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Extremals for sharp GNS inequalities on compact manifolds

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Abstract Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. In Ceccon and Montenegro (Math Z 258:851–873, 2008; J Diff Equ 254(6):2532–2555, 2013) showed that, for any $1 < p \leq 2$ and $1 \leq q < r < p^* = \frac{np}{n-p}$, there exists a constant B such that the sharp Gagliardo–Nirenberg inequality

$$\left(\int_M |u|^r \, dv_g\right)^{\frac{p}{r\theta}} \leq \left(A_{\mathrm{opt}} \int_M |\nabla_g u|^p \, dv_g + B \int_M |u|^p \, dv_g\right)\left(\int_M |u|^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}}.$$

holds for all $u \in C^\infty(M)$. In this work, assuming further $1 < p < 2$, $p < r$ and $1 \leq q \leq \frac{r}{r-p}$, we derive existence and compactness results of extremal functions corresponding to the saturated version of the above sharp inequality. Sobolev inequality can be seen as a limiting case as r tends to p^*.

Keywords Sharp Sobolev inequalities · De Giorgi–Nash–Moser estimates · Extremal functions · Compactness

Mathematics Subject Classification (2000) 32Q10 · 53C21

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1 Overview and main theorems

A lot of attention has been paid to so called sharp Gagliardo–Nirenberg inequalities. Such inequalities play a key role in the study of qualitative properties of some evolution PDEs (see, for example, [1,6,8,14,21,32,33]).

Let $1 < p < n$ and $1 \le q < r < p^*$, where $p^* = \frac{np}{n-p}$ denotes the Sobolev critical exponent. Denote by $D^{p,q}(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$||u||_{D^{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |u|^q \, \mathrm{d}x \right)^{\frac{1}{q}} .$$

The sharp Euclidean Gagliardo–Nirenberg inequality states that, for any function $u \in D^{p,q}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^r \, \mathrm{d}x \right)^{\frac{p}{r\theta}} \le A_0(p,q,r) \left(\int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x \right) \left(\int_{\mathbb{R}^n} |u|^q \, \mathrm{d}x \right)^{\frac{p(1-\theta)}{\theta q}} , \qquad (1)$$

where $\theta = \frac{np(r-q)}{r(q(p-n)+np)} \in (0,1)$ and $A_0(p,q,r)$ is the best possible constant in this inequality, which is well defined thanks to the Euclidean Sobolev inequality.

The inequality (1) was introduced independently by Gagliardo and Nirenberg in [20] and [27]. Some particular cases are quite known. Indeed, in the limit case $r = p^*$, (1) yields the well-known Euclidean Sobolev inequality introduced by Sobolev in [29]. The famous Nash inequality, introduced by Nash in [26], corresponds to $p = 2, q = 1$ and $\theta = n/(n+2)$. At last, the Moser inequality, introduced by Moser in [25], arises when $p = 2, q = 2$ and $\theta = n/(n+2)$. According to Bakry et al. [5], non-sharp inequalities of type (1) are all equivalent for $p \ge 1$ fixed and similar versions still hold when $p \ge n$, whereas the Sobolev embedding is not valid in this case.

Over the past years, Some studies have been devoted to the search for extremal functions of (1). Different methods have been employed in this endeavor for certain parameters p, q and r. Namely, Aubin [3] and Talenti [30] found extremal functions for Euclidean optimal Sobolev inequalities. Extremal functions to the sharp Nash inequality were found by Carlen and Loss in [9]. Besides, Cordero et al. [13] and Del Pino and Dolbeault [15] independently obtained extremal functions for the family of parameters $p < q \le \frac{p(n-1)}{n-p}$ and $r = \frac{p(q-1)}{p-1}$. In this case, the extremal functions are explicitly given by

$$u(x) = a \left(1 + b|x|^{\frac{p}{p-1}} \right)^{-\frac{p-1}{q-p}} ,$$

where a and b are positive constants. In particular, one easily sees that the set of extremals of (1) is not C^0-compact. The knowledge of extremal functions is open for several values of p, q and r.

Let (M, g) be a closed Riemannian manifold of dimension $n \ge 2$ and let $1 < p < n$ and $1 \le q < r < p^*$. Denote by $H^{1,p}(M)$ the Riemannian–Sobolev space defined as the completion of $C^\infty(M)$ under the norm

$$||u||_{H^{1,p}(M)} := \left(\int_M |\nabla_g u|^p \, \mathrm{d}v_g + \int_M |u|^p \, \mathrm{d}v_g \right)^{1/p} .$$

In [10], assuming $1 < p \leq 2$ and $p < r$, it is proved the existence of a constant B such that the Riemannian Gagliardo–Nirenberg inequality

$$\left(\int_M |u|^r \, dv_g \right)^{\frac{p}{r\theta}} \leq \left(A_0(p,q,r,g) \int_M |\nabla_g u|^p \, dv_g + B \int_M |u|^p \, dv_g \right) \times \left(\int_M |u|^q \, dv_g \right)^{\frac{p(1-\theta)}{\theta q}} . \tag{2}$$

holds for all $u \in H^{1,p}(M)$, where dv_g and ∇_g denote, respectively, the Riemannian volume element and the gradient operator of g and $A_0(p,q,r,g)$ stands for the first best possible constant in this inequality.

The case $r = p^*$ and $p = 2$ was proved to be valid for some B by Hebey and Vaugon [22] and, independently, by Aubin and Li [4] and Druet [16] when $1 < p < 2$, and generally non-valid for any B by Druet [17] when $p > 2$. The optimal Nash inequality, with $p = 2, q = 1$ and $\theta = n/(n+2)$, was obtained for some B by Humbert in [23] (see also [19]). Later, Brouttelande [7] extended its validity to $p = 2, 1 \leq q < r$ and $q \leq 2 \leq r < 2 + \frac{2}{n}q$. Closely related inequalities has been recently investigated by Chen and Sun in [12] for $p > 2$.

In a natural way, one then considers the sharp inequality

$$\left(\int_M |u|^r \, dv_g \right)^{\frac{p}{r\theta}} \leq \left(A_0(p,q,r,g) \int_M |\nabla_g u|^p \, dv_g + B_0(p,q,r,g) \int_M |u|^p \, dv_g \right) \left(\int_M |u|^q \, dv_g \right)^{\frac{p(1-\theta)}{\theta q}} \tag{3}$$

which is also valid for all $u \in H^{1,p}(M)$, where

$$B_0(p,q,r,g) := \min\{B \in \mathbb{R} : (2) \text{ is valid for all } u \in H^{1,p}(M)\},$$

and also the notion of extremal function as a non-zero function in $H^{1,p}(M)$ which satisfies (3) with equality.

For $r = p^*$, we refer the reader to the Druet and Hebey's book [18] which is an excellent survey concerning the whole program of sharp Sobolev inequalities such as validity of saturated inequalities, existence of extremals, among others.

Let $\mathcal{E}(p,q,r,g)$ be the set of all extremal functions $u \in H^{1,p}(M)$ such that $||u||_{L^r(M)} = 1$. A simple computation guarantees that each extremal function $u_0 \in \mathcal{E}(p,q,r,g)$ satisfies an equation of kind

$$-\Delta_{p,g} u_0 + a|u_0|^{p-2}u_0 + b|u_0|^{q-2}u_0 = c|u_0|^{r-2}u_0 \quad \text{on} \quad M$$

where $\Delta_{p,g} = -\mathrm{div}_g(|\nabla_g|^{p-2}\nabla_g)$ denotes the p-Laplace operator of g and a, b and c are positive constants. in particular, the elliptic regularity theory applied to this equation gives $\mathcal{E}(p,q,r,g) \subset C^0(M)$. Note also that, by the strong maximum principle, extremal functions can be assumed positive on M.

A first question is to know if $\mathcal{E}(p,q,r,g)$ is non-empty. Another important one concerns with topological properties satisfied by $\mathcal{E}(p,q,r,g)$ as, for example, if or not it is compact in the C^0-topology. This work answers positively both questions. The compactness is discussed into an uniform view point on the parameters p, q and r. Two ingredients are essential in order this: results on continuity of $A_0(p, q, r, g)$ and local boundedness of $B_0(p, q, r, g)$ with respect to p, q and r.

Namely, our main results are:

Theorem 1.1 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$ and let $1 \leq q < r < p^*$. The set $\mathcal{E}(p, q, r, g)$ is non-empty whenever $1 < p < 2, p < r$ and $1 \leq q \leq \frac{r}{r-p}$.*

Theorem 1.2 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. For fixed parameters $1 < p_1 \leq p_2 < 2$ and $1 \leq q_1 \leq q_2 < r_1 \leq r_2 < p_1^*$ with $p_2 < r_1, p_1 < r_2$ and $q_2 \leq \frac{r_2}{r_2 - p_1}$, the set $\{u \in \mathcal{E}(p, q, r, g) : \ p_1 \leq p \leq p_2, \ q_1 \leq q \leq q_2$ and $r_1 \leq r \leq r_2\}$ is compact in the C^0-topology. In particular, the same conclusion holds for each set $\mathcal{E}(p, q, r, g)$, where $1 < p < 2, p < r$ and $q \leq \frac{r}{r-p}$.*

The study of the continuity of $A_0(p, q, r, g)$ with respect to the triple (p, q, r) can be translated in terms of the continuity of $A_0(p, q, r)$ once these two best constants are equal whenever $p \leq r$.

When $p < r$, it is natural to hope that $A_0(p, q, r)$ continuously depends on (p, q, r). Indeed, according to [14],

$$A_0(p, q, r) = \frac{q-p}{p\sqrt{\pi}} \left(\frac{pq}{n(q-p)}\right)^{\frac{1}{p}} \left(\frac{np - q(n-p)}{pq}\right)^{\frac{1}{r}} \times \left(\frac{\Gamma\left(\frac{q(p-1)}{q-p}\right)\Gamma(\frac{n}{2}+1)}{\Gamma\left(\frac{p-1}{p(q-p)}(np - qn + qp)\right)\Gamma\left(\frac{n(p-1)}{p}+1\right)}\right)^{\frac{1}{n}}$$

for all $p < q < \frac{p(n-1)}{n-p}$ and $r = \frac{p(q-1)}{p-1}$.

In [2], Agueh showed that $A_0(p, q, r)$ can generally be splitted as

$$A_0(p, q, r) = D_0(p, q, r) m(p, q, r)^{\frac{nq-np-rp}{n(r-q)}},$$

where $D_0(p, q, r)$ is explicitly given in terms of Gamma functions and $m(p, q, r)$ is defined by

$$m(p, q, r) := \left\{E_{p,q}(u) : \ u \in D^{p,q}(\mathbb{R}^n) \quad \text{and} \quad ||u||_{L^r(\mathbb{R}^n)} = 1\right\}, \tag{4}$$

where

$$E_{p,q}(u) := \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^n} |u|^q \, \mathrm{d}x.$$

Using Corollary II.3 of [24], one concludes that the constant $m(p, q, r)$ is attained for a positive function, which is radially symmetric, non-increasing, tends to 0 as $|x| \to +\infty$ and satisfies the equation

$$-\Delta_p u + u^{q-1} = l(p, q, r) u^{r-1} \quad \text{in} \quad \mathbb{R}^n, \tag{5}$$

where $l(p, q, r)$ is a Lagrange multiplier. By using decaying properties for solutions of the above equation, we just establish the continuity of $m(p, q, r)$ for the range $1 < p < n$ and $1 \leq q < r < p^*$.

