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When does the heat equation have a solution with a sequence of similar level sets?

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Abstract In this paper, we consider an overdetermined Cauchy problem for the heat equation. We prove that if the problem has a non-trivial nonnegative solution with a certain sequence of similar level sets, then the solution must be radially symmetric.

Keywords Heat equation · Similar level sets · Overdetermined Cauchy problem

Mathematics Subject Classification Primary 35K05 · 35N20; Secondary 35J25 · 35N25

1 Introduction

Consider the unique bounded solution $u = u(x,t)$ of the Cauchy problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in} \quad \mathbb{R}^N \times (0,+\infty), \qquad u = g \geq 0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}, \tag{1.1}$$

where $N \geq 1$ and g is a non-trivial bounded nonnegative function. For the initial data g, we denote by G_0 the support of g, namely $G_0 = \operatorname{spt}(g)$. It is well known that if g is radially symmetric, then the solution u of (1.1) must be radially symmetric.

The overdetermined problems, which determine the shape of solutions using some additional information of solutions, are interesting ones in the study of qualitative properties of solutions of partial differential equations. In [8, Corollary 3.2, p.4829], problem (1.1), where g is replaced by a characteristic function of a bounded open set, is considered, and it is shown that if there exists a non-empty stationary isothermic surface of u, then, u must be radially symmetric. In this paper, we consider another type of overdetermination. Precisely, we con-

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sider the Cauchy problem (1.1) which has a solution with a certain sequence of similar level sets and prove the following.

Theorem 1.1 *Let $N \geq 1$ and let G_0 be a compact set. Suppose that there exists a bounded domain Ω in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$ and satisfying $G_0 \cup \{0\} \subset \Omega$ such that the solution u of (1.1) satisfies the following condition:*

$$(C) \quad \begin{cases} \text{there exist two sequences of positive numbers } \{t_n\}_{n=1}^{\infty} \text{ and } \{a_n\}_{n=1}^{\infty} \text{ such that} \\ t_n \uparrow \infty \text{ as } n \uparrow \infty \text{ and } u(t_n x, t_n) = a_n \text{ for all } n \in \mathbb{N} \text{ and } x \in \partial\Omega. \end{cases}$$

Then u must be radially symmetric with respect to the origin.

The overdetermination by only one stationary level surface of solutions of parabolic problems for diffusion equations has been considered since the paper [9] appeared. For instance, in [10], Magnanini and the second author of the present paper considered the initial-boundary value problem, where the initial value equals zero and the boundary value equals 1, for some nonlinear diffusion equation over a domain Ω with bounded boundary $\partial\Omega$, and also the Cauchy problem where the initial data is a characteristic function of the set $\mathbb{R}^N \setminus \Omega$. Then, they proved that if a solution u has a surface $\Gamma \subset \Omega$ of codimension 1 such that, for some function $a : (0, T) \to \mathbb{R}$, $u(x, t) = a(t)$ for every $(x, t) = \Gamma \times (0, T)$, where Γ is a so-called stationary level surface of u, then $\partial\Omega$ must be a sphere. See also [3,8,9,11].

The proof of Theorem 1.1 consists of two steps. In the first step, using condition (C) and the monotonicity of solutions on some exterior domain (see (2.2)), we see that $\partial\Omega$ is a sphere with center the origin (see Lemma 2.1 and Proposition 2.1), and we prove the radial symmetry of the solution in the second step.

Our argument in the first step is also applicable to some overdetermined elliptic boundary value problems over exterior domains in $\mathbb{R}^N$.

As an example, we consider the following boundary value problem for some fully nonlinear elliptic equation. Let $u \in C^1(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ be the unique viscosity solution of

$$\begin{cases} F(D^2u, Du, u) = 0 & \text{in} \quad \mathcal{D}, \\ u > 0 & \text{in} \quad \mathcal{D}, \\ u = 1 & \text{on} \quad \partial\Omega, \\ u \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{1.2}$$

where $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying $0 \in \Omega$ with C^1 boundary $\partial\Omega$, and $\mathcal{D} = \mathbb{R}^N \setminus \overline{\Omega}$ is also a domain. Here, the nonlinearity F satisfies the following:

(H1) (Regularity) F is a continuous function defined on $\mathcal{S}^N(\mathbb{R}) \times \mathbb{R}^N \times \mathbb{R}$, where $\mathcal{S}^N$ denotes the space of $N \times N$ symmetric (real) matrices. Furthermore, for any $R > 0$, there exists a positive constant C_1 such that

$$|F(M, p, u_1) - F(L, q, u_2)| \leq C_1\{|M - L| + |p - q| + |u_1 - u_2|\}$$

for all $M, L \in \mathcal{S}^N(\mathbb{R})$, $p, q \in \mathbb{R}^N$, and $u_1, u_2 \in [-R, R]$.

