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Mathematical treatment of the homogeneous Boltzmann equation for Maxwellian molecules in the presence of singular kernels

Emanuele Dolera

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Abstract This paper proves the existence of weak solutions to the the spatially homogeneous Boltzmann equation for Maxwellian molecules, when the initial data are chosen from the space of all Borel probability measures on $\mathbb{R}^3$ with finite second moments, and the (angular) collision kernel satisfies a very weak cutoff condition, namely $\int_{-1}^{1} x^2 b(x)\mathrm{d}x < +\infty$. For the equation at issue, the uniqueness of the solution corresponding to a specific initial datum has been recently established in Fournier and Guérin (J Stat Phys 131:749–781, 2008). Finally, conservation of momentum and energy is also proved for these weak solutions, without resorting to any boundedness of the entropy.

Keywords Boltzmann equation · Maxwellian molecules · Moments · Sum of random variables · Uniform integrability · Very weak cutoff · Weak solution

Mathematics Subject Classification 35Q20 · 82C40 · 60G50

Contents

E. Dolera (✉)
Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, via Campi 213/b, 41125 Modena, Italy
e-mail: emanuele.dolera@unimore.it; emanuele.dolera@unipv.it

1 Introduction and main results

This paper deals with the *spatially homogeneous Boltzmann equation* for *Maxwellian molecules* (SHBEMM), commonly written as

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = \int_{\mathbb{R}^3} \int_{S^2} [f(\mathbf{v}_*, t) f(\mathbf{w}_*, t) - f(\mathbf{v}, t) f(\mathbf{w}, t)] \times b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{w}, \quad (\mathbf{v}, t) \in \mathbb{R}^3 \times (0, +\infty) \tag{1}$$

with initial datum $f(\mathbf{v}, 0) = f_0(\mathbf{v})$. Existence and evolution of low-order moments of its solutions are the main topics at issue, in the event that *grazing collisions* are significantly taken into account. Uniqueness of the solution has been established in [15]. In spite of a vast literature on the subject, very few papers aim at minimizing as much as possible the set of hypotheses on both the initial datum and the collision kernel, as this work intends to do.

As to the symbols in (1), u_{S^2} denotes the uniform measure (i.e., the normalized Riemannian measure) on the unit sphere S^2, embedded in $\mathbb{R}^3$. The post-collisional velocities $\mathbf{v}_*$ and $\mathbf{w}_*$ are defined according to the $\boldsymbol{\omega}$ representation:

$$\mathbf{v}_* := \mathbf{v} + [(\mathbf{w}-\mathbf{v}) \cdot \boldsymbol{\omega}]\boldsymbol{\omega} \quad \mathbf{w}_* := \mathbf{w} - [(\mathbf{w}-\mathbf{v}) \cdot \boldsymbol{\omega}]\,\boldsymbol{\omega} \tag{2}$$

where $\cdot$ designates the standard scalar product. The solution $f(\mathbf{v}, t)$ is a probability density function, in the $\mathbf{v}$-variable, which characterizes the probability law of a single molecule's velocity, randomly chosen in a chaotic bath of like molecules. See [7,8,36] for an exhaustive explanation. The *(angular) collision kernel b* is an even measurable function from $[-1, 1]$ into $[0, +\infty]$, which plays a central role in the study of the Maxwellian molecules. Originally, this name was reserved for molecules repelling each other with a force inversely proportional to the fifth power of their distance, after Maxwell [21] had evaluated the exact expression of b in this peculiar case. Nowadays, the word Maxwellian indicates the presence of a generic kernel depending only on $\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|} \cdot \boldsymbol{\omega}$, as in (1). The present work deals with collision kernels satisfying

$$\int_{-1}^{1} x^2 b(x)\mathrm{d}x < +\infty, \tag{3}$$

i.e., a *very weak angular cutoff*, which is the weakest assumption on b considered so far in the literature, starting from [11]. The motivation for considering assumption (3) is manifold. It is well known that the form of b may influence the smoothness of the solution $f(\mathbf{v}, t)$ w.r.t. the $\mathbf{v}$-variable (the main reason adduced in [11] for studying singular kernels) and governs the way in which $f(\mathbf{v}, t)$ approaches the equilibrium when t goes to infinity (as shown in [12,26]). Therefore, it would be desirable to discover the minimal, essential conditions on b, which originate the aforesaid mathematical properties of the solutions. As far as physical arguments are concerned, it is worth noticing that the exact expression of the collision kernel has been evaluated only for a very narrow class of interactions, the most representative of which are those of pure repulsive type with potential energy $V_s(r) = \kappa r^{1-s}$, where $\kappa > 0$, $s > 2$ and r stands for the relative distance between two interacting particles. See, e.g., Subsection 1.4 and Section 3 of Chapter 2A of [36], Section 5 of Chapter 2 of [7], and Chapter 2 of [9]. The specific angular collision kernel relative to V_s, say b_s, possesses a unique singularity at $x = 0$ in such a way that $b(x) \sim |x|^{-\frac{s+1}{s-1}}$ and meets (3) for every $s > 2$. In particular, this is true for the Maxwellian interaction, which corresponds to $s = 5$. Thus, it can be reasonable to study Eq. (1) with b_s in place of b as a first approximation of the true Boltzmann equation with hard or soft interactions (so disregarding the presence of the kinetic collision kernel, according to the terminology used in Section 3 of Chapter 2A of [36]), to understand the influence of the angular collision kernel on the solutions, inasmuch it is a common belief among physicists that there is hardly any influence at all. Finally, condition (3) is equivalent, in the present setting, to the finiteness of the *collision kernel for momentum transfer*, a basic quantity in the theory of atomic collisions. The reader is referred to a specialized text in plasma physics, such as [31], for a precise definition and explicit computations via experimental measurements. Here, suffice it to say that such a finiteness has been proved to be a necessary condition for the RHS of (1) to make sense from a mathematical point of view. See [35], Annex I, Appendix A. The collision kernel for momentum transfer is also important in regard to the *asymptotics of grazing collisions*, an asymptotic regime in which the Boltzmann equation turns into a Landau equation of plasma physics. This theory, initiated in [10], is developed in [2,17,18,34] whence the need of a well-consolidated theory of the SHBEMM with a collision kernel satisfying (3). To complete the presentation, it remains to introduce the proper space for the initial data, to be considered throughout this paper as the class $\mathcal{P}_2(\mathbb{R}^3)$ of all Borel probability measures (p.m.'s) on $\mathbb{R}^3$ with finite second moments. Since an element of this space is not necessarily absolutely continuous and not constrained to any finite entropy condition, the first task of our work will consist in a weak reformulation of (1). The motivations to aim at such generality are both theoretical and practical: For example, in [6], it is expressly remarked that "in view of statistical physics, initial data are best chosen from the largest class, say the positive, finite Borel measures on $\mathbb{R}^3$," while in [25], the authors underline the importance of dropping finite entropy conditions "since no control of entropy can be expected in the explicit Euler scheme." In fact, some noteworthy papers, such as [1,6,25], proved important facts without assuming the finiteness of the entropy of the initial datum.

The very weak cutoff condition, in conjunction with a minimization of the hypotheses on the initial datum, leads to study a larger class of solutions than the usual one, arising in the context of integrable or at least not too singular collision kernels. Actually, this enlargement makes the problems of existence and uniqueness more challenging from a mathematical point of view and introduces new difficulties in determining the properties of such solutions. For example, a rigorous proof that they preserve momentum and energy, in the absence of extra condition on f_0, is still lacking. This fact has even been doubted in [11], where one wonders whether the energy may decrease. Besides, general initial data in $\mathcal{P}_2(\mathbb{R}^3)$ can be completely managed when b is summable (*Grad cutoff assumption*), thanks to a consolidate knowledge on the subject which started with the works [4,19,23,27,38] and culminated with [30,33]. The same extension in the *weak cutoff* case, which corresponds to assuming $\int_{-1}^{1} |x|b(x)\mathrm{d}x < +\infty$, is treated in [5,32,33]. Coming to the case of kernels satisfying (3), a general line of reasoning to tackle existence questions was devised by Arkeryd [3], who considered a sequence of integrable truncations of the kernel b, say $\{b_n\}_{n\geq 1}$, to obtain a sequence of auxiliary solutions approximating the real (unknown) solution. One of the main difficulties in this approach is to show some weak compactness of the approximating sequence, in order to get a converging subsequence. Actually, the more natural form of compactness in Boltzmann's equation can be derived from the boundedness of the entropy, as successfully done in pioneering works such as [11,14,18,34]. However, when the initial datum is only an element of $\mathcal{P}_2(\mathbb{R}^3)$, not constrained to a finite entropy condition, the only available form of compactness ought to be derived from the conservation of momentum and energy, as first proposed in [30] and then developed in [5]. In the wake of this line of research, the present work proposes a weak reformulation of (1) which fits the Arkeryd approach, with the contrivance to corroborate the weak compactness with a form of uniform integrability of the second absolute moments of the approximating solutions.

To start with this plan, the restricted class of collision kernels satisfying Grad's cutoff condition will be dealt with at first. Throughout the paper, this integrability condition will be written, without affecting the generality, as

$$\int_0^1 b(x)\mathrm{d}x = 1. \tag{4}$$

Under this condition, the standard weak reformulation, due to Maxwell, reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v},t) &= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [\psi(\mathbf{v}_*)+\psi(\mathbf{w}_*)-\psi(\mathbf{v})-\psi(\mathbf{w})]\,\mathbb{1}\{\mathbf{v}\neq\mathbf{w}\} \\ &\quad\times b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mu(\mathrm{d}\mathbf{v},t)\mu(\mathrm{d}\mathbf{w},t). \end{aligned}\tag{5}$$

It is derived from (1) by multiplying both sides by some regular function $\psi:\mathbb{R}^3\to\mathbb{C}$, integrating formally in the $\mathbf{v}$-variable and putting $f(\mathbf{v},t)\mathrm{d}\mathbf{v}=\mu(\mathrm{d}\mathbf{v},t)$. In this setting, the initial datum can be any Borel p.m. μ_0 on $\mathbb{R}^3$ ($\mu_0\in\mathcal{P}(\mathbb{R}^3)$, in symbols), and a solution is intended according to

Definition 1 (*Weak solution, the cutoff case*) When b satisfies (4), a weak solution of (1) is defined to be any family $\{\mu(\cdot,t)\}_{t\geq 0}$ of Borel p.m.'s on $\mathbb{R}^3$ such that

(i) $\mu(\cdot,0)=\mu_0(\cdot)$;
(ii) $t\mapsto\int_{\mathbb{R}^3}\psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v},t)$ is continuous on $[0,+\infty)$ and continuously differentiable on $(0,+\infty)$, for all ψ bounded and continuous ($\psi\in \mathrm{C}_b(\mathbb{R}^3;\mathbb{C})$ in symbols);
(iii) $\mu(\cdot,t)$ satisfies (5) for all $t>0$ and for all $\psi\in\mathrm{C}_b(\mathbb{R}^3;\mathbb{C})$.

Coming back to the original aim to study the SHBEMM under hypothesis (3), it has been easily understood that the RHS of (5) could loose a precise meaning (within the standard Lebesgue integration theory) in the presence of a too singular kernel b. See the comments in [18,34] and in Subsection 4.1 of Chapter 2B of [36]. In particular, the key idea contained in both [18,34] is to define a formal rule to retrieve the aforesaid meaning even when the integrand in the RHS of (5) is not Lebesgue-summable. This rule is explained in Sect. 2.2, which also provides a justification for the following

Definition 2 (*Weak solution, the singular case*) Let b satisfy (3), and let μ_0 be any element of $\mathcal{P}_2(\mathbb{R}^3)$. Then, a weak solution of (1) is defined to be any family $\{\mu(\cdot,t)\}_{t\geq 0}$ of Borel p.m.'s on $\mathbb{R}^3$ such that

(i) $\mu(\cdot,0)=\mu_0(\cdot)$;
(ii) $t\mapsto\int_{\mathbb{R}^3}\psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v},t)$ is continuous on $[0,+\infty)$ and continuously differentiable on $(0,+\infty)$, for all complex-valued ψ with bounded derivatives up to the order two ($\psi\in\mathrm{C}_b^2(\mathbb{R}^3;\mathbb{C})$ in symbols);
(iii) $\int_{\mathbb{R}^3}|\mathbf{v}|^2\mu(\mathrm{d}\mathbf{v},t)<+\infty$ for all $t\geq 0$;

(iv) $\mu(\cdot, t)$ satisfies, for all $t > 0$ and for all $\psi \in \mathrm{C}_b^2(\mathbb{R}^3; \mathbb{C})$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \psi(\mathbf{v}) \mu(\mathrm{d}\mathbf{v}, t) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{1} \mathrm{d}s \mathrm{d}\theta \mathrm{d}\xi\, \mu(\mathrm{d}\mathbf{v}, t) \mu(\mathrm{d}\mathbf{w}, t)\, 1\!\!1\{\mathbf{v} \neq \mathbf{w}\}$$
$$\times\, b(\xi)\xi^2(1 - s) \Bigg[\nabla\psi(\mathbf{v}_*(s\xi)) \cdot \frac{\mathrm{d}^2\mathbf{v}_*}{\mathrm{d}x^2}(s\xi) + \nabla\psi(\mathbf{w}_*(s\xi)) \cdot \frac{\mathrm{d}^2\mathbf{w}_*}{\mathrm{d}x^2}(s\xi)$$
$$+ \left(\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(s\xi) \right)^t \mathrm{Hess}[\psi](\mathbf{v}_*(s\xi)) \left(\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(s\xi) \right)$$
$$+ \left(\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(s\xi) \right)^t \mathrm{Hess}[\psi](\mathbf{w}_*(s\xi)) \left(\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(s\xi) \right) \Bigg]. \tag{6}$$

The RHS of (6) has now a precise mathematical meaning within the standard Lebesgue integration theory in view of point (iii) of this last definition, as shown in Lemma 6. Furthermore, when $\int_{-1}^{1} |x| b(x) \mathrm{d}x < +\infty$ holds, the RHS in (5) is well defined—without invoking (6)—for any test function ψ which is bounded and Lipschitz-continuous. Hence, in the weak cutoff context, any initial datum satisfying $\int_{\mathbb{R}^3} |\mathbf{v}| \mu_0(\mathrm{d}\mathbf{v}) < +\infty$ is allowed, with the proviso that condition (iii) of Definition 2 is relaxed to $\int_{\mathbb{R}^3} |\mathbf{v}| \mu(\mathbf{v}, t) < +\infty$ for all $t \geq 0$. See [32]. There are now the elements to state the new results, condensed in