The proof of Theorem 2.1 and also of the local boundedness of $B_0(p,q,r,g)$ are done by contradiction and are based on blow-up and concentration analyzes of minimizers associated to suitable functionals. Important additional difficulties arise in the concentration part when we seek to establish the desired contradiction. The ideas used for surrounding them are inspired in the recent paper [11]. Furthermore, our approach greatly simplifies that one made in the paper [10] devoted to the validity question for $p < r$.

The complete proof of Theorems 1.1 and 1.2 will be carried out into four sections. Section 2 is dedicated to the proof of a result on continuity of $A_0(p,q,r)$ which is stated as Theorem 2.1. In Sect. 3, we prove the bound of $B_0(p,q,r,g)$ under the assumptions of Theorem 1.2 which is stated as Theorem 3.1. Finally, the proofs of existence of extremals and of compactness are done in Sects. 4 and 5, respectively.

2 Continuous dependence of $A_0(p,q,r)$

In this section, it is proved the following theorem:

Theorem 2.1 *For each dimension $n \geq 2$, the best constant $A_0(p,q,r)$ is continuous on the set of parameters*

$$1 < p < n, \quad 1 \leq q < r < p^*. \tag{6}$$

In other words, given triples $(p_\alpha, q_\alpha, r_\alpha)$ converging to (p_0, q_0, r_0) as $\alpha \to +\infty$, if all these triples satisfy (6), then $A_0(p_\alpha, q_\alpha, r_\alpha)$ converges to $A_0(p_0, q_0, r_0)$ as $\alpha \to +\infty$.

Let $m(p,q,r)$ and $l(p,q,r)$ be defined as in (4) and (5), respectively. Given $\delta > 0$, one easily checks that these constants are bounded on all (p,q,r) satisfying (6) with $p \leq n - \delta$. Indeed, fixed a nonzero function $v \in C_0^\infty(\mathbb{R}^n)$, we have

$$0 \leq m(p,q,r) \leq E_{p,q}\left(\frac{v}{||v||_{L^r}}\right) \leq C_1(n,\delta) \tag{7}$$

for all triple (p,q,r) satisfying (6), where $C_1(n,\delta)$ is a positive constant depending only on n and δ. In particular, the claim follows from

$$0 \leq l(p,q,r) \leq \max\{p,q\} m(p,q,r) \leq \max\left\{n, \frac{n^2}{\delta}\right\} C_1(n,\delta).$$

Let us now describe a L^r decaying property satisfied by solutions of the problem (5).

Lemma 2.1 *Let p_0, q_0 and r_0 be fixed numbers satisfying (6). Then, for any $\delta_0 > 0$ small enough, there exist positive constants C_0 and ζ_0, depending only on n and δ_0 such that, for any (p,q,r) satisfying (6), $p \in [p_0 - \delta_0, n - \delta_0]$ and $q \in [1, p_0^* - \delta_0]$ and any positive radial minimizer $u \in D^{p,q}(\mathbb{R}^n)$ of $m(p,q,r)$, one has*

$$\int\limits_{|x|>\rho} |u|^r \, \mathrm{d}x \leq C_0 \rho^{-\zeta_0}$$

for all $\rho \geq 1$. In particular, the above decaying holds for (p,q,r) close enough to (p_0, q_0, r_0).

Proof of Lemma 2.1 Let $u \in D^{p,q}(\mathbb{R}^n)$ be a positive radial minimizer of $m(p,q,r)$. We next consider two distinct cases.

Assume first that $q > p$. By Hölder's inequality, one has

$$u^p(\rho) = -p \int\limits_{\rho}^{+\infty} u^{p-1} u' \mathrm{d}s = -p \int\limits_{\rho}^{+\infty} (us^{\frac{n-1}{q}})^{p-1} u' s^{\frac{n-1}{p}} s^{(n-1)(-\frac{p-1}{q}-\frac{1}{p})} \mathrm{d}s$$

$$\leq n||\nabla u||_{L^p(\mathbb{R}^n)} ||u||^{p-1}_{L^q(\mathbb{R}^n)} \left(\int\limits_{\rho}^{+\infty} s^{(n-1)(-\frac{p-1}{q}-\frac{1}{p})t} \mathrm{d}s \right)^{\frac{1}{t}},$$

where $t = pq/[(q-p)(p-1)]$. By (7), we then derive

$$u^p(\rho) \leq C_1(n, \delta_0) \left(\int\limits_{\rho}^{+\infty} s^{(n-1)(-t+1)} \mathrm{d}s \right)^{\frac{1}{t}}.$$

Because $q \leq p_0^* - \delta_0 < (p_0 - \delta_0)^* \leq p^*$, the above inequality yields

$$u(\rho) \leq C_2(n, \delta_0) \rho^{-\frac{n-1}{p} + \frac{n}{tp}} \tag{8}$$

for all $\rho > 0$ and (p, q, r) as in the statement of lemma, where $C_i(n, \delta_0), i = 1, 2$, are positive constants depending only on n and δ_0.

On the other hand, by Hölder's inequality,

$$\int\limits_{|x|>\rho} u^r \mathrm{d}x \leq \left(\int\limits_{|x|>\rho} u^q \mathrm{d}x \right)^{\frac{p^*-r}{p^*-q}} \left(\int\limits_{|x|>\rho} u^{p^*} \mathrm{d}x \right)^{\frac{r-q}{p^*-q}}.$$

By (7), the first right-hand side integral is bounded by a constant depending on n and δ_0. So, estimating the last integral with the aid (8), one obtains

$$\int\limits_{|x|>\rho} u^r \mathrm{d}x \leq C_0^* \rho^{-\zeta_0^*}$$

for all $\rho \geq 1$, where C_0^* and ζ_0^* are positive constants depending only on n and δ_0.

In the case that $q \leq p$, Hölder's inequality gives

$$u(\rho)^p = -p \int\limits_{\rho}^{+\infty} (us^{\frac{n-1}{p}})^{p-1} u' s^{\frac{n-1}{p}} s^{-(n-1)} \mathrm{d}s \leq n||\nabla u||_{L^p(\mathbb{R}^n)} ||u||^{p-1}_{L^p(\mathbb{R}^n)} \rho^{-(n-1)}.$$

Applying an interpolation with respect to q and p^* and also (7), one derives

$$u(\rho) \leq C_3(n, \delta_0) \rho^{-\frac{(n-1)}{p}},$$

where $C_3(n, \delta_0)$ is a positive constant depending only on n and δ_0.

Proceeding exactly as in the previous case, one gets

$$\int\limits_{|x|>\rho} u^r \mathrm{d}x \leq C_0^{**} \rho^{-\zeta_0^{**}}$$

for all $\rho \geq 1$, where C_0^{**} and ζ_0^{**} are positive constants depending only on n and δ_0.

Finally, letting $C_0 = \max\{C_0^*, C_0^{**}\}$ and $\zeta_0 = \min\{\zeta_0^*, \zeta_0^{**}\}$, we conclude the proof.

We now are ready to prove the main result of this section.

Proof of Theorem 2.1 Let $(p_\alpha, q_\alpha, r_\alpha)$ and (p_0, q_0, r_0) be triples satisfying (6) and such that $(p_\alpha, q_\alpha, r_\alpha)$ converges to (p_0, q_0, r_0) as $\alpha \to +\infty$. It suffices to show that there exists a subsequence, denoted also by $(p_\alpha, q_\alpha, r_\alpha)$, such that $A_0(p_\alpha, q_\alpha, r_\alpha)$ converges to $A_0(p_0, q_0, r_0)$ as $\alpha \to +\infty$.

Let $u_\alpha \in D^{p,q}(\mathbb{R}^n)$ be a positive radial minimizer for $m_\alpha = m(p_\alpha, q_\alpha, r_\alpha)$ such that $||u_\alpha||_{L^r(\mathbb{R}^n)} = 1$. Thanks to the boundedness of m_α, we can apply the Moser iterative scheme to the Eq. (5) on concentric balls of radii R. In particular, we find a positive constant $C_0(R)$ depending on R, so that

$$\|u_\alpha\|_{L^\infty(B_R)} \le C_0(R)$$

for $\alpha > 0$ large enough.

From the above estimate and elliptic regularity theory, one easily checks that (u_α) converges to u_0 in $C^1_{loc}(\mathbb{R}^n)$, modulo a subsequence. This fact and Lemma 2.1 readily yield

$$1 = \int\limits_{\mathbb{R}^n} u_\alpha^{r_\alpha}\, \mathrm{d}x = \int\limits_{B(0,\rho)} u_\alpha^{r_\alpha}\, \mathrm{d}x + \int\limits_{\mathbb{R}^n \setminus B(0,\rho)} u_\alpha^{r_\alpha}\, \mathrm{d}x \le \int\limits_{B(0,\rho)} u_\alpha^{r_\alpha}\, \mathrm{d}x + C_0 \rho^{-\zeta_0}\ .$$

Then, letting $\alpha \to +\infty$, one obtains

$$1 = \int\limits_{\mathbb{R}^n} u_\alpha^{r_\alpha}\, \mathrm{d}x \le \int\limits_{B(0,\rho)} u_0^{r_0}\, \mathrm{d}x + C_0 \rho^{-\zeta_0} \le \int\limits_{\mathbb{R}^n} u_0^{r_0}\, \mathrm{d}x + C_0 \rho^{-\zeta_0}$$

for all $\rho \ge 1$, so that

$$\int\limits_{\mathbb{R}^n} u_0^{r_0}\, \mathrm{d}x \ge 1.$$

Conversely,

$$\int\limits_{B(0,\rho)} u_0^{r_0}\, \mathrm{d}x = \lim_{\alpha \to +\infty} \int\limits_{B(0,\rho)} u_\alpha^{r}\, \mathrm{d}x \le 1,$$

so that

$$\int\limits_{\mathbb{R}^n} u_0^{r_0}\, \mathrm{d}x = 1.$$

Let now φ be any function in $C_0^\infty(\mathbb{R}^n)$. One knows that

$$\left(\int\limits_{\mathbb{R}^n} |\varphi|^{r_\alpha}\, \mathrm{d}x \right)^{\frac{p_\alpha}{r_\alpha \theta_\alpha}} \le A_0(p_\alpha, q_\alpha, r_\alpha) \left(\int\limits_{\mathbb{R}^n} |\nabla \varphi|^{p_\alpha}\, \mathrm{d}x \right) \left(\int\limits_{\mathbb{R}^n} |\varphi|^{q_\alpha}\, \mathrm{d}x \right)^{\frac{p_\alpha(1-\theta_\alpha)}{\theta_\alpha q_\alpha}}. \quad (9)$$

Letting $\alpha \to +\infty$, it follows that

$$\left(\int\limits_{\mathbb{R}^n} |\varphi|^{r_0}\, \mathrm{d}x \right)^{\frac{p_0}{r_0 \theta_0}} \le \liminf_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha) \left(\int\limits_{\mathbb{R}^n} |\nabla \varphi|^{p_0}\, \mathrm{d}x \right) \left(\int\limits_{\mathbb{R}^n} |\varphi|^{q_0}\, \mathrm{d}x \right)^{\frac{p_0(1-\theta_0)}{\theta_0 q_0}},$$

so that

$$A_0(p_0, q_0, r_0) \leq \liminf_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha). \tag{10}$$

