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On conformal minimal immersions of S^2 in HP^n with parallel second fundamental form

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Abstract In this paper, we determine all conformal minimal immersions of 2-spheres in quaternionic projective spaces HP^n with parallel second fundamental form.

Keywords Conformal minimal 2-spheres · Parallel second fundamental form · Classification · Quaternionic projective space

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1 Introduction

In 1976, Nakagawa and Takagi studied some properties about Kähler imbeddings of compact Hermitian symmetric spaces in the complex projective space CP^n and gave a classification of Kähler submanifolds in CP^n with parallel second fundamental form (cf. [8]). In 1984, Ros decided the compact Einstein Kähler submanifold in CP^n with parallel second fundamental form (cf. [9]). In 1985, Tsukada classified $2n$-dimensional totally complex submanifolds in HP^n with parallel second fundamental form into eight types (cf. [10,11]). Recently, we studied conformal minimal immersions of 2-spheres in CP^n and $G(k, N)$ with parallel second fundamental form, and obtained some geometric properties of them (cf. [6,7]).

In this paper, our interest is to study classification of conformal minimal immersions from S^2 to the quaternionic projective space HP^n with parallel second fundamental form by the theory of harmonic maps. Here, we regard HP^n as a totally geodesic totally real submani-

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folds in complex Grassmann manifolds $G(2, 2n+2)$ and obtain the following classification theorem:

Theorem 1.1 *Let $\varphi : S^2 \to HP^n$ be a linearly full conformal minimal immersion, and let K and B be its Gauss curvature and second fundamental form respectively. If B is parallel, then, up to a symplectic isometry of HP^n, it belongs to one of the following minimal immersions.*

(1) $\varphi = \left[(\sqrt{3}\overline{z} + \sqrt{3}\overline{z}^2 j,\ -1 + 2z\overline{z} - 2\overline{z}j - z\overline{z}^2 j)^T\right] : S^2 \to HP^1$, *with* $K = \frac{2}{3}$, $\|B\|^2 = \frac{8}{3}$*;*
(2) $\varphi = \left[(1,\ z)^T\right] : S^2 \to CP^1 \subset HP^1$, *with* $K = 2,\ B = 0$*;*
(3) $\varphi = \left[(1,\ \sqrt{2}z, z^2)^T\right] : S^2 \to CP^2 \subset HP^2$, *with* $K = 1,\ \|B\|^2 = 2$*;*
(4) $\varphi = \left[(1 - \frac{1}{2}\overline{z}^3 j,\ \sqrt{3}z + \frac{\sqrt{3}}{2}\overline{z}^2 j,\ \frac{3}{2}z^2,\ \frac{\sqrt{3}}{2}z^3)^T\right] : S^2 \to HP^3$, *with* $K = \frac{2}{3},\ \|B\|^2 = \frac{1}{3}$*;*
(5) $\varphi = \left[(-2\overline{z},\ \sqrt{2} - \sqrt{2}z\overline{z},\ 2z)^T\right] : S^2 \to CP^2 \subset HP^2$, *with* $K = \frac{1}{2},\ B = 0$*;*
(6) $\varphi = \left[(6\overline{z}^2,\ -6\overline{z} + 6z\overline{z}^2,\ \sqrt{6} - 4\sqrt{6}z\overline{z} + \sqrt{6}z^2\overline{z}^2,\ 6z - 6z^2\overline{z},\ 6z^2)^T\right] : S^2 \to CP^4 \subset HP^4$, *with* $K = \frac{1}{6},\ \|B\|^2 = \frac{2}{3}$.

Further, no two of the above six cases are congruent, i.e., there is no symplectic isometry which transforms one case into another.

2 Preliminaries

(A) For any $N = 1, 2, \ldots$, let $\langle , \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^N$ defined by $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$, where $z = (z_1, \ldots, z_N)^T$, $w = (w_1, \ldots, w_N)^T \in \mathbb{C}^N$ and $\bar{}$ denote complex conjugation. Let $\mathbb{H}$ denote the division ring of quaternions. Let j be a unit quaternion with $j^2 = -1$. Then, we have an identification of $\mathbb{C}^2$ with $\mathbb{H}$ given by making $(a, b) \in \mathbb{C}^2$ correspond to $a + bj \in \mathbb{H}$; let $n \in \{1, 2, \ldots\}$, and we have a corresponding identification of $\mathbb{C}^{2n}$ with $\mathbb{H}^n$. For any $a + bj \in \mathbb{H}$, the left multiplication by j is given by $j(a + bj) = -\overline{b} + \overline{a}j$; the conjugation is given by $\overline{a + bj} = \overline{a} - bj$; the positive-definite inner product is given by $\langle x, y\rangle_{\mathbb{H}} = Re(x\overline{y})$ for any $x, y \in \mathbb{H}$.

Let $\mathbf{J} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the conjugate linear map given by left multiplication by j, i.e.,

$$\mathbf{J}(z_1, z_2, \ldots, z_{2n-1}, z_{2n})^T = (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n}, \bar{z}_{2n-1})^T.$$

Then, $\mathbf{J}^2 = -id$ where id denotes the identity map on $\mathbb{C}^{2n}$. In fact, for any $v \in \mathbb{C}^{2n}$,

$$\mathbf{J}v = J_n\bar{v},$$

where $J_n = diag\underbrace{\left\{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \ldots, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\}}_{n}$.

By the above, we immediately have the following lemma (cf. [1]).

Lemma 2.1 *The operator* $\mathbf{J}$ *has the following properties:*

(i) $\langle \mathbf{J}v, \mathbf{J}w\rangle = \langle w, v\rangle$ *for all* $v, w \in \mathbb{C}^{2n}$*;*
(ii) $\langle \mathbf{J}v, v\rangle = 0$ *for all* $v \in \mathbb{C}^{2n}$*;*
(iii) $\partial \circ \mathbf{J} = \mathbf{J} \circ \overline{\partial},\ \overline{\partial} \circ \mathbf{J} = \mathbf{J} \circ \partial$*;*
(iv) $\mathbf{J}(\lambda v) = \overline{\lambda}\mathbf{J}v$ *for any* $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^{2n}$.

