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Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space

Juan de Dios Pérez

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Abstract We consider real hypersurfaces M in complex projective space equipped with both the Levi-Civita and generalized Tanaka-Webster connections. For any non-null constant k and any vector field X tangent to M, we can define an operator on M, $F_X^{(k)}$, related to both connections. We study commutativity problems of these operators and the structure Jacobi operator of M.

Keywords g-Tanaka-Webster connection · Complex projective space · Real hypersurface · kth Cho operator

Mathematics Subject Classification 53C15 · 53B25

1 Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the structure vector field on M. On M, there exists an almost contact metric structure (ϕ, ξ, η, g) induced by the Kaehlerian structure of $\mathbb{C}P^m$, where ϕ is the tangent component of J and η is an one form given by $\eta(X) = g(X, \xi)$ for any X tangent to M. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [5,12–14]. His classification contains 6 types of real hypersurfaces. Among them, we find type (A_1) real hypersurfaces that are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and type (A_2) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. We will call both types of real hypersurfaces type (A) real hypersurfaces.

J. D. Pérez (✉)
Departamento de Geometria y Topologia, Universidad de Granada, 18071 Granada, Spain
e-mail: jdperez@ugr.es

Ruled real hypersurfaces can be described as follows: take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ, there is a unique $\mathbb{C}P^{m-1}$ cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally, although globally it will in general have self-intersections and singularities. Equivalently, a ruled real hypersurface satisfies that the maximal holomorphic distribution on M, $\mathbb{D}$, given at any point by the vectors orthogonal to ξ, is integrable or $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples of ruled real hypersurfaces, see [6] or [8].

The Tanaka-Webster connection, [15–17], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR manifold. As a generalization of this connection, Tanno, [16], defined the generalized Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{1.1}$$

Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho defined the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $\mathbb{C}P^m$ given, see [3,4], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X, Y tangent to M where k is a nonzero real number. Then, $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the g-Tanaka-Webster connection coincides with the Tanaka-Webster connection.

Here, we can consider the tensor field of type (1,2) given by the difference in both connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M, see [7] Proposition 7.10, pages 234–235. We will call this tensor the kth Cho tensor on M. Associated to it, for any X tangent to M and any non-null real number k, we can consider the tensor field of type (1,1) $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any $Y \in TM$. This operator will be called the kth Cho operator corresponding to X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M.

The Jacobi operator R_X with respect to a unit vector field X is defined by $R_X = R(., X)X$, where R is the curvature tensor field on M. Then, we see that R_X is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields, which are solutions of the second-order differential equation (the Jacobi equation) $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}} Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ in M. The Jacobi operator with respect to the structure vector field ξ, R_ξ, is called the structure Jacobi operator on M.

The purpose of the present paper was to study real hypersurfaces M in $\mathbb{C}P^m$ such that the covariant and g-Tanaka-Webster derivatives of the structure Jacobi operator coincide. $\nabla R_\xi = \hat{\nabla}^{(k)} R_\xi$ is equivalent to the fact that, for any X tangent to M, $R_\xi F^{(k)}_X = F^{(k)}_X R_\xi$. The meaning of this condition is that every eigenspace of R_ξ is preserved by the kth Cho operator $F^{(k)}_X$ for any X tangent to M.

On the other hand, $TM = Span\{\xi\} \oplus \mathbb{D}$. Thus, we will obtain the following

Theorem 1 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Let k be a non-null constant. Then, $F^{(k)}_X R_\xi = R_\xi F^{(k)}_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

Theorem 2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Let k be a non-null constant. Then, $F^{(k)}_\xi R_\xi = R_\xi F^{(k)}_\xi$ if and only if M is locally congruent to either a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^m$ or to a type (A) real hypersurface with radius $r \neq \frac{\pi}{4}$.*

As a direct consequence of these Theorems, we have

Corollary *There do not exist real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that for a non-null constant k, $F_X^{(k)} R_\xi = R_\xi F_X^{(k)}$ for any X tangent to M.*

2 Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M, we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then, (ϕ, ξ, η, g) is an almost contact metric structure on M, see [1], that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1), we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J, we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M$, $\mathbb{D}(p) = \{X \in T_p M | g(X, \xi) = 0\}$. We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M.

From the above formulas, we have that the structure Jacobi operator on M is given by

$$R_\xi(X) = X - \eta(X)\xi + g(A\xi, \xi)AX - g(AX, \xi)A\xi \tag{2.7}$$

for any X tangent to M

In the sequel, we need the following results:

Theorem 2.1 [10] *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then, the following are equivalent:*

1. *M is locally congruent to either a geodesic hypersphere or a tube of radius r, $0 < r < \frac{\pi}{2}$ over a totally geodesic $\mathbb{C}P^n$, $0 < n < m - 1$.*

2. $\phi A = A\phi$.