On the other hand, as proved in Theorem 2.1 of [2], u_α is an extremal function for the inequality (9). Therefore,

$$\begin{aligned}
&\left(\int\limits_{B(0,\rho)} |\nabla u_0|^{p_0} \, \mathrm{d}x \right) \left(\int\limits_{B(0,\rho)} u_0^{q_0} \, \mathrm{d}x \right)^{\frac{p_0(1-\theta_0)}{\theta_0 q_0}} \\
&= \lim_{\alpha \to +\infty} \left(\int\limits_{B(0,\rho)} |\nabla u_\alpha|^{p_\alpha} \, \mathrm{d}x \right) \left(\int\limits_{B(0,\rho)} u_\alpha^{q_\alpha} \, \mathrm{d}x \right)^{\frac{p_\alpha(1-\theta_\alpha)}{\theta_\alpha q_\alpha}} \\
&\leq \liminf_{\alpha \to +\infty} \left(\int\limits_{\mathbb{R}^n} |\nabla u_\alpha|^{p_\alpha} \, \mathrm{d}x \right) \left(\int\limits_{\mathbb{R}^n} u_\alpha^{q_\alpha} \, \mathrm{d}x \right)^{\frac{p_\alpha(1-\theta_\alpha)}{\theta_\alpha q_\alpha}} \\
&= \liminf_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha)^{-1} \\
&= \left(\limsup_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha) \right)^{-1},
\end{aligned}$$

so that

$$\left(\int\limits_{\mathbb{R}^n} |\nabla u_0|^{p_0} \, \mathrm{d}x \right) \left(\int\limits_{\mathbb{R}^n} u_0^{q_0} \, \mathrm{d}x \right)^{\frac{p_0(1-\theta_0)}{\theta_0 q_0}} \leq \left(\limsup_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha) \right)^{-1}.$$

Since $u_0 \in D^{p_0,q_0}(\mathbb{R}^n)$ and $\|u_0\|_{L^r} = 1$, one has

$$\limsup_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha) \leq A_0(p_0, q_0, r_0). \tag{11}$$

Finally, from (10) and (11), we conclude that

$$\lim_{\alpha \to +\infty} A_0(p_\alpha, q_\alpha, r_\alpha) = A_0(p_0, q_0, r_0).$$

3 Boundedness of $B_0(p, q, r, g)$

Our goal in this section is to establish the following result on bound of $B_0(p, q, r, g)$:

Theorem 3.1 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. For fixed parameters $1 < p_1 < p_2 < 2$ and $1 \leq q_1 < q_2 < r_1 < r_2 < p_1^*$ with $p_2 < r_1$, there exists a constant $K > 0$ such that $B_0(p, q, r, g) \leq K$ for all $p_1 \leq p \leq p_2, q_1 \leq q \leq q_2$ and $r_1 \leq r \leq r_2$.*

The proof of this theorem is done into several claims and, in order to make the simpler notations, we denote $\alpha = (p, q, r)$, $\alpha_0 = (p_0, q_0, r_0)$, $\theta = \theta(p, q, r)$ and $\theta_0 = \theta(p_0, q_0, r_0)$. Here we assume α converges to α_0.

From now on, several possibly different positive constants independent of α will be denoted by c or c_i, $i = 1, 2, \ldots$

Let $\kappa \in (0, 1)$ be a fixed number. From the definition of $B_0(p, q, r, g)$, we have

$$\nu_\alpha = \inf_{u \in E} J_\alpha(u) < A_0(p, q, r)^{-1}, \tag{12}$$

where $E = \{u \in H^{1,p}(M) : \ ||u||_{L^r(M)} = 1\}$ and

$$J_\alpha(u) = \left(\int_M |\nabla_g u|^p \, \mathrm{d}v_g + C_\alpha \int_M |u|^p \, \mathrm{d}v_g \right) \left(\int_M |u|^q \, \mathrm{d}v_g \right)^{\frac{p(1-\theta)}{\theta q}}$$

with $C_\alpha = \frac{B_0(p,q,r,g)}{A_0(p,q,r)} \kappa$.

Since J_α is of class C^1, by using standard variational arguments, we find a minimizer $u_\alpha \in E$ of J_α, i.e.

$$J_\alpha(u_\alpha) = \nu_\alpha = \inf_{u \in E} J_\alpha(u). \tag{13}$$

One may assume $u_\alpha \geq 0$, since $\nabla_g |u_\alpha| = \pm \nabla_g u_\alpha$. Each minimizer u_α satisfies the Euler–Lagrange equation

$$A_\alpha \Delta_{p,g} u_\alpha + C_\alpha A_\alpha u_\alpha^{p-1} + \frac{1-\theta}{\theta} \nu_\alpha ||u_\alpha||_{L^q(M)}^{-q} u_\alpha^{q-1} = \frac{\nu_\alpha}{\theta} u_\alpha^{r-1} \ \text{ on } \ M, \tag{14}$$

where $\Delta_{p,g} = -\mathrm{div}_g(|\nabla_g|^{p-2}\nabla_g)$ is the p-Laplace operator of g and

$$A_\alpha = \left(\int_M u_\alpha^q \, \mathrm{d}v_g \right)^{\frac{p(1-\theta)}{\theta q}}.$$

By the elliptic regularity theory [31], it follows that u_α is of class $C^1(M)$.

The proof is now carried out by contradiction. namely, assume $B_0(p, q, r, g)$ is not bounded as $\alpha \to \alpha_0$.

Thanks to Theorem 2.1, up to a subsequence, we have

$$\lim_{\alpha \to \alpha_0} C_\alpha = +\infty,$$

where $\alpha_0 = (p_0, q_0, r_0)$ with $p_1 \leq p_0 \leq p_2$, $q_1 \leq q_0 \leq q_2$ and $r_1 \leq r_0 \leq r_2$.

From (12) and (13), one gets

$$C_\alpha A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g < A_0(p, q, r)^{-1},$$

so that

$$A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g \to 0. \tag{15}$$

One also knows that

$$A_0(p, q, r)^{-1} \leq A_\alpha \left(\int_M |\nabla_g u_\alpha|^p \, \mathrm{d}v_g + C_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g \right) + \frac{\kappa}{A_0(p, q, r)} A_\alpha \int_M u_\alpha^p \mathrm{d}v_g.$$

Letting $\alpha \to \alpha_0$ and evoking again Theorem 2.1, one obtains

$$\liminf_{\alpha \to \alpha_0} J_\alpha(u_\alpha) \geq A(p_0, q_0, r_0)^{-1}.$$

So, by (12), one has

$$\lim_{\alpha\to\alpha_0} \nu_\alpha = \lim_{\alpha\to\alpha_0} J_\alpha(u_\alpha) = A(p_0, q_0, r_0)^{-1}. \tag{16}$$

Finally, we assert that

$$\lim_{\alpha\to\alpha_0} A_\alpha = 0. \tag{17}$$

Otherwise, if $\limsup_{\alpha\to\alpha_0} A_\alpha > 0$, up to a subsequence, we can assume $\lim_{\alpha\to\alpha_0} A_\alpha > 0$. Then, by (12) and (15) (instead of using that $p \le r$), there exists a constant $c > 0$ such that

$$||u_\alpha||_{H^{1,p}(M)} \le c$$

for α close enough to α_0.

Because $p_0 < r_0 < p_0^*$ and p and r tend respectively to p_0 and r_0, we can choose $t < p_0$ and s so that $p, r < s < t^*$. So, one easily deduces that (u_α) is bounded in $H^{1,t}(M)$ for α close enough to α_0 and, by compactness, $u_\alpha \to u$ in $L^s(M)$. Therefore,

$$||u_\alpha - u||_{L^p(M)} \to 0$$

and

$$||u_\alpha - u||_{L^r(M)} \to 0$$

as $\alpha \to \alpha_0$.

From the first above limit and (15),

$$||u_\alpha||_{L^p(M)} \to ||u||_{L^{p_0}(M)} = 0$$

and from the second one,

$$1 = ||u_\alpha||_{L^r(M)} \to ||u||_{L^{r_0}(M)}$$

as $\alpha \to \alpha_0$. This contradiction concludes the claim (17).

Let $x_\alpha \in M$ be a maximum point of u_α, i.e

$$u_\alpha(x_\alpha) = ||u_\alpha||_{L^\infty(M)}. \tag{18}$$

Claim 1 We assert that

$$\lim_{\sigma\to+\infty} \lim_{\alpha\to\alpha_0} \int_{B(x_\alpha,\sigma a_\alpha)} u_\alpha^r \, dv_g = 1, \tag{19}$$

where

$$a_\alpha = A_\alpha^{\frac{r}{np-nr+pr}}. \tag{20}$$

Proof of Claim 1 By (17), it is clear that $a_\alpha \to 0$ as $\alpha \to \alpha_0$.

For $x \in B(0, \sigma)$, set

$$\begin{aligned} h_\alpha(x) &= g(\exp_{x_\alpha}(a_\alpha x)),\\ \varphi_\alpha(x) &= a_\alpha^{\frac{n}{r}} u_\alpha(\exp_{x_\alpha}(a_\alpha x)). \end{aligned} \tag{21}$$

Joining (14) and the definition of θ, one easily checks that

$$\Delta_{p,h_\alpha}\varphi_\alpha + C_\alpha a_\alpha^p \varphi_\alpha^{p-1} + \frac{1-\theta}{\theta}\nu_\alpha\, \varphi_\alpha^{q-1} = \frac{\nu_\alpha}{\theta}\varphi_\alpha^{r-1} \quad \text{on} \quad B(0,\sigma). \tag{22}$$

A Moser's iterative scheme applied to (22) (see [28]) produces

$$a_\alpha^n ||u_\alpha||^r_{L^\infty(M)} = \sup_{B(0,\frac{\sigma}{2})} \varphi_\alpha^r \leq c \int\limits_{B(0,\sigma)} \varphi_\alpha^r \, \mathrm{d}h_\alpha = c \int\limits_{B(x_\alpha,\sigma a_\alpha)} u_\alpha^r \, \mathrm{d}v_g \leq c$$

for α close enough to α_0. This estimate together with

$$1 = \int\limits_M u_\alpha^r \, \mathrm{d}v_g \leq ||u_\alpha||^{r-q}_{L^\infty(M)} \int\limits_M u_\alpha^q \mathrm{d}v_g = \left(||u_\alpha||_{L^\infty(M)} \, a_\alpha^{\frac{n}{r}} \right)^{r-q}$$

yield

$$1 \leq ||u_\alpha||_{L^\infty(M)} \, a_\alpha^{\frac{n}{r}} \leq c. \tag{23}$$

In particular, there exists a constant $c > 0$ such that

$$\int\limits_{B(0,\sigma)} \varphi_\alpha^r \, \mathrm{d}h_\alpha \geq c \tag{24}$$

for α close enough to α_0.

On the other hand, we have

$$\int\limits_{B(0,\sigma)} \varphi_\alpha^p \, \mathrm{d}x \leq c \int\limits_{B(0,\sigma)} \varphi_\alpha^p \, \mathrm{d}h_\alpha = a_\alpha^{\frac{np}{r}-n} \int\limits_{B(x_\alpha,\sigma a_\alpha)} u_\alpha^p \, \mathrm{d}v_g \leq c(\sigma) a_\alpha^{\frac{np}{r}} ||u_\alpha||^p_{L^\infty(M)} \leq c(\sigma),$$

with $c(\sigma) \to +\infty$ as $\sigma \to +\infty$.