(H2) (Ellipticity) There exists a constant $C_2 > 0$ such that

$$F(M + L, p, u) - F(M, p, u) \geq C_2 \mathrm{Tr}(L)$$

for all $M, L \in \mathcal{S}^N(\mathbb{R})$ with $L \geq 0$, $p \in \mathbb{R}^N$, and $u \in \mathbb{R}$.

(H3) (Symmetry) For any $M \in \mathcal{S}^N(\mathbb{R})$, $A \in \mathcal{O}^N(\mathbb{R})$, $p \in \mathbb{R}^N$, and $u \in (0, \infty)$,

$$F(M, p, u) = F({}^tAMA, {}^tAp, u),$$

where $\mathcal{O}^N(\mathbb{R})$ denotes the set of N-dimensional orthogonal matrices and ${}^tA$ denotes the transpose of $A \in \mathcal{O}^N(\mathbb{R})$.

(H4) (Homogeneity) There exists some constant $\beta < 0$ such that, for any $\mu > 1$, the function

$$u_\mu(x) := \mu^\beta u(\mu^{-1}x) \tag{1.3}$$

is a solution of problem (1.2), where Ω is replaced by

$$\Omega_\mu = \{\mu x \in \mathbb{R}^N : x \in \Omega\}$$

and where the boundary condition $u = 1$ on $\partial\Omega$ is replaced by

$$u_\mu(x) = \mu^\beta \quad \text{on} \quad \partial\Omega_\mu.$$

Then the following holds.

Theorem 1.2 *Suppose that F satisfies* (H1)–(H4) *and moreover F is nonincreasing in $u > 0$. Let $u \in C^1(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ be a viscosity solution of* (1.2). *Assume that there exists a constant $\lambda > 1$ such that $\overline{\Omega} \subset \Omega_\lambda$ and*

$$u(x) = \lambda^\beta \quad for\ all \quad x \in \partial\Omega_\lambda. \tag{1.4}$$

Then $\partial\Omega$ is a sphere with center the origin and u must be radially symmetric.

Such overdetermination by only one level surface of solutions of elliptic boundary value problems over bounded domains has been considered. In [5], Enache and the second author of the present paper studied the overdetermination by only one level surface which is similar to the boundary. To be precise, they considered some boundary value problem for some fully nonlinear elliptic problem in a bounded domain Ω, and proved that, if there exist constants $\lambda \in (0, 1)$ and α such that $u(x) = \alpha$ on $\partial\Omega_\lambda$, then Ω must be the interior of an N-dimensional ellipsoid. See [5, Theorem 2.1]. In [3,13], the overdetermination by only one level surface which is parallel to the boundary was considered. By applying the method of moving planes directly to the problems as in [10, Proof of Theorem 1.2, pp. 941–942], the authors proved that the underlying domain must be a ball.

The paper is organized as follows. In Sect. 2, we give some preliminary proposition and prove the key lemma of this paper. Using this lemma and the proposition, we prove the main theorems in Sect. 3. In Sect. 4, we give two remarks on condition (C) concerning a sequence of similar level sets.

2 Preliminaries

We prepare several notations. For each $r > 0$ and $z \in \mathbb{R}^N$, denote by $B_r(z)$, the open ball in $\mathbb{R}^N$ with radius r and center z. For each C^1 domain $\Omega \subset \mathbb{R}^N$, at $p \in \partial\Omega$, $T_p(\partial\Omega)$ and $\nu(p)$ denote the tangent space of $\partial\Omega$ and the outer unit normal vector to $\partial\Omega$, respectively.