Theorem 3 *Let b satisfy (3), and let μ_0 be any element of $\mathcal{P}_2(\mathbb{R}^3)$. Then, there exists a unique solution $\{\mu(\cdot, t)\}_{t \geq 0}$ of (1) with initial datum μ_0, in the sense of* Definition 2. *This solution can be obtained upon defining $B_n := \int_0^1 [b(x) \wedge n] \mathrm{d}x$ and $\{\mu_n(\cdot, t)\}_{t \geq 0}$ as the unique solution of (1), in the sense of* Definition 1, *with $[b(x) \wedge n]/B_n$ and μ_0 as collision kernel and initial datum, respectively. Indeed, there exists an increasing subsequence $\{n_l\}_{l \geq 1}$ of positive integers such that $\lim_{l \to +\infty} \int_{\mathbb{R}^3} \psi(\mathbf{v}) \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) = \int_{\mathbb{R}^3} \psi(\mathbf{v}) \mu(\mathrm{d}\mathbf{v}, t)$ for all $\psi \in \mathrm{C}_b(\mathbb{R}^3; \mathbb{C})$ and $t \geq 0$. Moreover, if R is any orthogonal 3×3 matrix and $f_R(\mathbf{v}) := R\mathbf{v}$, then $\{\mu(\cdot, t) \circ f_R^{-1}\}_{t \geq 0}$ is the solution of (1) with $\mu_0 \circ f_R^{-1}$ as initial datum. In addition, momentum and kinetic energy are preserved, i.e.,*

$$\int_{\mathbb{R}^3} \mathbf{v} \mu(\mathrm{d}\mathbf{v}, t) = \int_{\mathbb{R}^3} \mathbf{v} \mu_0(\mathrm{d}\mathbf{v}) := \overline{\mathbf{V}} \quad \text{and} \quad \int_{\mathbb{R}^3} |\mathbf{v}|^2 \mu(\mathrm{d}\mathbf{v}, t) = \int_{\mathbb{R}^3} |\mathbf{v}|^2 \mu_0(\mathrm{d}\mathbf{v}) \tag{7}$$

are in force for all $t \geq 0$, and

$$\lim_{t \to +\infty} \left[\int_{\mathbb{R}^3} v_i v_j \mu(\mathrm{d}\mathbf{v}, t) - \overline{V}_i \overline{V}_j \right] = \frac{\delta_{ij}}{3} \int_{\mathbb{R}^3} |\mathbf{v} - \overline{\mathbf{V}}|^2 \mu_0(\mathrm{d}\mathbf{v}) \tag{8}$$

holds for every $i, j \in \{1, 2, 3\}$, δ_{ij} standing for the Kronecker delta. Finally, there exists a positive constant $C(\mu_0)$ and a continuous, non-decreasing function $q : [0, +\infty) \to [0, +\infty)$, with $\lim_{x \to +\infty} q(x) = +\infty$, which are both determinable only on the basis of the knowledge of μ_0, such that

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 q(|\mathbf{v}|) \mu(\mathrm{d}\mathbf{v}, t) \leq C(\mu_0) \tag{9}$$

is valid for all $t \geq 0$, leading to

$$\lim_{R \to +\infty} \sup_{t \geq 0} \int_{|\mathbf{v}| \geq R} |\mathbf{v}|^2 \mu(\mathrm{d}\mathbf{v}, t) = 0. \tag{10}$$

To comment on this statement, it seems to be the first rigorous result of existence for the SHBEMM, which, at the same time, is valid for kernels satisfying (3) and initial data restricted to the sole condition of finite energy, and gives a genuine physical meaning to the solutions, thanks to the conservation of momentum and kinetic energy encapsulated in (7). The uniqueness of the solution for this general case of the SHBEMM has been established in the recent paper [15]. The property expressed by (8) can be seen as a weak form of *propagation of chaos*, as well as a macroscopic version of the principle of *equipartition of the energy*. The validity of (9)–(10) expresses the desired uniform integrability previously invoked to retrieve the proper form of compactness in the Arkeryd approach. From a physical standpoint, (10) shows that the amount of energy actually due to the tails of the distribution remains uniformly small in time. This property is explicitly remarked in [6] with a view to proving relaxation to equilibrium, but the authors confine themselves to proving its validity in the presence of smooth kernels with cutoff. The elimination of these extra conditions on b, to comprehend all the cases of physical relevance, seems a novelty of this paper, which completes the fruitful line of reasoning set forth in [6].

The proof of Theorem 3 relies heavily on a probabilistic representation of the solutions recently proposed in [12]. For the sake of completeness, it is also summed up in Sect. 2.1 of this paper. Though the analogies between kinetic theory and probability are well known in the case of Maxwellian molecules, starting from the pioneering works by McKean [22,23], some decisive improvements concerning the properties of the solutions have come only after the specific formulation of the representation in [12]. In fact, it is slightly different from the original one in [23] and enjoys the property of being particulary effective for generalizing inequalities of Povzner-Elmroth-type about the uniform boundedness of the moments. Cf. Proposition 6 in [12] and Lemma 8 of the present paper. It is also interesting to remark the role of a very popular formula about the use of the Fourier transform, known as *Bobylev's identity*, in the proof of Theorem 3, since many works on the SHBEMM take advantage from it. Actually, the Fourier representation has nothing to do with the weak form (6), but is crucially hidden into the probabilistic representation borrowed from [12], as shown in Sect. 2.1 of that paper.

A last remark is about the placement of these results in the literature. Actually, the main points of Theorem 3 can be found in various works, which prove them under somewhat different hypotheses. See [3,5,11,14,19,23,27,28,30,32–34,38]. Other papers even consider these statements as folklore. A particular mention is reserved to the recent paper [28], appeared when the present article was a first draft, as the statements of existence and uniqueness contained therein are rather similar to those in Theorem 3, even if [28] starts from a different weak formulation. It also contains an adaptation of the Arkeryd strategy to the same context of Theorem 3. Truthfully, my original aim was twofold: to complete some points expressly mentioned in [12], and to seize this opportunity to deal with those points within a framework more general than the required one. I have decided to carry through my own work even after the publication of [28] since I found the short proof therein not completely satisfactory. In fact, it seems not clear what kind of integral the author is adopting: on the one hand, Lemma 2.2 turns out to be false if integrals are of Lebesgue-type (see the remark after Lemma 9 of the present paper). On the other hand, if they are thought of as improper Riemann integrals, then there is a crucial—i.e., the core of the proof—exchange of limit with integral, immediately after (23) therein lacking in explanation. These shortcomings are not easy to restore, since this would require a kind of uniform convergence of the approximating sequence not yet proved. In any case, the problem can be solved, as I do here, by following a different strategy that shows, in addition, that the solutions conserve momentum and energy.

The rest of the paper is organized as follows: section 2 contains some complements that, on the one hand, are aimed at introducing concepts and notation to be used in the real proof and, on the other hand, provide a justification for (6). Section 3 is devoted to the proof of Theorem 3 along with the proof of two technical results formulated in Section 2.

2 Complements to the introduction

2.1 The Wild sum and the probabilistic representation

Assume that (4) holds and, on noting that $\int_{S^2} b(\mathbf{u}\cdot\boldsymbol{\omega})u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = 1$ for all $\mathbf{u} \in S^2$, rewrite the SHBEMM as $\frac{\partial}{\partial t} f(\mathbf{v},t) = Q[f(\cdot,t), f(\cdot,t)](\mathbf{v}) - f(\mathbf{v})$ with

$$Q[p,q](\mathbf{v}) := \int_{\mathbb{R}^3}\int_{S^2} p(\mathbf{v}_*)q(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{w}.$$

The bilinear operator Q sends the couple (p,q) of probability densities into a new probability density on $\mathbb{R}^3$. Then, to include initial data that are not absolutely continuous p.m.'s, define the operator $\mathcal{Q}$, which sends a pair (ζ,η) in $\mathcal{P}(\mathbb{R}^3)\times\mathcal{P}(\mathbb{R}^3)$ into a new element of $\mathcal{P}(\mathbb{R}^3)$ according to

$$\mathcal{Q}[\zeta,\eta](\mathrm{d}\mathbf{v}) := \text{w-lim}_{n\to\infty} Q[p_n,q_n](\mathbf{v})\mathrm{d}\mathbf{v} \tag{11}$$

where p_n (q_n, respectively) denotes the density of ζ_n (η_n, respectively), $\{\zeta_n\}_{n\geq 1}$ and $\{\eta_n\}_{n\geq 1}$ being two sequences of absolutely continuous p.m.'s such that ζ_n (η_n, respectively) converges weakly to ζ (η, respectively). Recall that a statement as "ζ_n converges weakly to ζ" ($\zeta_n \Rightarrow \zeta$, in symbols) means that $\int_{\mathbb{R}^3}\psi(\mathbf{v})\zeta_n(\mathrm{d}\mathbf{v}) \to \int_{\mathbb{R}^3}\psi(\mathbf{v})\zeta(\mathrm{d}\mathbf{v})$ for every $\psi \in \mathrm{C}_b(\mathbb{R}^3;\mathbb{C})$. The following result shows that $\mathcal{Q}[\zeta,\eta]$ is well defined.

Lemma 4 *Let b meet* (4)*. Then, the limit in* (11) *exists and is independent of the choice of the approximating sequences* $\{\zeta_n\}_{n\geq1}$ *and* $\{\eta_n\}_{n\geq1}$*, and*

$$\int_{\mathbb{R}^3}\psi(\mathbf{v})\mathcal{Q}[\zeta,\eta](\mathrm{d}\mathbf{v}) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\psi(\mathbf{v}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\zeta(\mathrm{d}\mathbf{v})\eta(\mathrm{d}\mathbf{w}) \tag{12}$$

holds with $\psi \in \mathrm{C}_b(\mathbb{R}^3;\mathbb{C})$. *Moreover, if R and* f_R *are as in* Theorem 3*, then*

$$\mathcal{Q}\left[\zeta\circ f_R^{-1},\eta\circ f_R^{-1}\right] = \mathcal{Q}[\zeta,\eta]\circ f_R^{-1}. \tag{13}$$

The proof of this lemma is deferred to Sect. 3.1. A remarkable corollary is the Bobylev identity [4], namely

$$\hat{\mathcal{Q}}[\zeta,\eta](\boldsymbol{\xi}) = \int_{S^2}\hat{\zeta}(\boldsymbol{\xi}-(\boldsymbol{\xi}\cdot\boldsymbol{\omega})\boldsymbol{\omega})\hat{\eta}((\boldsymbol{\xi}\cdot\boldsymbol{\omega})\boldsymbol{\omega})b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega}), \tag{14}$$

which is valid for every $\boldsymbol{\xi} \in \mathbb{R}^3\setminus\{\mathbf{0}\}$. Here, $\hat{}$ denotes the Fourier transform according to $\hat{\zeta}(\boldsymbol{\xi}) := \int_{\mathbb{R}^3} \mathrm{e}^{i\boldsymbol{\xi}\cdot\mathbf{x}}\zeta(\mathrm{d}\mathbf{x})$. To proceed, one can put

$$\begin{aligned}\mathcal{Q}_1[\mu_0] &:= \mu_0\\ \mathcal{Q}_n[\mu_0] &:= \frac{1}{n-1}\sum_{j=1}^{n-1}\mathcal{Q}\left[\mathcal{Q}_j[\mu_0],\mathcal{Q}_{n-j}[\mu_0]\right] \quad \text{for } n\geq 2,\end{aligned} \tag{15}$$

to state the following

Proposition 5 *When b satisfies* (4) *and μ_0 is in $\mathcal{P}(\mathbb{R}^3)$, the unique solution of* (1)*, in the meaning of* Definition 1*, is given by the* Wild sum

$$\mu(\cdot, t) := \sum_{n=1}^{+\infty} e^{-t}(1-e^{-t})^{n-1}\mathcal{Q}_n[\mu_0](\cdot) \quad (t \geq 0). \tag{16}$$

The proof of this proposition is provided in Sect. 3.2. The last tool to introduce is the probabilistic representation of $\mu(\cdot, t)$ borrowed from [12]. Here is only a short presentation of both ideas and notation. The reader is referred to Subsection 1.5 of [12], which shows in addition how to deduce it by cleverly manipulating the Wild sum and the Bobylev identity. The core of the representation, valid only upon assuming (4), is encapsulated in the identity

$$\hat{\mu}(\rho\mathbf{u}, t) = \mathsf{E}_t\left[\mathrm{e}^{i\rho S(\mathbf{u})}\right] \quad (\rho \in \mathbb{R}, \mathbf{u} \in S^2, t \geq 0) \tag{17}$$

where $S(\mathbf{u})$ is a random sum of weighted random variables and E_t is an expectation, for every $t \geq 0$. To define these two objects, consider the sample space $\Omega := \mathbb{N} \times \mathbb{T} \times [0, \pi]^\infty \times (0, 2\pi)^\infty \times (\mathbb{R}^3)^\infty$ endowed with the σ-algebra $\mathscr{F} := 2^{\mathbb{N}} \otimes 2^{\mathbb{T}} \otimes \mathscr{B}([0, \pi]^\infty) \otimes \mathscr{B}((0, 2\pi)^\infty) \otimes \mathscr{B}((\mathbb{R}^3)^\infty)$ where X^∞ stands for the set of all sequences $(x_1, x_2, \dots)$ with elements in X, 2^X is the power set and $\mathscr{B}(X)$ the Borel class on X. Then, $\mathbb{T} := \mathsf{X}_{n\geq 1}\mathbb{T}(n)$ and $\mathbb{T}(n)$ is the (finite) set of all *McKean binary trees* with n leaves, whose generic element will be indicated as $\mathfrak{t}_n$. Denoting by ν, $\{\tau_n\}_{n\geq 1}$, $\{\phi_n\}_{n\geq 1}$, $\{\vartheta_n\}_{n\geq 1}$, $\{\mathbf{V}_n\}_{n\geq 1}$ the coordinate random variables of Ω, consider for any $t \geq 0$ the unique probability distribution (p.d.) P_t on $(\Omega, \mathscr{F})$, which makes these random elements stochastically independent, consistently with the following marginal p.d.'s:

(a) $\mathsf{P}_t[\nu = n] = \mathrm{e}^{-t}(1 - \mathrm{e}^{-t})^{n-1}$ for $n = 1, 2, \dots$, with the proviso that $0^0 := 1$.
(b) $\{\tau_n\}_{n\geq 1}$ is a Markov sequence driven by the initial condition $\mathsf{P}_t[\tau_1 = \mathfrak{t}_1] = 1$ and the transition probabilities

$$\mathsf{P}_t[\tau_{n+1} = \mathfrak{t}_{n,k} \mid \tau_n = \mathfrak{t}_n] = \frac{1}{n} \quad \text{for } k = 1, \dots, n$$
$$\mathsf{P}_t[\tau_{n+1} = \mathfrak{t}_{n+1} \mid \tau_n = \mathfrak{t}_n] = 0 \quad \text{if } \mathfrak{t}_{n+1} \notin \mathbb{G}(\mathfrak{t}_n)$$

where, for a given $\mathfrak{t}_n$, $\mathfrak{t}_{n,k}$ indicates the germination of $\mathfrak{t}_n$ at its k-th leaf, obtained by appending a two-leaved tree to the k-th leaf of $\mathfrak{t}_n$, and $\mathbb{G}(\mathfrak{t}_n)$ is the subset of $\mathbb{T}(n+1)$ containing all the germinations of $\mathfrak{t}_n$.
(c) The elements of $\{\phi_n\}_{n\geq 1}$ are independent and identically distributed (i.i.d.) random numbers with p.d. $\beta(\mathrm{d}\varphi) := \frac{1}{2}b(\cos\varphi)\sin\varphi \mathrm{d}\varphi$, with $\varphi \in [0, \pi]$.
(d) The elements of $\{\vartheta_n\}_{n\geq 1}$ are i.i.d. with uniform p.d. on $(0, 2\pi)$.
(e) The elements of $\{\mathbf{V}_n\}_{n\geq 1}$ are i.i.d. with p.d. μ_0, the initial datum for (1).