Theorem 2.2 [9] *If ξ is a principal curvature vector with corresponding principal curvature α and $X \in \mathbb{D}$ is principal with principal curvature λ, then ϕX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

3 Proof of Theorem 1

If we suppose that $F_X^{(k)} R_\xi = R_\xi F_X^{(k)}$ for any $X \in \mathbb{D}$, we get

$$\begin{aligned} g(Y, \phi AX)\xi + \eta(A\xi)g(\phi AX, AY)\xi - \eta(AY)g(\phi AX, A\xi)\xi \\ +\eta(Y)\phi AX + \eta(Y)\eta(A\xi)A\phi AX - \eta(Y)\eta(A\phi AX)A\xi = 0 \end{aligned} \tag{3.1}$$

for any $X \in \mathbb{D}$, $Y \in TM$. Let us suppose that M is non-Hopf. Thus, locally we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$, α and β are functions on M and $\beta \neq 0$. We also call $\mathbb{D}_U$ to the orthogonal complementary distribution in $\mathbb{D}$ to the one spanned by $U, \phi U$.

If we take $X = Y = \phi U$ in (3.1), we get

$$g(AU, \phi U) = 0. \tag{3.2}$$

And taking $Y = \xi$ in (3.1), we obtain

$$\phi AX + \alpha A\phi AX - \alpha\beta g(\phi AX, U)\xi - \beta^2 g(\phi AX, U)U = 0 \tag{3.3}$$

for any $X \in \mathbb{D}$. In particular, from (3.2) and (3.3), we have

$$\phi AU + \alpha A\phi AU = 0. \tag{3.4}$$

The scalar product of (3.3) and U yields

$$(\beta^2 - 1)g(A\phi U, X) - \alpha g(A\phi AU, X) = 0 \tag{3.5}$$

for any $X \in \mathbb{D}$. Thus, $(\beta^2 - 1)A\phi U - \alpha A\phi AU$ has not a component in $\mathbb{D}$, and taking its scalar product with ξ, it follows

$$(\beta^2 - 1)A\phi U - \alpha A\phi AU = 0. \tag{3.6}$$

From (3.4) and (3.6), we get

$$\phi AU = (1 - \beta^2)A\phi U. \tag{3.7}$$

Therefore, we can write $A\phi U = \delta\phi U + \omega Z_1$, where $Z_1 \in \mathbb{D}_U$ is a unit vector field. The scalar product of (3.3) and $Y \in \mathbb{D}_U$ yields $A\phi Y + \alpha A\phi Y$ has not component in $\mathbb{D}$. Then,

$$A\phi Y + \alpha A\phi AY = -\alpha\beta g(A\phi U, Y)\xi \tag{3.8}$$

for any $Y \in \mathbb{D}_U$. Taking $Y = \phi Z_1$, we obtain $-AZ_1 + \alpha A\phi A\phi Z_1 = 0$. Its scalar product with ξ gives

$$\alpha\beta\omega(\beta^2 - 1) = 0. \tag{3.9}$$

As $\beta \neq 0$, the following cases appear
Case 1. $\alpha = 0$.
Case 2. $\beta^2 = 1$. In this case, from (3.7), $AU = \beta\xi$.
Case 3. $\omega = 0$, thus $\mathbb{D}_U$ is A-invariant.

Case 1. $\alpha = 0$. From (3.4) $\phi AU = 0$, that is, $AU = \beta\xi$ and $A\xi = \beta U$ and from (3.6) $(\beta^2 - 1)A\phi U = 0$. So we have the following subcases

Subcase 1.1. Let us suppose that $\beta^2 \neq 1$. Then, $A\phi U = 0$. Moreover, from (3.8) for any $Y \in \mathbb{D}_U$ $A\phi Y = 0$. That means that M is a minimal ruled hypersurface.

Subcase 1.2. $\alpha = 0$, $\beta^2 = 1$. We can suppose $\beta = 1$, maybe after changing ξ by $-\xi$. As above, $A\phi Y = 0$ for any $Y \in \mathbb{D}_U$, $AU = \xi$, $A\xi = U$. Then, $AZ_1 = 0$ and $\omega = g(A\phi U, Z_1) = 0$. Thus, $A\phi U = \delta\phi U$.

By the Codazzi equation $g((\nabla_\xi A)U - (\nabla_U A)\xi, \phi U) = 1$ yields

$$\delta g(\nabla_\xi U, \phi U) + g(\nabla_U U, \phi U) = 0. \tag{3.10}$$

From $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, \xi) = 0$, we obtain

$$g(\nabla_\xi \phi U, U) = -3\delta. \tag{3.11}$$

From (3.10) and (3.11), we have

$$g(\nabla_U U, \phi U) = -3\delta^2. \tag{3.12}$$

As $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, \xi) = -2$, it follows

$$g(\nabla_U U, \phi U) = -2 \tag{3.13}$$

and from $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, U) = 0$, we get

$$\delta g(\nabla_U \phi U, U) + 2\delta = 0. \tag{3.14}$$

From (3.13) and (3.14), we have $\delta = 0$. Therefore, M is still a minimal ruled real hypersurface.

Case 2. $\beta^2 = 1$. As above, we suppose $\beta = 1$. As the case $\alpha = 0$ has been studied, we suppose $\alpha \neq 0$. Then, from (3.6), $A\phi AU = 0$, and from (3.4), $\phi AU = 0$. Therefore, $A\xi = \alpha\xi + U$, $AU = \xi$. Moreover, we know that $-AZ_1 + \alpha A\phi AZ_1 = 0$. Taking its scalar product with ϕU, we get $\omega + \alpha\omega g(A\phi Z_1, \phi Z_1) = 0$. Supposing $\omega \neq 0$, we have $g(A\phi Z_1, \phi Z_1) = -\frac{1}{\alpha}$.

