |u(t)|t^{n-1}\,dt$$

we obtain straightforwardly

$$\inf_{u\in\mathscr{W}_{rad}}\frac{\|\Delta u\|_1}{\|u\|_1} \ge \frac{4n}{R^2}.$$

Let us now prove the reverse inequality. Let u be the solution of the problem

$$\begin{cases} -\Delta u = \delta_0 & \text{in} \quad B_R \\ \quad u = 0 & \text{on} \quad B_R, \end{cases}$$

where δ_0 is a Dirac mass concentrated in 0. Hence, u has the form

$$u(r) = \begin{cases} \frac{1}{2}(R-r) & \text{if} \quad n = 1, \\ \frac{1}{2\pi}\log\left(\frac{R}{r}\right) & \text{if} \quad n = 2, \\ \frac{1}{n(n-2)\omega_n}\left(r^{2-n} - R^{2-n}\right) & \text{if} \quad n \ge 3. \end{cases}$$

In particular, $u \in BL_0(B_R)$ and

$$|\Delta u|(\Omega) = 1$$

so that (see also Proposition 3.2)

$$|\Delta u|(\mathbb{R}^n) = 2.$$

On the other hand, if $n = 1$,

$$\|u\|_1 = 2\int_0^R \frac{R-r}{2}\,dr = \frac{R^2}{2},$$

if $n = 2$

$$\begin{aligned}\|u\|_1 &= \int_0^R \log\left(\frac{R}{r}\right)\cdot r\,dr\\ &= \int_0^R \frac{1}{r}\cdot\frac{r^2}{2}\,dr = \frac{R^2}{4},\end{aligned}$$

and if $n \geq 3$

$$\begin{aligned}\|u\|_1 &= \frac{1}{n-2}\int_0^R \left(r^{2-n} - R^{2-n}\right) r^{n-1}\,dr\\ &= \frac{1}{n-2}\int_0^R \left(r - R^{2-n}r^{n-1}\right)\,dr\\ &= \frac{1}{n-2}\left(\frac{R^2}{2} - \frac{R^2}{n}\right) = \frac{R^2}{2n}.\end{aligned}$$

Hence,

$$\inf_{u\in\mathscr{W}_{rad}(B_R)} \frac{\|\Delta u\|_1}{\|u\|_1} \leq \frac{|Du|}{\|u\|_1} = \frac{4n}{R^2}.$$

$\square$

7 Toward a Faber–Krahn-type inequality: proof of Theorem 1.2

In this section, we discuss the validity of a Faber–Krahn-type inequality for $\Lambda^c_{1,1}(\Omega)$: does

$$\Lambda^c_{1,1}(\Omega) \geq \Lambda^c_{1,1}(\Omega^\#) \qquad \text{or, equivalently,} \qquad \frac{\Lambda^c_{1,1}(\Omega)}{\Lambda^c_{1,1}(\Omega^\#)} \geq 1$$

hold? As recalled in the Introduction, the question resembles the well-known conjecture about the validity of a Faber–Krahn-type inequality for the first eigenvalue of the clamped plate, proposed by Payne (see [19]) and then investigated, among other authors, by Szegö (see [20]) in the 2-dimensional case, and by Talenti in [22]; the conjecture was solved in the cases $n = 2$ and $n = 3$, as recalled above, but it is still open if $n \geq 4$.

The main obstacle in proving a Faber–Krahn-type inequality is that in the clamped case we cannot assure neither the positivity of the minimizers, nor their superharmonicity; indeed, it is true that $u \in BL_0 \Rightarrow |u| \in BL_0$, but the total variation of $\Delta|u|$ may increase, and also comparison arguments cannot be applied as in the non-clamped case. Applying a comparison argument, separately, to the positive and negative parts of the function u, we can prove the following result

Proposition 7.1 *Let $\Omega \subset \mathbb{R}^n$ be such that $|\Omega| = \omega_n R^n$. Then*

$$\Lambda^c_{1,1}(\Omega) \geq \frac{1}{2^{\frac{n-2}{n}}}\frac{4n}{R^2} \tag{17}$$

for $n \geq 3$. If $n = 2$,

$$\Lambda^c_{1,1}(\Omega) \geq \frac{8}{R^2}$$

and equality holds if $\Omega = B_R$. If $n = 1$ and $\Omega = (-R, R)$, then

$$\Lambda^c_{1,1}(\Omega) = \frac{4}{R^2}.$$

Proof Let $u \in W^{2,1}_{\Delta,0}(\Omega)$, and denote with u^+, u^-, respectively, its positive and negative part. We apply the estimates obtained in [10, Theorem 1] for the decreasing rearrangement of u^+ and u^-; for the sake of clarity, we briefly recall the notations and the proof of the inequality we need. Let $N_{B_R}(r)$ denote the Green function of the Laplacian in the ball B_R,

$$N_{B_R}(r) = \begin{cases} \dfrac{r^{2-n} - R^{2-n}}{n(n-2)\omega_n}, \; n \geq 3 \\[2ex] \dfrac{1}{2\pi} \log\left(\dfrac{R}{r}\right), \quad n = 2 \end{cases} \qquad 0 < r < R$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. For any bounded domain Ω, we denote with $N^*_{|\Omega|}(t)$ the decreasing rearrangement of the fundamental function N_{B_R}, associated to the ball with radius R such that $|\Omega| = |B_R| = \omega_n R^n$; that is,