Whence, E_t is defined as the expectation w.r.t. P_t. As for $S(\mathbf{u})$, consider the array $\boldsymbol{\pi} := \{\pi_{j,n} \mid j = 1, \dots, n; n \in \mathbb{N}\}$ of $[-1, 1]$-valued random numbers obtained by setting $\pi_{j,n} := \pi^*_{j,n}(\tau_n, (\phi_1, \dots, \phi_{n-1}))$ for $j = 1, \dots, n$ and n in $\mathbb{N}$. The $\pi^*_{j,n}$s are functions on $\mathbb{T}(n) \times [0, \pi]^{n-1}$ given by $\pi^*_{1,1} \equiv 1$ and, for $n \geq 2$,

$$\pi^*_{j,n}(\mathfrak{t}_n, \boldsymbol{\varphi}) := \begin{cases} \pi^*_{j,n_l}(\mathfrak{t}^l_n, \boldsymbol{\varphi}^l)\cos\varphi_{n-1} & \text{for } j = 1, \dots, n_l \\ \pi^*_{j-n_l,n_r}(\mathfrak{t}^r_n, \boldsymbol{\varphi}^r)\sin\varphi_{n-1} & \text{for } j = n_l + 1, \dots, n \end{cases}$$

for every $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^l, \boldsymbol{\varphi}^r, \varphi_{n-1})$ in $[0, \pi]^{n-1}$, with $\boldsymbol{\varphi}^l := (\varphi_1, \dots, \varphi_{n_l-1})$ and $\boldsymbol{\varphi}^r := (\varphi_{n_l}, \dots, \varphi_{n-2})$ where $\mathfrak{t}^l_n$ and $\mathfrak{t}^r_n$ symbolize the two trees, of n_l and n_r leaves, respectively, obtained by deleting the root node of $\mathfrak{t}_n$. Apropos of the $\pi_{j,n}$'s, one can show, for every $n \in \mathbb{N}$, the validity of the identity

$$\sum_{j=1}^{n} \pi_{j,n}^2 = 1. \tag{18}$$

Another constituent of the desired representation is the array $\mathbf{O} := \{O_{j,n} \mid j = 1, \dots, n; n \in \mathbb{N}\}$ of random matrices $O_{j,n}$, taking values in the Lie group $\mathbb{SO}(3)$ of orthogonal matrices with positive determinant, defined by $O_{j,n} := O^*_{j,n}(\tau_n, (\phi_1, \dots, \phi_{n-1}), (\vartheta_1, \dots, \vartheta_{n-1}))$ for $j = 1, \dots, n$ and n in $\mathbb{N}$. The $O^*_{j,n}$s are $\mathbb{SO}(3)$-valued functions obtained by putting $O^*_{1,1} \equiv \mathrm{Id}_{3\times 3}$ and, for $n \geq 2$,

$$O^*_{j,n}(\mathfrak{t}_n, \boldsymbol{\varphi}, \boldsymbol{\theta}) := \begin{cases} M^l(\varphi_{n-1}, \theta_{n-1}) O^*_{j,n_l}(\mathfrak{t}^l_n, \boldsymbol{\varphi}^l, \boldsymbol{\theta}^l) & \text{for } j = 1, \dots, n_l \\ M^r(\varphi_{n-1}, \theta_{n-1}) O^*_{j-n_l,n_r}(\mathfrak{t}^r_n, \boldsymbol{\varphi}^r, \boldsymbol{\theta}^r) & \text{for } j = n_l + 1, \dots, n \end{cases}$$

for every $\mathfrak{t}_n$ in $\mathbb{T}(n)$, $\boldsymbol{\varphi}$ in $[0, \pi]^{n-1}$ and $\boldsymbol{\theta}$ in $(0, 2\pi)^{n-1}$. Here, $\boldsymbol{\theta}^l := (\theta_1, \dots, \theta_{n_l-1})$ and $\boldsymbol{\theta}^r := (\theta_{n_l}, \dots, \theta_{n-2})$, and finally,

$$M^l(\varphi, \theta) := \begin{pmatrix} -\cos\theta\cos\varphi & \sin\theta & \cos\theta\sin\varphi \\ -\sin\theta\cos\varphi & -\cos\theta & \sin\theta\sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$M^r(\varphi, \theta) := \begin{pmatrix} \sin\theta & \cos\theta\sin\varphi & -\cos\theta\cos\varphi \\ -\cos\theta & \sin\theta\sin\varphi & -\sin\theta\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{pmatrix}.$$

As a final step, choose a non-random measurable function B from S^2 onto $\mathbb{SO}(3)$ such that $B(\mathbf{u})\mathbf{e}_3 = \mathbf{u}$ for every $\mathbf{u}$ in S^2, and define the random functions $\boldsymbol{\psi}_{j,n} : S^2 \to S^2$ through the relation $\boldsymbol{\psi}_{j,n}(\mathbf{u}) := B(\mathbf{u}) O_{j,n} \mathbf{e}_3$ for $j = 1, \dots, n$ and n in $\mathbb{N}$, with $\mathbf{e}_3 := (0, 0, 1)^t$, to get

$$S(\mathbf{u}) := \sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j. \tag{19}$$

2.2 Justification for the weak form (6)

Starting from (5), fix $\mathbf{v} \neq \mathbf{w}$ and rewrite the integral

$$\int_{S^2} [\psi(\mathbf{v}_*) + \psi(\mathbf{w}_*) - \psi(\mathbf{v}) - \psi(\mathbf{w})] b \left(\frac{\mathbf{w} - \mathbf{v}}{|\mathbf{w} - \mathbf{v}|} \cdot \boldsymbol{\omega} \right) u_{S^2}(d\boldsymbol{\omega}) \tag{20}$$

by the (formal) change in variable $\boldsymbol{\omega} \leftrightarrow (\theta, \varphi)$ given by

$$\boldsymbol{\omega}(\theta, \varphi, \mathbf{u}) := \cos\theta \sin\varphi \mathbf{a}(\mathbf{u}) + \sin\theta \sin\varphi \mathbf{b}(\mathbf{u}) + \cos\varphi \mathbf{u},$$

where $(\theta, \varphi) \in (0, 2\pi) \times [0, \pi]$, $\mathbf{u} := \frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}$ and $\{\mathbf{a}(\mathbf{u}), \mathbf{b}(\mathbf{u}), \mathbf{u}\}$ is an orthonormal basis of $\mathbb{R}^3$. The identities in (2) become

$$\mathbf{v}_* = \mathbf{v} + |\mathbf{w} - \mathbf{v}| \cos\varphi \boldsymbol{\omega}(\theta, \varphi, \mathbf{u}) \quad \mathbf{w}_* = \mathbf{w} - |\mathbf{w} - \mathbf{v}| \cos\varphi \boldsymbol{\omega}(\theta, \varphi, \mathbf{u})$$

and so, putting $x = \cos\varphi$, Taylor's formula with integral remainder yields

$$\psi(\mathbf{v}_*(x)) = \psi(\mathbf{v}) + x\left(\nabla\psi(\mathbf{v})\cdot\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(0)\right) + x^2\int_0^1(1-s)\frac{\mathrm{d}^2\psi(\mathbf{v}_*)}{\mathrm{d}x^2}(sx)\mathrm{d}s$$

$$\psi(\mathbf{w}_*(x)) = \psi(\mathbf{w}) + x\left(\nabla\psi(\mathbf{w})\cdot\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(0)\right) + x^2\int_0^1(1-s)\frac{\mathrm{d}^2\psi(\mathbf{w}_*)}{\mathrm{d}x^2}(sx)\mathrm{d}s.$$

Then, define the expression (20) as iterated integral by *first integrating w.r.t. to θ and then w.r.t. φ*. At this stage, observe that

$$\int_0^{2\pi}\left(\nabla\psi(\mathbf{v})\cdot\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(0)\right)\mathrm{d}\theta = \int_0^{2\pi}\left(\nabla\psi(\mathbf{w})\cdot\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(0)\right)\mathrm{d}\theta = 0$$

holds, since $\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(0)$ and $\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(0)$ are given, up to a factor $\pm|\mathbf{w}-\mathbf{v}|$, by $\cos\theta\,\mathbf{a}(\mathbf{u}) + \sin\theta\,\mathbf{b}(\mathbf{u})$. Whence, the first-order terms in the above Taylor expansion will not contribute to the reformulation of the RHS of (5). As for the second-order terms, suffice it to observe that the chain rule for the second derivative gives

$$\frac{\mathrm{d}^2\psi(\mathbf{v}_*)}{\mathrm{d}x^2}(x) = \nabla\psi(\mathbf{v}_*(x))\cdot\frac{\mathrm{d}^2\mathbf{v}_*}{\mathrm{d}x^2}(x) + \left(\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(x)\right)^t \mathrm{Hess}[\psi](\mathbf{v}_*(x))\left(\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(x)\right)$$

$$\frac{\mathrm{d}^2\psi(\mathbf{w}_*)}{\mathrm{d}x^2}(x) = \nabla\psi(\mathbf{w}_*(x))\cdot\frac{\mathrm{d}^2\mathbf{w}_*}{\mathrm{d}x^2}(x) + \left(\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(x)\right)^t \mathrm{Hess}[\psi](\mathbf{w}_*(x))\left(\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(x)\right)$$

for all $x \in (-1, 1)$. Hence, for a given $\psi \in \mathrm{C}_b^2(\mathbb{R}^3; \mathbb{C})$, the RHS of (5) turns into the RHS of (6), which is now well defined in view of the following

Lemma 6 *Let b satisfy (3), and let χ belong to $\mathcal{P}_2(\mathbb{R}^3)$. Then,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-1}^{1}\int_0^{2\pi}\int_0^1 \left[\left|\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(s\xi)\right|^2 + \left|\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(s\xi)\right|^2 + \left|\frac{\mathrm{d}^2\mathbf{v}_*}{\mathrm{d}x^2}(s\xi)\right| + \left|\frac{\mathrm{d}^2\mathbf{w}_*}{\mathrm{d}x^2}(s\xi)\right|\right]$$
$$\times\, 1\!\!1\{\mathbf{v}\neq\mathbf{w}\}(1-\mathrm{s})\xi^2\mathrm{b}(\xi)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\xi\,\chi(\mathrm{d}\mathbf{v})\chi(\mathrm{d}\mathbf{w}) < +\infty.$$

proof of this lemma is deferred to Sect. 3.5.

3 Proofs

This section gathers the proofs of the various statements, which are scattered through the rest of the paper. Precisely, Sect. 3.1 provides the proof of Lemma 4. Sections 3.2–3.4 are aimed at proving Theorem 3 under the additional hypothesis (4). Section 3.5 contains the proof of Lemma 6. Finally, the real proof of Theorem 3 is developed in Sect. 3.6.

3.1 Proof of Lemma 4

The following are basic facts in the theory of the Boltzmann equation. First, the map $\mathrm{T}_{\boldsymbol{\omega}} : (\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{v}_*, \mathbf{w}_*)$ is a linear diffeomorphism of $\mathbb{R}^6$ with $\mathrm{T}_{\boldsymbol{\omega}} \circ \mathrm{T}_{\boldsymbol{\omega}} = \mathrm{Id}_{\mathbb{R}^6}$, for every $\boldsymbol{\omega} \in S^2$. Second, $(\mathbf{w}-\mathbf{v})\cdot\boldsymbol{\omega} = -(\mathbf{w}_*-\mathbf{v}_*)\cdot\boldsymbol{\omega}$ and $|\mathbf{w}-\mathbf{v}| = |\mathbf{w}_*-\mathbf{v}_*|$ hold for every $(\mathbf{v}, \mathbf{w}, \boldsymbol{\omega}) \in \mathbb{R}^6 \times S^2$. Under assumption (4), the combination of these basic relations with