We first prove the following proposition:

Proposition 2.1 *Let $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be a bounded C^1 domain containing the origin. If every point vector $p \in \partial\Omega$ is parallel to the outer unit normal vector $\nu(p)$ to $\partial\Omega$, then $\partial\Omega$ must be a sphere with center the origin.*

Proof Since Ω is a bounded C^1 domain, $\partial\Omega$ has finitely many connected components, and each component is a C^1 closed hypersurface embedded in $\mathbb{R}^N$. Let Γ be a component of $\partial\Omega$. Then, for any $p \in \Gamma$, we have

$$p \perp T_p(\Gamma). \tag{2.1}$$

Let $p = p(t)$ be a regular curve on Γ. Then, by (2.1) we obtain

$$p(t) \perp \frac{\mathrm{d}}{\mathrm{d}t} p(t) \quad \text{for} \quad \text{all} \quad t,$$

namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(|p(t)|^2) = 0 \quad \text{for} \quad \text{all} \quad t.$$

Therefore, we see that there exists a positive constant C such that $|p(t)| = C$ for all t. This implies that Γ is a sphere with center the origin. Therefore, since Ω is a domain containing the origin, we see that $\partial\Omega$ must be a sphere with center the origin, and Proposition 2.1 follows. □

Next, we prove the key lemma for the proofs of Theorems 1.1 and 1.2.

Lemma 2.1 *Let $N \geq 2$ and let $u \in C^1(\mathbb{R}^N \times (0, \infty))$, and let Ω be a bounded domain in $\mathbb{R}^N$ containing the origin. Suppose that there exists a half-space H of $\mathbb{R}^N$ including $\overline{\Omega}$ such that*

$$\frac{\partial u}{\partial l} < 0 \quad \text{in} \quad \left(\mathbb{R}^N \setminus H\right) \times (0, \infty), \tag{2.2}$$

where l is the outer unit normal vector to ∂H and suppose the following condition:

$$(C) \quad \begin{cases} \text{there exist two sequences of positive numbers } \{t_n\}_{n=1}^{\infty} \text{ and } \{a_n\}_{n=1}^{\infty} \text{ such that} \\ t_n \uparrow \infty \text{ as } n \uparrow \infty, \text{ and } u(t_n x, t_n) = a_n \text{ for all } n \in \mathbb{N} \text{ and } x \in \partial\Omega. \end{cases}$$

If $p \in \partial\Omega$ and $l \in T_p(\partial\Omega)$, then $p \cdot l \leq 0$.

Proof Without loss of generality, set $l = (1, 0, \ldots, 0)$. Then, since $0 \in \Omega$ and $\overline{\Omega} \subset H$, there exists a positive constant λ satisfying $H = \{x \in \mathbb{R}^N : x_1 < \lambda\}$. Suppose that there exists a point $p \in \partial\Omega$ such that

$$p \cdot l = p_1 > 0 \quad \text{and} \quad l \in T_p(\partial\Omega).$$

Hence, by condition (C), we have

$$\frac{\partial u}{\partial x_1}(t_n p, t_n) = 0 \tag{2.3}$$

for all $n \in \mathbb{N}$. Since $p_1 > 0$ and $t_n \uparrow \infty$ as $n \uparrow \infty$, by (2.3), we see that there exists a sufficiently large number n_* such that $t_{n_*} p_1 > \lambda$ and

$$\frac{\partial u}{\partial x_1}(t_{n_*} p, t_n) = 0,$$

which contradicts (2.2). □

3 Proofs of Theorems 1.1 and 1.2

The purpose of this section is to prove Theorems 1.1 and 1.2. We first prove Theorem 1.2.

Proof of Theorem 1.2. By (1.3), (1.4) and the uniqueness of the solution of (1.2), we see that

$$u(x) = \lambda^{\beta} u(\lambda^{-1} x) \text{ for all } x \in \mathbb{R}^N \setminus \overline{\Omega_\lambda}.$$

Therefore, setting

$$t_n = \lambda^n \text{ and } a_n = \lambda^{\beta(n+1)} \text{ for every } n \in \mathbb{N}$$

yields that the solution u of (1.2) satisfies

$$t_n \uparrow \infty \quad \text{as} \quad n \uparrow \infty, \quad \text{and} \quad u(t_n x) = a_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \partial\Omega_\lambda.$$