Let $G(2, 2n+2)$ denote the Grassmann manifold of all complex 2-dimensional subspaces of $\mathbb{C}^{2n+2}$ with its standard Kähler structure. The quaternionic projective space HP^n is the set of all one-dimensional quaternionic subspaces of $\mathbb{H}^{n+1}$. Throughout the above, we shall regard HP^n as the totally geodesic submanifold of $G(2, 2n+2)$ given by

$$HP^n = \{V \in G(2, 2n+2) : \mathbf{J}V = V\}.$$

Let $Sp(n+1) = \{g \in GL(n+1; \mathbb{H}),\ g^*g = I_{n+1}\}$ be the symplectic isometry group of HP^n, here I_{n+1} is the identity matrix. The explicit description is that the following diagram commutes:

$$\begin{array}{ccc} Sp(n+1) & \xrightarrow{i_1} & U(2n+2) \\ \pi_1 \big\downarrow & & \pi_2 \big\downarrow \\ HP^n & \xrightarrow{i_2} & G(2, 2n+2) \end{array}$$

where i_1, i_2 are inclusions and π_1, π_2 are projections, and $i_1(g) = U$, for $1 \le a, b \le n+1$

$$\begin{cases} U^{2a-1}_{2b-1} = A^a_b, & U^{2a-1}_{2b} = -\overline{D}^a_b, \\ U^{2a}_{2b-1} = D^a_b, & U^{2a}_{2b} = \overline{A}^a_b, \end{cases}$$

where $A = (A^a_b),\ D = (D^a_b) \in M_{n+1}(\mathbb{C}),\ g = A + \overline{D}j \in Sp(n+1)$;

$$\pi_1(g) = g \cdot [1, 0, \ldots, 0]^T \in HP^n;$$

$$\pi_2(U) = U \cdot \begin{bmatrix} 1, 0, 0, \ldots, 0 \\ 0, 1, 0, \ldots, 0 \end{bmatrix}^T \in G(2, 2n+2);$$

$$i_2\left([z_1 + \overline{z}_2 j, \ldots, z_{2n+1} + \overline{z}_{2n+2} j]^T\right) = \begin{bmatrix} z_1, & z_2, & \ldots, & z_{2n+1}, & z_{2n+2} \\ -\overline{z}_2, & \overline{z}_1, & \ldots, & -\overline{z}_{2n+2}, & \overline{z}_{2n+1} \end{bmatrix}^T.$$

Here, we take the standard metric on $G(2, 2n+2)$ as described in section 2 of [7]; then, the metric induced by i_2 is twice as much as the standard metric on HP^n.

Thus, a harmonic map from S^2 to HP^n is precisely a harmonic map from S^2 to $G(2, 2n+2)$ which has image in $i_2(HP^n)$.

Then, for any $g \in Sp(n+1)$, the action of g on HP^n induces an action of $i_1(g)$ on CP^{2n+1}, where $i_1(g) \in U(2n+2)$ commutes with $\mathbf{J}$. We shall retain g to also denote $i_1(g)$. Then

$$Sp(n+1) = \{g \in U(2n+2),\ g \circ \mathbf{J} = \mathbf{J} \circ g\} = \left\{g \in U(2n+2),\ gJ_{n+1}g^T = J_{n+1}\right\}.$$

In the following, we deal with the symplectic isometry of HP^n through the corresponding symplectic isometry of CP^{2n+1}.

(B) In this section, we give general expression of some geometric quantities about conformal minimal immersions from S^2 to HP^n (cf. [7]).

Let M be a simply connected domain in the unit sphere S^2 and let $(z, \overline{z})$ be complex coordinates on M. We take the metric $ds^2_M = dz d\overline{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}.$$

We consider the complex Grassmann manifold $G(2, N)$ as the set of Hermitian orthogonal projections from $\mathbb{C}^N$ onto a 2-dimensional subspace in $\mathbb{C}^N$. Then, a map $\psi : S^2 \to G(2, N)$ is a Hermitian orthogonal projection onto a 2-dimensional subbundle $\underline{\psi}$ of the trivial bundle

$\underline{\mathbb{C}}^N = M \times \mathbb{C}^N$ given by setting the fibre of $\underline{\psi}$ at x, $\underline{\psi}_x$, equal to $\psi(x)$ for all $x \in M$. We say that $\underline{\psi}$ is a harmonic subbundle if ψ is harmonic (cf. [3]).

Let $\varphi : S^2 \to HP^n$ be a conformal minimal immersion. The map $i_2 \circ \varphi : S^2 \to G(2, 2n+2)$ may be represented via the local sections of the subbundle $\underline{Im}(i_2 \circ \varphi)$ by the projection map (cf. [7], (2.10)):

$$i_2 \circ \varphi = XX^* + (\mathbf{J}X)(\mathbf{J}X)^*,$$

where $X \in \underline{Im}(i_2 \circ \varphi)$ is a unit column vector in $\mathbb{C}^{2n+2}$, and X and $\mathbf{J}X$ are naturally orthogonal.

Denote $i_2 \circ \varphi$ by φ_0 (in the following, we will use this notation all the time). Suppose that the metric induced by φ_0 is $ds^2 = \lambda^2 dz d\overline{z}$. Let K and B be its Gauss curvature and second fundamental form, respectively. From section 2 and 3 of [7], we have

$$\begin{cases} \lambda^2 = tr\partial\varphi_0\overline{\partial}\varphi_0, \\ K = -\frac{2}{\lambda^2}\partial\overline{\partial}\log\lambda^2, \\ \|B\|^2 = 4trPP^*. \end{cases} \tag{2.1}$$

where $P = \partial\left(A_z/\lambda^2\right)$ with $A_z = (2\varphi_0 - I)\,\partial\varphi_0$, and I is the identity matrix, then $P^* = \overline{\partial}\left(A_z^*/\lambda^2\right)$, $A_z^* = -A_{\overline{z}}$.

3 The proof of main theorem

We recall that an immersion of S^2 in HP^n is conformal and minimal if and only if it is harmonic (cf. [4], Sec 10.6). Thus, we shall consider the immersive harmonic maps from S^2 to HP^n with parallel second fundamental form for the reducible and irreducible cases to give the proof of Theorem 1.1 in Sect. 1. At first, we state a conclusion about parallel minimal immersions of 2-spheres in $G(k, N)$ as follows:

Lemma 3.1 ([7]) *Let $\varphi : S^2 \to G(k, N)$ be a conformal minimal immersion with the second fundamental form B. Then B is parallel if and only if the equation*

$$\frac{\lambda^2}{16}\|B\|^2\left(8K + \|B\|^2\right) + 2tr[A_z, P][A_z^*, P^*] - 5tr[A_z, A_z^*][P, P^*] = 0 \tag{3.1}$$

holds.

(I) Let $\varphi : S^2 \to HP^n$ be a linearly full reducible harmonic map, then by ([1], Proposition 3.7) we know that φ is a quaternionic mixed pair or a quaternionic Frenet pair. In the following, we discuss the two cases of φ with parallel second fundamental form, respectively.