Fubini and Tonelli's theorems entails that $Q[p, q](\mathbf{v})$ is itself a probability density function, if p and q are both so, and that

$$\int_{\mathbb{R}^3} \psi(\mathbf{v}) Q[p, q](\mathbf{v})\mathrm{d}\mathbf{v} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \psi(\mathbf{v}_*) b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) p(\mathbf{v})q(\mathbf{w})u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}$$

holds for every $\psi \in \mathrm{C}_b(\mathbb{R}^3; \mathbb{C})$. This equation can be taken as the core of the proof, after showing that

$$H_\psi : (\mathbf{v}, \mathbf{w}) \mapsto \begin{cases} \int_{S^2} \psi(\mathbf{v}_*) b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) & \text{if } \mathbf{v} \neq \mathbf{w} \\ \psi(\mathbf{v}) & \text{if } \mathbf{v} = \mathbf{w} \end{cases}$$

is bounded and continuous. Indeed, the continuity at some $(\mathbf{v}_0, \mathbf{v}_0)$ is obvious and can be checked directly. Then, to prove this continuity claim also at some $(\mathbf{v}_0, \mathbf{w}_0)$ with $\mathbf{v}_0 \neq \mathbf{w}_0$, change the coordinate in the spherical integral as in Sect. 2.2 to obtain

$$H_\psi(\mathbf{v}_0, \mathbf{w}_0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \psi\Big(\mathbf{v}_0 + |\mathbf{w}_0 - \mathbf{v}_0|\cos\varphi\Big[\cos\theta\sin\varphi\mathbf{a}_0 + \sin\theta\sin\varphi\mathbf{b}_0 + \cos\varphi\mathbf{u}_0\Big]\Big) b(\cos\varphi)\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta$$

where $\mathbf{u}_0 := \frac{\mathbf{w}_0-\mathbf{v}_0}{|\mathbf{w}_0-\mathbf{v}_0|}$ and $\{\mathbf{a}_0, \mathbf{b}_0, \mathbf{u}_0\}$ is an orthonormal basis of $\mathbb{R}^3$. Since it is always possible to define two measurable functions $\mathbf{a}, \mathbf{b} : S^2 \to S^2$ which are continuous in a neighborhood of $\mathbf{u}_0$, satisfy $\mathbf{a}(\mathbf{u}_0) = \mathbf{a}_0$ and $\mathbf{b}(\mathbf{u}_0) = \mathbf{b}_0$, and are such that $\{\mathbf{a}(\mathbf{u}), \mathbf{b}(\mathbf{u}), \mathbf{u}\}$ is an orthonormal basis of $\mathbb{R}^3$ for every $\mathbf{u} \in S^2$, the continuity claim about H_ψ follows by a dominated convergence argument. Now, the limit of the sequence $a_n := \int_{\mathbb{R}^3} \psi(\mathbf{v})Q[p_n, q_n](\mathbf{v})\mathrm{d}\mathbf{v}$, as n goes to infinity, exists for every $\psi \in \mathrm{C}_b(\mathbb{R}^3; \mathbb{C})$, by virtue of the identity $a_n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} H_\psi(\mathbf{v}, \mathbf{w})\zeta_n(\mathrm{d}\mathbf{v})\eta_n(\mathrm{d}\mathbf{w})$, and the fact that these integrals are converging to $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} H_\psi(\mathbf{v}, \mathbf{w})\zeta(\mathrm{d}\mathbf{v})\eta(\mathrm{d}\mathbf{w})$. To conclude that the limit is of the form $\int_{\mathbb{R}^3} \psi(\mathbf{v})\lambda(\mathrm{d}\mathbf{v})$, for some specific $\lambda \in \mathcal{P}(\mathbb{R}^3)$, which will be henceforth denoted by $\mathcal{Q}[\zeta, \eta]$, one can choose ψ as $\psi_{\boldsymbol{\xi}}(\mathbf{v}) := \mathrm{e}^{i\boldsymbol{\xi}\cdot\mathbf{v}}$ and invoke Lévy's continuity theorem (cf. Theorem 5.22 in [20]). The only point that deserves some care in this application is the continuity of $\lim_{n\to+\infty}\int_{\mathbb{R}^3} \mathrm{e}^{i\boldsymbol{\xi}\cdot\mathbf{v}} Q[p_n, q_n](\mathbf{v})\mathrm{d}\mathbf{v}$ w.r.t. $\boldsymbol{\xi}$, which is tantamount to saying that $\boldsymbol{\xi} \mapsto \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} H_{\psi_{\boldsymbol{\xi}}}(\mathbf{v}, \mathbf{w})\zeta(\mathrm{d}\mathbf{v})\eta(\mathrm{d}\mathbf{w})$ is continuous. Since the check of this fact boils down to an obvious application of the dominated convergence theorem, the existence of the limit in (11) is guaranteed along with its independence of the approximating sequences, for $\int_{\mathbb{R}^3} \psi(\mathbf{v})\mathcal{Q}[\zeta, \eta](\mathrm{d}\mathbf{v})$ has been shown to depend on (ζ, η) only.

As for (13), the weak continuity of $\mathcal{Q}$ w.r.t. (ζ, η) reduces the problem to check the validity of $Q[p_S, q_S](\mathbf{v}) = Q[p, q](S\mathbf{v})$ for every orthogonal matrix $S \in \mathbb{O}(3)$, where p_S (q_S, respectively) denotes the density function $p(S\mathbf{v})$ ($q(S\mathbf{v})$, respectively). Then, the change in variable $\boldsymbol{\omega} \leftrightarrow S\boldsymbol{\omega}$ in the spherical integral entails that $Q[p_S, q_S](\mathbf{v})$ is equal to

$$\int_{\mathbb{R}^3}\int_{S^2} p(S\mathbf{v} + [(S\mathbf{w} - S\mathbf{v})\cdot\boldsymbol{\omega}]\,\boldsymbol{\omega}) \times q(S\mathbf{w} - [(S\mathbf{w} - S\mathbf{v})\cdot\boldsymbol{\omega}]\,\boldsymbol{\omega}) b\left(\frac{S\mathbf{w} - S\mathbf{v}}{|S\mathbf{w} - S\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{w}$$

and, to conclude, it is enough to change again the coordinates in the integral on $\mathbb{R}^3$ according to $\mathbf{z} = S\mathbf{w}$.

Finally, to prove (14), consider again $H_{\psi_{\boldsymbol{\xi}}}$ with $\boldsymbol{\xi} \neq \mathbf{0}$. An application of the change in variable $\boldsymbol{\omega} \leftrightarrow R\boldsymbol{\omega}$, where $R \in \mathbb{O}(3)$ is such that $R^t \frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ shows that $H_{\psi_{\boldsymbol{\xi}}}(\mathbf{v}, \mathbf{w}) = \mathrm{e}^{i\boldsymbol{\xi}\cdot\mathbf{v}} \int_{S^2} \mathrm{e}^{i(\boldsymbol{\xi}\cdot\boldsymbol{\omega})[(\mathbf{w}-\mathbf{v})\cdot\boldsymbol{\omega}]} b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\boldsymbol{\omega}\right)$.

3.2 Existence and uniqueness in the cutoff case

This subsection provides, at the same time, the proof of Proposition 5 and the proof of existence and uniqueness in Theorem 3, under the assumption (4). Of course, the term "solution" is here intended according to Definition 1.

As to the existence, observe that the validity of $\sum_{n=1}^{+\infty} \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{n-1} = 1$ for every $t \in [0, +\infty)$ entails that the series in (16) is a mixture, which defines a family $\{\mu(\cdot, t)\}_{t\geq 0}$ of Borel p.m.'s on $\mathbb{R}^3$ such that $\mu(\cdot, t) = \mu_0(\cdot)$. Then, consider $F_\psi(t) := \int_{\mathbb{R}^3} \psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v}, t)$ with $\psi \in \mathrm{C}_b(\mathbb{R}^3; \mathbb{C})$ not identically zero, and put $a_n := \int_{\mathbb{R}^3} \psi(\mathbf{v})\mathcal{Q}_n[\mu_0](\mathrm{d}\mathbf{v})$ and $||\,\psi\,||_\infty := \sup_{\mathbf{v}\in\mathbb{R}^3} |\psi(\mathbf{v})|$. Since the radius of convergence of the power series $\sum_{n=0}^{+\infty} a_{n+1} z^n$ is at least 1, in view of $\limsup_{n\to\infty} \sqrt[n]{|a_{n+1}|} \leq \limsup_{n\to\infty} \sqrt[n]{||\,\psi\,||_\infty} = 1$, it follows that the convergence of the series is uniform when $t \in [-\log(1+M), -\log(1-M)]$, for every $M \in (0, 1)$, and F_ψ is analytic in $(-\log 2, +\infty)$. This proves point (ii) of Definition 1 while, as far as point (iii) is concerned, the analyticity allows the exchange of the time derivative with the series. Whence,

$$\frac{\mathrm{d}}{\mathrm{d}t} F_\psi(t) = -F_\psi(t) + \sum_{n=1}^{+\infty} n\mathrm{e}^{-2t}(1-\mathrm{e}^{-t})^{n-1} \int_{\mathbb{R}^3} \psi(\mathbf{v})\mathcal{Q}_{n+1}[\mu_0](\mathrm{d}\mathbf{v})$$

$$= -F_\psi(t) + \sum_{n=1}^{+\infty}\sum_{k=1}^{n} \mathrm{e}^{-2t}(1-\mathrm{e}^{-t})^{n-1} \int_{\mathbb{R}^3} \psi(\mathbf{v})\mathcal{Q}\left[\mathcal{Q}_k[\mu_0], \mathcal{Q}_{n+1-k}[\mu_0]\right](\mathrm{d}\mathbf{v}).$$

Moreover, $-F_\psi(t)$ coincides, by definition, with

$$\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \mathrm{e}^{-2t}(1-\mathrm{e}^{-t})^{n+m-1} \frac{1}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [-\psi(\mathbf{v}) - \psi(\mathbf{w})]1\!\!1\{\mathbf{v} \neq \mathbf{w}\}$$
$$\times\, b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathcal{Q}_n[\mu_0](\mathrm{d}\mathbf{v})\mathcal{Q}_m[\mu_0](\mathrm{d}\mathbf{w})$$

while the other summand in the expression giving $\frac{\mathrm{d}}{\mathrm{d}t}F_\psi$ is equal to

$$\sum_{n=1}^{+\infty}\sum_{k=1}^{n} \mathrm{e}^{-2t}(1-\mathrm{e}^{-t})^{n-1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \psi(\mathbf{v}_*)1\!\!1\{\mathbf{v} \neq \mathbf{w}\}$$
$$\times\, b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathcal{Q}_k[\mu_0](\mathrm{d}\mathbf{v})\mathcal{Q}_{n+1-k}[\mu_0](\mathrm{d}\mathbf{w})$$
$$= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \mathrm{e}^{-2t}(1-\mathrm{e}^{-t})^{n+m-1} \frac{1}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [\psi(\mathbf{v}_*) + \psi(\mathbf{w}_*)]1\!\!1\{\mathbf{v} \neq \mathbf{w}\}$$
$$\times\, b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathcal{Q}_n[\mu_0](\mathrm{d}\mathbf{v})\mathcal{Q}_m[\mu_0](\mathrm{d}\mathbf{w})$$

by virtue of (12). This completes the proof of the existence.

The proof of uniqueness relies on the fact that any solution $\overline{\mu}(\cdot, t)$ with initial datum μ_0 must coincide with that solution given by the Wild sum, denoted by $\mu(\cdot, t)$, with the same initial datum. To this aim, it is useful to pass to the following reformulation [equivalent to (5)]

$$\int_{\mathbb{R}^3} \psi(\mathbf{v})\overline{\mu}(\mathrm{d}\mathbf{v}, t) = \mathrm{e}^{-t}\int_{\mathbb{R}^3} \psi(\mathbf{v})\mu_0(\mathrm{d}\mathbf{v}) + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \mathrm{e}^{-(t-s)}\psi(\mathbf{v}_*)\mathbb{1}\{\mathbf{v} \neq \mathbf{w}\}$$
$$\times\, b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\overline{\mu}(\mathrm{d}\mathbf{v}, s)\overline{\mu}(\mathrm{d}\mathbf{w}, s)\mathrm{d}s \quad (\forall\, t \geq 0, \forall\, \psi \in \mathrm{C}_b(\mathbb{R}^3; \mathbb{C})) \tag{21}$$

which can be obtained by integrating the two sides of (5) in time. From this identity, one gets the key relation

$$\int_{\mathbb{R}^3} \psi(\mathbf{v})\overline{\mu}(\mathrm{d}\mathbf{v}, t) \geq \sum_{n=1}^{2^N} \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{n-1}\int_{\mathbb{R}^3}\psi(\mathbf{v})\mathcal{Q}_n[\mu_0](\mathrm{d}\mathbf{v}) \tag{22}$$

for every $N \in \mathbb{N}_0$ and fixed $t \geq 0$ and $\psi \in \mathrm{C}_b(\mathbb{R}^3; [0, +\infty))$. The proof of (22) is by induction. Indeed, it holds for $N = 0$ as a direct consequence of (15) and (21). Then, if (22) is valid for a certain $N \in \mathbb{N}_0$, one can consider the RHS of (21) and conclude that it is not less than the same expression with $\sum_{n=1}^{2^N} \mathrm{e}^{-s}(1-\mathrm{e}^{-s})^{n-1}\mathcal{Q}_n[\mu_0]$ in place of $\overline{\mu}(\cdot, s)$. This is true by virtue of the inductive hypothesis and the fact that $(\mathbf{v}, \mathbf{w}) \mapsto H_\psi(\mathbf{v}, \mathbf{w})$ is bounded and continuous, as proved in the previous subsection. Now, the RHS of (21) with $\sum_{n=1}^{2^N} \mathrm{e}^{-s}(1-\mathrm{e}^{-s})^{n-1}\mathcal{Q}_n[\mu_0]$ in place of $\overline{\mu}(\cdot, s)$ turns out to be

$$\mathrm{e}^{-t}\int_{\mathbb{R}^3}\psi(\mathbf{v})\mu_0(\mathrm{d}\mathbf{v}) + \sum_{n=1}^{2^N}\sum_{m=1}^{2^N}\int_0^t \mathrm{e}^{-(t-s)}\mathrm{e}^{-2s}(1-\mathrm{e}^{-s})^{n+m-2}\mathrm{d}s$$
$$\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\psi(\mathbf{v}_*)\mathbb{1}\{\mathbf{v}\neq\mathbf{w}\}b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathcal{Q}_n[\mu_0](\mathrm{d}\mathbf{v})\mathcal{Q}_m[\mu_0](\mathrm{d}\mathbf{w})$$

which coincides with $\sum_{n=2}^{2^{N+1}} \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{n-1}\int_{\mathbb{R}^3}\psi(\mathbf{v})\mathcal{Q}_n[\mu_0](\mathrm{d}\mathbf{v})$, thanks to (12), the identity $\int_0^t \mathrm{e}^{-(t-s)}\mathrm{e}^{-2s}(1-\mathrm{e}^{-s})^{n+m-2}\mathrm{d}s = \frac{1}{n+m-1}\mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{n+m-1}$ and (15). Therefore, the validity of (22) for every $N \in \mathbb{N}_0$ follows and then, taking the limit as N goes to infinity, one gets $\int_{\mathbb{R}^3}\psi(\mathbf{v})\overline{\mu}(\mathrm{d}\mathbf{v}, t) \geq \int_{\mathbb{R}^3}\psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v}, t)$ for every $t \geq 0$ and $\psi \in \mathrm{C}_b(\mathbb{R}^3; [0, +\infty))$. But this last inequality is tantamount to asserting that $\overline{\mu}(\cdot, t) = \mu(\cdot, t)$ for every $t \geq 0$.

Finally, the check that $\{\mu(\cdot, t)\circ f_R^{-1}\}_{t\geq 0}$ coincides with the solution of (1) with $\mu_0 \circ f_R^{-1}$ as initial datum is an obvious consequence of (12) and (15)–(16).