On the other hand, by applying an argument similar to that in the proof of [4, Theorem 1.3] with Aleksandrov's reflection principle (see [7], for example), the maximum principle and Hopf's boundary point lemma (see [2] and [4, Proposition 2.6]), we see that, for each direction $l \in \partial B_1(0)$, if a half-space H of $\mathbb{R}^N$ includes $\overline{\Omega}$ and has the outer unit normal vector l to ∂H, then the solution of (1.2) satisfies that

$$\frac{\partial u}{\partial l} < 0 \quad \text{on} \quad \partial H.$$

Moreover, since every half-space including the above half-space satisfies the same conditions, we notice that (2.2) of Lemma 2.1 holds true also for the solution of (1.2). Therefore, we can apply Lemma 2.1 to every direction $l \in \partial B_1(0)$ and conclude that every point vector $p \in \partial\Omega$ is parallel to the outer unit normal vector $\nu(p)$ to $\partial\Omega$. Hence, by Proposition 2.1, $\partial\Omega$ must be a sphere with center the origin. Therefore, since for every $A \in \mathcal{O}^N(\mathbb{R})$, the function $u(Ax)$ also satisfies (1.2) by (H3), then, by the uniqueness of the solution of (1.2), $u(x) \equiv u(Ax)$ and hence u must be radially symmetric. The proof of Theorem 1.2 is complete. ☐

Next, we prove Theorem 1.1. Let H be an arbitrary half-space of $\mathbb{R}^N$ including G_0. Then, by Aleksandrov's reflection principle, the maximum principle and Hopf's boundary point lemma, we have

$$\frac{\partial u}{\partial l} < 0 \quad \text{on} \quad \partial H \times (0, \infty), \tag{3.1}$$

where l is the outer unit normal vector to ∂H. Moreover, u satisfies (2.2) of Lemma 2.1. Therefore, by condition (C), we can use Lemma 2.1 and hence by Proposition 2.1 $\partial\Omega$ must be a sphere with center the origin for $N \geq 2$.

We first prove Theorem 1.1 with $N = 1$.

Proof of Theorem 1.1 for N = 1. Since $0 \in G_0 \subset \Omega$, we can set $\Omega = (a, b)$ for some $a < 0 < b$. Then, it follows from condition (C) that

$$u(t_n a, t_n) = u(t_n b, t_n)(= a_n) \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.2}$$

Let us show that $a + b = 0$. Suppose that $a + b > 0$. Then, since $t_n \uparrow \infty$ as $n \uparrow \infty$, there exists $m \in \mathbb{N}$ such that

$$\frac{t_m(a+b)}{2} > b. \tag{3.3}$$

Consider the function $w = w(x, t)$ defined by

$$w(x, t) = u(x, t) - u(t_m a + t_m b - x, t).$$

Then, we have from (3.3) the following:

$$\begin{array}{lll} \partial_t w = \partial_x^2 w & \text{in} & (t_m(a+b)/2, +\infty) \times (0, +\infty), \\ w = 0 & \text{on} & \{t_m(a+b)/2\} \times (0, +\infty), \\ w \leq 0 \quad \text{and} \quad w \not\equiv 0 & \text{on} & (t_m(a+b)/2, +\infty) \times \{0\}. \end{array}$$

Thus, it follows from the strong maximum principle that

$$w < 0 \text{ in } (t_m(a+b)/2, +\infty) \times (0, +\infty),$$

which contradicts the fact that $w(t_m b, t_m) = 0$ because of (3.2). Therefore, we conclude that $a + b \leq 0$. By the same argument, we also conclude that $a + b \geq 0$.

Here, we can put $\Omega = (-b, b)$. For $(x, t) \in [0, \infty) \times [0, \infty)$, consider the functions $v = v(x, t)$ and $v_0 = v_0(x)$ defined by

$$v(x, t) = u(x, t) - u(-x, t) \quad \text{and} \quad v_0(x) = g(x) - g(-x).$$

It suffices to prove

$$v_0(x) = 0 \qquad \text{for almost every} \quad x \in [0, \infty). \tag{3.4}$$

Indeed, if (3.4) holds, then $g(x) = g(|x|)$ for almost every $x \in \mathbb{R}$. This together with the uniqueness of the solution yields the conclusion of Theorem 1.1 with $N = 1$.