Moreover,

$$\int\limits_{B(0,\sigma)} |\nabla \varphi_\alpha|^p \, \mathrm{d}x \leq c \int\limits_{B(0,\sigma)} |\nabla_{h_\alpha} \varphi_\alpha|^p \, \mathrm{d}h_\alpha = A_\alpha \int\limits_{B(x_\alpha,\sigma a_\alpha)} |\nabla_g u_\alpha|^p \, \mathrm{d}v_g \leq A_0(p,q,r)^{-1}. \tag{25}$$

Let $1 < t < p_0$. For α close enough to α_0, the above inequalities imply that (φ_α) is bounded in $H^{1,t}(B(0,\sigma))$ for each $\sigma > 0$. So, modulo a subsequence, we derive the pointwise convergence $\varphi_\alpha \to \varphi$ almost everywhere in $\mathbb{R}^n$. By Fatou's Lemma,

$$\int\limits_{B(0,\sigma)} \varphi^{q_0} \, \mathrm{d}x = \liminf_{\alpha \to \alpha_0} \int\limits_{B(0,\sigma)} \varphi_\alpha^q \, \mathrm{d}h_\alpha = \liminf_{\alpha \to \alpha_0} \frac{\int_{B(x_\alpha,\sigma a_\alpha)} u_\alpha^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} \leq 1, \tag{26}$$

$$\int\limits_{B(0,\sigma)} \varphi^{r_0} \, \mathrm{d}x = \liminf_{\alpha \to \alpha_0} \int\limits_{B(0,\sigma)} \varphi_\alpha^r \, \mathrm{d}h_\alpha = \liminf_{\alpha \to \alpha_0} \int\limits_{B(x_\alpha,\sigma a_\alpha)} u_\alpha^r \, \mathrm{d}v_g \leq 1. \tag{27}$$

In particular,

$$\varphi \in L^{q_0}(\mathbb{R}^n) \cap L^{r_0}(\mathbb{R}^n).$$

In addition, proceeding as before, it is possible to choose $t < p_0$ and s so that $q, r < s < t^*$. Thus, for any $\sigma > 0$, we can assume

$$||\varphi_\alpha - \varphi||_{L^q(B(0,\sigma))} \to 0 \tag{28}$$

and

$$||\varphi_\alpha - \varphi||_{L^r(B(0,\sigma))} \to 0 \tag{29}$$

as $\alpha \to \alpha_0$.

Let $\eta \in C_0^1(\mathbb{R})$ be a cutoff function such that $\eta = 1$ on $[0, \frac{1}{2}]$, $\eta = 0$ on $[1, \infty)$ and $0 \leq \eta \leq 1$. Set now $\eta_{\alpha,\sigma}(x) = \eta((\sigma a_\alpha)^{-1} d_g(x, x_\alpha))$. Taking $u_\alpha \eta_{\alpha,\sigma}^p$ as a test function in (14), one gets

$$A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g + A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-2} \nabla_g u_\alpha \cdot \nabla_g (\eta_{\alpha,\sigma}^p) u_\alpha \, \mathrm{d}v_g + C_\alpha A_\alpha \int\limits_M u_\alpha^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g$$
$$+ \frac{1-\theta}{\theta} \nu_\alpha ||u_\alpha||_{L^q(M)}^{-q} \int\limits_M u_\alpha^q \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g = \frac{\nu_\alpha}{\theta} \int\limits_M u_\alpha^r \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g. \tag{30}$$

next we show that

$$\lim_{\sigma \to +\infty} \lim_{\alpha \to \alpha_0} A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-2} \nabla_g u_\alpha \cdot \nabla_g (\eta_{\alpha,\sigma}^p) u_\alpha \, \mathrm{d}v_g = 0. \tag{31}$$

Indeed, it suffices to guarantee that

$$\lim_{\sigma \to \infty} \lim_{\alpha \to \alpha_0} A_\alpha \int\limits_M u_\alpha^p |\nabla_g \eta_{\alpha,\sigma}|^p \, \mathrm{d}v_g = 0. \tag{32}$$

Thanks to the inequality $|\nabla_g \eta_{\alpha,\sigma}| \leq \frac{c}{\sigma a_\alpha}$ and (20), one obtains

$$\begin{aligned} A_\alpha \int\limits_M u_\alpha^p |\nabla_g \eta_{\alpha,\sigma}|^p \, \mathrm{d}v_g &\leq c \frac{A_\alpha}{\sigma^p a_\alpha^p} \int\limits_{B(x_\alpha, \sigma a_\alpha)} u_\alpha^p \, \mathrm{d}v_g \\ &\leq c \frac{A_\alpha}{\sigma^p a_\alpha^p} \left(\int\limits_M u_\alpha^r \, \mathrm{d}v_g \right)^{\frac{p}{r}} \left(\int\limits_{B(x_\alpha, \sigma a_\alpha)} \mathrm{d}v_g \right)^{1-\frac{p}{r}} \\ &= c \sigma^{\frac{nr-np-pr}{r}} \end{aligned}$$

which clearly converges to 0 as $\alpha \to \alpha_0$ and $\sigma \to +\infty$.

Replacing (31) in (30), one arrives at

$$\theta_0 A(p_0, q_0, r_0) \lim_{\sigma \to +\infty} \lim_{\alpha \to \alpha_0} \left(A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g \right)$$
$$+ (1-\theta_0) \lim_{\sigma \to +\infty} \lim_{\alpha \to \alpha_0} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} \leq \lim_{\sigma \to +\infty} \lim_{\alpha \to \alpha_0} \int\limits_M u_\alpha^r \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g, \tag{33}$$

where $\theta_0 = \theta(p_0, q_0, r_0)$. In order to rewrite this inequality in a more suitable format, we first remark that

$$\left| \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} - \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} \right| \leq \frac{\int_{B(x_\alpha, \sigma a_\alpha) \setminus B(x_\alpha, \sigma a_\alpha/2)} u_\alpha^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} = \int\limits_{B(0,\sigma) \setminus B(0,\sigma/2)} \varphi_\alpha^q \, \mathrm{d}h_\alpha.$$

So, thanks to (28) and the fact that $\varphi \in L^{q_0}(\mathbb{R}^n)$, one has

$$\lim_{\sigma \to +\infty} \lim_{\alpha \to \alpha_0} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} = \lim_{\sigma \to +\infty} \lim_{\alpha \to \alpha_0} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g}.$$

Estimating

$$\left|\int_M u_\alpha^r \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g - \int_M u_\alpha^r \eta_{\alpha,\sigma}^r \, \mathrm{d}v_g\right| \leq \int_{B(x_\alpha,\sigma a_\alpha)\setminus B(x_\alpha,(\sigma a_\alpha)/2)} u_\alpha^r \, \mathrm{d}v_g = \int_{B(0,\sigma)\setminus B(0,\sigma/2)} \varphi_\alpha^r \, \mathrm{d}h_\alpha$$

and arguing in a similar way, by (29), one gets

$$\lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\int_M u_\alpha^r \eta_{\alpha,\sigma}^r \, \mathrm{d}v_g = \lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\int_M u_\alpha^r \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g.$$

Consequently, (33) can be rewritten as

$$\begin{aligned}&\theta_0 A(p_0,q_0,r_0)\lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\left(A_\alpha\int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g\right)\\&+(1-\theta_0)\lim_{\sigma+\to\infty}\lim_{\alpha\to\alpha_0}\frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g}\\&\leq \lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\int_M u_\alpha^r \eta_{\alpha,\sigma}^r \, \mathrm{d}v_g.\end{aligned}\tag{34}$$

On the other hand,

$$\begin{aligned}\left(\int_M u_\alpha^r \eta_{\alpha,\sigma}^r \, \mathrm{d}v_g\right)^{\frac{p}{r\theta}} \leq &\left(A_0(p,q,r)\int_M |\nabla_g(u_\alpha\eta_{\alpha,\sigma})|^p \, \mathrm{d}v_g\right.\\&\left.+B_0(p,q,r,g)\int_M u_\alpha^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g\right)\left(\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\end{aligned}$$

and the definition of A_α lead to

$$\begin{aligned}\left(\int_M u_\alpha^r \eta_{\alpha,\sigma}^r \, \mathrm{d}v_g\right)^{\frac{p}{r\theta}} \leq &(A_0(p,q,r)+\varepsilon)\left(\int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g\right)\left(\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\&+c(\varepsilon)\, A_\alpha\int_M u_\alpha^p |\nabla_g \eta_{\alpha,\sigma}|^p \, \mathrm{d}v_g + C_\alpha A_\alpha \int_M u_\alpha^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g.\end{aligned}\tag{35}$$

Using then (14) and (32) and letting $\alpha\to\alpha_0$, $\sigma\to+\infty$ and $\varepsilon\to 0$, one gets

$$\begin{aligned}&\left(\lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\left(\int_M u_\alpha^r \eta_{\alpha,\sigma}^r \, \mathrm{d}v_g\right)\right)^{\frac{p_0}{r_0\theta_0}}\\&\leq A(p_0,q_0,r_0)\lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\left(A_\alpha\int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, \mathrm{d}v_g\right)\\&\times\lim_{\sigma\to+\infty}\lim_{\alpha\to\alpha_0}\left(\frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g}\right)^{\frac{p_0(1-\theta_0)}{\theta_0 q_0}}.\end{aligned}\tag{36}$$

Let

$$X = A(p_0, q_0, r_0) \lim_{\sigma\to+\infty} \lim_{\alpha\to\alpha_0} \left(A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \eta^p_{\alpha,\sigma} \mathrm{d}v_g \right),$$

$$Y = \lim_{\sigma\to+\infty} \lim_{\alpha\to\alpha_0} \frac{\int_M u^q_\alpha \eta^q_{\alpha,\sigma} \mathrm{d}v_g}{\int_M u^q_\alpha \mathrm{d}v_g},$$

and

$$Z = \lim_{\sigma\to+\infty} \lim_{\alpha\to\alpha_0} \int\limits_M u^r_\alpha \eta^r_{\alpha,\sigma} \mathrm{d}v_g.$$

It is clear that $X, Y, Z \le 1$ and (34) and (36) take the form

$$\begin{cases} \theta_0 X + (1-\theta_0) Y \le Z \\ Z \le X^{\frac{r_0\theta_0}{p_0}} Y^{\frac{r_0(1-\theta_0)}{q_0}} \end{cases} \tag{37}$$

By (24), we have $Z > 0$, so that $X, Y > 0$.

In order to end the proof of (19), it suffices to show that $Z = 1$. By Young's inequality, (37) immediately yields

$$\begin{cases} X^{\theta_0} Y^{1-\theta_0} \le Z \\ Z \le X^{\frac{r_0\theta_0}{p_0}} Y^{\frac{r_0(1-\theta_0)}{q_0}} \end{cases}$$

But these two inequalities give

$$X^{\theta_0} Y^{1-\theta_0} \le X^{\frac{r_0\theta_0}{p_0}} Y^{\frac{r_0(1-\theta_0)}{q_0}} \le X^{\theta_0} Y^{\frac{r_0(1-\theta_0)}{q_0}},$$

so that $Y = 1$. Therefore, by (20) and (23),

$$\int\limits_{M\setminus B(x_\alpha,\sigma a_\alpha)} u^r_\alpha \,\mathrm{d}v_g \le ||u_\alpha||^{r-q}_{L^\infty(M)} a_\alpha^{\frac{n(r-q)}{r}} \frac{\int_{M\setminus B(x_\alpha,\sigma a_\alpha)} u^q_\alpha \,\mathrm{d}v_g}{\int_M u^q_\alpha \,\mathrm{d}v_g} \le c \frac{\int_{M\setminus B(x_\alpha,\sigma a_\alpha)} u^q_\alpha \,\mathrm{d}v_g}{\int_M u^q_\alpha \,\mathrm{d}v_g},$$

which implies that

$$\lim_{\sigma\to+\infty} \lim_{\alpha\to\alpha_0} \int\limits_{M\setminus B(x_\alpha,\sigma a_\alpha)} u^r_\alpha \,\mathrm{d}v_g = 0.$$

Thus, it follows that $Z = 1$.