3.3 Some preparatory results

This subsection contains two technical lemmata, which will come in useful later on. The first statement consists in a refinement of a classical result about uniform integrability, whose original form is contained, e.g., in Section 7.VI of [13] or in Section 2.II of [24]. The inspiration for the following refined version has come from the contents of Section 3 of [16] and from Lemma 2 of [33].

Lemma 7 *Let γ be a Borel p.m. on $[0, +\infty)$ such that $\int_0^{+\infty} x\gamma(\mathrm{d}x) < +\infty$. Then, there exists a function $G : [0, +\infty) \to [0, +\infty)$, depending on γ, with the following properties:*

(i) $\int_0^{+\infty} G(x)\gamma(\mathrm{d}x) < +\infty$;
(ii) *G is strictly increasing and continuous, with* $G(0) = 0$;
(iii) $\lim_{x\to+\infty} G(x)/x = +\infty$;
(iv) *there exists a discrete set* $\Delta \subset (0, +\infty)$ *such that* $G \in C^1((0, +\infty)\setminus\Delta)$;
(v) *there exists a constant* $\lambda_1 > 1$ *such that* $G(x) \leq xG'(x) \leq \lambda_1 G(x)$ *for all* $x \in (0, +\infty)\setminus\Delta$;
(vi) *there exists a constant* $\lambda_2 > 1$ *such that* $G'(2x) \leq \lambda_2 G'(x)$ *for all* $x \in (0, +\infty)\setminus\Delta$.

Proof Put $g(x) := 1\!\!1_{[0,1)}(x) + \sum_{n=0}^{\infty} A_n 1\!\!1_{[2^n,2^{n+1})}(x)$ for every $x \in [0, +\infty)$, where $\{A_n\}_{n\geq 0}$ is a suitable sequence of real numbers, to be determined from the knowledge of γ. General properties of this sequence must be the following:

(a) $1 \leq A_n \leq A_{n+1}$ for all $n \in \mathbb{N}_0$;
(b) $\lim_{n\to+\infty} A_n = +\infty$;
(c) $\sup_{n\in\mathbb{N}_0} A_{n+1}/A_n < +\infty$.

Define also $G(x) := \int_0^x g(y)\mathrm{d}y$ for every $x \in [0, +\infty)$ and $\Delta := \{2^n \mid n \in \mathbb{N}_0\}$. At this stage, note that the above setting is enough to guarantee, independently of the specific determination of $\{A_n\}_{n\geq 0}$, the validity of the points from (ii) to (iv), as well as the inequality $G(x) \leq xG'(x)$ for all $x \in (0, +\infty)\setminus\Delta$. Point (vi) holds true after putting $\lambda_2 := \max\{A_0, \sup_{n\in\mathbb{N}_0} A_{n+1}/A_n\}$. As to the remaining inequality at point (v), one shows that it is in force for all $x \in (0, +\infty)\setminus\Delta$ with $\lambda_1 := 2\lambda_2$. Indeed, when $x \in (0, 1)$ suffice it to know that $\lambda_1 > 1$. When $x \in (1, 2)$, the fact that $\lambda_1 \geq A_0$ yields $\lambda_1 G(x) - xg(x) \geq A_0(A_0 - 1)(x - 1) \geq 0$. When $x \in [2^m, 2^{m+1})$ for some integer $m \in \mathbb{N}$ observe that the thesis is equivalent to the validity of $\lambda_1 \left[2^m A_m - 1 - \sum_{n=0}^{m-1} 2^n A_n\right] \leq A_m(\lambda_1 - 1)x$. Since the RHS is minimum when $x = 2^m$, it is enough to test this inequality in correspondence with this minimum point, reducing the problem to checking that $\sup_{m\geq 1}(2^m A_m)/(1 + \sum_{n=0}^{m-1} 2^n A_n) \leq \lambda_1$, which follows in view of

$$\frac{2^m A_m}{1 + \sum_{n=0}^{m-1} 2^n A_n} \leq \frac{2^m \lambda_2 A_{m-1}}{2^{m-1} A_{m-1}} = 2\lambda_2.$$

After showing that the validity of (a)–(c) entails that (ii)–(vi) are in force, the conclusion of the proof focuses on the specific determination of $\{A_n\}_{n\geq 0}$ in conformity with (a)–(c) and (i). Accordingly, consider the distribution function $\Gamma(x) := \gamma([0, x])$, for every $x \in [0, +\infty)$, and define $\Gamma^*(x) := 1 - \Gamma(x)$. Next, integrate by parts to obtain $\int_0^z G(x)\gamma(\mathrm{d}x) = -\Gamma^*(z)G(z) + \int_0^z \Gamma^*(x)g(x)\mathrm{d}x$ for all $z \in [0, +\infty)$. As for the latter integral, write

$$\int_0^z \Gamma^*(x)g(x)\mathrm{d}x \leq \int_0^1 \Gamma^*(x)\mathrm{d}x + \sum_{n=0}^{\infty} A_n \int_{2^n}^{2^{n+1}} \Gamma^*(x)\mathrm{d}x$$
$$\leq 1 + \sum_{n=0}^{\infty} A_n \left(\sum_{k=2^n}^{2^{n+1}-1} \alpha_k\right) = 1 + \sum_{k=1}^{\infty} A_{n(k)}\alpha_k \tag{23}$$

where $\alpha_k := \Gamma^*(k)$ and $n(k)$ is the only integer such that $2^{n(k)} \leq k < 2^{n(k)+1}$. Then, choose a sequence of positive integers $\{r_n\}_{n\geq 1}$ such that $r_n \leq r_{n+1}$ and $\int_{r_n}^{+\infty} x\gamma(\mathrm{d}x) \leq 2^{-n}$ for every $n \in \mathbb{N}$, which is possible by virtue of the hypothesis $\int_0^{+\infty} x\gamma(\mathrm{d}x) < +\infty$. Whence, $2^{-n} \geq \int_{r_n}^{+\infty} x\gamma(\mathrm{d}x) \geq \sum_{k=r_n}^{+\infty} k(\alpha_k - \alpha_{k+1}) \geq \sum_{k=r_n}^{+\infty} \alpha_k$, proving that the series $\sum_{n=1}^{\infty}\sum_{k=r_n}^{+\infty} \alpha_k$ is convergent. By inverting the summation order, put the last series in the form $\sum_{n=1}^{+\infty} B_n \alpha_n$ with $B_n := \sum_{j=1}^{+\infty} 1\!\!1\{r_j \le n\}$, which shows that $B_n \le B_{n+1}$ is in force for every $n \in \mathbb{N}$, and $\lim_{n\to+\infty} B_n = +\infty$. Next, introduce a new sequence $\{B_n^*\}_{n\ge1}$ by setting $B_1^* := B_1$ and $B_n^* := \min\{B_m \mid B_m > B_n\}$, which satisfies $\lim_{n\to+\infty} B_n^* = +\infty$ and $n - 1 \le B_n^* < B_{n+1}^*$ for all $n \in \mathbb{N}$. With a view to the determination of $\{A_n\}_{n\ge0}$, consider the function $h : [0, +\infty) \to [0, +\infty)$ defined by $h(0) = 0$, $h(n) = B_n^* + 1$ for all $n \in \mathbb{N}$ and by a linear interpolation in correspondence with the remaining values of x. This function turns out to be continuous and strictly increasing, and meets $h(x) \ge x$ for all $x \in [0, +\infty)$. Its inverse h^{-1} is again strictly increasing, diverges at infinity, and meets $h^{-1}(x) \le x$ for all $x \in [0, +\infty)$, so that one can finally put $A_0 = A_1$ and $A_n := h^{-1}(B_n + 1) + 1$, for every $n \in \mathbb{N}$. At this stage, points (a)–(b) are automatically satisfied while, as to point (c), note that

$$\sup_{n\in\mathbb{N}_0} \frac{A_{n+1}}{A_n} = \sup_{n\in\mathbb{N}_0} \frac{h^{-1}(B_{n+1}^* + 1) + 1}{h^{-1}(B_n^* + 1) + 1} = \sup_{n\in\mathbb{N}_0} \frac{n+2}{n+1} = 2.$$

The validity of point (i) follows from

$$\sum_{k=1}^{\infty} A_{n(k)}\alpha_k \le \sum_{k=1}^{\infty} A_k\alpha_k \le \sum_{k=1}^{\infty} B_k\alpha_k + 2\sum_{k=1}^{\infty} \alpha_k < +\infty$$

which shows, via (23), that $\int_0^{+\infty} \Gamma^*(x)g(x)\mathrm{d}x < +\infty$. The conclusion ensues from the above-mentioned integration by parts, which gives $\int_0^{+\infty} G(x)\gamma(\mathrm{d}x) \le \int_0^{+\infty} \Gamma^*(x)g(x)\mathrm{d}x$. □

As a straightforward corollary of points (ii)–(vi) of this lemma, one can show the following additional properties, they are as follows:

(i') $G(2x) \le \lambda_3 G(x)$ for all $x \in [0, +\infty)$, with $\lambda_3 := 2\lambda_2$;
(ii') $G(x) = x\mathfrak{G}(x)$, where $\mathfrak{G} : [0, +\infty) \to [0, +\infty)$ is non-decreasing;
(iii') $G(x) \le G(1)x^{\lambda_1}$ for all $x \in [1, +\infty)$;
(iv') $G(\sum_{i=1}^{2^m} x_i) \le \lambda_3^m \sum_{i=1}^{2^m} G(x_i)$ for all $m \in \mathbb{N}$ and $x_1, \dots, x_{2^m} \in [0, +\infty)$.

Indeed, (i') follows from

$$G(2x) = \int_0^{2x} G'(y)\mathrm{d}y = 2\int_0^{x} G'(2y)\mathrm{d}y \le 2\lambda_2 \int_0^{x} G'(y)\mathrm{d}y = \lambda_3 G(x).$$

The next point (ii') is an obvious consequence of $G(x) \le xG'(x)$. Then, (iii') emanates by virtue of $xG'(x) \le \lambda_1 G(x)$ and (iv') can be deduced, by means of an easy induction argument, from

$$G(x_1 + x_2) \le G(2\max\{x_1, x_2\}) \le \lambda_3 G(\max\{x_1, x_2\}) \le \lambda_3[G(x_1) + G(x_2)].$$

Now, a close link between Lemma 7 and the sum $S(\mathbf{u})$ appearing in (19) is established by means of the following

Lemma 8 *Let the initial datum μ_0 satisfy $\mathfrak{m}_2 := \int_{\mathbb{R}^3} |\mathbf{v}|^2\mu_0(\mathrm{d}\mathbf{v}) < +\infty$. Then, there exists a positive constant $C_1(\mu_0)$, depending only on μ_0, such that*

$$\mathrm{E}_t[G_*(S(\mathbf{u}))] \le C_1(\mu_0) \tag{24}$$

is valid for every $t \ge 0$ and $\mathbf{u} \in S^2$, where $G_*(x) := G(x^2)$ and G is the same function provided by Lemma 7 when $\gamma(A) := \int_{\mathbb{R}^3} 1\!\!1\{|\mathbf{v}|^2 \in A\}\mu_0(\mathrm{d}\mathbf{v})$.

Proof Since $\int_0^{+\infty} x\gamma(\mathrm{d}x) = \mathfrak{m}_2$, the hypothesis in Lemma 7 is fulfilled and G is given accordingly. Then, introduce the σ-algebra $\mathscr{H} := \sigma\big(\nu,\ \{\tau_n\}_{n\geq 1},\ \{\phi_n\}_{n\geq 1}, \{\vartheta_n\}_{n\geq 1}\big)$ and invoke the structure of the probabilistic representation set forth in Sect. 2.1 to have, for all $m \in \mathbb{N}$, $\mathbf{u} \in S^2$ and $j = 1, \ldots, \nu$,

$$\begin{aligned}
&\mathsf{E}_t[G_*(2^m \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j) \mid \mathscr{H}] = \mathsf{E}_t[G(2^{2m} \pi_{j,\nu}^2 (\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2) \mid \mathscr{H}] \\
&\leq \lambda_3^{2m} \mathsf{E}_t[G(\pi_{j,\nu}^2 (\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2) \mid \mathscr{H}] \\
&= \lambda_3^{2m} \pi_{j,\nu}^2 \mathsf{E}_t[(\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2 \mathfrak{G}(\pi_{j,\nu}^2 (\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2) \mid \mathscr{H}] \\
&\leq \lambda_3^{2m} \pi_{j,\nu}^2 \mathsf{E}_t[|\mathbf{V}_j|^2 \mathfrak{G}(|\mathbf{V}_j|^2)] = \lambda_3^{2m} \pi_{j,\nu}^2 \mathsf{E}_t[G_*(|\mathbf{V}_j|)]
\end{aligned}$$

since $|\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j| \leq |\mathbf{V}_j|$. Thus, (18) entails

$$\sum_{j=1}^{\nu} \mathsf{E}_t[G_*(2^m \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j) \mid \mathscr{H}] \leq \lambda_3^{2m} \int_0^{+\infty} G(x)\gamma(\mathrm{d}x) \tag{25}$$

for all $m \in \mathbb{N}$ and $\mathbf{u} \in S^2$. After defining $G_{*,l}(x) := \min\{G_*(x), l\}$ for $l \in \mathbb{N}$ and checking that $\mathsf{E}_t[G_{*,l}(S(\mathbf{u})) \mid \mathscr{H}] \leq l$, apply Lemma 2.4 in [29] to obtain

$$\mathsf{E}_t[G_{*,l}(S(\mathbf{u})) \mid \mathscr{H}] = \int_0^{+\infty} \mathsf{P}_t[|S(\mathbf{u})| \geq x \mid \mathscr{H}]\mathrm{d}G_{*,l}(x). \tag{26}$$

Now, the conclusion hinges on the remark that $A \mapsto \mathsf{P}_t[S(\mathbf{u}) \in A \mid \mathscr{H}]$ is a (random) p.m. having the structure of probability law of a sum of independent random variables, which establishes a link with the subject of Chapter 2 of [29] and allows the use of formula (2.33) therein, with $y = x2^{-m}$, to get

$$\begin{aligned}
\mathsf{P}_t[|S(\mathbf{u})| \geq x \mid \mathscr{H}] &\leq \sum_{j=1}^{\nu} \mathsf{P}_t[|\pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j| \geq x2^{-m} \mid \mathscr{H}] \\
&\quad + 2\mathrm{e}^{2^m} \left(1 + \frac{x^2}{2^m \sum_{j=1}^{\nu} \pi_{j,\nu}^2 \mathsf{E}_t[(\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2 \mid \mathscr{H}]}\right)^{-2^m}.
\end{aligned}$$