$$N^*_{|\Omega|}(t) := N^*_{B_R}(t) = N_{B_R}\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right) = \begin{cases} \dfrac{t^{-\frac{n-2}{n}} - |\Omega|^{-\frac{n-2}{n}}}{n(n-2)\omega_n^{2/n}}, \; n \geq 3 \\[2ex] \dfrac{1}{4\pi} \log\left(\dfrac{|\Omega|}{t}\right), \qquad n = 2 \end{cases} \qquad 0 < t < |\Omega|$$

The estimates proved in [5] and [2] can be rephrased as

$$u \in W^{2,1}_{\Delta}(B_R) \quad \text{and } u \text{ radial} \quad \Longrightarrow \quad u^*(t) \leq N^*_{B_R}(t)\|\Delta u\|_1,$$

and

$$u \in W^{2,1}_{\Delta,0}(B_R) \quad \text{and } u \text{ radial} \quad \Longrightarrow \quad u^*(t) \leq \frac{1}{2} N^*_{B_R}(t)\|\Delta u\|_1,$$

Then, applying Talenti's comparison theorem to the two problems

$$\begin{cases} -\Delta w_\varepsilon = (-\Delta u)^+ & \text{in} \quad \Omega_\varepsilon, \\ \quad w_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \qquad \begin{cases} -\Delta w_\varepsilon = (-\Delta u)^- & \text{in} \quad \Omega'_\varepsilon, \\ \quad w_\varepsilon = 0 & \text{on} \quad \partial\Omega'_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \{x : u(x) > \varepsilon\}$ and $\Omega'_\varepsilon = \{x : u(x) < -\varepsilon\}$, and defining $u_\varepsilon = (u - \varepsilon)|_{\Omega_\varepsilon}$, $u'_\varepsilon = (-u - \varepsilon)|_{\Omega'_\varepsilon}$ one has

$$u^*_\varepsilon(t) \leq w^*_\varepsilon(t) \leq N^*_{|\Omega_\varepsilon|}(t) \int_{\Omega_\varepsilon} f^+ dx;$$

here, we have used the monotonicity of the decreasing rearrangement. Now, let $\varepsilon \to 0$: We obtain

$$(u^+)^*(t) \leq \frac{\|\Delta u\|_1}{2} N^*_{|\Omega^+|}(t),$$
$$(u^-)^*(t) \leq \frac{\|\Delta u\|_1}{2} N^*_{|\Omega^-|}(t),$$

as in [10].

Suppose now that $|\Omega^+| = \omega_n R_1^n$, $|\Omega^-| = \omega_n R_2^n$ and $|\Omega| = \omega_n R^n$. Then, $R_1^n + R_2^n \leq R^n$; moreover, since

$$\|u\|_1 = \int_0^\infty u^*(s)\, ds$$

we have that

$$\|u^+\|_1 \leq \frac{\|\Delta u\|_1}{2} \|N_{|\Omega^+|}\|_1 = \frac{R_1^2}{4n} \|\Delta u\|_1,$$
$$\|u^-\|_1 \leq \frac{\|\Delta u\|_1}{2} \|N_{|\Omega^-|}\|_1 = \frac{R_2^2}{4n} \|\Delta u\|_1.$$

Hence, if $n \geq 3$,

$$\|u\|_1 \leq \frac{R_1^2 + R_2^2}{4n} \|\Delta u\|_1 \leq \frac{R_1^2 + (R^n - R_1^n)^{\frac{2}{n}}}{4n} \|\Delta u\|_1 \leq 2^{\frac{n-2}{n}} \frac{R^2}{4n} \|\Delta u\|_1, \tag{18}$$

since the function $g(z) = z^2 + (1 - z^n)^{\frac{2}{n}}$ has $2^{\frac{n-2}{n}}$ as maximum value on $[0, 1]$. If $n = 2$ we have

$$\|u\|_1 \leq \frac{R_1^2 + R_2^2}{8} \|\Delta u\|_1 \leq \frac{R^2}{8} \|\Delta u\|_1$$

and thus

$$\inf_{u \in W^{2,1}_{\Delta,0}(\Omega)} \frac{\|\Delta u\|_1}{\|u\|_1} \geq \frac{8}{R^2}.$$

If $\Omega = B_R$, since

$$\inf_{u \in W^{2,1}_{\Delta,0}(B_R)} \frac{\|\Delta u\|_1}{\|u\|_1} \leq \inf_{u \in \mathscr{W}_{rad}(B_R)} \frac{\|\Delta u\|_1}{\|u\|_1} \leq \frac{8}{R^2}$$

by Proposition 6.2, so that equality holds. Finally, for $n = 1$,

$$\|u\|_1 \leq \frac{R_1^2 + R_2^2}{4} \|u''\|_1 \leq \frac{R_1^2 + (R - R_1)^2}{4} \|u''\|_1 \leq \frac{R^2}{4} \|u''\|_1$$

and thus

$$\inf_{u \in W^{2,1}_{\Delta,0}(\Omega)} \frac{\|u''\|_1}{\|u\|_1} \geq \frac{4}{R^2}$$

since the function $g(z) = z^2 + (1 - z)^2$ has a maximum equal to 1 for $z = 0$ or $z = 1$. Again by Proposition 6.2, equality holds. □