Since $G_0 \subset \Omega = (-b, b)$, we see that $\mathrm{spt}(v_0) \subset (-b, b)$. This implies that v satisfies

$$v(x, t) = (4\pi t)^{-\frac{1}{2}} \int_{-b}^{b} \mathrm{e}^{-\frac{(x-y)^2}{4t}} v_0(y)\, \mathrm{d}y \tag{3.5}$$

for all $(x, t) \in \mathbb{R} \times (0, +\infty)$. Furthermore, by (3.2) with $a = -b$, we see that

$$v(t_n b, t_n) = 0 \quad \text{for every} \quad n \in \mathbb{N}.$$

This together with (3.5) yields

$$\int_{-b}^{b} v_0(y) \mathrm{e}^{\frac{by}{2}} \mathrm{e}^{-\frac{y^2}{4t_n}}\, \mathrm{d}y = 0 \;\; \text{for every} \;\; n \in \mathbb{N}. \tag{3.6}$$

Put

$$w_0(y) = v_0(y) \mathrm{e}^{\frac{by}{2}}. \tag{3.7}$$

Then, by (3.6), we have

$$\int_{0}^{b^2} \frac{w_0(\sqrt{s}) + w_0(-\sqrt{s})}{2\sqrt{s}} \mathrm{e}^{-\frac{s}{4t_n}}\, \mathrm{d}s = 0 \;\; \text{for every } n \in \mathbb{N}.$$

Since $t_n \to \infty$ as $n \to \infty$, by the analyticity of the exponential function, we obtain

$$\int_{0}^{b^2} \frac{w_0(\sqrt{s}) + w_0(-\sqrt{s})}{2\sqrt{s}} \mathrm{e}^{-\lambda s}\, \mathrm{d}s = 0, \quad \lambda \in \mathbb{R}.$$

This together with the injectivity of the Laplace transform yields

$$w_0(\sqrt{s}) + w_0(-\sqrt{s}) = 0 \;\; \text{for almost every } s > 0. \tag{3.8}$$

By (3.7) and (3.8), we have

$$v_0(\sqrt{s}) \mathrm{e}^{\frac{b\sqrt{s}}{2}} + v_0(-\sqrt{s}) \mathrm{e}^{-\frac{b\sqrt{s}}{2}} = 0 \quad \text{for almost every} \quad s > 0.$$

This implies that $v_0(\sqrt{s}) = 0$ for almost every $s > 0$. Thus, we have (3.4), and Theorem 1.1 with $N = 1$ follows. □

Next, we prove Theorem 1.1 for the case $N \geq 2$. Before beginning the proof, we recall the following lemma, which follows from the Funk-Hecke formula (see [1, Theorem 2.22, p. 36] or [12, Theorem 6, p. 20]) and Rodrigues' formula (see [1, Theorem 2.23, p. 37] or [12, Theorem 5, p. 17]).

Lemma 3.1 *Let $L \neq 0$ be a real constant. For $f \in L^2(S^{N-1})$, set*

$$\mathcal{L}f(\omega) = \int_{S^{N-1}} e^{L\alpha\cdot\omega} f(\alpha)\, d\sigma(\alpha) \;\; for\ every\ \omega \in S^{N-1},$$

where $d\sigma(\alpha)$ denotes the area element of the $(N-1)$-dimensional unit sphere S^{N-1} in $\mathbb{R}^N$. Then the set $\{\mathcal{L}f : f \in L^2(S^{N-1})\}$ is dense in $L^2(S^{N-1})$.

Proof Let $p = p(x)$ be an arbitrary harmonic homogeneous polynomial of degree $k \geq 0$ in $\mathbb{R}^N$. Then, it follows from the Funk-Hecke formula that

$$\mathcal{L}p(\omega) = \lambda p(\omega) \quad \text{for every } \omega \in S^{N-1} \text{ and } \lambda = |S^{N-2}| \int_{-1}^{1} \mathrm{e}^{Lt} P_k(t)(1-t^2)^{\frac{N-3}{2}}\, dt,$$

where $|S^{N-2}|$ denotes the volume of the $(N-2)$-dimensional unit sphere in $\mathbb{R}^{N-1}$ and $P_k(t)$ denotes the Legendre polynomial of degree k in $\mathbb{R}^N$. Moreover, Rodrigues' formula gives

$$P_k(t) = (-1)^k \frac{\Gamma\left(\frac{N-1}{2}\right)}{2^k \Gamma\left(k + \frac{N-1}{2}\right)} (1-t^2)^{\frac{3-N}{2}} \left(\frac{d}{dt}\right)^k (1-t^2)^{k+\frac{N-3}{2}}.$$