The combination of this last inequality with (26) leads to the analysis of two terms, the former of which can be bounded as

$$\begin{aligned}
&\sum_{j=1}^{\nu} \int_0^{+\infty} \mathsf{P}_t[|\pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j| \geq x2^{-m} \mid \mathscr{H}]\mathrm{d}G_{*,l}(x) \\
&\quad \leq \sum_{j=1}^{\nu} \mathsf{E}_t[G_*(2^m \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j) \mid \mathscr{H}]
\end{aligned}$$

for every $m, l \in \mathbb{N}$, which produces a finite upper bound thanks to (25). As to the latter term, observe in advance that $\sum_{j=1}^{\nu} \pi_{j,\nu}^2 \mathsf{E}_t[(\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2 \mid \mathscr{H}] \leq \mathfrak{m}_2$ holds in view of the combination of (18) and the inequality $|\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j| \leq |\mathbf{V}_j|$. Thus, property (iii') of G entails

$$\int_0^{+\infty} \left(1 + \frac{x^2}{2^m \sum_{j=1}^{\nu} \pi_{j,\nu}^2 \mathsf{E}_t[(\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2 \mid \mathscr{H}]}\right)^{-2^m} \mathrm{d}G_{*,l}(x)$$
$$\leq \int_0^{+\infty} \left(1 + \frac{x^2}{2^m \mathfrak{m}_2}\right)^{-2^m} \mathrm{d}G_*(x) \leq G(1)\left[1 + 2\lambda_1 (2^m \mathfrak{m}_2)^{2^m} \int_1^{+\infty} x^{-2^{m+1}+2\lambda_1+1} \mathrm{d}x\right]$$

which produces again a finite upper bound for every $l \in \mathbb{N}$, provided that m is chosen in such a way that $-2^{m+1} + 2\lambda_1 + 1 < -1$. After putting $C_1(\mu_0) := \lambda_3^{2m} \int_0^{+\infty} G(x)\gamma(\mathrm{d}x) + 2\mathrm{e}^{2^m} G(1)\left[1 + \frac{\lambda_1 (2^m \mathfrak{m}_2)^{2^m}}{2^m - \lambda_1 - 1}\right]$ with a suitable choice of m (e.g., $m = [\log_2(\lambda_1 + 1)] + 1$, where $[x]$ denotes the integral part of x), one finally has $\mathsf{E}_t[G_{*,l}(S(\mathbf{u})) \mid \mathscr{H}] \leq C_1(\mu_0)$ for every $l \in \mathbb{N}$, which entails (24). □

3.4 Evolution of the moments in the cutoff case

Consider the sum $S(\mathbf{u})$ in (19) and combine Lyapunov and Cauchy-Schwartz's inequalities with (18) to obtain

$$\mathsf{E}_t[(S(\mathbf{u}))^2] \leq \mathsf{E}_t\left[\nu \sum_{j=1}^{\nu} \pi_{j,\nu}^2 (\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j)^2\right] \leq \mathsf{E}_t\left[\nu \sum_{j=1}^{\nu} \pi_{j,\nu}^2 |\mathbf{V}_j|^2\right]$$
$$= \mathsf{E}_t\left[\mathsf{E}_t\left[\nu \sum_{j=1}^{\nu} \pi_{j,\nu}^2 |\mathbf{V}_j|^2 \mid \mathscr{G}\right]\right] = \mathsf{E}_t[\nu]\mathfrak{m}_2$$

for every $\mathbf{u} \in S^2$, where $\mathscr{G} := \sigma\big(\nu,\ \{\tau_n\}_{n\geq 1},\ \{\phi_n\}_{n\geq 1}\big)$. Thus, the finiteness of the first two moments of $S(\mathbf{u})$ follows from $\mathsf{E}_t[\nu] = \mathrm{e}^t$. To prove the former identity in (7), note that (17) entails $\mathsf{E}_t[S(\mathbf{u})] = \mathbf{u} \cdot \int_{\mathbb{R}^3} \mathbf{v}\mu(\mathrm{d}\mathbf{v}, t)$ for every $t \geq 0$ and $\mathbf{u} \in S^2$. Moreover,

$$\mathsf{E}_t[S(\mathbf{u})] = \mathsf{E}_t\left[\mathsf{E}_t\left[\sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \mathbf{V}_j \mid \mathscr{H}\right]\right] = \mathsf{E}_t\left[\sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u})\right] \cdot \overline{\mathbf{V}}$$
$$= \mathsf{E}_t\left[\mathsf{E}_t\left[\sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \mid \mathscr{G}\right]\right] \cdot \overline{\mathbf{V}} = \mathsf{E}_t\left[\sum_{j=1}^{\nu} \pi_{j,\nu} \mathsf{E}_t\left[\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \mid \mathscr{G}\right]\right] \cdot \overline{\mathbf{V}}$$

holds with $\overline{\mathbf{V}} := \int_{\mathbb{R}^3} \mathbf{v}\mu_0(\mathrm{d}\mathbf{v})$ and $\mathscr{H} := \sigma\big(\nu,\ \{\tau_n\}_{n\geq 1},\ \{\phi_n\}_{n\geq 1}, \{\vartheta_n\}_{n\geq 1}\big)$. To conclude, combine the identity $\mathsf{E}_t\Big[\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \mid \mathscr{G}\Big] = \pi_{j,\nu}\mathbf{u}$, which emanates from (111) in [12], with (18) to get $\mathsf{E}_t\Big[\sum_{j=1}^{\nu} \pi_{j,\nu} \mathsf{E}_t\Big[\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \mid \mathscr{G}\Big]\Big] = \mathbf{u}$. Whence, $\mathbf{u} \cdot \int_{\mathbb{R}^3} \mathbf{v}\mu(\mathrm{d}\mathbf{v}, t) = \mathbf{u} \cdot \overline{\mathbf{V}}$ for every $t \geq 0$ and $\mathbf{u} \in S^2$, which amounts to the desired result. To proceed, note that $\sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot \overline{\mathbf{V}} = \mathbf{u} \cdot \overline{\mathbf{V}}$ is valid for every $\mathbf{u} \in S^2$, since $\delta_{\overline{\mathbf{V}}}$ is a stationary solution of (1). Whence,

$$\mathsf{E}_t[(S(\mathbf{u}))^2] = \mathsf{E}_t\left[\left(\sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot (\mathbf{V}_j - \overline{\mathbf{V}}) + \mathbf{u} \cdot \overline{\mathbf{V}}\right)^2\right]$$

$$= \mathsf{E}_t \left[\left(\sum_{j=1}^{\nu} \pi_{j,\nu} \boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot (\mathbf{V}_j - \overline{\mathbf{V}}) \right)^2 \right] + (\mathbf{u} \cdot \overline{\mathbf{V}})^2$$

$$= \mathsf{E}_t \left[\sum_{j=1}^{\nu} \pi_{j,\nu}^2 [\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot (\mathbf{V}_j - \overline{\mathbf{V}})]^2 \right] + (\mathbf{u} \cdot \overline{\mathbf{V}})^2. \quad (27)$$

At this stage, an application of (187) in [12] with $k = 2$ shows that

$$\mathsf{E}_t \left[\sum_{j=1}^{\nu} \pi_{j,\nu}^2 [\boldsymbol{\psi}_{j,\nu}(\mathbf{u}) \cdot (\mathbf{V}_j - \overline{\mathbf{V}})]^2 \right] = \frac{1}{3} \int_{\mathbb{R}^3} |\mathbf{v} - \overline{\mathbf{V}}|^2 \mu_0(\mathrm{d}\mathbf{v})$$

$$+ \mathsf{E}_t \left[\sum_{j=1}^{\nu} \pi_{j,\nu}^2 \zeta_{j,\nu} \right] \cdot \left[\int_{\mathbb{R}^3} \{(\mathbf{u} \cdot (\mathbf{v} - \overline{\mathbf{V}}))^2 - \frac{1}{3} |\mathbf{v} - \overline{\mathbf{V}}|^2 \} \mu_0(\mathrm{d}\mathbf{v}) \right] \quad (28)$$

holds, where the $\zeta_{j,n}$'s are given by $\zeta_{j,n} := \zeta_{j,n}^*(\tau_n, (\phi_1, \dots, \phi_{n-1}))$ and the $\zeta_{j,n}^*$'s are defined on $\mathbb{T}(n) \times [0, \pi]^{n-1}$ by putting $\zeta_{1,1}^* \equiv 1$ and, for $n \geq 2$,

$$\zeta_{j,n}^*(\mathfrak{t}_n, \boldsymbol{\varphi}) := \begin{cases} \zeta_{j,n_l}^*(\mathfrak{t}_n^l, \boldsymbol{\varphi}^l) \cdot \left(\frac{3}{2} \cos^2 \varphi_{n-1} - \frac{1}{2} \right) & \text{for } j = 1, \dots, n_l \\ \zeta_{j-n_l,n_r}^*(\mathfrak{t}_n^r, \boldsymbol{\varphi}^r) \cdot \left(\frac{3}{2} \sin^2 \varphi_{n-1} - \frac{1}{2} \right) & \text{for } j = n_l + 1, \dots, n \end{cases}$$

for every $\boldsymbol{\varphi}$ in $[0, \pi]^{n-1}$. The same techniques contained in Appendix A.1 of [12], used to get (106) therein, show that $\mathsf{E}_t \left[\sum_{j=1}^{\nu} \pi_{j,\nu}^2 \zeta_{j,\nu} \right] = \mathrm{e}^{-(1-f_1(b))t}$ for every $t \geq 0$, with $f_1(b) := \frac{3}{2} \int_0^{\pi} (\sin^4 \varphi + \cos^4 \varphi) \beta(\mathrm{d}\varphi) - \frac{1}{2}$. At this stage, the proof of the latter identity in (7) follows from (27)–(28), which give

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mu(\mathrm{d}\mathbf{v}, t) = \sum_{i=1}^{3} \mathsf{E}_t[(S(\mathbf{e}_i))^2] = \int_{\mathbb{R}^3} |\mathbf{v} - \overline{\mathbf{V}}|^2 \mu_0(\mathrm{d}\mathbf{v}) + \sum_{i=1}^{3} (\mathbf{e}_i \cdot \overline{\mathbf{V}})^2 = \mathfrak{m}_2$$

for every $t \geq 0$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ being the canonical basis of $\mathbb{R}^3$. The proof of (8) in the case $i = j$ is even simpler, since it follows directly from the combination of (27)–(28) with $\mathbf{u} = \mathbf{e}_i$. When $i \neq j$, start from the remark that $\int_{\mathbb{R}^3} v_i v_j \mu(\mathrm{d}\mathbf{v}, t)$ can be written as $\int_{\mathbb{R}^3} \left(\frac{\sqrt{2}}{2} v_i + \frac{\sqrt{2}}{2} v_j \right)^2 \mu(\mathrm{d}\mathbf{v}, t) - \frac{1}{2} \int_{\mathbb{R}^3} (v_i^2 + v_j^2) \mu(\mathrm{d}\mathbf{v}, t)$. Then, define $\mathbf{u}_{ij} := \frac{\sqrt{2}}{2} \mathbf{e}_i + \frac{\sqrt{2}}{2} \mathbf{e}_j$ and invoke again (27)–(28) to obtain

$$\int_{\mathbb{R}^3} v_i v_j \mu(\mathrm{d}\mathbf{v}, t) = \mathsf{E}_t[(S(\mathbf{u}_{ij}))^2] - \frac{1}{2} \left\{ \mathsf{E}_t[(S(\mathbf{e}_i))^2] + \mathsf{E}_t[(S(\mathbf{e}_j))^2] \right\} = \epsilon_{ij}(t) + \overline{V}_i \overline{V}_j$$

for every $t \geq 0$, where $\epsilon_{ij}(t) := \mathrm{e}^{-(1-f_1(b))t} \int_{\mathbb{R}^3} (v_i - \overline{V}_i)(v_j - \overline{V}_j) \mu_0(\mathrm{d}\mathbf{v})$. This completes the proof of (8).

To prove (9), consider Lemma 8 with the same γ, G and G_* and define

$$q(x) := \begin{cases} \frac{1}{x^3} \int_0^x G_*(y) \mathrm{d}y & \text{if } x > 0 \\ \frac{1}{3} & \text{if } x = 0 \end{cases}$$

and $F_*(x) := x^2 q(x)$. This q meets the requirements of the theorem since $\lim_{x \to +\infty} q(x) = +\infty$ holds after a straightforward application of l'Hôpital's rule, while property (ii') of G

shows that q is non-decreasing. Then, after noting that $F_*(x) \le G_*(x)$ for all $x \in [0, +\infty)$ thanks to the fact that G_* is non-decreasing, the combination of Lemma 8 with property iv') of G gives

$$\int_{\mathbb{R}^3} F_*(|\mathbf{v}|)\mu(\mathrm{d}\mathbf{v}, t) \le \int_{\mathbb{R}^3} G(|\mathbf{v}|^2)\mu(\mathrm{d}\mathbf{v}, t) = \mathsf{E}_t[G(S(\mathbf{e}_1)^2 + S(\mathbf{e}_2)^2 + S(\mathbf{e}_3)^2)]$$
$$\le 3\lambda_3^2 \sup_{\mathbf{u}\in S^2} \mathsf{E}_t[G_*(S(\mathbf{u}))] \le 3\lambda_3^2 C_1(\mu_0) := C(\mu_0)$$

which is the desired conclusion. Finally, since (9) is in force, then

$$\int_{|\mathbf{v}|\ge R} |\mathbf{v}|^2\mu(\mathrm{d}\mathbf{v}, t) \le \frac{1}{q(R)}\int_{\mathbb{R}^3} |\mathbf{v}|^2 q(|\mathbf{v}|)\mu(\mathrm{d}\mathbf{v}, t) \le \frac{C(\mu_0)}{q(R)}$$

holds for every $t \ge 0$, and the validity of (10) follows.