Taking $X = Y \in \mathbb{D}_U$ in (3.1), we obtain $\phi AY + \alpha A\phi AY + \omega g(Y, Z_1)A\xi = 0$. Its scalar product with ϕU gives $\alpha g(A\phi AY, \phi U) = 0 = -\alpha\omega g(Y, A\phi Z_1)$. As $\alpha\omega \neq 0$, $g(Y, A\phi Z_1) = 0$ for any $Y \in \mathbb{D}_U$. This yields $A\phi Z_1 = 0$ and $0 = -\frac{1}{\alpha}$. This is a contradiction, and we have $\omega = 0$, $A\xi = \alpha\xi + U$, $AU = \xi$ and $A\phi U = \delta\phi U$.

This yields $\mathbb{D}_U$ is A-invariant and ϕ-invariant, and we arrive to Case 3. As also $\phi AU = (1 - \beta^2)A\phi U$, we have two possible subcases:

Subcase 3.1. $\beta^2 = 1$. In this case, $AU = \beta\xi$.

Subcase 3.2. $\beta^2 \neq 1$ and $AU = \beta\xi + \sigma U$, where $\sigma = (1 - \beta^2)\delta$.

If we take $Y = \phi X \in \mathbb{D}_U$ in (3.1) for $X \in \mathbb{D}_U$ such that $AX = \lambda X$, we have $\lambda + \alpha\lambda g(\phi X, A\phi X) = 0$. This yields that either any eigenvalue in $\mathbb{D}_U$ is 0 or that if there exists a non-null eigenvalue λ in $\mathbb{D}_U$, $\alpha \neq 0$ and $\lambda = -\frac{1}{\alpha}$. In this case, the eigenspace corresponding to this eigenvalue is ϕ-invariant.

Let us suppose that for any $Y \in \mathbb{D}_U$ $AY = 0$. As $g((\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y, \xi) = -2$, we obtain

$$g([\phi Y, Y], U) = -\frac{2}{\beta}. \tag{3.15}$$

And as $g((\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y, U) = 0$, it follows

$$\sigma g([\phi Y, Y], U) = 0. \tag{3.16}$$

From (3.15) and (3.16), we get $\sigma = 0$. If also $\beta^2 \neq 1$, $\delta = 0$ and our real hypersurface should be ruled.

Let us then suppose that $\beta^2 = 1$. As above, we will take $\beta = 1$ and $\sigma = 0$. If we develop $g((\nabla_Y A)\phi U - (\nabla_{\phi U} A)Y, \xi) = 0$, we get

$$g(\nabla_Y \phi U, U) = g(\nabla_{\phi U} Y, U) \tag{3.17}$$

and from $g((\nabla_Y A)\phi U - (\nabla_{\phi U} A)Y, U) = 0$, it follows

$$\delta g(\nabla_Y \phi U, U) = 0. \tag{3.18}$$

From (3.17) and (3.18), suppose $g(\nabla_Y \phi U, U) = g(\nabla_{\phi U} Y, U) = 0$. As $g((\nabla_{\phi U} A)U - (\nabla_U A)\phi U, \xi) = 2$, we obtain

$$g(\nabla_U \phi U, U) = 2 + \alpha\delta \tag{3.19}$$

and from $g((\nabla_{\phi U} A)U - (\nabla_U A)\phi U, U) = 0$, we have

$$2\delta + \delta g(\nabla_U \phi U, U) = 0. \tag{3.20}$$

If $\delta \neq 0$, from (3.20) $g(\nabla_U \phi U, U) = -2$ and from (3.19)

$$\alpha\delta = -4. \tag{3.21}$$

Now, $g((\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U, U) = 1$ gives $\delta g(\nabla_\xi \phi U, U) = 2 - \alpha\delta$. From (3.21)

$$\delta g(\nabla_\xi \phi U, U) = 6. \tag{3.22}$$

But from $g((\nabla_\xi A)U - (\nabla_U A)\xi, \phi U) = 1$, we obtain

$$-\delta g(\nabla_\xi U, \phi U) = 0. \tag{3.23}$$

From (3.22) and (3.23), we arrive to a contradiction. Thus, $\delta = 0$ and M is also a ruled real hypersurface.

Therefore, we have only to study the following case: $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \sigma U$, $A\phi U = \delta\phi U$, $\mathbb{D}_U$ is A-invariant, and there exists $Z \in \mathbb{D}_U$ such that $AZ = -\frac{1}{\alpha}Z$, $A\phi Z = -\frac{1}{\alpha}\phi Z$. As $(1-\beta^2)A\phi U = \phi AU$, two subcases appear

Subcase 1. $\beta^2 = 1$, and then $\sigma = 0$.

Subcase 2. $\beta^2 \neq 1$, $\sigma = (1-\beta^2)\delta$.