(Ia) If φ is a linearly full quaternionic Frenet pair, then

$$\underline{\varphi}_0 = \underline{f}_n^{(2n+1)} \oplus \underline{f}_{n+1}^{(2n+1)}, \tag{3.2}$$

where $\underline{f}_0^{(2n+1)}, \underline{f}_1^{(2n+1)}, \ldots, \underline{f}_{2n+1}^{(2n+1)} : S^2 \to CP^{2n+1}$ is the harmonic sequence generated by a linearly full totally $\mathbf{J}$-isotropic map $\underline{f}_0^{(2n+1)}$.

Firstly, we recall ([1], §3) that a full holomorphic map $\underline{f}_0^{(2n+1)} : S^2 \to CP^{2n+1}$ in the following harmonic sequence satisfying $\underline{f}_{2n+1}^{(2n+1)} = \mathbf{J}\underline{f}_0^{(2n+1)}$ is said to be *totally* $\mathbf{J}$*-isotropic*,

$$0 \xleftarrow{A_0''} \underline{f}_0^{(2n+1)} \xrightarrow{A_0'} \cdots \xrightarrow{A_{n-1}'} \underline{f}_n^{(2n+1)} \xrightarrow{A_n'} \underline{f}_{n+1}^{(2n+1)} \xrightarrow{A_{n+1}'} \cdots \xrightarrow{A_{2n}'} \underline{f}_{2n+1}^{(2n+1)} \xrightarrow{A_{2n+1}'} 0,$$

where $A'_j(v) = \pi_{f_j^{(2n+1)\perp}}(\partial v)$, $A''_j(v) = \pi_{f_j^{(2n+1)\perp}}(\overline{\partial} v)$ for $v \in C^\infty(\underline{f}_j^{(2n+1)})$, here $\pi_{f_j^{(2n+1)\perp}}$ denotes orthogonal projection onto bundle $\underline{f}_j^{(2n+1)\perp}$ and $C^\infty(\underline{f}_j^{(2n+1)})$ denotes the vector space of smooth sections of bundle $\underline{f}_j^{(2n+1)}$, $j = 0, \ldots, 2n+1$.

Let $f_0^{(2n+1)}$ be a holomorphic section of $\underline{f}_0^{(2n+1)}$, i.e., $\overline{\partial} f_0^{(2n+1)} = 0$, and let $f_j^{(2n+1)}$ be a local section of $\underline{f}_j^{(2n+1)}$ such that

$$f_j^{(2n+1)} = \pi_{f_{j-1}^{(2n+1)\perp}}\left(\partial f_{j-1}^{(2n+1)}\right)$$

for $j = 1, \ldots, 2n+1$. Then, we have some formulas as follows (cf. [2]):

$$\begin{aligned}
\partial f_j^{(2n+1)} &= f_{j+1}^{(2n+1)} + \partial \log |f_j^{(2n+1)}|^2 f_j^{(2n+1)}, \ j = 0, \ldots, 2n,\\
\overline{\partial} f_j^{(2n+1)} &= -l_{j-1}^{(2n+1)} f_{j-1}^{(2n+1)}, \ j = 1, \ldots, 2n+1,\\
\partial\overline{\partial} \log |f_j^{(2n+1)}|^2 &= l_j^{(2n+1)} - l_{j-1}^{(2n+1)},\\
\partial\overline{\partial} \log l_j^{(2n+1)} &= l_{j+1}^{(2n+1)} - 2l_j^{(2n+1)} + l_{j-1}^{(2n+1)}, \ j = 0, \ldots, 2n,
\end{aligned}$$

where $l_j^{(2n+1)} = |f_{j+1}^{(2n+1)}|^2/|f_j^{(2n+1)}|^2$ for $j = 0, \ldots, 2n+1$, and $l_{-1}^{(2n+1)} = l_{2n+1}^{(2n+1)} = 0$.

Since $\underline{f}_0^{(2n+1)}$ is totally **J**-isotropic, in a similar fashion to ([2], Lemma 7.1) we obtain

$$l_j^{(2n+1)} = l_{2n-j}^{(2n+1)}. \tag{3.3}$$

And set $\mathbf{J} f_0^{(2n+1)} = \tau_0 f_{2n+1}^{(2n+1)}$, then

$$|\tau_0|^2 = \frac{|f_0^{(2n+1)}|^2}{|f_{2n+1}^{(2n+1)}|^2}, \quad \mathbf{J} f_j^{(2n+1)} = (-1)^j \tau_0 \frac{|f_{2n+1}^{(2n+1)}|^2}{|f_{2n+1-j}^{(2n+1)}|^2} f_{2n+1-j}^{(2n+1)},$$

where $j = 0, \ldots, n$.

Obviously, φ_0 belongs to the following harmonic sequence (cf. [3])

$$0 \overset{A''_0}{\longleftarrow} \underline{f}_0^{(2n+1)} \overset{A''_1}{\longleftarrow} \cdots \overset{A''_{n-1}}{\longleftarrow} \underline{f}_{n-1}^{(2n+1)} \overset{A''_{\varphi_0}}{\longleftarrow} \underline{\varphi}_0 \overset{A'_{\varphi_0}}{\longrightarrow} \underline{f}_{n+2}^{(2n+1)} \overset{A'_{n+2}}{\longrightarrow} \cdots \overset{A'_{2n}}{\longrightarrow} \underline{f}_{2n+1}^{(2n+1)} \overset{A'_{2n+1}}{\longrightarrow} 0, \tag{3.4}$$

where $A'_{\varphi_0}(v) = \pi_{\varphi_0^\perp}(\partial v)$, $A''_{\varphi_0}(v) = \pi_{\varphi_0^\perp}(\overline{\partial} v)$ for $v \in C^\infty(\underline{\varphi}_0)$, here $\pi_{\varphi_0^\perp}$ denotes orthogonal projection onto bundle $\underline{\varphi}_0^\perp$ and $C^\infty(\underline{\varphi}_0)$ denotes the vector space of smooth sections of bundle $\underline{\varphi}_0$.