A key tool in the proof of Theorem 3.1 consists of the following uniform estimate:

Claim 2 There exists a constant $c > 0$, independent of p, q and r, such that

$$d_g(x, x_\alpha)^p u_\alpha(x)^{r-p} \le c a_\alpha^{\frac{np-nr+pr}{r}}$$

for $x \in M$ and α close enough to α_0.

Proof of Claim 2 Suppose, by contradiction, that the above assertion is false.

Set

$$f_\alpha(x) = d_g(x, x_\alpha)^p u_\alpha(x)^{r-p} a_\alpha^{\frac{nr-np-pr}{r}}.$$

If $y_\alpha \in M$ is a maximum point of f_α, then $f_\alpha(y_\alpha) = ||f_\alpha||_{L^\infty(M)} \to +\infty$ when $\alpha \to \alpha_0$. By (23), we have

$$f_\alpha(y_\alpha) \leq c\left(\frac{u_\alpha(y_\alpha)}{||u_\alpha||_{L^\infty(M)}}\right)^{r-p} d_g(x_\alpha, y_\alpha)^p ||u_\alpha||^{\frac{pr}{n}}_{L^\infty(M)} \leq c d_g(x_\alpha, y_\alpha)^p ||u_\alpha||^{\frac{pr}{n}}_{L^\infty(M)},$$

so that

$$d_g(x_\alpha, y_\alpha)||u_\alpha||^{\frac{r}{n}}_{L^\infty(M)} \to +\infty. \tag{38}$$

For any fixed $\sigma > 0$ and $\varepsilon \in (0, 1)$, we next show that

$$B(y_\alpha, \varepsilon\, d_g(x_\alpha, y_\alpha)) \cap B\left(x_\alpha, \sigma||u_\alpha||^{-\frac{r}{n}}_{L^\infty(M)}\right) = \emptyset \tag{39}$$

for α close enough to α_0. Note that this claim follows readily from

$$d_g(x_\alpha, y_\alpha) \geq \sigma||u_\alpha||^{-\frac{r}{n}}_{L^\infty(M)} + \varepsilon d_g(x_\alpha, y_\alpha).$$

On the other hand, the above inequality is equivalent to

$$d_g(x_\alpha, y_\alpha)(1-\varepsilon)||u_\alpha||^{\frac{r}{n}}_{L^\infty(M)} \geq \sigma,$$

which is clearly satisfied since $d_g(x_\alpha, y_\alpha)||u_\alpha||^{\frac{r}{n}}_{L^\infty(M)} \to +\infty$ and $1-\varepsilon > 0$.

We assert that exists a constant $c > 0$ such that

$$u_\alpha(x) \leq c u_\alpha(y_\alpha) \tag{40}$$

for $x \in B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$ and α close enough to α_0. In fact, for $x \in B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$, we have

$$d_g(x, x_\alpha) \geq d_g(x_\alpha, y_\alpha) - d_g(x, y_\alpha) \geq (1-\varepsilon) d_g(x_\alpha, y_\alpha).$$

Thus,

$$\begin{aligned} d_g(y_\alpha, x_\alpha)^p u_\alpha(y_\alpha)^{r-p} a_\alpha^{\frac{nr-np-pr}{r}} &= f_\alpha(y_\alpha) \geq f_\alpha(x) = d_g(x, x_\alpha)^p u_\alpha(x)^{r-p} a_\alpha^{\frac{nr-np-pr}{r}} \\ &\geq (1-\varepsilon)^p d_g(y_\alpha, x_\alpha)^p u_\alpha(x)^{r-p} a_\alpha^{\frac{nr-np-pr}{r}}, \end{aligned}$$

so that

$$u_\alpha(x) \leq \left(\frac{1}{1-\varepsilon}\right)^{\frac{p}{r-p}} u_\alpha(y_\alpha)$$

for $x \in B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$ and α close enough to α_0. This proves our claim.

Since $f(y_\alpha) \to +\infty$, one has

$$A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}} \to 0.$$

So, we can define

$$\begin{aligned} h_\alpha(x) &= g(\exp_{y_\alpha}(A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}} x)) \\ \psi_\alpha(x) &= u_\alpha(y_\alpha)^{-1} u_\alpha(\exp_{y_\alpha}(A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}} x)) \end{aligned}$$

for each $x \in B(0, 2)$ and α close enough to α_0.

By (14), one easily checks that

$$\Delta_{p,h_\alpha}\psi_\alpha + C_\alpha A_\alpha u_\alpha(y_\alpha)^{p-r}\psi_\alpha^{p-1} + \frac{1-\theta}{\theta}\nu_\alpha ||u_\alpha||_{L^q(M)}^{-q} u_\alpha(y_\alpha)^{q-r}\psi_\alpha^{q-1}$$
$$= \frac{\nu_\alpha}{\theta}\psi_\alpha^{r-1} \quad \text{on} \quad B(0,2). \tag{41}$$

In particular,

$$\int\limits_{B(0,2)} |\nabla_{h_\alpha}\psi_p|^{p-2}\nabla_{h_\alpha}\psi_\alpha \cdot \nabla_{h_\alpha}\phi \; dv_{h_\alpha} \le c \int\limits_{B(0,2)} \psi_\alpha^{r-1}\phi \; dv_{h_\alpha}$$

for all positive test function $\phi \in C_0^1(B(0,2))$. So, a Moser's iterative scheme combined with (23) furnishes

$$1 = \sup_{B\left(0,\frac{1}{4}\right)} \psi_\alpha^r \le c \int\limits_{B\left(0,\frac{1}{2}\right)} \psi_\alpha^r \; dv_{h_\alpha}$$
$$= c\left(A_\alpha^{\frac{\theta q}{p(1-\theta)}} u_\alpha(y_\alpha)^{r-q}\right)^{-\frac{n(1-\theta)}{\theta q}} \int\limits_{B\left(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r \; dv_g$$
$$\le c\left(\frac{||u_\alpha||_{L^\infty(M)}}{u_\alpha(y_\alpha)}\right)^{\frac{np-rn+pr}{p}} \int\limits_{B\left(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r \; dv_g .$$

For simplicity, rewrite this last inequality as

$$0 < c \le m_\alpha^\varrho \int\limits_{B\left(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r \; dv_g, \tag{42}$$

where $m_\alpha = \frac{||u_\alpha||_{L^\infty(M)}}{u_\alpha(y_\alpha)}$ and $\varrho = \frac{np-rn+pr}{p}$.

By (20), (23) and (38), one has $B(y_\alpha, \frac{1}{2}A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}}) \subset B(y_\alpha, \varepsilon d(x_\alpha, y_\alpha))$ for α close enough to α_0. Therefore, (19) and (39) imply

$$\int\limits_{B\left(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r \; dv_g \to 0,$$

so that $m_\alpha \to +\infty$ as $\alpha \to \alpha_0$.

Our main goal now is to establish a contradiction to (42).

At first, by (23) and (40), one has

$$m_\alpha^\varrho \int\limits_{D_\alpha} u_\alpha^r \; dv_g \le m_\alpha^\varrho ||u_\alpha||_{L^\infty(D_\alpha)}^r (A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}})^n \le c m_\alpha^\varrho u_\alpha(y_\alpha)^r (A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}})^n \le c, \tag{43}$$

where $D_\alpha = B(y_\alpha, A_\alpha^{\frac{1}{p}}u_\alpha(y_\alpha)^{\frac{p-r}{p}})$.

Consider the function $\eta_\alpha(x) = \eta(A_\alpha^{-\frac{1}{p}} d_g(x, y_\alpha) u_\alpha(y_\alpha)^{\frac{r-p}{p}})$, where $\eta \in C_0^1(\mathbb{R})$ is a cutoff function satisfying $\eta = 1$ on $[0, \frac{1}{2}]$, $\eta = 0$ on $[1, \infty)$ and $0 \le \eta \le 1$. Taking $u_\alpha \eta_\alpha^p$ as a test function in (14), one has

$$A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, \mathrm{d}v_g + p A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-2} u_\alpha \eta_p^{p-1} \nabla_g u_\alpha \cdot \nabla_g \eta_\alpha \, \mathrm{d}v_g + C_\alpha A_\alpha \\ \times \int\limits_M u_\alpha^p \eta_\alpha^p \, \mathrm{d}v_g + \frac{1-\theta}{\theta} \nu_\alpha ||u_\alpha||_{L^q(M)}^{-q} \int\limits_M u_\alpha^q \eta_\alpha^p \, \mathrm{d}v_g = \frac{\nu_\alpha}{\theta} \int\limits_M u_\alpha^r \eta_\alpha^p \, \mathrm{d}v_g.$$

From Hölder and Young inequalities, the above second term can be estimated as

$$\left| \int\limits_M |\nabla_g u_\alpha|^{p-2} u_\alpha \eta_\alpha^{p-1} \nabla_g u_\alpha \cdot \nabla_g \eta_\alpha \, \mathrm{d}v_g \right| \le \varepsilon \int\limits_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, \mathrm{d}v_g + c_\varepsilon \int\limits_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, \mathrm{d}v_g.$$

Also, by (23) and (40), we have

$$\begin{aligned} A_\alpha \int\limits_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, \mathrm{d}v_g &\le A_\alpha (A_\alpha^{-\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{r-p}{p}})^p \int\limits_{D_\alpha} u_\alpha^p \, \mathrm{d}v_g \\ &\le c u_\alpha(y_\alpha)^r (A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}})^n \le c m_\alpha^{-\varrho}. \end{aligned} \tag{44}$$

Putting these inequalities into (43), one gets

$$A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, \mathrm{d}v_g + c \, C_\alpha A_\alpha \int\limits_M u_\alpha^p \eta_\alpha^p \, \mathrm{d}v_g + c \nu_\alpha ||u_\alpha||_{L^q(M)}^{-q} \int\limits_M u_\alpha^q \eta_\alpha^p \, \mathrm{d}v_g \le c m_\alpha^{-\varrho}. \tag{45}$$

On the other hand, the sharp Riemannian Gagliardo–Nirenberg inequality gives

$$\begin{aligned} &\left(\int\limits_{B(y_\alpha, \frac{1}{2} A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}})} u_\alpha^r \, \mathrm{d}v_g \right)^{\frac{p}{r\theta}} \le \left(\int\limits_M (u_\alpha \eta_\alpha^p)^r \, \mathrm{d}v_g \right)^{\frac{p}{r\theta}} \\ &\le c \left(\int\limits_M |\nabla_g u_\alpha|^p \eta_\alpha^{p^2} \, \mathrm{d}v_g \right) \left(\int\limits_M (u_\alpha \eta_\alpha^p)^q \, \mathrm{d}v_g \right)^{\frac{p(1-\theta)}{\theta q}} \\ &\quad + c \left(\int\limits_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, \mathrm{d}v_g + c \, C_\alpha \int\limits_M (u_\alpha \eta_\alpha^p)^p \, \mathrm{d}v_g \right) \left(\int\limits_M (u_\alpha \eta_\alpha^p)^q \, \mathrm{d}v_g \right)^{\frac{p(1-\theta)}{\theta q}}. \end{aligned} \tag{46}$$