From $g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, \xi) = -2$, we obtain

$$\beta g([\phi Z, Z], U) = \frac{2}{\alpha^2} \tag{3.24}$$

and from $g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, U) = 0$, we get

$$\left(\frac{1}{\alpha} + \sigma\right) g([\phi Z, Z], U) = \frac{2\beta}{\alpha}. \tag{3.25}$$

From (3.24) and (3.25), we obtain

$$1 + \alpha\sigma = \alpha^2\beta^2. \tag{3.26}$$

In Subcase 1, as $\beta^2 = 1$ and $\sigma = 0$, we should obtain $\alpha^2 = 1$. Changing, if necessary, ξ by $-\xi$, we can take $\alpha = 1$. This case cannot occur by Proposition 3.2, page 1607 in [11]. Therefore, we have $\beta^2 \neq 1$ and from (3.26) $1 + \alpha\delta(1-\beta^2) = \alpha^2\beta^2$. Thus,

$$\delta = \frac{\alpha^2\beta^2 - 1}{\alpha(1-\beta^2)}. \tag{3.27}$$

Now, $g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, \phi U) = 0$ yields

$$\left(\frac{1}{\alpha} + \delta\right) g([Z, \phi Z], \phi U) = 0. \tag{3.28}$$

Let us suppose that $\delta = -\frac{1}{\alpha}$. Then, $\sigma = \frac{\beta^2-1}{\alpha}$ and from (3.26) $\alpha^2 = 1$. As above, we suppose $\alpha = 1$. Thus, $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = -\phi U$, and there exists a unit $Z \in \mathbb{D}_U$ such that $AZ = -Z$, $A\phi Z = -\phi Z$.

Suppose that there exists a unit $W \in \mathbb{D}_U$ such that $AW = A\phi W = 0$. From $g((\nabla_W A)\xi - (\nabla_\xi A)W, \xi) = 0$, we obtain $g(\nabla_\xi W, U) = 0$, and from $g((\nabla_W A)\xi - (\nabla_\xi A)W, U) = 0$, we get $W(\beta) + (\beta^2 - 1)g(\nabla_\xi W, U) = 0$. Thus, $W(\beta) = 0$. This fact and the proof of Proposition 3.3, page 1608 in [11], yield $grad(\beta) = -(2\beta^2 + 1)\phi U$. The same proof yields this case cannot occur. Therefore, $\delta \neq -\frac{1}{\alpha}$ and $g([Z, \phi Z], \phi U) = 0$.

Then, from $g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, Z) = g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, \phi Z) = 0$, we get

$$Z(\alpha) = (\phi Z)(\alpha) = 0. \tag{3.29}$$

From $g((\nabla_Z A)\xi - (\nabla_\xi A)Z, \xi) = 0$, it follows

$$Z(\alpha) + \beta g(\nabla_\xi Z, U) = 0. \tag{3.30}$$

From (3.29) and (3.30), we obtain

$$g(\nabla_\xi Z, U) = 0. \tag{3.31}$$

As $g((\nabla_Z A)\xi - (\nabla_\xi A)Z, U) = 0$, we have, bearing in mind (3.31),

$$Z(\beta) = 0. \tag{3.32}$$

From $g((\nabla_\xi A)U - (\nabla_U A)\xi, \xi) = 0$, we get

$$\xi(\beta) = U(\alpha) \tag{3.33}$$

and as $g((\nabla_\xi A)U - (\nabla_U A)\xi, U) = 0$, it follows

$$\xi(\sigma) = U(\beta). \tag{3.34}$$

Now, $g((\nabla_Z A)U - (\nabla_U A)Z, \xi) = 0$ yields

$$Z(\beta) + \sigma g(\nabla_U Z, U) = 0 \tag{3.35}$$

and from (3.26) and $g((\nabla_Z A)U - (\nabla_U A)Z, U) = 0$, we obtain

$$Z(\sigma) + \alpha\beta^2 g(\nabla_U Z, U) = 0. \tag{3.36}$$

From (3.32) and (3.36), we have $\sigma g(\nabla_U Z, U) = 0$. This and (3.36) yield $Z(\sigma) + \frac{1}{\alpha} g(\nabla_U Z, U) = 0$. As $Z(\alpha) = Z(\beta) = 0$, from (3.26) $Z(\sigma) = 0$. Therefore,

$$g(\nabla_U Z, U) = 0. \tag{3.37}$$

As $g((\nabla_Z A)U - (\nabla_U A)Z, \phi U) = 0$, this gives

$$(\sigma - \delta)g(\nabla_Z U, \phi U) + \left(\delta + \frac{1}{\alpha}\right) g(\nabla_U Z, \phi U) = 0 \tag{3.38}$$

and $g((\nabla_Z A)U - (\nabla_U A)Z, Z) = 0$ yields

$$U\left(\frac{1}{\alpha}\right) = \left(\sigma + \frac{1}{\alpha}\right) g(\nabla_Z Z, U). \tag{3.39}$$

From $g((\nabla_Z A)\xi - (\nabla_\xi A)Z, Z) = 0$, we obtain

$$\xi\left(\frac{1}{\alpha}\right) = \beta g(\nabla_Z Z, U) \tag{3.40}$$

and from $g((\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z, U) = 0$, we get

$$(\delta - \sigma) g(\nabla_Z \phi U, U) + \left(\sigma + \frac{1}{\alpha}\right) g(\nabla_{\phi U} Z, U) = 0. \tag{3.41}$$