From (3.2), we have $\varphi_0 = \frac{f_n^{(2n+1)}(f_n^{(2n+1)})^*}{|f_n^{(2n+1)}|^2} + \frac{f_{n+1}^{(2n+1)}(f_{n+1}^{(2n+1)})^*}{|f_{n+1}^{(2n+1)}|^2}$. Then by (2.1), (3.3) and a series of calculations, we obtain

$$\begin{cases}
\lambda^2 = 2l_{n-1}^{(2n+1)},\\
K = 2 - \frac{l_n^{(2n+1)} + l_{n-2}^{(2n+1)}}{l_{n-1}^{(2n+1)}},\\
\|B\|^2 = 2\frac{l_n^{(2n+1)} + l_{n-2}^{(2n+1)}}{l_{n-1}^{(2n+1)}},\\
tr[A_z, P][A_z^*, P^*] = -l_n^{(2n+1)},\\
tr[A_z, A_z^*][P, P^*] = \frac{1}{2} l_{n-2}^{(2n+1)}.
\end{cases} \tag{3.5}$$

Now, we prove that *if* $\varphi : S^2 \to HP^n$ *is a linearly full quaternionic Frenet pair with parallel second fundamental form, then, up to* $Sp(n+1)$, *it belongs to the following case: (1)* $\varphi = \left[(\sqrt{3}\overline{z} + \sqrt{3}\overline{z}^2 j, -1 + 2z\overline{z} - 2\overline{z}j - z\overline{z}^2 j)^T\right] : S^2 \to HP^1$, *with* $K = \frac{2}{3}, ||B||^2 = \frac{8}{3}$.

If φ is a linearly full quaternionic Frenet pair with parallel second fundamental form, then applying Lemma 3.1 and substituting (3.5) into (3.1), we get

$$3l_{n-2}^{(2n+1)}l_{n-1}^{(2n+1)} + 4l_{n-1}^{(2n+1)}l_{n}^{(2n+1)} - 3\left(l_n^{(2n+1)} + l_{n-2}^{(2n+1)}\right)^2 = 0. \tag{3.6}$$

Since the second fundamental form of the map φ is parallel, its Gauss curvature is a constant (cf. [7], Theorem 4.5). We know up to $U(2n+2)$, $\underline{f}_0^{(2n+1)}$ is a Veronese surface by ([5], Lemma 4.1). Then from [2], we have $\underline{f}_0^{(2n+1)}, \underline{f}_1^{(2n+1)}, \dots, \underline{f}_{2n+1}^{(2n+1)}$ is the Veronese sequence in CP^{2n+1}, up to $U(2n+2)$. So, from ([2], Section 5), we get

$$|f_i^{(2n+1)}|^2 = \frac{(2n+1)!i!}{(2n+1-i)!}(1+z\overline{z})^{2n+1-2i},\ l_j^{(2n+1)} = \frac{(j+1)(2n+1-j)}{(1+z\overline{z})^2}, \tag{3.7}$$

where $i = 0, \dots, 2n+1,\ j = 0, \dots, 2n$.

Substituting (3.7) into (3.6), we get

$$(n-1)(n+3)(5n^2 + 10n - 4) = 0,$$

which implies $n = 1$, since n is a positive integer. Hence,

$$\underline{\varphi}_0 = U\underline{V}_1^{(3)} \oplus \mathbf{J}U\underline{V}_1^{(3)}, \tag{3.8}$$

where $\underline{V}_1^{(3)}$ is a Veronese surface in $CP^3 \subset CP^{2n+1}$ with the standard expression given in ([2], §5), and $U \in U(2n+2)$ satisfies $J_{n+1}\overline{UV_0^{(3)}} = \lambda U V_3^{(3)}$ (λ is a parameter).

Set $U^T J_{n+1} U = \overline{W}$, then we immediately get

$$\overline{W}V_0^{(3)} = \overline{\lambda V_3^{(3)}},\ W^T = -W,\ W^*W = I, \tag{3.9}$$

where I is the identity matrix.

Define a set

$$G_W \triangleq \left\{U \in U(2n+2),\ UWU^T = J_{n+1}\right\}.$$

For a given W, the following can be easily checked

(i) $\forall\, g \in Sp(n+1),\ U \in G_W$, we have that $gU \in G_W$;
(ii) $\forall\, U, V \in G_W,\ \exists\, g = UV^* \in Sp(n+1)\ s.t.\ U = gV$.

Then, we discuss the type of W to get the type of the corresponding U. From ([2], section 5), we get

$$V_0^{(3)} = (1,\ \sqrt{3}z,\ \sqrt{3}z^2,\ z^3,\ 0, \dots, 0)^T,$$
$$V_3^{(3)} = \frac{6}{(1+z\overline{z})^3}(-\overline{z}^3,\ \sqrt{3}\overline{z}^2,\ -\sqrt{3}\overline{z},\ 1,\ 0, \dots, 0)^T. \tag{3.10}$$

Then, by (3.9) and (3.10) we get the type of $\overline{W}$ as follows:

$$\overline{W} = \begin{bmatrix} 0 & 0 & 0 & 1 & \\ 0 & 0 & -1 & 0 & \\ 0 & 1 & 0 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 0 & \\ & & \mathbf{0} & & * \end{bmatrix}. \tag{3.11}$$

From $\overline{WU}^T = U^T J_{n+1}$, the corresponding $U = \left[e_1, e_2, \ldots, e_{2n+1}, e_{2n+2}\right]^T$ satisfy

$$e_{2p} = \overline{W}\overline{e}_{2p-1}, \quad p = 1, \ldots, n+1, \tag{3.12}$$

where e_i are unit column vectors in $\mathbb{C}^{2n+2}$.

Without loss of generality, in this case we choose

$$\begin{cases} e_1 = (1, 0, 0, 0, \ldots, 0)^T, \\ e_3 = (0, 1, 0, 0, \ldots, 0)^T. \end{cases} \tag{3.13}$$

By(3.11)–(3.13), we get $e_2 = \overline{W}\overline{e}_1 = (0, 0, 0, -1, \ldots, 0)^T$ and $e_4 = \overline{W}\overline{e}_3 = (0, 0, 1, 0, \ldots, 0)^T$, obviously $\{e_1, e_2, e_3, e_4\}$ are mutually orthogonal. Next, we choose a unit column vector $e_5 = (0, 0, 0, 0, *)^T \in \mathbb{C}^{2n+2}$, which satisfies $\{e_1, e_2, e_3, e_4, e_5\}$ are mutually orthogonal. Set $e_6 = \overline{W}\overline{e}_5$, then $\{e_1, e_2, e_3, e_4, e_6\}$ are mutually orthogonal. Since $\langle e_6,\ e_5\rangle = e_5^T W^T e_5 = -tr(e_5 e_5^T W) = 0$, then $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ are mutually orthogonal.