Therefore, integrating by parts k times on the definition of the number λ yields that

$$\lambda = |S^{N-2}| \frac{\Gamma\left(\frac{N-1}{2}\right)}{2^k \Gamma\left(k + \frac{N-1}{2}\right)} L^k \int_{-1}^{1} \mathrm{e}^{Lt} (1-t^2)^{\frac{N-3}{2}}\, dt \neq 0.$$

This implies that the linear space $\{\mathcal{L}f : f \in L^2(S^{N-1})\}$ contains all the spherical harmonics because any spherical harmonic is given by restricting a harmonic homogeneous polynomial onto S^{N-1}. Therefore, the conclusion holds true. □

Now, we are ready to prove Theorem 1.1 for the case $N \geq 2$.

Proof of Theorem 1.1 for $N \geq 2$. First of all, since we already know that $\partial\Omega$ is a sphere with center the origin, there exists a constant $R > 0$ such that $\Omega = B_R(0)$. Take $A \in \mathcal{O}^N(\mathbb{R})$ arbitrarily. Then, it follows from (C) that

$$u(t_n x, t_n) - u(t_n Ax, t_n) = 0, \qquad x \in \partial B_R(0), \tag{3.9}$$

for all $n \in \mathbb{N}$. Since $G_0 \subset B_R(0)$, by (3.9), we have

$$\int_{|y|\leq R} \mathrm{e}^{\frac{x\cdot y}{2}} \mathrm{e}^{-\frac{|y|^2}{4t_n}} (g(y) - g(Ay))\, \mathrm{d}y = 0, \qquad x \in \partial B_R(0),$$

for all $n \in \mathbb{N}$. This, together with the analyticity of the exponential function and the fact that $t_n \to \infty$ as $n \to \infty$, implies that

$$\int_{|y|\leq R} \mathrm{e}^{\frac{x\cdot y}{2}} \mathrm{e}^{-s|y|^2} (g(y) - g(Ay))\, \mathrm{d}y = 0, \qquad x \in \partial B_R(0), \tag{3.10}$$

for all $s \in \mathbb{R}$. By setting $x = R\alpha$ for $\alpha \in S^{N-1}(= \partial B_1(0))$, we have from (3.10) that

$$\int_0^R r^{N-1} e^{-sr^2} \int_{S^{N-1}} e^{\frac{Rr}{2}\alpha\cdot\omega}(g(r\omega) - g(rA\omega))\, d\sigma(\omega)\, dr = 0, \qquad \alpha \in S^{N-1},$$

for all $s \in \mathbb{R}$. This together with the injectivity of the Laplace transform yields

$$\int_{S^{N-1}} e^{\frac{Rr}{2}\alpha\cdot\omega}(g(r\omega) - g(rA\omega))\, d\sigma(\omega) = 0, \qquad \alpha \in S^{N-1}, \tag{3.11}$$

for almost every $r > 0$. Let $f \in L^2(S^{N-1})$. Then, by (3.11), we obtain

$$\int_{S^{N-1}} \int_{S^{N-1}} e^{\frac{Rr}{2}\alpha\cdot\omega}(g(r\omega) - g(rA\omega)) f(\alpha)\, d\sigma(\omega)\, d\sigma(\alpha) = 0$$

for almost every $r > 0$. Thus, setting $L = \frac{Rr}{2}$ for almost every fixed $r > 0$ in Lemma 3.1 yields that for almost every fixed $r > 0$

$$\int_{S^{N-1}} \mathcal{L}f(\omega)(g(r\omega) - g(rA\omega))\, d\sigma(\omega) = 0 \quad \text{for every} \quad f \in L^2(S^{N-1}).$$

Then, by Lemma 3.1, we have that for almost every fixed $r > 0$,

$$g(r\omega) = g(rA\omega) \ \text{ for almost every } \omega \in S^{N-1}.$$

This yields that $g(x) = g(Ax)$ for almost every $x \in \mathbb{R}^N$ and hence by the uniqueness of the solution, we have that

$$u(x,t) = u(Ax,t) \ \text{ for all } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Therefore, since $A \in \mathcal{O}^N(\mathbb{R})$ is arbitrary, we see that u is radially symmetric with respect to the origin. This completes the proof of Theorem 1.1 for $N \geq 2$. □