We also have from $g((\nabla_Z A)\phi U) - (\nabla_{\phi U} A)Z, \xi) = 0$

$$g([\phi U, Z], U) = 0. \tag{3.42}$$

Thus, from (3.41) and (3.42), we have a homogeneous system of linear equations where $g(\nabla_Z \phi U, U)$ and $g(\nabla_{\phi U} Z, U)$ are unknown. The determinant of its matrix of coefficients is $\delta + \frac{1}{\alpha}$. As $\delta \neq -\frac{1}{\alpha}$, we obtain

$$g(\nabla_Z \phi U, U) = g(\nabla_{\phi U} Z, U) = 0. \tag{3.43}$$

As $g((\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z, \phi Z) = 0$, we have $(\delta + \frac{1}{\alpha}) g(\nabla_Z \phi U, \phi Z) = 0$. As $\delta \neq -\frac{1}{\alpha}$, $g(\nabla_Z \phi U, \phi Z) = 0$. By (2.3), this gives $g(\nabla_Z U, Z) = 0$. From (3.39) and (3.40), it follows

$$\xi(\alpha) = U(\alpha) = 0. \tag{3.44}$$

From $g((\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z, Z) = 0$, we have $(\delta + \frac{1}{\alpha}) g(\nabla_Z \phi U, Z) + (\phi U)(\frac{1}{\alpha}) = 0$. Now, from (2.3)

$$(\phi U)\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha} + \delta\right) g(\nabla_Z U, \phi Z). \tag{3.45}$$

Developing $g((\nabla_Z A)U - (\nabla_U A)Z, \phi Z) = 0$ and bearing in mind (3.26), we get

$$\alpha^2 \beta^2 g(\nabla_Z U, \phi Z) = \beta. \tag{3.46}$$

Now, from (3.45) and (3.46), we obtain

$$(\phi U)(\alpha) = -\frac{1 + \alpha\delta}{\alpha\beta} = \frac{\beta(1 - \alpha^2)}{\alpha(1 - \beta^2)}. \tag{3.47}$$

From (3.33) and (3.44), we have

$$\xi(\beta) = U(\beta) = 0. \tag{3.48}$$

The equality $g((\nabla_\xi A)U - (\nabla_U A)\xi, \phi U) = 1$ yields

$$\beta g(\nabla_U U, \phi U) = \beta^2 + \sigma^2 - \alpha\sigma - 1. \tag{3.49}$$

From $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, \xi) = -2$, we arrive to $-2\delta\sigma + \alpha\sigma + \alpha\delta - \beta g(\nabla_U \phi U, U) - (\phi U)(\beta) = 2$. This and (3.49) yield

$$(\phi U)(\beta) = -2\delta\sigma + \alpha\delta + \beta^2 + \sigma^2 + 1. \tag{3.50}$$

Bearing in mind all these facts, we arrive to

$$grad(\alpha) = \rho\phi U$$
$$grad(\beta) = \theta\phi U \tag{3.51}$$

where $\rho = -(\frac{1+\alpha\delta}{\alpha\beta})$ and $\theta = -2\delta\sigma + \alpha\delta + \beta^2 + \sigma^2 + 1$. As $g(\nabla_X grad(\alpha), Y) = g(\nabla_Y grad(\alpha), X)$ for any X, Y tangent to M, we have, taking $X = \xi$, $\xi(\rho)g(\phi U, Y) + \rho g(\nabla_\xi \phi U, Y) = -\rho g(U, AY)$. If $Y = \phi U$, this yields $\xi(\rho) = 0$. Thus, $\rho g(\nabla_\xi \phi U, Y) = -\rho g(U, AY)$, for any Y tangent to M. As $\rho \neq 0$, taking $Y = U$, we get

$$g(\nabla_\xi \phi U, U) = -\sigma. \tag{3.52}$$

From $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, \xi) = 0$ and bearing in mind (3.52), we have

$$(\phi U)(\alpha) = -3\beta\delta + \alpha\beta - \beta\sigma. \tag{3.53}$$

From (3.47) and (3.53) $2 + \alpha\delta + \alpha\sigma - 3\alpha\beta^2\delta + \alpha\sigma\beta^2 = 0$, or equivalently

$$\alpha^2(2\beta^2 - 3\beta^4 - \beta^6) + \beta^2(2 + \beta^2) - 1 = 0. \tag{3.54}$$

If $2 - 3\beta^2 - \beta^4 = 0$, we should have $\beta^4 + 2\beta^2 - 1 = 0$. Both equalities yield $\beta^2 = 1$, that is impossible. From (3.54), we have

$$\alpha^2 = \frac{1 - \beta^2(2 + \beta^2)}{2\beta^2 - 3\beta^4 - \beta^6}. \tag{3.55}$$

If we take the derivative of (3.54) in the direction of ϕU and bear in mind (3.47), (3.48), and (3.54), we find that β is a root of a polynomial with constant coefficients. Therefore, β is constant. From (3.55), α is also constant, which is impossible.