Generally, suppose $\{e_1, e_2, \ldots, e_{2p-3}, e_{2p-2} = \overline{W}\overline{e}_{2p-3}\}$ $(p \geq 3)$ are mutually orthogonal, we choose a unit column vector $e_{2p-1} = (0, 0, 0, 0, *)^T \in \mathbb{C}^{2n+2}$ such that $\{e_1, e_2, \ldots, e_{2p-3}, e_{2p-2}, e_{2p-1}\}$ are mutually orthogonal. Set $e_{2p} = \overline{W}\overline{e}_{2p-1}$, then

$$\left\langle e_{2p},\ e_{2p-1}\right\rangle = e_{2p-1}^T W^T e_{2p-1} = -tr(e_{2p-1} e_{2p-1}^T W) = 0,$$

and for any $2 \leq q \leq p$,

$$\begin{aligned} \left\langle e_{2p},\ e_{2q-3}\right\rangle &= e_{2p-1}^T W^T e_{2q-3} = -e_{2p-1}^T W e_{2q-3} = -e_{2p-1}^T \overline{e}_{2q-2} \\ &= -\left\langle e_{2p-1},\ e_{2q-2}\right\rangle = 0, \left\langle e_{2p},\ e_{2q-2}\right\rangle \\ &= e_{2p-1}^T W^T \overline{W}\overline{e}_{2q-3} = e_{2p-1}^T \overline{e}_{2q-3} = \left\langle e_{2p-1},\ e_{2q-3}\right\rangle = 0. \end{aligned}$$

Thus $\{e_1, e_2, \ldots, e_{2p-3}, e_{2p-2}, e_{2p-1}, e_{2p}\}$ are mutually orthogonal.

So, we can choose $n-1$ proper unit column vectors $e_{2p+1} = (0, 0, 0, 0, *)^T \in \mathbb{C}^{2n+2}$ $(2 \leq p \leq n)$ such that $\{e_1, e_2, \ldots, e_{2n+1}, e_{2n+2} = \overline{W}\overline{e}_{2n+1}\}$ are mutually orthogonal, and the type of the corresponding U is as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -1 & \\ 0 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \\ & & \mathbf{0} & & * \end{bmatrix}. \tag{3.14}$$

Thus, we have

$$UV_1^{(3)} = \frac{-\sqrt{3}}{1+z\bar{z}}(\sqrt{3}\bar{z},\ \sqrt{3}z^2,\ -1+2z\bar{z},\ -2z+z^2\bar{z},\ 0,\dots,\ 0)^T,$$
$$\mathbf{J}UV_1^{(3)} = \frac{-\sqrt{3}}{1+z\bar{z}}(-\sqrt{3}\bar{z}^2,\ \sqrt{3}z,\ 2\bar{z}-z\bar{z}^2,\ -1+2z\bar{z},\ 0,\dots,\ 0)^T.$$

Obviously, in this case φ is congruent to the case (1) with $K = \frac{2}{3},\ \|B\|^2 = \frac{8}{3}$.

(Ib) If φ is a linearly full quaternionic mixed pair, then

$$\underline{\varphi}_0 = \underline{f}_0^{(m)} \oplus \mathbf{J}\underline{f}_0^{(m)}, \tag{3.15}$$

where $\underline{f}_0^{(m)} : S^2 \to CP^m \subseteq CP^{2n+1}$ $(n \le m \le 2n+1)$ is holomorphic and $\underline{f}_1^{(m)} \perp \mathbf{J}\underline{f}_0^{(m)}$. Obviously, φ_0 belongs to the following harmonic sequence

$$0 \xleftarrow{A''_m} \mathbf{J}\underline{f}_m^{(m)} \xleftarrow{A''_{m-1}} \dots \xleftarrow{A''_1} \mathbf{J}\underline{f}_1^{(m)} \xleftarrow{A''_{\varphi_0}} \underline{\varphi}_0 \xrightarrow{A'_{\varphi_0}} \underline{f}_1^{(m)} \xrightarrow{A'_1} \dots \xrightarrow{A'_{m-1}} \underline{f}_m^{(m)} \xrightarrow{A'_m} 0. \tag{3.16}$$

As in the case (Ia), let $f_0^{(m)}$ be a holomorphic section of $\underline{f}_0^{(m)}$, i.e., $\bar{\partial} f_0^{(m)} = 0$, and $f_j^{(m)}$ $(j = 1,\dots,m)$ satisfy the corresponding formulas. From (3.15), we have $\varphi_0 = \frac{f_0^{(m)}(f_0^{(m)})^*}{|f_0^{(m)}|^2} + \frac{(\mathbf{J}f_0^{(m)})(\mathbf{J}f_0^{(m)})^*}{|f_0^{(m)}|^2}$. Then by (2.1) and a series of calculations, we obtain

$$\begin{cases} \lambda^2 = 2l_0^{(m)}, \\ K = 2 - \frac{l_1^{(m)}}{l_0^{(m)}}, \\ \|B\|^2 = 2\frac{l_1^{(m)}}{l_0^{(m)}}, \\ tr[A_z, P][A_z^*, P^*] = -\frac{1}{4}\frac{\left|\left\langle f_2^{(m)}, \mathbf{J}f_1^{(m)}\right\rangle\right|^2}{|f_1^{(m)}|^4}, \\ tr[A_z, A_z^*][P, P^*] = \frac{1}{2}\left(l_1^{(m)} - \frac{\left|\left\langle f_2^{(m)}, \mathbf{J}f_1^{(m)}\right\rangle\right|^2}{|f_1^{(m)}|^4}\right). \end{cases} \tag{3.17}$$

Now, we prove that *if $\varphi : S^2 \to HP^n$ is a linearly full quaternionic mixed pair with parallel second fundamental form, then, up to $Sp(n+1)$, it belongs to one of the following three cases:*

(2) $\varphi = \left[(1,\ z)^T\right] : S^2 \to CP^1 \subset HP^1$, *with* $K = 2,\ B = 0$;

(3) $\varphi = \left[(1,\ \sqrt{2}z,\ z^2)^T\right] : S^2 \to CP^2 \subset HP^2$, with $K = 1,\ ||B||^2 = 2$;

(4) $\varphi = \left[(1 - \frac{1}{2}\bar{z}^3 j,\ \sqrt{3}z + \frac{\sqrt{3}}{2}\bar{z}^2 j,\ \frac{3}{2}z^2,\ \frac{\sqrt{3}}{2}z^3)^T\right] : S^2 \to HP^3$, *with* $K = \frac{2}{3},\ ||B||^2 = \frac{1}{3}$.