3.5 Proof of Lemma 6

Observing that

$$\mathbf{v}_*(x) = \mathbf{v} + |\mathbf{w} - \mathbf{v}|x\left[\sqrt{1-x^2}(\cos\theta\mathbf{a}(\mathbf{u}) + \sin\theta\mathbf{b}(\mathbf{u})) + x\mathbf{u}\right]$$
$$\mathbf{w}_*(x) = \mathbf{w} - |\mathbf{w} - \mathbf{v}|x\left[\sqrt{1-x^2}(\cos\theta\mathbf{a}(\mathbf{u}) + \sin\theta\mathbf{b}(\mathbf{u})) + x\mathbf{u}\right]$$

hold for every $\mathbf{v} \ne \mathbf{w}$ and $x \in (-1, 1)$, one gets

$$\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x} = -\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x} = |\mathbf{w} - \mathbf{v}|\left[\frac{1-2x^2}{(1-x^2)^{1/2}}(\cos\theta\mathbf{a}(\mathbf{u}) + \sin\theta\mathbf{b}(\mathbf{u})) + 2x\mathbf{u}\right]$$
$$\frac{\mathrm{d}^2\mathbf{v}_*}{\mathrm{d}x^2} = -\frac{\mathrm{d}^2\mathbf{w}_*}{\mathrm{d}x^2} = |\mathbf{w} - \mathbf{v}|\left[\frac{-3x+2x^3}{(1-x^2)^{3/2}}(\cos\theta\mathbf{a}(\mathbf{u}) + \sin\theta\mathbf{b}(\mathbf{u})) + 2\mathbf{u}\right].$$

Whence,

$$\left|\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}\right|^2 = \left|\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}\right|^2 = |\mathbf{w} - \mathbf{v}|^2\left[\frac{(1-2x^2)^2}{1-x^2} + 4x^2\right]$$
$$\left|\frac{\mathrm{d}^2\mathbf{v}_*}{\mathrm{d}x^2}\right| = \left|\frac{\mathrm{d}^2\mathbf{w}_*}{\mathrm{d}x^2}\right| = |\mathbf{w} - \mathbf{v}|\left[\frac{(-3x+2x^3)^2}{(1-x^2)^3} + 4\right]^{1/2}.$$

At this stage, for the first derivatives, one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1}\left[\left|\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(s\xi)\right|^2 + \left|\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(s\xi)\right|^2\right] 1\!\!1\{\mathbf{v} \ne \mathbf{w}\}(1-s)\xi^2 b(\xi)$$
$$\times\, \mathrm{d}s\mathrm{d}\theta\mathrm{d}\xi\,\chi(\mathrm{d}\mathbf{v})\chi(\mathrm{d}\mathbf{w}) \le 2\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\mathbf{w}-\mathbf{v}|^2\chi(\mathrm{d}\mathbf{v})\chi(\mathrm{d}\mathbf{w})\int_{-1}^{1}\xi^2 b(\xi)\mathrm{d}\xi$$
$$\times \int_0^1 (1-s)\sup_{\xi\in[0,1]}\left(\frac{(1-2s^2\xi^2)^2}{1-s^2\xi^2} + 4s^2\xi^2\right)\mathrm{d}s$$

the RHS being finite in view of (3), the condition $\int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v}) < +\infty$ and the fact that, for every s in (0, 1),

$$(1-s) \sup_{\xi\in[0,1]} \left(\frac{(1-2s^2\xi^2)^2}{1-s^2\xi^2} + 4s^2\xi^2 \right) \le 14.$$

Analogously, for the second derivatives, one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} \left[\left| \frac{\mathrm{d}^2\mathbf{v}_*}{\mathrm{d}x^2}(s\xi) \right| + \left| \frac{\mathrm{d}^2\mathbf{w}_*}{\mathrm{d}x^2}(s\xi) \right| \right] 1\!\!1\{\mathbf{v}\neq\mathbf{w}\}(1-s)\xi^2 b(\xi)$$
$$\times \mathrm{d}s\mathrm{d}\theta\mathrm{d}\xi\, \chi(\mathrm{d}\mathbf{v})\chi(\mathrm{d}\mathbf{w}) \le 2\pi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\mathbf{w}-\mathbf{v}|\chi(\mathrm{d}\mathbf{v})\chi(\mathrm{d}\mathbf{w}) \int_{-1}^{1} \xi^2 b(\xi)\mathrm{d}\xi$$
$$\times \int_0^1 (1-s) \sup_{\xi\in[0,1]} \left(\frac{(-3s\xi + 2s^3\xi^3)^2}{(1-s^2\xi^2)^3} + 4 \right)^{1/2} \mathrm{d}s$$

and again the RHS is finite in view of (3), the condition $\int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v}) < +\infty$ and the fact that, for every s in (0, 1),

$$(1-s) \sup_{\xi\in[0,1]} \left(\frac{(-3s\xi + 2s^3\xi^3)^2}{(1-s^2\xi^2)^3} + 4 \right)^{1/2} \le \frac{2\sqrt{13}}{\sqrt{1-s}} + 2.$$

3.6 Proof of Theorem 3

Before getting to the heart of the matter, it is worth explaining the structure of this concluding subsection that contains the proof of the existence of a solution, following the Arkeryd approach. To make this strategy working, two forms of uniform continuity—encapsulated in (30) and (32), respectively—are deduced by exploiting the properties of the approximating solutions established in Sects. 3.2 and 3.4. Moreover, after showing the existence of a converging subsequence via the Ascoli–Arzelà theorem, the uniform integrability conditions (9)–(10) will play a key role in proving that the limit is indeed a solution, according to the Definition 2, and satisfies (7)–(10).

To start with the real proof, note that $[b(x) \wedge n]/B_n$ meets (4) for all $n \ge n_0 := \min\{n \in \mathbb{N} \mid B_n > 0\}$. Therefore, the Cauchy problem relative to (1), with $[b(x) \wedge n]/B_n$ and μ_0 as collision kernel and initial datum, respectively, admits a unique solution $\{\mu_n(\cdot, t)\}_{t\ge 0}$, which possesses all the properties established in Sects. 3.2 and 3.4. In particular, (7) yields

$$\int_{\mathbb{R}^3} \mathbf{v}\mu_n(\mathrm{d}\mathbf{v}, t) = \overline{\mathbf{V}} \quad \text{and} \quad \int_{\mathbb{R}^3} |\mathbf{v}|^2 \mu_n(\mathrm{d}\mathbf{v}, t) = \mathfrak{m}_2 \tag{29}$$

for all $t \ge 0$ and $n \ge n_0$, with $\overline{\mathbf{V}} := \int_{\mathbb{R}^3} \mathbf{v}\mu_0(\mathrm{d}\mathbf{v})$ and $\mathfrak{m}_2 := \int_{\mathbb{R}^3} |\mathbf{v}|^2\mu_0(\mathrm{d}\mathbf{v})$. Then, (29) leads to the former important inequality, namely

$$|\hat{\mu}_n(\boldsymbol{\xi}_2, t) - \hat{\mu}_n(\boldsymbol{\xi}_1, t)| \le |\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1| \sup_{\boldsymbol{\xi}\in\mathbb{R}^3} \left|\nabla_{\boldsymbol{\xi}}\hat{\mu}_n(\boldsymbol{\xi}, t)\right| \le |\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1| \int_{\mathbb{R}^3} |\mathbf{v}|\mu_n(\mathrm{d}\mathbf{v}, t)$$
$$\le |\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1| \left(\int_{\mathbb{R}^3} |\mathbf{v}|^2\mu_n(\mathrm{d}\mathbf{v}, t) \right)^{1/2} = \mathfrak{m}_2^{1/2} |\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1| \tag{30}$$

valid for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^3$, with t and n as above. The latter key inequality easily follows from a result borrowed from [28] (precisely, Lemma 2.2), restated here in a slightly different form.

Lemma 9 *Let χ belong to $\mathcal{P}_2(\mathbb{R}^3)$ and b satisfy (4). Then,*

$$\Big| \int_{S^2} [\hat{\chi}(\boldsymbol{\xi}_+)\hat{\chi}(\boldsymbol{\xi}_-) - \hat{\chi}(\boldsymbol{\xi})] b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \Big| \leq \frac{3}{2}\overline{B}\, |\boldsymbol{\xi}|^2 \int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v}) \tag{31}$$

holds for all $\boldsymbol{\xi} \neq \mathbf{0}$, with $\boldsymbol{\xi}_+ := \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\omega})\boldsymbol{\omega}$, $\boldsymbol{\xi}_- := (\boldsymbol{\xi} \cdot \boldsymbol{\omega})\boldsymbol{\omega}$ and $\overline{B} := \int_0^1 x^2 b(x)\mathrm{d}x$.

It is worth noting that the original formulation in [28] deals with collision kernels satisfying (3), but, in that case, (31) turns out to be false if $\int_{S^2}$ is intended as a standard Lebesgue integral. Indeed, it is enough to choose χ as a Gaussian probability law with zero means and covariance matrix $V = (v_{i,j})_{1 \leq i,j \leq 3}$, with $v_{2,2} = v_{3,3} = 1$, $v_{2,3} = v_{3,2} = 1/2$, $v_{i,j} = 0$ otherwise, and $b(x) = |x|^{-5/2}$, to verify that $\int_{S^2} |\hat{\chi}(\boldsymbol{\xi}_+)\hat{\chi}(\boldsymbol{\xi}_-) - \hat{\chi}(\boldsymbol{\xi})| b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = +\infty$. This counterexample can be easily reformulated also in the different parametrization used in [28]. Therefore, due to its relevance, the original proof of this lemma is shortly reproduced below.

Proof of Lemma 9 Define $\boldsymbol{\zeta} := \left(\boldsymbol{\xi}_+ \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ and $\tilde{\boldsymbol{\xi}}_+ := 2\boldsymbol{\zeta} - \boldsymbol{\xi}_+$ to write

$$\begin{aligned} \int_{S^2} [\hat{\chi}(\boldsymbol{\xi}_+)\hat{\chi}(\boldsymbol{\xi}_-) - \hat{\chi}(\boldsymbol{\xi})] b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) &= \frac{1}{2}\int_{S^2} [\hat{\chi}(\boldsymbol{\xi}_+) + \hat{\chi}(\tilde{\boldsymbol{\xi}}_+) - 2\hat{\chi}(\boldsymbol{\zeta})] \\ \times b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) + \int_{S^2} [\hat{\chi}(\boldsymbol{\zeta}) - \hat{\chi}(\boldsymbol{\xi})] b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \\ + \int_{S^2} \hat{\chi}(\boldsymbol{\xi}_+)[\hat{\chi}(\boldsymbol{\xi}_-) - 1] b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \end{aligned}$$

for all $\boldsymbol{\xi} \neq \mathbf{0}$. Upon assuming that $\int_{\mathbb{R}^3} \mathbf{v}\chi(\mathrm{d}\mathbf{v}) = \mathbf{0}$—which does not affect the generality, for the replacement of $\hat{\chi}(\boldsymbol{\xi})$ with $\hat{\chi}(\boldsymbol{\xi}) \exp\{-i\boldsymbol{\xi} \cdot \int_{\mathbb{R}^3} \mathbf{v}\chi(\mathrm{d}\mathbf{v})\}$ does not change the LHS of (31)—invoke the elementary inequality $|\hat{\chi}(\boldsymbol{\xi}) - 1| \leq \frac{1}{2}|\boldsymbol{\xi}|^2 \int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v})$ to obtain

$$\begin{aligned} \frac{1}{2}|\hat{\chi}(\boldsymbol{\xi}_+) + \hat{\chi}(\tilde{\boldsymbol{\xi}}_+) - 2\hat{\chi}(\boldsymbol{\zeta})| &\leq \frac{1}{2}|\boldsymbol{\xi}_+ - \boldsymbol{\zeta}|^2 \int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v}) \\ |\hat{\chi}(\boldsymbol{\zeta}) - \hat{\chi}(\boldsymbol{\xi})| &\leq \frac{1}{2}|\boldsymbol{\zeta} - \boldsymbol{\xi}|^2 \int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v}) \\ |\hat{\chi}(\boldsymbol{\xi}_-) - 1| &\leq \frac{1}{2}(\boldsymbol{\xi} \cdot \boldsymbol{\omega})^2 \int_{\mathbb{R}^3} |\mathbf{v}|^2 \chi(\mathrm{d}\mathbf{v}), \end{aligned}$$

so that the conclusion is reached by noting that $|\boldsymbol{\xi}_+ - \boldsymbol{\zeta}| \leq |\boldsymbol{\xi} \cdot \boldsymbol{\omega}|$, $|\boldsymbol{\zeta} - \boldsymbol{\xi}| \leq |\boldsymbol{\xi} \cdot \boldsymbol{\omega}|$ and $\int_{S^2} \left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right)^2 b\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = \overline{B}$. □

At this stage, by the Bobylev identity (14), one has

$$\frac{\partial}{\partial t}\hat{\mu}_n(\boldsymbol{\xi}, t) = \int_{S^2} [\hat{\mu}_n(\boldsymbol{\xi}_+, t)\hat{\mu}_n(\boldsymbol{\xi}_-, t) - \hat{\mu}_n(\boldsymbol{\xi}, t)] \frac{b(\boldsymbol{\xi}/|\boldsymbol{\xi}| \cdot \boldsymbol{\omega}) \wedge n}{B_n} u_{S^2}(\mathrm{d}\boldsymbol{\omega})$$

for $\boldsymbol{\xi} \neq \mathbf{0}$, $t > 0$ and $n \geq n_0$, which, combined with Lemma 9 and (29), gives

$$|\hat{\mu}_n(\boldsymbol{\xi}, B_n t_2) - \hat{\mu}_n(\boldsymbol{\xi}, B_n t_1)| = \left| \int_{B_n t_1}^{B_n t_2} \left[\frac{\partial}{\partial s} \hat{\mu}_n(\boldsymbol{\xi}, s) \right] \mathrm{d}s \right| \leq \frac{3}{2} \overline{B}\, \mathfrak{m}_2 |\boldsymbol{\xi}|^2 |t_2 - t_1| \tag{32}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^3$, $t_1, t_2 \geq 0$ and $n \geq n_0$, corresponding to the latter key inequality.