Thanks to (44) and (45), we can estimate each term of the right-hand side of (46). Indeed,

$$\begin{aligned}
&\left(\int_M |\nabla_g u_\alpha|^p \eta_\alpha^{p^2}\, \mathrm{d}v_g\right)\left(\int_M (u_\alpha \eta_\alpha^p)^q\, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\
&\le \left(A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_\alpha^p\, \mathrm{d}v_g\right)\left(||u_\alpha||_{L^q(M)}^{-q} \int_M u_\alpha^q \eta_\alpha^p\, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\
&\le c m_\alpha^{-\varrho\left(1+\frac{p(1-\theta)}{\theta q}\right)},\\
&\left(\int_M |\nabla_g \eta_\alpha|^p u_\alpha^p\, \mathrm{d}v_g\right)\left(\int_M (u_\alpha \eta_\alpha^p)^q\, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\
&\le A_\alpha \int_M |\nabla_g \eta_\alpha|^p u_\alpha^p\, \mathrm{d}v_g \left(||u_\alpha||_{L^q(M)}^{-q} \int_M u_\alpha^q \eta_\alpha^p\, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\
&\le c m_\alpha^{-\varrho\left(1+\frac{p(1-\theta)}{\theta q}\right)}
\end{aligned}$$

and

$$\begin{aligned}
&C_\alpha \left(\int_M (u_\alpha \eta_\alpha^p)^p\, \mathrm{d}v_g\right)\left(\int_M (u_\alpha \eta_\alpha^p)^q\, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\
&\le C_\alpha A_\alpha \int_M u_\alpha^p \eta_\alpha^p\, \mathrm{d}v_g \left(||u_\alpha||_{L^q(M)}^{-q} \int_M u_\alpha^q \eta_\alpha^p\, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}\\
&\le c m_\alpha^{-\varrho\left(1+\frac{p(1-\theta)}{\theta q}\right)}.
\end{aligned}$$

Replacing these three estimates in (46), one gets

$$\left(\int_{B\left(y_p, \frac{1}{2} A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r\, \mathrm{d}v_g\right)^{\frac{p}{r\theta}} \le c\, m_\alpha^{-\varrho\left(1+\frac{p(1-\theta)}{\theta q}\right)},$$

so that

$$m_\alpha^\varrho \int_{B\left(y_\alpha, \frac{1}{2} A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r\, \mathrm{d}v_g \le c\, m_\alpha^{\varrho\left(1-\frac{r\theta}{p}-\frac{r(1-\theta)}{q}\right)}.$$

Since $m_\alpha \to +\infty$ and

$$\lim_{\alpha \to \alpha_0} \left(1 - \frac{r\theta}{p} - \frac{r(1-\theta)}{q}\right) < c < 0,$$

we derive

$$m_\alpha^\varrho \int\limits_{B\left(y_\alpha,\, \frac{1}{2} A_\alpha^{\frac{1}{p}} u_\alpha(y_\alpha)^{\frac{p-r}{p}}\right)} u_\alpha^r \,\mathrm{d}v_g \to 0.$$

But this notedly contradicts (42).

Proof of Theorem 3.1 In order to establish the desired contradiction, we will perform several integral estimates by using the Claim 2. Assume, without loss of generality, that the radius of injectivity of M is >1.

Let $\eta \in C_0^1(\mathbb{R})$ be a cutoff function as in the above proof and define $\eta_{\alpha,\delta}(x) = \eta(\frac{d_g(x,x_\alpha)}{\delta})$ for $0 < \delta \leq 1$. In normal coordinates around x_α, the sharp Euclidean Gagliardo–Nirenberg inequality furnishes

$$\left(\int\limits_{B(0,\delta)} u_\alpha^r \eta_{\alpha,\delta}^r \,\mathrm{d}x\right)^{\frac{p}{r\theta}} \leq A_0(p,q,r)\left(\int\limits_{B(0,\delta)} |\nabla(u_\alpha \eta_{\alpha,\delta})|^p \,\mathrm{d}x\right)\left(\int\limits_{B(0,\delta)} u_\alpha^q \eta_{\alpha,\delta}^q \,\mathrm{d}x\right)^{\frac{p(1-\theta)}{\theta q}}.$$

Expanding the metric g on these same coordinates, one locally gets

$$(1 - c d_g(x,x_\alpha)^2)\,\mathrm{d}v_g \leq \mathrm{d}x \leq (1 + c d_g(x,x_\alpha)^2)\,\mathrm{d}v_g \tag{47}$$

and

$$|\nabla(u_\alpha \eta_{\alpha,\delta})|^p \leq |\nabla_g(u_\alpha \eta_{\alpha,\delta})|^p (1 + c d_g(x,x_\alpha)^2). \tag{48}$$

Thanks to these expansions, one arrives at

$$\begin{aligned}\left(\int\limits_{B(0,\delta)} u_\alpha^r \eta_{\alpha,\delta}^r \,\mathrm{d}x\right)^{\frac{p}{r\theta}} \leq & \left(A_0(p,q,r)\, A_\alpha \int\limits_M |\nabla_g(u_\alpha \eta_{\alpha,\delta})|^p \,\mathrm{d}v_g\right.\\ & \left. + c A_\alpha \int\limits_M |\nabla_g(u_\alpha \eta_{\alpha,\delta})|^p d_g(x,x_\alpha)^2 \,\mathrm{d}v_g\right)\\ & \times \left(\frac{\int_{B(0,\delta)} u_\alpha^q \eta_{\alpha,\delta}^q \,\mathrm{d}x}{\int_M u_\alpha^q \,\mathrm{d}v_g}\right)^{\frac{p(1-\theta)}{\theta q}}.\end{aligned}$$

Using now the inequalities

$$|\nabla_g(u_\alpha \eta_{\alpha,\delta})|^p \leq |\nabla_g u_\alpha|^p \eta_{\alpha,\delta}^p + c|\eta_{\alpha,\delta} \nabla_g u_p|^{p-1} |u_\alpha \nabla_g \eta_{\alpha,\delta}| + c|u_\alpha \nabla_g \eta_{\alpha,\delta}|^p$$

and

$$A_0(p,q,r)\left(A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \,\mathrm{d}v_g\right) \leq 1 - A_0(p,q,r)\left(C_\alpha A_\alpha \int\limits_M u_\alpha^p \,\mathrm{d}v_g\right),$$

we then derive

$$\left(\int\limits_{B(0,\delta)} u_\alpha^r \eta_{\alpha,\delta}^r \,\mathrm{d}x\right)^{\frac{p}{r\theta}} \le \Bigg(1 - c_1 C_\alpha A_\alpha \int\limits_M u_\alpha^p \,\mathrm{d}v_g$$

$$+c_2 F_\alpha + c_2 G_\alpha + c_3 \delta^{-p} A_\alpha \int\limits_{M\backslash B(x_\alpha,\frac{\delta}{2})} u_\alpha^p \,\mathrm{d}v_g\Bigg)$$

$$\times\left(\frac{\int_{B(0,\delta)} u_\alpha^q \eta_{\alpha,\delta}^q \,\mathrm{d}x}{\int_M u_\alpha^q \,\mathrm{d}v_g}\right)^{\frac{p(1-\theta)}{\theta q}}, \tag{49}$$

where

$$F_\alpha = A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \eta_{\alpha,\delta}^p d_g(x, x_\alpha)^2 \,\mathrm{d}v_g$$

and

$$G_\alpha = A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-1} \eta_{\alpha,\delta}^{p-1} u_\alpha |\nabla_g \eta_{\alpha,\delta}| \,\mathrm{d}v_g.$$

In order to estimate F_α and G_α, let $\zeta_{\alpha,\delta}(x) = 1 - \eta(\frac{2}{\delta} d_g(x, x_\alpha))$, where η is a cutoff function as above. Taking $u_\alpha \zeta_{\alpha,\delta}^p$ as a test function in (14), one gets

$$A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \zeta_{\alpha,\delta}^p \,\mathrm{d}v_g \le c \int\limits_M u_\alpha^r \zeta_{\alpha,\delta}^p \,\mathrm{d}v_g + c A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-1} \zeta_{\alpha,\delta}^{p-1} |\nabla_g \zeta_{\alpha,\delta}| u_\alpha \,\mathrm{d}v_g.$$

By Young's inequality, one has

$$A_\alpha \int\limits_M |\nabla_g u_\alpha|^p \zeta_{\alpha,\delta}^p \,\mathrm{d}v_g \le c\delta^{-p} A_\alpha \int\limits_{M\backslash B(x_\alpha,\frac{\delta}{2})} u_\alpha^p \,\mathrm{d}v_g + c \int\limits_{M\backslash B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^r \,\mathrm{d}v_g,$$

so that

$$G_\alpha \le A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-1} \zeta_{\alpha,\delta}^p u_\alpha \,\mathrm{d}v_g \le c\delta^{-p} A_\alpha \int\limits_{M\backslash B(x_\alpha,\frac{\delta}{2})} u_\alpha^p \,\mathrm{d}v_g + c \int\limits_{M\backslash B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^r \,\mathrm{d}v_g. \tag{50}$$

Using further the fact that $p < 2$, one has

$$\int\limits_M |\nabla_g u_\alpha|^{p-1} \eta_{\alpha,\delta}^p u_\alpha d_g(x, x_\alpha) \,\mathrm{d}v_g \le \varepsilon \int\limits_M |\nabla_g u_\alpha|^p \eta_{\alpha,\delta}^p d_g(x, x_\alpha)^2 \,\mathrm{d}v_g$$

$$+c_\varepsilon \int\limits_M u_\alpha^p d_g(x, x_\alpha)^{2-p} \,\mathrm{d}v_g. \tag{51}$$

Besides, taking $u_\alpha d_g(\cdot, x_\alpha)^2 \eta^p_{\alpha,\delta}$ as a test function in (14), one gets

$$\begin{aligned} &A_\alpha \int_M |\nabla_g u_\alpha|^p \eta^p_{\alpha,\delta} d_g(x,x_\alpha)^2 \, \mathrm{d}v_g + \frac{\int_M u_\alpha^q \eta^p_{\alpha,\delta} d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} \\ &\le c \int_{B(x_\alpha,\delta)} u_\alpha^r d_g(x,x_\alpha)^2 \, \mathrm{d}v_g + cA_\alpha \int_M |\nabla_g u_\alpha|^{p-1} \eta^p_{\alpha,\delta} u_\alpha d_g(x,x_\alpha) \, \mathrm{d}v_g + cG_\alpha. \end{aligned} \tag{52}$$

Joining now (50), (51) and (52), one obtains

$$\begin{aligned} F_\alpha \le c &\int_{B(x_\alpha,\delta)} u_\alpha^r d_g(x,x_\alpha)^2 \, \mathrm{d}v_g + c \int_{M\setminus B(x_\alpha,\frac{\delta}{4})} u_\alpha^r \, \mathrm{d}v_g \\ &+ c\delta^{-p} A_\alpha \int_{M\setminus B(x_\alpha,\frac{\delta}{2})} u_\alpha^p \, \mathrm{d}v_g + c\delta^{2-p} A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g \, . \end{aligned}$$