4 Remarks on condition (*C*)

In this last section, we give two remarks on condition (C) concerning a sequence of similar level sets. First one means that the order of dependence for the sequence $\{t_n\}_{n=1}^{\infty}$ with respect to spatial variables in condition (C) is important. Second one says that if the solution u has similar level sets continuously with time, then we can easily carry out the second step of the proof of Theorem 1.1.

Remark 4.1 There exists a solution of (1.1) with $N = 3$ which is not radially symmetric even if it has a sequence of similar level sets.

Let a be a positive constant and $v_0 \in C_0^{\infty}(\mathbb{R})$ be a nonnegative even function satisfying

$$\operatorname{spt}(v_0) = [-a, a] \quad \text{and} \quad v_0' < 0 \quad \text{if} \quad s \in (0, a). \tag{4.1}$$

Put

$$v(s,t) = (4\pi t)^{-\frac{1}{2}} \int_{-a}^{a} e^{-\frac{(s-\mu)^2}{4t}} v_0(\mu)\, d\mu, \qquad w(s,t) = \frac{\partial^3}{\partial s^3} v(s,t), \tag{4.2}$$

for $(s, t) \in \mathbb{R} \times (0, \infty)$. Then, since v is a even function in s, w is a odd function in s. Furthermore, we set $r = |x|$ for $x \in \mathbb{R}^3$ and put

$$f(r, t) = \frac{\partial}{\partial r}\left(\frac{w(r, t)}{r}\right) = \left(r\frac{\partial w}{\partial r} - w\right) r^{-2}. \tag{4.3}$$

Then, we have the following lemma:

Lemma 4.1 *Let f be the function given in (4.3). Then there exists a positive function $r(t)$ for $t > 0$ such that*

$$f(r(t), t) = 0, \qquad r(t) = O(t^{\frac{1}{2}}) \quad as \quad t \to \infty.$$

Proof By (4.1) and (4.2), we can take three positive functions $r_2(t)$, $r_3(t)$ and $r_4(t)$ for $t > 0$ such that $r_2(t) < r_3(t) < r_4(t)$ and

$$\frac{\partial^3}{\partial s^3} v(r_3(t), t) = 0, \qquad \frac{\partial^3}{\partial s^3} v(s, t) < 0 \quad \text{if} \quad s > r_3(t), \tag{4.4}$$

$$\frac{\partial^4}{\partial s^4} v(r_2(t), t) = \frac{\partial^4}{\partial s^4} v(r_4(t), t) = 0, \tag{4.5}$$

$$\frac{\partial^4}{\partial s^4} v(s, t) < 0 \quad \text{if} \quad r_2(t) < r < r_4(t), \text{ and } \frac{\partial^4}{\partial s^4} v(s, t) > 0 \quad \text{for} \quad s > r_4(t), \tag{4.6}$$

for all $t > 0$. Put

$$h(s, t) = s^2 f(s, t) = s\frac{\partial^4 v}{\partial s^4} - \frac{\partial^3 v}{\partial s^3}$$

Then, since $h(s, t) > 0$ for $s \geq r_4(t)$ and $h(s, t) < 0$ for $s \in [r_2(t), r_3(t)]$, by applying the intermediate value theorem, we can take a positive function $r(t)$ for $t > 0$ such that

$$f(r(t), t) = 0, \qquad r_3(t) < r(t) < r_4(t). \tag{4.7}$$

On the other hands, by (4.2), we obtain

$$(4\pi t)^{\frac{1}{2}} \frac{\partial^3 v}{\partial s^3} = -\frac{1}{8t^3} \int_{-a}^{a} (s - \mu)\{-6t + (s - \mu)^2\} \mathrm{e}^{-\frac{(s-\mu)^2}{4t}} v_0(\mu)\, \mathrm{d}\mu, \tag{4.8}$$