If φ is a linearly full quaternionic mixed pair with parallel second fundamental form, then applying Lemma 3.1 and substituting (3.17) into (3.1), we get

$$\frac{\left|\left\langle f_2^{(m)},\ \mathbf{J}f_1^{(m)}\right\rangle\right|^2}{|f_1^{(m)}|^4}=\frac{3}{4}l_1^{(m)}\left(\frac{l_1^{(m)}}{l_0^{(m)}}-1\right). \tag{3.18}$$

Since the metric $ds^2=2l_0^{(m)}dzd\bar{z}$ induced by φ is of constant curvature, and the metric induced by $\underline{f}_0^{(m)}$ is $ds^2=l_0^{(m)}dzd\bar{z}$, then it follows from ([2], Theorem 5.4) that $\underline{f}_0^{(m)},\underline{f}_1^{(m)},\dots,\underline{f}_m^{(m)}$ is the Veronese sequence in $CP^m\subset CP^{2n+1}$, up to $U(2n+2)$. Then from (3.7) and (3.18) we get

$$\left|\left\langle f_2^{(m)},\ \mathbf{J}f_1^{(m)}\right\rangle\right|^2=\frac{3m(m-1)(m-2)}{2}(1+z\bar{z})^{2m-6}. \tag{3.19}$$

We denote by r the isotropy order of φ (cf. [3], §3A). If r is finite, then $r=2s$ $(1\le s\le n+1)$ by ([1], Proposition 3.2). Otherwise, $r=\infty$, in which case φ is called strongly isotropic (cf. [1], section 2C).

If $m=1$, observing (3.17), we find $K=2,\ B=0$. It belongs to the case of totally geodesic. If $m=2$, since $r\ge 2$, which implies $\underline{f}_2^{(2)}\perp\mathbf{J}\underline{f}_0^{(2)}$ by (3.16), then we have $\left\langle f_2^{(2)},\ \mathbf{J}f_1^{(2)}\right\rangle=\partial\left\langle f_2^{(2)},\ \mathbf{J}f_0^{(2)}\right\rangle=0$, which implies (3.19) holds. Hence, its second fundamental form is parallel. In fact, the above two cases are both strongly isotropic.

If $m\ge 3$, from (3.19) we find $\left\langle f_3^{(m)},\ \mathbf{J}f_0^{(m)}\right\rangle=-\left\langle f_2^{(m)},\ \mathbf{J}f_1^{(m)}\right\rangle\neq 0$, which implies in this case $r=2$. In the following, we discuss the above three cases, respectively.

Case Ib1, $m=1$.

In this case, we have

$$\underline{\varphi}_0=U\underline{V}_0^{(1)}\oplus\mathbf{J}U\underline{V}_0^{(1)}, \tag{3.20}$$

where $\underline{V}_0^{(1)}$ is a Veronese surface in $CP^1\subset CP^{2n+1}$ with the standard expression given in ([2], section 5), and $U\in U(2n+2)$ satisfies $tr\left(V_1^{(1)}{V_0^{(1)}}^T U^T J_{n+1}U\right)=0$, as this expresses the orthogonality of $Jf_0^{(1)}$ and $f_1^{(1)}$.

Similarly, we get the type of $\overline{W}=U^TJ_{n+1}U\in U(2n+2)$ as follows:

$$\overline{W}=\begin{bmatrix}0&0&a_{13}&a_{14}&\cdots&a_{1,2n+2}\\0&0&a_{23}&a_{24}&\cdots&a_{2,2n+2}\\-a_{13}&-a_{23}&0&a_{34}&\cdots&a_{3,2n+2}\\-a_{14}&-a_{24}&-a_{34}&0&\cdots&a_{4,2n+2}\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\-a_{1,2n+2}&-a_{2,2n+2}&-a_{3,2n+2}&-a_{4,2n+2}&\cdots&0\end{bmatrix}. \tag{3.21}$$

As in case (1a), by (3.12),(3.13), and (3.21), we get

$$\begin{cases}e_2=\overline{W}\bar{e}_1=\left(0,0,-a_{13},-a_{14},\dots,-a_{1,2n+2}\right)^T,\\e_4=\overline{W}\bar{e}_3=\left(0,0,-a_{23},-a_{24},\dots,-a_{2,2n+2}\right)^T.\end{cases}$$

Since $\overline{W}$ in (3.21) is a unitary matrix, $\{e_1,e_2,e_3,e_4\}$ are mutually orthogonal. Similarly, we can choose $n-1$ proper unit column vectors $e_{2p+1}=(0,0,*)^T\in\mathbb{C}^{2n+2}$ $(2\le p\le n)$

such that $\{e_1, e_2, \ldots, e_{2n+1}, e_{2n+2} = \overline{W}\overline{e}_{2n+1}\}$ are mutually orthogonal, and the type of the corresponding U is as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -a_{13} & \cdots & -a_{1,2n+2} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & -a_{2,2n+2} \\ & \mathbf{0} & & * & \end{bmatrix}. \tag{3.22}$$

Thus, we have

$$U V_0^{(1)} = (1,\ 0,\ z,\ 0,\ 0, \ldots, 0)^T,$$
$$\mathbf{J} U V_0^{(1)} = (0, 1,\ 0,\ \bar{z},\ 0, \ldots, 0)^T.$$

Obviously, in this case φ is congruent to the case (2) with $K = 2,\ B = 0$.

Case Ib2, $m = 2$.

In this case, we have

$$\underline{\varphi}_0 = U \underline{V}_0^{(2)} \oplus \mathbf{J} U \underline{V}_0^{(2)}, \tag{3.23}$$

where $\underline{V}_0^{(2)}$ is a Veronese surface in $CP^2 \subset CP^{2n+1}$ with the standard expression given in ([2], §5), and $U \in U(2n+2)$ satisfies $tr\left(V_1^{(2)} V_0^{(2)T} U^T J_{n+1} U\right) = 0$, as this expresses the orthogonality of $Jf_0^{(2)}$ and $f_1^{(2)}$.

Similarly, we get the type of $\overline{W} = U^T J_{n+1} U \in U(2n+2)$ as follows:

$$\overline{W} = \begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} & \cdots & a_{1,2n+2} \\ 0 & 0 & 0 & a_{24} & a_{25} & \cdots & a_{2,2n+2} \\ 0 & 0 & 0 & a_{34} & a_{35} & \cdots & a_{3,2n+2} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & \cdots & a_{4,2n+2} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & \cdots & a_{5,2n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1,2n+2} & -a_{2,2n+2} & -a_{3,2n+2} & -a_{4,2n+2} & -a_{5,2n+2} & \cdots & 0 \end{bmatrix}. \tag{3.24}$$

And the type of the corresponding U is as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -a_{14} & \cdots & -a_{1,2n+2} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -a_{24} & \cdots & -a_{2,2n+2} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -a_{34} & \cdots & -a_{3,2n+2} \\ & \mathbf{0} & & & * & \end{bmatrix}. \tag{3.25}$$

Thus, we have

$$U V_0^{(2)} = (1,\ 0,\ \sqrt{2}z,\ 0,\ z^2,\ 0,\ 0, \ldots, 0)^T,$$
$$\mathbf{J} U V_0^{(2)} = (0,\ 1,\ 0,\ \sqrt{2}\bar{z},\ 0,\ \bar{z}^2,\ 0, \ldots, 0)^T.$$

Obviously, in this case φ is congruent to the case (3) with $K = 1,\ \|B\|^2 = 2$.