Remark 7.2 Note that, under dilation, $\Lambda^c_{1,1}(\Omega)$ scales as follows

$$\Lambda^c_{1,1}(t\Omega) = \min_{u \in BL_0(t\Omega)} \frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1} = \min_{v \in BL_0(\Omega)} \frac{t^{n-2}|\Delta v|(\mathbb{R}^n)}{t^n \|v\|_1}$$
$$= \frac{1}{t^2} \min_{v \in BL_0(\Omega)} \frac{|\Delta v|(\mathbb{R}^n)}{\|v\|_1} = \frac{1}{t^2} \Lambda^c_{1,1}(\Omega)$$

so that the lower bound (17) can be written in the scaling-invariant form

$$|\Omega|^{2/n} \Lambda^c_{1,1}(\Omega) \geq \frac{1}{2^{\frac{n-2}{n}}} \cdot 4n\omega_n^{2/n}$$

where ω_n is the Lebesgue measure of the n-dimensional unit ball.

As a consequence of Proposition 7.1, we obtain the following corollary, which concludes the proof of (9) in Theorem 1.2.

Corollary 7.3 *For any bounded domain with boundary of class $C^{1,1}$,*

$$\Lambda^c_{1,1}(\Omega) \geq \Lambda^c_{1,1}(\Omega^\#) = 2 \cdot \Lambda_{1,1}(\Omega^\#) \quad \text{if} \quad n = 1, 2$$

$$\Lambda^c_{1,1}(\Omega) \geq \frac{1}{2^{\frac{n-2}{n}}} \cdot \Lambda^c_{1,1}(\Omega^\#) \qquad \text{if} \quad n \geq 3.$$

Proof If $n = 1, 2$, the previous proposition states that

$$\Lambda^c_{1,1}(\Omega) \geq 2 \cdot \frac{2n}{R^2} = 2 \cdot \Lambda_{1,1}(\Omega^\#)$$

On the other hand, by Proposition 3.2,

$$\Lambda^c_{1,1}(\Omega^\#) \leq 2 \cdot \Lambda_{1,1}(\Omega^\#)$$

which yields directly the claim. The case $n \geq 3$ follows by the same arguments. □

Remark 7.4 If $n \geq 3$, the inequality

$$\inf_{u \in W^{2,1}_{\Delta,0}(\Omega)} \frac{\|\Delta u\|_1}{\|u\|_1} \geq \frac{1}{2^{\frac{n-2}{n}}} \frac{4n}{R^2}$$

is not optimal. Indeed, if it were optimal, looking at (18) one could easily deduce that Ω is the union of two disjoint, equal balls B_1 and B_2. Let $u = v_1 + v_2$ be the optimal function. Then,

$$\frac{|\Delta u|(\mathbb{R}^n)}{\|u\|_1} = \frac{|\Delta v_1|(\mathbb{R}^n) + |\Delta v_2|(\mathbb{R}^n)}{\|v_1\|_1 + \|v_2\|_1} \geq \frac{4n}{R_1^2} = 2^{\frac{2}{n}} \frac{4n}{R^2},$$

a contradiction.

Let us now conclude the proof of Theorem 1.2, discussing the minimization problem (4) restricted to the positive cone of $W^{2,1}_{\Delta,0}(\Omega)$.

We define

$$\mathscr{W}^+(\Omega) := \left\{ u \in W^{2,1}_{\Delta,0}(\Omega) \,|\, u \geq 0 \right\}.$$

Proposition 7.5 *Let $\Omega \subset \mathbb{R}^n$, such that $|\Omega| = |B_R|$. Then,*

$$\inf_{u \in \mathscr{W}^+(\Omega)} \frac{\|\Delta u\|_1}{\|u\|_1} \geq \frac{4n}{R^2},$$

and equality holds if $\Omega = B_R$.

Proof The proof follows the same notations as in Proposition 7.1. If $u \in \mathscr{W}^+(\Omega)$, by the estimates in [5] one obtains

$$u^*(t) \leq \frac{1}{2} N^*_{|\Omega|}(t) \|\Delta u\|_1$$

Now, since $u \geq 0$,

$$\|u\|_1 = \int_\Omega u(x)\,dx = \int_0^{|\Omega|} u^*(t)\,dt \leq \frac{\|\Delta u\|_1}{2} \int_0^{|\Omega|} N^*_{|\Omega|}(t)dt = \frac{\|\Delta u\|_1}{4n} \frac{|\Omega|^{2/n}}{\omega_n^{2/n}} = \frac{R^2}{4n} \|\Delta u\|_1.$$

Then,

$$\Lambda_{1,1}^{c,+}(\Omega) \geq \frac{4n}{R^2}.$$

Reasoning as in the previous corollary, the proof can be easily concluded. □

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