On the other hand, the Claim 2 gives

$$\int_{B(x_\alpha,\delta)} u_\alpha^r d_g(x,x_\alpha)^2 \, \mathrm{d}v_g \le c\delta^{2-p} A_\alpha \int_M u_\alpha^p \mathrm{d}v_g \tag{53}$$

and

$$\begin{aligned} \int_{M\setminus B(x_\alpha,\frac{\delta}{4})} u_\alpha^r \, \mathrm{d}v_g &\le 16 \int_{M\setminus B(x_\alpha,\frac{\delta}{4})} u_\alpha^p u_\alpha^{r-p} d_g(x,x_\alpha)^2 \, \mathrm{d}v_g \\ &\le c\delta^{p-2} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \mathrm{d}v_g. \end{aligned} \tag{54}$$

Consequently,

$$\begin{aligned} &F_\alpha \le c\delta^{2-p} A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g + c\delta^{-p} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \mathrm{d}v_g \quad \text{and} \\ &G_\alpha \le c\delta^{-p} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g. \end{aligned} \tag{55}$$

Putting these two estimates in (49), one arrives at

$$\begin{aligned} \left(\int_{B(x_\alpha,\delta)} u_\alpha^r \eta^r_{\alpha,\delta} \, \mathrm{d}x \right)^{\frac{p}{r\theta}} \le &\left(1 - \left(c_1 \, C_\alpha + c\delta^{2-p}\right) A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g \right. \\ &\left. + c\delta^{-p} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g \right) \left(\frac{\int_{B(x_\alpha,\delta)} u_\alpha^q \eta^q_{\alpha,\delta} \, \mathrm{d}x}{\int_M u_\alpha^q \, \mathrm{d}v_g} \right)^{\frac{p(1-\theta)}{\theta q}} . \end{aligned} \tag{56}$$

However, by (48), we have

$$\begin{aligned}\left(\int_M u_\alpha^r \eta_{\alpha,\delta}^r \, \mathrm{d}x\right)^{\frac{p}{r\theta}} &\geq \left(\int_M u_\alpha^r \eta_{\alpha,\delta}^r \, \mathrm{d}v_g - c\int_M u_\alpha^r \eta_{\alpha,\delta}^r d_g(x,x_\alpha)^2 \, \mathrm{d}v_g\right)^{\frac{p}{r\theta}} \\ &\geq 1 - c\int_{M\setminus B(x_\alpha,\delta)} u_\alpha^r \, \mathrm{d}v_g - c\int_M u_\alpha^r \eta_{\alpha,\delta}^r d_g(x,x_\alpha)^2 \, \mathrm{d}v_g\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\int_{B(x_\alpha,\delta)} u_\alpha^q \eta_{\alpha,\delta}^q \, \mathrm{d}x}{\int_M u_\alpha^q \, \mathrm{d}v_g}\right)^{\frac{p(1-\theta)}{\theta q}} &\leq \left(\frac{\int_M u_\alpha^q \eta_{\alpha,\delta}^q \, \mathrm{d}v_g + c\int_M u_\alpha^q \eta_{\alpha,\delta}^q d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g}\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\leq \left(\frac{\int_M u_\alpha^q \eta_{\alpha,\delta}^q \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g}\right)^{\frac{p(1-\theta)}{\theta q}} + c\frac{\int_M u_\alpha^q \eta_{\alpha,\delta}^q d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} \\ &\leq 1 + c\frac{\int_M u_\alpha^q \eta_{\alpha,\delta}^q d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g}.\end{aligned}$$

Replacing these two inequalities in (56) and using the fact that $p < 2$, one gets

$$\begin{aligned}0 \leq -C_\alpha A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g + \frac{\int_M u_\alpha^q \eta_{\alpha,\delta}^q d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} + c\int_M u_\alpha^r \eta_{\alpha,\delta}^r d_g(x,x_\alpha)^2 \, \mathrm{d}v_g \\ +c\int_{M\setminus B(x_\alpha,\delta)} u_\alpha^r \, \mathrm{d}v_g + c\delta^{2-p} A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g + c\delta^{-p} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g.\end{aligned}$$

By (53) and (54), we then derive

$$\begin{aligned}C_\alpha A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g \leq c\frac{\int_M u_\alpha^q \eta_{p,q,\delta}^p d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} + c\delta^{2-p} A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g \\ +c\delta^{-p} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g.\end{aligned} \tag{57}$$

Plugging (50), (51), (53) and (54) in (52), one obtains

$$\frac{\int_M u_\alpha^q \eta_{\alpha,\delta}^p d_g(x,x_\alpha)^2 \, \mathrm{d}v_g}{\int_M u_\alpha^q \, \mathrm{d}v_g} \leq c\delta^{2-p} A_\alpha \int_M u_\alpha^p \, \mathrm{d}v_g + c\delta^{-p} A_\alpha \int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g.$$

Introducing now this inequality in (57), one gets

$$C_\alpha \leq c\delta^{2-p} + c(\delta)\frac{\int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g}{\int_M u_\alpha^p \, \mathrm{d}v_g} \leq c\delta^{2-p} + c(\delta), \tag{58}$$

where $c(\delta) \to +\infty$ as $\delta \to 0^+$. But this is a contradiction, since $\lim_{\alpha\to\alpha_0} C_\alpha = +\infty$.

4 Proof of Theorem 1.1

In this section, we furnish the proof of the existence of an extremal function for parameters p, q and r as in Theorem 1.1.

Given $\alpha \in (0, 1)$, consider the functional

$$J_\alpha(u) = \left(\int_M |\nabla_g u|^p \, dv_g + C_\alpha \int_M |u|^p \, dv_g \right) \left(\int_M |u|^q \, dv_g \right)^{\frac{p(1-\theta)}{\theta q}}$$

constrained to $E = \{u \in H^{1,p}(M) : \ ||u||_{L^r(M)} = 1\}$, where $C_\alpha = \frac{B_0(p,q,r,g)}{A_0(p,q,r)}\alpha$.

The definition of $B_0(p, q, r, g)$ yields

$$\nu_\alpha = \inf_{u \in E} J_\alpha(u) < A_0(p, q, r)^{-1}. \tag{59}$$

In a standard way, one knows that ν_α is attained by a nonnegative function $u_\alpha \in E$ of C^1 class. In particular, u_α satisfies the Euler-Lagrange equation

$$A_\alpha \Delta_{p,g} u_\alpha + C_\alpha A_\alpha u_\alpha^{p-1} + \frac{1-\theta}{\theta} \nu_\alpha ||u_\alpha||_{L^q(M)}^{-q} u_\alpha^{q-1} = \frac{\nu_\alpha}{\theta} u_\alpha^{r-1} \quad \text{on} \quad M \tag{60}$$

where

$$A_\alpha = \left(\int_M u_\alpha^q \, dv_g \right)^{\frac{p(1-\theta)}{\theta q}}.$$

We assert that

$$\lim_{\alpha \to 1^-} A_\alpha > 0.$$

If so, the conclusion of Theorem 1.1 follows. In fact, the above claim and (59) imply that the sequence (u_α) is bounded in $H^{1,p}(M)$. So, up to a subsequence, (u_α) converges weakly to u_0 in $H^{1,p}(M)$ and also strongly in $L^p(M)$, $L^q(M)$ and $L^r(M)$, so that $u_0 \in E$. Moreover, letting $\alpha \to 1^-$ in the inequality

$$J_\alpha(u_\alpha) < A_0(p, q, r)^{-1},$$

one readily concludes that u_0 is extremal for (3).

Instead, assume

$$\lim_{\alpha \to 1^-} A_\alpha = 0.$$

In this case, since $p \leq r$,

$$A_\alpha \int_M u_\alpha^p \, dv_g \to 0, \tag{61}$$

which in turn implies that

$$\lim_{\alpha \to 1^-} \nu_\alpha = A_0(p, q, r)^{-1}. \tag{62}$$

Because (60) is quite similar to (14), proceeding in the same spirit of the proof of Theorem 3.1, we achieve the following conclusions:

Let $x_\alpha \in M$ be a maximum point of u_α. Then, for any $\sigma > 0$, the concentration property of u_α around x_α holds, namely

$$\lim_{\sigma \to \infty} \lim_{\alpha \to 1^-} \int\limits_{B(x_\alpha, \sigma a_\alpha)} u_\alpha^r \, \mathrm{d}v_g = 1, \tag{63}$$

where

$$a_\alpha = A_\alpha^{\frac{r}{np-nr+pr}}$$

with $a_\alpha \to 0$ as $\alpha \to 1^-$.

The above concentration leads to a uniform estimate referred as distance type lemma. namely, there exists a constant $c > 0$, independent of α, such that

$$d_g(x, x_\alpha)^p u_\alpha(x)^{r-p} \le c a_\alpha^{\frac{np-nr+pr}{r}} \tag{64}$$

for all $x \in M$ and α close enough to 1^-.

As before, using (62), (63) and (64), with natural adaptations one arrives at [see (58)]

$$C_\alpha \le c\delta^{2-p} + c(\delta) \frac{\int_{M \setminus B\left(x_\alpha, \frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g}{\int_M u_\alpha^p \, \mathrm{d}v_g} \tag{65}$$

for $\delta > 0$ small enough, where $c(\delta) \to +\infty$ as $\delta \to 0^+$.

We assert that

$$\lim_{\alpha \to 1^-} \frac{\int_{M \setminus B\left(x_\alpha, \frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g}{\int_M u_\alpha^p \, \mathrm{d}v_g} = 0$$

whenever $p, q < r < p^*$ and $1 \le q \le \frac{r}{r-p}$.

At first, an integration of the Eq. (14) on M furnishes, for any nonnegative function $h \in C^1(M)$,

$$A_\alpha \int\limits_M |\nabla_g u_\alpha|^{p-2} \nabla_g u_\alpha \cdot \nabla_g h \, \mathrm{d}v_g \le c \int\limits_M u_\alpha^{r-1} h \, \mathrm{d}v_g.$$

On the other hand, the claim 2 yields, for any nonnegative function $h \in C^1(M \setminus B(x_\alpha, \lambda))$,

$$\int\limits_M u_\alpha^{r-1} h \, \mathrm{d}v_g \le c_\lambda A_\alpha \int\limits_M u_\alpha^{p-1} h \, \mathrm{d}v_g$$

for some constant $c_\lambda > 0$. Thus,

$$\int\limits_M |\nabla_g u_\alpha|^{p-2} \nabla_g u_\alpha \cdot \nabla_g h \, \mathrm{d}v_g \le c \int\limits_M u_\alpha^{p-1} h \, \mathrm{d}v_g$$

for all nonnegative function $h \in C^1(M \setminus B(x_\alpha, \lambda))$. A Moser's iteration then produces

$$||u_\alpha||_{L^\infty\left(M \setminus B\left(x_\alpha, \frac{\delta}{4}\right)\right)} \le c||u_\alpha||_{L^p(M)}.$$

We now analyze two distinct cases: $q \le p < r$ and $p < q < r$.