Case Ib3, $m \geq 3$.

In this case, the trivial bundle $S^2 \times \mathbb{C}^{2n+2}$ over S^2 has a corresponding decomposition $S^2 \times \mathbb{C}^{2n+2} = S^2 \times \mathbb{C}^{m+1} \oplus S^2 \times \mathbb{C}^{2n-m+1}$. From (3.16) we set $\mathbf{J} f_0^{(m)} = x_3 f_3^{(m)} + x_4 f_4^{(m)} + \cdots + x_m f_m^{(m)} + V$, where x_i $(i = 3, \ldots, m)$ are complex coefficients and bundle $\underline{V} \subset S^2 \times \mathbb{C}^{2n-m+1}$. Then, it follows from $\partial \mathbf{J} f_0^{(m)} = 0$ that

$$\begin{cases} \partial x_3 + x_3 \partial \log |f_3^{(m)}|^2 = 0, \\ \partial x_i + x_{i-1} + x_i \partial \log |f_i^{(m)}|^2 = 0,\ (i = 4, \ldots, m), \\ \partial V = 0. \end{cases} \tag{3.26}$$

And we have $\left\langle f_3^{(m)},\ \mathbf{J} f_0^{(m)} \right\rangle = \bar{x}_3 |f_3^{(m)}|^2$. Then, from ([2], §5) and (3.19) we get

$$|x_3|^2 |f_3^{(m)}|^4 = \frac{3m(m-1)(m-2)}{2} (1 + z\bar{z})^{2m-6}. \tag{3.27}$$

By (3.26) and (3.27), we find $\partial \bar{\partial} \log \left(|x_3|^2 |f_3^{(m)}|^4 \right) = \frac{2m-6}{(1+z\bar{z})^2} = 0$, which implies $m = 3$, i.e.,

$$\underline{\varphi}_0 = U \underline{V}_0^{(3)} \oplus \mathbf{J} U \underline{V}_0^{(3)}, \tag{3.28}$$

where $\underline{V}_0^{(3)}$ is a Veronese surface in $CP^3 \subset CP^{2n+1}$ with the standard expression given in ([2], §5), and $U \in U(2n+2)$ satisfies $tr\left(V_1^{(3)} {V_0^{(3)}}^T U^T J_{n+1} U \right) = 0$ and $tr\left(V_3^{(3)} {V_0^{(3)}}^T U^T J_{n+1} U \right) \neq 0$, as these express $r = 2$.

Similarly, we get the type of $\overline{W} = U^T J_{n+1} U \in U(2n+2)$ as follows:

$$\overline{W} = \begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} & \cdots & a_{1,2n+2} \\ 0 & 0 & -a_{14} & 0 & a_{25} & \cdots & a_{2,2n+2} \\ 0 & a_{14} & 0 & 0 & a_{35} & \cdots & a_{3,2n+2} \\ -a_{14} & 0 & 0 & 0 & a_{45} & \cdots & a_{4,2n+2} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & \cdots & a_{5,2n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1,2n+2} & -a_{2,2n+2} & -a_{3,2n+2} & -a_{4,2n+2} & -a_{5,2n+2} & \cdots & 0 \end{bmatrix}, \tag{3.29}$$

where $a_{14} \neq 0$.

In this case, **if** $|a_{14}|^2 = 1$, as in the case (Ia) we choose (3.13), then choose $n-1$ proper unit column vectors $e_{2p+1} = (0, 0, 0, 0, *)^T \in \mathbb{C}^{2n+2}$ $(2 \leq p \leq n)$ such that the type of the corresponding $U \in U(2n+2)$ is as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -a_{14} & \\ 0 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & a_{14} & 0 & \\ & & \mathbf{0} & & * \end{bmatrix}. \tag{3.30}$$

Thus, we have

$$UV_0^{(3)} = (1,\ -a_{14}z^3,\ \sqrt{3}z,\ \sqrt{3}a_{14}z^2,\ 0,\ 0, \dots, 0)^T,$$
$$\mathbf{J}UV_0^{(3)} = (\overline{a}_{14}\overline{z}^3,\ 1,\ -\sqrt{3}\overline{a}_{14}\overline{z}^2,\ \sqrt{3}\overline{z},\ 0,\ 0, \dots, 0)^T.$$

If $|a_{14}|^2 \neq 1$, then we choose

$$\begin{cases} e_1 = (1, 0, 0, 0, 0, \dots, 0)^T, \\ e_3 = (0, 1, 0, 0, 0, \dots, 0)^T, \\ e_5 = \frac{1}{\sqrt{1-|a_{14}|^2}}(0, 0, 1-|a_{14}|^2, 0, \overline{a}_{14}a_{25}, \dots, \overline{a}_{14}a_{2,2n+2})^T, \\ e_7 = \frac{1}{\sqrt{1-|a_{14}|^2}}(0, 0, 0, 1-|a_{14}|^2, -\overline{a}_{14}a_{15}, \dots, -\overline{a}_{14}a_{1,2n+2})^T. \end{cases} \tag{3.31}$$

And we choose $n-3$ proper unit column vectors $e_{2p+1} = (0, 0, 0, 0, *)^T \in \mathbb{C}^{2n+2}$ $(4 \leq p \leq n)$ such that the type of the corresponding $U \in U(2n+2)$ is as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -a_{14} & -a_{15} & \cdots & -a_{1,2n+2} \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{14} & 0 & -a_{25} & \cdots & -a_{2,2n+2} \\ 0 & 0 & \sqrt{1-|a_{14}|^2} & 0 & \frac{\overline{a}_{14}a_{25}}{\sqrt{1-|a_{14}|^2}} & \cdots & \frac{\overline{a}_{14}a_{2,2n+2}}{\sqrt{1-|a_{14}|^2}} \\ 0 & 0 & 0 & 0 & \frac{-a_{35}}{\sqrt{1-|a_{14}|^2}} & \cdots & \frac{-a_{3,2n+2}}{\sqrt{1-|a_{14}|^2}} \\ 0 & 0 & 0 & \sqrt{1-|a_{14}|^2} & \frac{-\overline{a}_{14}a_{15}}{\sqrt{1-|a_{14}|^2}} & \cdots & \frac{-\overline{a}_{14}a_{1,2n+2}}{\sqrt{1-|a_{14}|^2}} \\ 0 & 0 & 0 & 0 & \frac{-a_{45}}{\sqrt{1-|a_{14}|^2}} & \cdots & \frac{-a_{4,2n+2}}{\sqrt{1-|a_{14}|^2}} \\ & & \mathbf{0} & & & * & \end{bmatrix}. \tag{3.32}$$