Thus, we have proved that if M is not Hopf, it is locally congruent to a ruled real hypersurface. It is easy to see that these real hypersurfaces satisfy (3.1).

Let us now suppose that M is a Hopf real hypersurface with $A\xi = \alpha\xi$ and that M satisfies (3.1). Then, we have for any $X \in \mathbb{D}$ that $\phi AX + \alpha A\phi AX = 0$. If $\alpha = 0$, we get $\phi AX = 0$. Thus, $AX = 0$ for any $X \in \mathbb{D}$ and M should be totally geodesic, which is impossible.

Suppose now that $\alpha \neq 0$ and take a unit $X \in \mathbb{D}$ such that $AX = \lambda X$. From (3.1), we get either $\lambda = 0$ or $A\phi X = -\frac{1}{\alpha}\phi X$. Applying the same reasoning to ϕX, we obtain that the eigenspaces in $\mathbb{D}$ are ϕ-invariant and correspond to the eigenvalues 0 and $-\frac{1}{\alpha}$. This is impossible by Theorem 2.2, and we finish the proof.

4 Proof of Theorem 2

If we suppose that $R_\xi F_\xi^{(k)} = F_\xi^{(k)} R_\xi$, we get

$$g(Y, \phi A\xi)\xi + g(A\xi, \xi)g(\phi A\xi, AY)\xi + \eta(Y)\phi A\xi + \eta(Y)\eta(A\xi)A\phi A\xi$$
$$- \eta(Y)\eta(A\phi A\xi)A\xi - k\phi R_\xi(Y) + kR_\xi(\phi Y) = 0 \tag{4.1}$$

for any $Y \in TM$. Let us suppose that M is non-Hopf. Thus, we write $A\xi = \alpha\xi + \beta U$ for a unit $U \in \mathbb{D}$ and functions α and β on M, β being nonvanishing. From (4.1), we have

$$\beta g(Y, \phi U)\xi + \alpha\beta g(\phi U, AY)\xi + \beta\eta(Y)\phi U$$
$$+\alpha\beta\eta(Y)A\phi U = k\phi R_\xi(Y) - kR_\xi(\phi Y) \tag{4.2}$$

for any $Y \in TM$. Taking $Y = \xi$ in (4.2), we obtain

$$\beta\phi U + \alpha\beta A\phi U = 0. \tag{4.3}$$

As $\beta \neq 0$, this yields

$$\begin{aligned} \alpha &\neq 0, \\ A\phi U &= -\frac{1}{\alpha}\phi U. \end{aligned} \tag{4.4}$$

If now we take $Y = U$ in (4.2), as $k \neq 0$, we get $\phi R_\xi(U) = R_\xi(\phi U)$. This yields $\alpha\phi AU = (\beta^2 - 1)\phi U$, that is, $\phi AU = \frac{\beta^2-1}{\alpha}\phi U$. By applying ϕ to such an equality, it follows

$$AU = \beta\xi + \frac{\beta^2 - 1}{\alpha}U. \tag{4.5}$$

From (4.4) and (4.5), we obtain that $\mathbb{D}_U$ is ϕ-invariant and A-invariant. Take a unit $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$. Introducing this Y in (4.2), we get $\phi R_\xi(Y) = R_\xi(\phi Y)$. This yields $\alpha\lambda\phi Y = \alpha A\phi Y$. As $\alpha \neq 0$, $A\phi Y = \lambda\phi Y$. Therefore, the eigenspaces in $\mathbb{D}_U$ are ϕ-invariant.

The Codazzi equation gives $(\nabla_\xi A)\phi Y - (\nabla_{\phi Y} A)Y = -2\xi$. Taking its scalar product with ϕY, respectively, with Y, we have

$$Y(\lambda) = (\phi Y)(\lambda) = 0. \tag{4.6}$$

Its scalar product with ξ implies

$$\beta g([\phi Y, Y], U) = 2\lambda^2 - 2\alpha\lambda - 2 \tag{4.7}$$

and its scalar product with U gives

$$\left(\lambda - \frac{\beta^2 - 1}{\alpha}\right) g([\phi Y, Y], U) = 2\beta\lambda. \tag{4.8}$$

From (4.7) and (4.8), we get

$$(\alpha\lambda + 1)(\lambda^2 - \alpha\lambda - 1) = \beta^2(\lambda^2 - 1). \tag{4.9}$$

From $g((\nabla_{\phi U} A)Y - (\nabla_Y A)\phi U, \phi Y) = 0$, we have $(\lambda + \frac{1}{\alpha})g(\nabla_Y \phi U, \phi Y) = 0$. Then, either $g(\nabla_Y \phi U, \phi Y) = g(\nabla_Y U, Y) = 0$, where we have applied (2.3) or $\lambda = -\frac{1}{\alpha}$. In this second case from (4.9), we have $0 = \beta^2(\lambda^2 - 1)$. As $\beta \neq 0$, this yields $\lambda^2 = 1$ and $\alpha^2 = 1$. Changing, if necessary, ξ by $-\xi$, we can suppose $\alpha = 1$ and then $\lambda = -1$.