Assume the first above situation. From the Claim 2 and integration of (14), we have

$$\int\limits_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g \le c A_\alpha^{\frac{p-q}{r-p}} \int\limits_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^q \, \mathrm{d}v_g \le c||u_\alpha||_{L^\infty(M\setminus B(x_\alpha,\frac{\delta}{4}))} \int\limits_M u_\alpha^{q-1} \, \mathrm{d}v_g$$

$$\le c \left(\int\limits_M u_\alpha^p \, \mathrm{d}v_g \right)^{\frac{1}{p}} \left(\int\limits_M u_\alpha^q \, \mathrm{d}v_g \right) \left(\int\limits_M u_\alpha^{r-1} \, \mathrm{d}v_g \right) \le c \left(\int\limits_M u_\alpha^p \mathrm{d}v_g \right)^{\frac{q+1}{p}} .$$

Therefore,

$$\frac{\int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g}{\int_M u_\alpha^p \, \mathrm{d}v_g} \le c \left(\int\limits_M u_\alpha^p \, \mathrm{d}v_g \right)^{\frac{q-p+1}{p}} \to 0$$

as $\alpha \to 1^-$, since $p < 2$ and $q \ge 1$ imply $q - p + 1 > 0$.

Assume now the second case. Using Hölder's inequality and arguing in a similar manner as above, one gets

$$\int\limits_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g \le c \left(\int\limits_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^q \, \mathrm{d}v_g \right)^{\frac{p}{q}}$$

$$\le c \left(\left(\int\limits_M u_\alpha^p \, \mathrm{d}v_g \right)^{\frac{1}{p}} \left(\int\limits_M u_\alpha^q \, \mathrm{d}v_g \right) \left(\int\limits_M u_\alpha^{r-1} \, \mathrm{d}v_g \right) \right)^{\frac{p}{q}} .$$

If $r - 1 < p$, by Hölder's inequality,

$$\frac{\int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g}{\int_M u_\alpha^p \, \mathrm{d}v_g} \le c \left(\int\limits_M u_\alpha^q \, \mathrm{d}v_g \right)^{\frac{p}{q}} \left(\int\limits_M u_\alpha^p \, \mathrm{d}v_g \right)^{\frac{r}{q}-1} \to 0$$

Otherwise, if $r - 1 \ge p$, then an interpolation argument combined the normalization $||u_\alpha||_r = 1$ yields

$$\int\limits_M u_\alpha^{r-1} \mathrm{d}v_g \le c \left(\int\limits_M u_\alpha^p \mathrm{d}v_g \right)^{\frac{1}{r-p}} .$$

Thus,

$$\int\limits_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g \le c \left(\left(\int\limits_M u_\alpha^q \, \mathrm{d}v_g \right) \left(\int\limits_M u_\alpha^p \, \mathrm{d}v_g \right)^{\frac{1}{p}+\frac{1}{r-p}} \right)^{\frac{p}{q}} ,$$

so that

$$\frac{\int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p \, \mathrm{d}v_g}{\int_M u_\alpha^p \, \mathrm{d}v_g} \le c \left(\int_M u_\alpha^q \, \mathrm{d}v_g\right)^{\frac{p}{q}} \left(\int_M u_\alpha^p \mathrm{d}v_g\right)^{\left(\frac{1}{r-p}+\frac{1-q}{p}\right)\frac{p}{q}} \to 0$$

as $\alpha \to 1^-$, since the inequality $q \le \frac{r}{r-p}$ is equivalent to $\frac{1}{r-p} + \frac{1-q}{p} \ge 0$.

So, taking the limit in (65), one obtains

$$\frac{B_0(p,q,r,g)}{A_0(p,q,r)} \le c\delta^{2-p}$$

for all $\delta > 0$ small enough.

Finally, the facts that $p < 2$ and

$$B_0(p,q,r,g) \ge v_g(M)^{-\frac{p}{n}} > 0\,, \tag{66}$$

which can be easily checked by replacing a constant function in (3), lead to the desired contradiction.

5 Proof of Theorem 1.2

In this last section, we present the proof of the compactness theorem.

Let $\alpha = (p,q,r)$. Consider a sequence (u_α) formed by extremal functions $u_\alpha \in \mathcal{E}(p,q,r,g)$ for parameters $p_1 \le p \le p_2, q_1 \le q \le q_2$ and $r_1 \le r \le r_2$. Without loss of generality, assume (α) converges to $\alpha_0 = (p_0, q_0, r_0)$.

It is clear that u_α satisfies

$$1 = \left(\int_M |u_\alpha|^r \, \mathrm{d}v_g\right)^{\frac{p}{\theta r}} = \left(A_0(p,q,r)\int_M |\nabla_g u_\alpha|^p \, \mathrm{d}v_g \right.$$
$$\left. + B_0(p,q,r,g)\int_M |u_\alpha|^p \, \mathrm{d}v_g\right)\left(\int_M |u_\alpha|^q \, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}$$

and is a C^1 solution of the equation

$$A_0(p,q,r)A_\alpha \Delta_{p,g} u_\alpha + B_0(p,q,r,g)A_\alpha u_\alpha^{p-1} + \frac{1-\theta}{\theta}||u_\alpha||_{L^q(M)}^{-q} u_\alpha^{q-1} = \frac{1}{\theta}u_\alpha^{r-1} \text{ on } M, \tag{67}$$

where

$$A_\alpha = \left(\int_M u_\alpha^q \, \mathrm{d}v_g\right)^{\frac{p(1-\theta)}{\theta q}}.$$

As in the proof of Theorem 1.1, we show that

$$\lim_{\alpha\to\alpha_0} A_\alpha > 0. \tag{68}$$

Assuming the above assertion is true, we prove that (u_α) is weakly compact in a certain sense. Precisely, consider $t < p_0$ so that $p, q, r \leq s < t^*$. This chosen guarantees, up to a subsequence, that $u_\alpha \rightharpoonup u_0$ in $H^{1,t}(M)$ and $u_\alpha \to u_0$ in $L^s(M)$. In particular,

$$||u_\alpha - u_0||_{L^p(M)} \to 0,$$
$$||u_\alpha - u_0||_{L^q(M)} \to 0$$

and

$$||u_\alpha - u_0||_{L^r(M)} \to 0$$

as $\alpha \to \alpha_0$, so that $||u_0||_{L^{r_0}(M)} = 1$.

On the other hand, by Theorems 2.1 and 3.1,

$$\int_M |\nabla_g u_0|^t \, \mathrm{d}v_g \leq \liminf_{\alpha\to\alpha_0} \int_M |\nabla_g u_\alpha|^t \, \mathrm{d}v_g \leq \liminf_{\alpha\to\alpha_0} \left(v_g(M)^{1-\frac{t}{p}} \left(\int_M |\nabla_g u_\alpha|^p \, \mathrm{d}v_g \right)^{\frac{t}{p}} \right)$$
$$= \left[\left(\left(\int_M |u_0|^{q_0} \, \mathrm{d}v_g \right)^{-\frac{p_0(1-\theta_0)}{\theta_0 q_0}} - B_0 \int_M |u_0|^{p_0} \, \mathrm{d}v_g \right) A(p_0, q_0, r_0)^{-1} \right]^{\frac{t}{p_0}},$$

where $B_0 := \lim_{\alpha\to\alpha_0} B_0(p, q, r, g)$. Letting $t \to p_0^-$, by Fatou's Lemma, one has

$$\int_M |\nabla_g u_0|^{p_0} \, \mathrm{d}v_g \leq \left(\left(\int_M |u_0|^{q_0} \, \mathrm{d}v_g \right)^{-\frac{p_0(1-\theta_0)}{\theta_0 q_0}} - B_0 \int_M |u_0|^{p_0} \, \mathrm{d}v_g \right) A(p_0, q_0, r_0)^{-1}.$$

Thus,

$$\left(\int_M |u_0|^r \, \mathrm{d}v_g \right)^{\frac{p}{\theta r}} = 1 \geq \left(A(p_0, q_0, r_0) \int_M |\nabla_g u_0|^{p_0} \, \mathrm{d}v_g \right.$$
$$\left. + B_0 \int_M u_0^{p_0} \mathrm{d}v_g \right) \left(\int_M u_0^{q_0} \mathrm{d}v_g \right)^{\frac{p_0(1-\theta_0)}{\theta_0 q_0}},$$

so that, by (3), one has $B_0 \leq B(p_0, q_0, r_0, g)$. On the other hand, for fixed u, passing the limit in

$$\left(\int_M |u|^r \, \mathrm{d}v_g \right)^{\frac{p}{r\theta}} \leq \left(A_0(p, q, r, g) \int_M |\nabla_g u|^p \, \mathrm{d}v_g \right.$$
$$\left. + B_0(p, q, r, g) \int_M |u|^p \, \mathrm{d}v_g \right) \left(\int_M |u|^q \, \mathrm{d}v_g \right)^{\frac{p(1-\theta)}{\theta q}},$$

one gets

$$\left(\int\limits_M |u|^{r_0}\,\mathrm{d}v_g\right)^{\frac{p_0}{r_0\theta_0}} \le \left(A_0(p_0,q_0,r_0,g)\int\limits_M |\nabla_g u|^{p_0}\,\mathrm{d}v_g + B_0\int\limits_M |u|^{p_0}\,\mathrm{d}v_g\right)\left(\int\limits_M |u|^{q_0}\,\mathrm{d}v_g\right)^{\frac{p_0(1-\theta_0)}{\theta_0 q_0}},$$

so that $B(p_0,q_0,r_0)\le B_0$. So, we conclude that $B_0 = B(p_0,q_0,r_0)$ and u_0 is a corresponding extremal function. This end the weak compactness.

In order to attain the C^0 compactness, note that (67) and (68) yield

$$\int\limits_M |\nabla_g u_\alpha|^{p-2}\nabla_g u\cdot\nabla_g h\,\mathrm{d}v_g \le c\int\limits_M u_\alpha^{r-1} h\,\mathrm{d}v_g$$

for all nonnegative function $h\in C^1(M)$. Evoking now a Moser's iterative scheme, one obtains

$$||u_\alpha||_{L^\infty(M)} = \sup_{x\in M} u_\alpha(x) \le c||u_\alpha||_{L^r(M)} \le c,$$

for some constant $c>0$, which is independent of α. The conclusion follows then from the classical elliptic theory.

Finally, it only remains to show (68). Suppose by contradiction that

$$\lim_{\alpha\to\alpha_0} A_\alpha = 0.$$

Then,

$$A_\alpha\int\limits_M u_\alpha^p\,\mathrm{d}v_g \to 0.$$

The assumptions imply that $p\le p_2<r_1\le r\le r_2<p^*$ and $1\le q\le q_2\le\frac{r_2}{r_2-p_1}\le\frac{r}{r-p}$. Thanks to these inequalities, the same strategy of proof of Theorem 3.1 yields the Claims 1 and 2. As before, these claims produce

$$B_0(p,q,r,g)\le c\delta^{2-p} + c(\delta)\frac{\int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p\,\mathrm{d}v_g}{\int_M u_\alpha^p\,\mathrm{d}v_g},$$

with $c(\delta)\to+\infty$ as $\delta\to 0^+$. Proceeding now in the same spirit of the proof of Theorem 1.1, one concludes that

$$\lim_{\alpha\to\alpha_0}\frac{\int_{M\setminus B\left(x_\alpha,\frac{\delta}{4}\right)} u_\alpha^p\,\mathrm{d}v_g}{\int_M u_\alpha^p\,\mathrm{d}v_g} = 0.$$

Using the facts that $p\le p_2<2$, $\delta>0$ can be taken small enough and the lower estimate (66) holds for $B_0(p,q,r,g)$, we derive a clear contradiction as $\alpha\to\alpha_0$.

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