After selecting a sequence $T := \{t_k\}_{k\geq 1}$ dense in $[0, +\infty)$, one can and note that $\{\hat{\mu}_n(\boldsymbol{\xi}, B_n t_k)\}_{n\geq n_0, k\in\mathbb{N}}$ is a *uniformly bounded* and *equicontinuous* family of complex-valued functions. The former property follows from $|\hat{\mu}_n(\boldsymbol{\xi}, B_n t_k)| \leq 1$, while the latter is a consequence of (30). Hence, the combination of the Ascoli–Arzelà theorem with the Lévy continuity theorem and the Cantor diagonal argument entails the existence of two sequences: the former, $\{\mu(\cdot, t_k)\}_{k\geq 1}$, is composed of Borel p.m.'s on $\mathbb{R}^3$ and the latter, $\{n_l\}_{l\geq 1}$, is an increasing sequence of positive integers such that, for all $t_k \in T$, $\mu_{n_l}(\cdot, B_{n_l} t_k) \Rightarrow \mu(\cdot, t_k)$ as $l \to +\infty$. Then, for any other $t \in [0, +\infty)\backslash T$, take any subsequence $\{t_{k_r}\}_{r\geq 1} \subset T$ converging to t, and consider $\text{w-lim}_{r\to+\infty} \mu(\cdot, t_{k_r})$. This limit exists and is independent of the choice of the approximating sequence $\{t_{k_r}\}_{r\geq 1}$, for (32) yields $|\hat{\mu}(\boldsymbol{\xi}, t^{''}) - \hat{\mu}(\boldsymbol{\xi}, t^{'})| \leq \frac{3}{2}\overline{B}\, \mathfrak{m}_2 |\boldsymbol{\xi}|^2 |t^{''} - t^{'}|$ for all $\boldsymbol{\xi} \in \mathbb{R}^3$ and $t^{'}, t^{''} \in T$. Thus, $\{\hat{\mu}(\boldsymbol{\xi}, t_{k_r})\}_{r\geq 1}$ is a Cauchy sequence in $\mathbb{C}$, which converges to some $g_t(\boldsymbol{\xi})$ for any fixed $\boldsymbol{\xi} \in \mathbb{R}^3$, and $\boldsymbol{\xi} \mapsto g_t(\boldsymbol{\xi})$ is continuous by $|g_t(\boldsymbol{\xi}_2) - g_t(\boldsymbol{\xi}_1)| \leq \mathfrak{m}_2^{1/2} |\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1|$, which obviously emanates from the combination of (30) with (32). A further application of the Lévy continuity theorem shows that, for all $t \in [0, +\infty)\backslash T$, there exists $\mu(\cdot, t) \in \mathcal{P}(\mathbb{R}^3)$ such that $g_t(\boldsymbol{\xi}) = \hat{\mu}(\boldsymbol{\xi}, t)$ for all $\boldsymbol{\xi} \in \mathbb{R}^3$ and that $\mu(\cdot, t_{k_r}) \Rightarrow \mu(\cdot, t)$, as $r \to +\infty$. In conclusion, $\mu(\cdot, t)$ satisfies

(A) $\mu_{n_l}(\cdot, B_{n_l} t) \Rightarrow \mu(\cdot, t)$ as $l \to +\infty$, for all $t \geq 0$;
(B) $|\hat{\mu}(\boldsymbol{\xi}, t^{''}) - \hat{\mu}(\boldsymbol{\xi}, t^{'})| \leq \frac{3}{2}\overline{B}\, \mathfrak{m}_2 |\boldsymbol{\xi}|^2 |t^{''} - t^{'}|$ for all $\boldsymbol{\xi} \in \mathbb{R}^3$ and $t^{'}, t^{''} \in [0, +\infty)$.

$\{\mu(\cdot, t)\}_{t\geq 0}$ is the obvious candidate as solution of (1). Indeed, $\mu(\cdot, 0) = \mu_0(\cdot)$ by (A), while $\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mu(\mathrm{d}\mathbf{v}, t) < +\infty$ and (7) are in force for all $t \geq 0$ as a consequence of Lemma 1 in [33], whose hypotheses are met in view of (A) and

$$\sup_{\substack{l\in\mathbb{N} \\ t\geq 0}} \int_{|\mathbf{v}|\geq R} |\mathbf{v}|^2 \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) \leq \frac{1}{q(R)} \sup_{\substack{l\in\mathbb{N} \\ t\geq 0}} \int_{\mathbb{R}^3} |\mathbf{v}|^2 q(|\mathbf{v}|) \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) \leq \frac{C(\mu_0)}{q(R)}, \tag{33}$$

which emanates from (9). After fixing $\psi \in \mathrm{C}_b^2(\mathbb{R}^3; \mathbb{C})$, Definition 2 entails

$$\begin{aligned} \int_{\mathbb{R}^3} \psi(\mathbf{v}) \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) &= \int_{\mathbb{R}^3} \psi(\mathbf{v}) \mu_0(\mathrm{d}\mathbf{v}) \\ &+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{-1}^1 A_\psi(\mathbf{v}, \mathbf{w}, \xi) \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} \tau) \mu_{n_l}(\mathrm{d}\mathbf{w}, B_{n_l} \tau) \xi^2 [b(\xi) \wedge n_l] \mathrm{d}\xi \mathrm{d}\tau \end{aligned} \tag{34}$$

for all $t \geq 0$ and $l \in \mathbb{N}$, where

$$\begin{aligned} A_\psi(\mathbf{v}, \mathbf{w}, \xi) := \frac{1}{8\pi} \int_0^{2\pi} \int_0^1 \mathbb{1}\{\mathbf{v} \neq \mathbf{w}\}(1 - s) \Big[\nabla\psi(\mathbf{v}_*(s\xi)) \cdot \frac{\mathrm{d}^2 \mathbf{v}_*}{\mathrm{d}x^2}(s\xi) \\ + \nabla\psi(\mathbf{w}_*(s\xi)) \cdot \frac{\mathrm{d}^2 \mathbf{w}_*}{\mathrm{d}x^2}(s\xi) + \left(\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(s\xi) \right)^t \mathrm{Hess}[\psi](\mathbf{v}_*(s\xi)) \left(\frac{\mathrm{d}\mathbf{v}_*}{\mathrm{d}x}(s\xi) \right) \end{aligned}$$

$$+\left(\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(s\xi)\right)^t \mathrm{Hess}[\psi](\mathbf{w}_*(s\xi))\left(\frac{\mathrm{d}\mathbf{w}_*}{\mathrm{d}x}(s\xi)\right)\Bigg]\mathrm{d}s\mathrm{d}\theta.$$

The bounds provided in Sect. 3.5 give

$$|A_\psi(\mathbf{v},\mathbf{w},\xi)| \le K_\psi(1+|\mathbf{v}-\mathbf{w}|^2) \tag{35}$$

for all $(\mathbf{v},\mathbf{w},\xi)\in\mathbb{R}^6\times[-1,1]$ with a suitable positive constant K_ψ, while the same argument contained in Sect. 3.1 shows that $(\mathbf{v},\mathbf{w})\mapsto\int_{-1}^1 A_\psi(\mathbf{v},\mathbf{w},\xi)\xi^2 b(\xi)\mathrm{d}\xi$ is continuous on $\mathbb{R}^6$. The key point consists now in exploiting (A) to take the limit of both sides of (34) as $n_l\to+\infty$, with particular attention to the multiple integral on the RHS, which will be proved to converge to

$$\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-1}^1 A_\psi(\mathbf{v},\mathbf{w},\xi)\mu(\mathrm{d}\mathbf{v},\tau)\mu(\mathrm{d}\mathbf{w},\tau)\xi^2 b(\xi)\mathrm{d}\xi\mathrm{d}\tau.$$

Indeed, thanks to the dominated convergence theorem, combined with (3), (29), and (35), it is enough to show that both the quantities

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-1}^1 A_\psi(\mathbf{v},\mathbf{w},\xi)\mu_{n_l}(\mathrm{d}\mathbf{v},B_{n_l}\tau)\mu_{n_l}(\mathrm{d}\mathbf{w},B_{n_l}\tau)\xi^2[b(\xi)-(b(\xi)\wedge n_l)]\mathrm{d}\xi\Big|$$

and

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\int_{-1}^1 A_\psi(\mathbf{v},\mathbf{w},\xi)\xi^2 b(\xi)\mathrm{d}\xi\Big)[\mu_{n_l}(\mathrm{d}\mathbf{v},B_{n_l}\tau)\mu_{n_l}(\mathrm{d}\mathbf{w},B_{n_l}\tau)-\mu(\mathrm{d}\mathbf{v},\tau)\mu(\mathrm{d}\mathbf{w},\tau)]\Big|$$

go to zero for all $\tau\ge 0$, as $n_l\to+\infty$. Apropos of the former, use (29) and (35) to bound it from above by $K_\psi(1+4\mathfrak{m}_2)\int_{-1}^1\xi^2[b(\xi)-(b(\xi)\wedge n_l)]\mathrm{d}\xi$, which goes to zero by (3). As to the latter, note in advance that $\mu_{n_l}(\cdot,B_{n_l}\tau)\otimes\mu_{n_l}(\cdot,B_{n_l}\tau)\Rightarrow\mu(\cdot,\tau)\otimes\mu(\cdot,\tau)$ thanks to Theorem 4.29 in [20], and that

$$\lim_{R\to+\infty}\sup_{l\in\mathbb{N}}\int_{|\mathbf{v}|^2+|\mathbf{w}|^2\ge R}(|\mathbf{v}|^2+|\mathbf{w}|^2)\mu_{n_l}(\mathrm{d}\mathbf{v},B_{n_l}\tau)\mu_{n_l}(\mathrm{d}\mathbf{w},B_{n_l}\tau)=0$$

in view of Lemma 1 in [33]. Thus, an application of Theorem 7.12 in [37] leads to the desired conclusion. Whence,

$$\int_{\mathbb{R}^3}\psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v},t)=\int_{\mathbb{R}^3}\psi(\mathbf{v})\mu_0(\mathrm{d}\mathbf{v})$$
$$+\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-1}^1 A_\psi(\mathbf{v},\mathbf{w},\xi)\mu(\mathrm{d}\mathbf{v},\tau)\mu(\mathrm{d}\mathbf{w},\tau)\xi^2 b(\xi)\mathrm{d}\xi\mathrm{d}\tau \tag{36}$$

holds for all $t\ge 0$, by which $t\mapsto\int_{\mathbb{R}^3}\psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v},t)$ turns out to be continuous on $[0,+\infty)$. Lastly, take a sequence $\{\tau_k\}_{k\ge1}\subset[0,+\infty)$ converging to some given $\tau\in[0,+\infty)$, and mimic the above argument to get $\mu(\cdot, \tau_k) \otimes \mu(\cdot, \tau_k) \Rightarrow \mu(\cdot, \tau) \otimes \mu(\cdot, \tau)$ and

$$\lim_{k \to +\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(\int_{-1}^{1} A_\psi(\mathbf{v}, \mathbf{w}, \xi)\xi^2 b(\xi)\mathrm{d}\xi \Big) \mu(\mathrm{d}\mathbf{v}, \tau_k)\mu(\mathrm{d}\mathbf{w}, \tau_k)$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(\int_{-1}^{1} A_\psi(\mathbf{v}, \mathbf{w}, \xi)\xi^2 b(\xi)\mathrm{d}\xi \Big) \mu(\mathrm{d}\mathbf{v}, \tau)\mu(\mathrm{d}\mathbf{w}, \tau).$$

This continuity and (36) entails that $t \mapsto \int_{\mathbb{R}^3} \psi(\mathbf{v})\mu(\mathrm{d}\mathbf{v}, t)$ is continuously differentiable on $(0, +\infty)$ and that (5)–(6) are in force for any fixed $\psi \in \mathrm{C}_b^2(\mathbb{R}^3; \mathbb{C})$.

Then, consider the additional properties of $\mu(\cdot, t)$. First, the identities proved in Sect. 3.4 give

$$\int_{\mathbb{R}^3} v_i^2 \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) - \overline{V}_i^2 = e_{n_l}(t) \left[\int_{\mathbb{R}^3} \left\{ (v_i - \overline{V}_i)^2 - \frac{1}{3}|\mathbf{v} - \overline{\mathbf{V}}|^2 \right\} \mu_0(\mathrm{d}\mathbf{v}) \right] + \frac{1}{3}\int_{\mathbb{R}^3} |\mathbf{v} - \overline{\mathbf{V}}|^2 \mu_0(\mathrm{d}\mathbf{v}) \tag{37}$$

$$\int_{\mathbb{R}^3} v_i v_j \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) - \overline{V}_i \overline{V}_j = e_{n_l}(t) \int_{\mathbb{R}^3} (v_i - \overline{V}_i)(v_j - \overline{V}_j)\mu_0(\mathrm{d}\mathbf{v}) \tag{38}$$

where

$$e_{n_l}(t) = \exp\left\{ -\frac{3}{2} \left[2 \int_0^1 x^2(1 - x^2) \frac{b(x) \wedge n_l}{B_{n_l}} \mathrm{d}x \right] B_{n_l} t \right\}.$$

The uniform integrability of the second absolute moments of the μ_{n_l}'s, encapsulated in (33), yields $\lim_{l \to +\infty} \int_{\mathbb{R}^3} v_i v_j \mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) = \int_{\mathbb{R}^3} v_i v_j \mu(\mathrm{d}\mathbf{v}, t)$ for all $i, j \in \{1, 2, 3\}$ and $t \geq 0$, while an obvious application of the monotone convergence theorem shows that $\lim_{l \to +\infty} e_{n_l}(t) = \exp\{-\frac{3}{2}[2\int_0^1 x^2(1 - x^2)b(x)\mathrm{d}x]t\}$ for all $t \geq 0$. Hence, (37)–(38) pass to the limit as $l \to +\infty$, and (8) follows. Apropos of the extension of (9), write

$$\int_{\mathbb{R}^3} \min\{|\mathbf{v}|^2 q(|\mathbf{v}|), m\}\mu(\mathrm{d}\mathbf{v}, t) = \lim_{l \to +\infty} \int_{\mathbb{R}^3} \min\{|\mathbf{v}|^2 q(|\mathbf{v}|), m\}\mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t)$$
$$\leq \sup_{\substack{l \in \mathbb{N} \\ t \geq 0}} \int_{\mathbb{R}^3} |\mathbf{v}|^2 q(|\mathbf{v}|)\mu_{n_l}(\mathrm{d}\mathbf{v}, B_{n_l} t) \leq C(\mu_0)$$

for all $m \in \mathbb{N}$. Thus, the monotone convergence theorem shows that (9) is still valid with the same q and $C(\mu_0)$ as in Sect. 3.4, and (10) follows as before. In addition, take R and f_R as in the statement of the theorem and remember from Sect. 3.2 that $\{\mu_{n_l}(\cdot, B_{n_l} t) \circ f_R^{-1}\}_{t \geq 0}$ solves (1) with $\frac{b \wedge n_l}{B_{n_l}}$ and $\mu_0 \circ f_R^{-1}$ as collision kernel and initial datum, respectively. Since the continuous mapping theorem (cf. Theorem 4.27 in [20]) yields $\mu_{n_l}(\cdot, B_{n_l} t) \circ f_R^{-1} \Rightarrow \mu(\cdot, t) \circ f_R^{-1}$ for all $t \geq 0$, as $l \to +\infty$, then $\mu(\cdot, t) \circ f_R^{-1}$ is a solution of (1) with b and $\mu_0 \circ f_R^{-1}$ as collision kernel and initial datum, respectively.

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