The scalar product of $(\nabla_\xi A)Y - (\nabla_Y A)\xi = \phi Y$ and Y gives

$$\xi(\lambda) - \beta g(\nabla_Y U, Y) = 0. \tag{4.10}$$

As either $\lambda = -1$ or $g(\nabla_Y U, Y) = 0$, we always have

$$\xi(\lambda) = 0. \tag{4.11}$$

Developing $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$ and taking its scalar product with ϕU, we get

$$\alpha U\left(\frac{1}{\alpha}\right) = \beta^2 g(\nabla_{\phi U}\phi U, U) \tag{4.12}$$

The same procedure applied to $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, \phi U) = 0$ yields

$$\xi\left(\frac{1}{\alpha}\right) = \beta g(\nabla_{\phi U}\phi U, U). \tag{4.13}$$

From (4.12) and (4.13), we obtain

$$\alpha U(\alpha) = \beta\xi(\alpha). \tag{4.14}$$

From $g((\nabla_\xi A)U - (\nabla_U A)\xi, \xi) = 0$, $\xi(\beta) = U(\alpha)$ and from (4.14), we have

$$\beta\xi(\alpha) = \alpha\xi(\beta). \tag{4.15}$$

By derivating (4.9) in the direction of ξ and bearing in mind (4.11) and (4.15), it follows

$$(\lambda(\lambda^2 - \alpha\lambda - 1) - \lambda(\alpha\lambda + 1))\xi(\alpha) = \frac{2\beta^2}{\alpha}(\lambda^2 - 1)\xi(\alpha). \tag{4.16}$$

If we suppose $\xi(\alpha) \neq 0$ and bear in mind (4.9), from (4.16), we have

$$\alpha\lambda^3 - 2\alpha\lambda + 2\lambda^2 - 2 = 0. \tag{4.17}$$

Derivating (4.17) in the direction of ξ, we obtain $(\lambda^3 - 2\lambda)\xi(\alpha) = 0$. As we suppose $\xi(\alpha) \neq 0$, we have $\lambda(\lambda^2 - 2) = 0$. If $\lambda = 0$ from (4.9), it follows $\beta^2 = 1$ and β should be constant. From (4.15), $\xi(\alpha) = 0$, and we arrive to a contradiction. Therefore, $\lambda^2 = 2$. From (4.9), we obtain $1 - 2\alpha^2 = \beta^2$. By derivating this equality in the direction of ξ and bearing in mind (4.15) and that we suppose $\xi(\alpha) \neq 0$, we get $-2\alpha^2 = \beta^2$ and we have a new contradiction. This proves

$$\xi(\alpha) = \xi(\beta) = U(\alpha) = 0. \tag{4.18}$$

The equality $g((\nabla_\xi A)Y - (\nabla_Y A)\xi, \xi) = 0$ yields

$$Y(\alpha) = -\beta g(\nabla_\xi Y, U). \tag{4.19}$$

Analogously, from $g((\nabla_\xi A)Y - g(\nabla_Y A)\xi, U) = 0$, we obtain

$$Y(\beta) = \left(\lambda - \frac{\beta^2 - 1}{\alpha}\right) g(\nabla_\xi Y, U). \tag{4.20}$$

From (4.19) and (4.20), we get

$$\beta Y(\beta) = \left(\frac{\beta^2 - 1}{\alpha} - \lambda\right) Y(\alpha). \tag{4.21}$$

As $Y(\lambda) = 0$, from (4.9), it follows

$$(\lambda(\lambda^2 - \alpha\lambda - 1) - \lambda(\alpha\lambda + 1))Y(\alpha) = (\lambda^2 - 1)2\beta Y(\beta) \tag{4.22}$$

and from (4.21) and (4.22), if we suppose $Y(\alpha) \neq 0$, we have

$$\alpha(3\lambda^2 - 2\alpha\lambda^2 - 4\lambda) = 2(\alpha\lambda + 1)(\lambda^2 - \alpha\lambda - 1) - 2(\lambda^2 - 1). \tag{4.23}$$

Derivating once again in the direction of Y and bearing in mind that we suppose $Y(\alpha) \neq 0$, we obtain $\lambda^3 = 0$, that is, $\lambda = 0$ and $\beta^2 = 1$. Therefore, β is constant and $Y(\beta) = 0$.

From $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, \xi) = 0$, we have

$$(\phi U)(\alpha) = \frac{3\beta}{\alpha} + \alpha\beta + \beta g(\nabla_\xi U, \phi U). \tag{4.24}$$