$$\begin{aligned}(4\pi t)^{\frac{1}{2}} \frac{\partial^4 v}{\partial s^4} &= \frac{1}{16t^4} \int_{-a}^{a} \{12t^2 - 12t(s - \mu)^2 + (s - \mu)^4\} \mathrm{e}^{-\frac{(s-\mu)^2}{4t}} v_0(\mu)\, \mathrm{d}\mu \\ &> \frac{1}{16t^4} \int_{-a}^{a} (s - \mu)^2\{(s - \mu)^2 - 12t\} \mathrm{e}^{-\frac{(s-\mu)^2}{4t}} v_0(\mu)\, \mathrm{d}\mu.\end{aligned} \tag{4.9}$$

for all $(s, t) \in \mathbb{R} \times (0, \infty)$. By (4.8), we see that

$$\begin{aligned}&\frac{\partial^3 v}{\partial s^3} < 0 \quad \text{for} \quad s > a + \sqrt{6t}, \\ &\frac{\partial^3 v}{\partial s^3} > 0 \quad \text{for} \quad a < s < \sqrt{5t} - a \quad \text{with} \quad t > \frac{4a^2}{5}.\end{aligned}$$

These together with (4.4) yield

$$\sqrt{5t} - a \leq r_3(t) \leq a + \sqrt{6t} \quad \text{for} \quad t > \frac{4a^2}{5}. \tag{4.10}$$

Furthermore, by (4.9), we have

$$\frac{\partial^4 v}{\partial s^4} > 0 \quad \text{for} \quad s > a + \sqrt{12t}.$$

This together with (4.5) implies that

$$r_4(t) \leq a + \sqrt{12t}. \tag{4.11}$$

Combining (4.7), (4.10) and (4.11) yields that

$$\sqrt{5t} - a \leq r_3(t) < r(t) < r_4(t) \leq a + \sqrt{12t}$$

for all $t > (4a^2)/5$. This implies that $r(t) = O(t^{\frac{1}{2}})$ as $t \to \infty$; thus, Lemma 4.1 follows. □

On the other hand, by (4.1) and (4.3), we can take a nonnegative radially symmetric function $\psi(|x|) \in C_0^\infty(\mathbb{R}^3)$ such that

$$\psi(|x|) + f(|x|, 0)\frac{x_1}{|x|} \geq 0$$

for all $x \in \mathbb{R}^3$. We set

$$u(x,t) = (4\pi t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \psi(|y|)\,\mathrm{d}y + f(|x|,t)\frac{x_1}{|x|} := u_{\mathrm{rad}}(|x|,t) + f(|x|,t)\frac{x_1}{|x|}. \tag{4.12}$$

Then, the function u is a solution of (1.1) with $N = 3$ and

$$g(x) = \psi(|x|) + f(|x|, 0)\frac{x_1}{|x|}.$$

By Lemma 4.1 and (4.12), we see that if $|x| = r(t)$, then there exists a function $c(t)$ such that

$$u(x,t) = u_{\mathrm{rad}}(r(t), t) = c(t).$$

This implies that the solution u is not radially symmetric even if it has a sequence of similar level sets.

Remark 4.2 Instead of (C), suppose that there exists a function $a = a(t)$ for $t > 0$ such that

$$u((1+t)x, t) = a(t)$$

for all $(x,t) \in \partial\Omega \times (0,\infty)$. Then, we can use the maximum principle and the unique continuation theorem and get the same conclusion of Theorem 1.1.

Indeed, as in the proof of Theorem 1.1, by Aleksandrov's reflection principle, the maximum principle, Hopf's boundary point lemma, Proposition 2.1 and Lemma 2.1, we see that $\partial\Omega$ must be a sphere with center the origin for $N \geq 2$. Say $\partial\Omega = \partial B_R(0)$ for some $R > 0$. Take $A \in \mathcal{O}^N(\mathbb{R})$ arbitrarily. Then

$$u((1+t)x, t) - u((1+t)Ax, t) = 0, \qquad x \in \partial B_R(0),\ t > 0.$$

Since $u((1+0)x, 0) - u((1+0)Ax, 0) = 0$ if $x \notin B_R(0)$, by the maximum principle, we get

$$u(x,t) - u(Ax,t) = 0 \quad \text{if } |x| > (1+t)R.$$

Hence, it follows from the unique continuation theorem (see [6]) that $u(x,t) - u(Ax,t)$ equals zero identically, which gives the conclusion.

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