And $\beta^2 = 1$ and $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, U) = -1$ yield

$$g(\nabla_\xi U, \phi U) = -\alpha. \tag{4.25}$$

From (4.24) and (4.25), we conclude

$$(\phi U)(\alpha) = \frac{3\beta}{\alpha}. \tag{4.26}$$

As $g((\nabla_\xi A)U - (\nabla_U A)\xi, \phi U) = 1$, we get

$$\frac{1}{\alpha} g(\nabla_\xi U, \phi U) - \beta g(\nabla_U U, \phi U) = 0 \tag{4.27}$$

and from the Codazzi equation $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, U) = 0$, it follows

$$g(\nabla_U U, \phi U) = 2\beta. \tag{4.28}$$

From (4.27) and (4.28), bearing in mind that $\beta^2 = 1$, we have

$$g(\nabla_\xi U, \phi U) = 2\alpha. \tag{4.29}$$

Now, from (4.27) and (4.29), α should vanish. As this is a contradiction, we arrive to

$$Y(\alpha) = Y(\beta) = 0 \tag{4.30}$$

By linearity, we have $X(\alpha) = X(\beta) = 0$ for any $X \in \mathbb{D}_U$.
The Codazzi equation $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, U) = -1$ yields

$$(\phi U)(\beta) = \frac{\beta^2 - 1}{\alpha^2} + \beta^2 + \frac{\beta^2}{\alpha} g(\nabla_\xi U, \phi U). \tag{4.31}$$

As $g((\nabla_\xi A)U - (\nabla_U A)\xi, \phi U) = 1$, we have

$$\frac{\beta^2}{\alpha} g(\nabla_\xi U, \phi U) - \beta g(\nabla_U U, \phi U) = \frac{\beta^2 - 1}{\alpha^2} \tag{4.32}$$

and $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, U) = 0$ shows

$$\beta g(\nabla_U U, \phi U) + \beta^2 - 3 - 2(\phi U)(\beta) - \frac{\beta^2 - 1}{\alpha\beta}(\phi U)(\alpha) = 0. \tag{4.33}$$

From (4.31), (4.32), and (4.33), we have

$$\beta g(\nabla_U U \phi U) - \frac{\beta^2 - 1}{\alpha} g(\nabla_\xi U, \phi U) + \frac{\beta^2 - 1}{\alpha^2} - 4 = 0. \tag{4.34}$$

Now, from (4.33) and (4.34), it follows $g(\nabla_\xi U, \phi U) = -4\alpha$ and $g(\nabla_U U, \phi U) = \frac{1-\beta^2}{\alpha^2\beta} - 4\beta$. Then, from (4.24) and (4.32), we get

$$(\phi U)(\alpha) = 3\beta \left(\frac{1 - \alpha^2}{\alpha} \right) \tag{4.35}$$

and

$$(\phi U)(\beta) = -3\beta^2 + \frac{\beta^2 - 1}{\alpha^2}. \tag{4.36}$$

From all the facts we have until now, we obtain $grad(\alpha) = \omega\phi U$, where $\omega = 3\beta(\frac{1-\alpha^2}{\alpha})$. As $g(\nabla_X grad(\alpha), Y) = g(\nabla_Y grad(\alpha), X)$ for any $X, Y \in TM$, we get

$X(\omega)g(\phi U, Y) - Y(\omega)g(\phi U, X) + \omega(g(\nabla_X \phi U, Y) - g(\nabla_Y \phi U, X)) = 0$. Taking $Y = \xi$, this yields $\omega(g(\nabla_X \phi U, \xi) - g(\nabla_\xi \phi U, X)) = 0$ for any $X \in TM$. Thus, either $\omega = 0$ or $g(\nabla_X \phi U, \xi) = g(\nabla_\xi \phi U, X)$ for any $X \in TM$. If we take $X = U$, we have $-g(U, AU) = g(\nabla_\xi \phi U, U)$. Then, $4\alpha^2 + \beta^2 = 1$. This yields $4\alpha(\phi U)(\alpha) + \beta(\phi U)(\beta) = 0$. From (4.35) and (4.36), we have $9\alpha^2 + \beta^2 = 1$. Therefore, $\alpha = 0$, which is impossible. So we have $\omega = 0$ and $\alpha^2 = 1$. From (4.35) $(\phi U)(\alpha) = 0$ and from (4.36) $(\phi U)(\beta) = -(2\beta^2 + 1)$. Then, from (4.18) and the fact that $g((\nabla_\xi A)U - (\nabla_U A)\xi, U) = 0$, we have $U(\beta) = 0$ and then $grad(\beta) = -(2\beta^2 + 1)\phi U$.

Applying the same reasoning to $grad(\beta)$, $-(1 + 2\beta^2)(g(\nabla_X \phi U, \xi) - g(\nabla_\xi \phi U, X)) = 0$ for any $X \in TM$. This yields $g(\nabla_X \phi U, \xi) = g(\nabla_\xi \phi U, X)$ for any $X \in TM$. Taking $X = U$, it follows $4\alpha^2 + \beta^2 = 1$ and being $\alpha^2 = 1$, $\beta^2 = -3$, which is impossible and proves that M must be Hopf.

If M is Hopf with $A\xi = \alpha\xi$, from (4.1), we get $\phi R_\xi = R_\xi \phi$. Let $Y \in \mathbb{D}$ a unit vector field such that $AY = \lambda Y$. Therefore, $\alpha\lambda Y = \alpha A\phi Y$. Then, either $\alpha = 0$ and M is locally congruent to a tube of radius $\frac{\pi}{4}$ around a complex submanifold of $\mathbb{C}P^m$, see [2], or $A\phi = \phi A$ and from Theorem 2.1, M is locally congruent to a type (A) real hypersurface.

It is very easy to see that these real hypersurfaces satisfy (4.1), and we finish the proof. □

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