Thus, we have

$$UV_0^{(3)} = (1,\ -a_{14}z^3,\ \sqrt{3}z,\ \sqrt{3}a_{14}z^2, \sqrt{3-3|a_{14}|^2}z^2,\ 0, \sqrt{1-|a_{14}|^2}z^3,\ 0,\ 0, \dots, 0)^T,$$
$$\mathbf{J}UV_0^{(3)} = (\overline{a}_{14}\overline{z}^3, 1,\ -\sqrt{3}\overline{a}_{14}\overline{z}^2,\ \sqrt{3}\overline{z},\ 0, \sqrt{3-3|a_{14}|^2}\overline{z}^2,\ 0,\ \sqrt{1-|a_{14}|^2}\overline{z}^3,\ 0, \dots, 0)^T.$$

From (3.30) and (3.32), we have $\left\langle f_3^{(3)},\ \mathbf{J} f_0^{(3)}\right\rangle = \left\langle U V_3^{(3)},\ \mathbf{J} U V_0^{(3)}\right\rangle = 6a_{14}$. On the other hand, from (3.27) we get $\left|\left\langle f_3^{(3)},\ \mathbf{J} f_0^{(3)}\right\rangle\right|^2 = 9$. So $a_{14} = \frac{1}{2} e^{\sqrt{-1}\theta}$ $(0 \le \theta \le 2\pi)$. Hence, in this case φ is congruent to the case (4) with $K = \frac{2}{3},\ \|B\|^2 = \frac{1}{3}$.

(II) Let $\varphi : S^2 \to HP^n$ be an irreducible linearly full harmonic map. At first, we state a conclusion about parallel minimal immersions of 2-spheres in $G(k, N)$ as follows:

Lemma 3.2 *([7]) Let $\varphi : S^2 \to G(k, N)$ be a conformal minimal immersion with the second fundamental form B. Suppose that B is parallel, then the following equations*

$$\begin{cases} \lambda^2(2K + ||B||^2)A_z^* + 4\left[[A_z, A_z^*], A_z^*\right] = 0, \\ \lambda^2(\frac{||B||^2}{4} - K)P + \left[[A_z, A_z^*], P\right] = 0 \end{cases} \tag{3.33}$$

hold.

By [1], we know φ_0 belongs to the following harmonic sequence

$$0 \longleftarrow \cdots \xleftarrow{A''_{\varphi_2}} \underline{\varphi}_{-2} \xleftarrow{A''_{\varphi_1}} \underline{\varphi}_{-1} \xleftarrow{A''_{\varphi_0}} \underline{\varphi}_0 \xrightarrow{A'_{\varphi_0}} \underline{\varphi}_1 \xrightarrow{A'_{\varphi_1}} \underline{\varphi}_2 \xrightarrow{A'_{\varphi_2}} \cdots \longrightarrow 0, \tag{3.34}$$

where $\underline{\varphi}_0 = \underline{i_2 \circ \varphi}$, $\underline{\varphi}_{-1} = \mathbf{J}\underline{\varphi}_1$, $\underline{\varphi}_{-2} = \mathbf{J}\underline{\varphi}_2$.
We choose a unit column vector $X \in \underline{\varphi}_0$, then we have

$$\underline{\varphi}_0 = \underline{X} \oplus \mathbf{J}\underline{X}, \quad \underline{\varphi}_1 = span\left\{\underline{X}_1, \underline{Y}_1\right\}, \tag{3.35}$$

where $X_1 = \partial X - \langle \partial X, X\rangle X - \langle \partial X, \mathbf{J}X\rangle \mathbf{J}X$ and $Y_1 = \partial \mathbf{J}X - \langle \partial \mathbf{J}X, X\rangle X - \langle \partial \mathbf{J}X, \mathbf{J}X\rangle \mathbf{J}X$. Here, X_1 and Y_1 are not orthogonal in general.

Let

$$E = [X, \mathbf{J}X]^* \partial [X, \mathbf{J}X]. \tag{3.36}$$

Then, from (3.35), we have

$$\begin{cases} \partial[X, \mathbf{J}X] = [X, \mathbf{J}X]E + [X_1, Y_1], \\ \overline{\partial}[X, \mathbf{J}X] = -[X, \mathbf{J}X]E^* + [-\mathbf{J}Y_1, \mathbf{J}X_1], \end{cases} \tag{3.37}$$

where $[X_1, Y_1]^*[X, \mathbf{J}X] = [-\mathbf{J}Y_1, \mathbf{J}X_1]^*[X, \mathbf{J}X] = 0$.
From (3.37) and the identity $\partial\overline{\partial} = \overline{\partial}\partial$, we get

$$\overline{\partial}E + \partial E^* + [E, E^*] = \begin{bmatrix} |X_1|^2 - |Y_1|^2, & 2\langle Y_1, X_1\rangle \\ 2\langle X_1, Y_1\rangle, & |Y_1|^2 - |X_1|^2 \end{bmatrix}. \tag{3.38}$$

From (3.35), we have $\varphi_0 = XX^* + (\mathbf{J}X)(\mathbf{J}X)^*$. Then by (2.1), a straightforward calculation shows

$$\begin{cases} \lambda^2 = 2(|X_1|^2 + |Y_1|^2), \\ A_z = (\mathbf{J}X)(\mathbf{J}X_1)^* - X(\mathbf{J}Y_1)^* - X_1X^* - Y_1(\mathbf{J}X)^*. \end{cases} \tag{3.39}$$

Since φ_0 is harmonic, the corresponding equivalent condition $\overline{\partial}A_z + A_zA_z^* - A_z^*A_z = 0$ (cf. [12]) implies

$$\begin{cases} \frac{\langle\overline{\partial}X_1, X_1\rangle}{|X_1|^2} = -\frac{\langle\overline{\partial}Y_1, Y_1\rangle}{|Y_1|^2} = \left\langle\overline{\partial}X, X\right\rangle, \\ \frac{\langle\overline{\partial}X_1, Y_1\rangle}{|Y_1|^2} = \left\langle\overline{\partial}X, \mathbf{J}X\right\rangle, \quad \frac{\langle\overline{\partial}Y_1, X_1\rangle}{|X_1|^2} = \left\langle\overline{\partial}\mathbf{J}X, X\right\rangle. \end{cases} \tag{3.40}$$

Now, we prove that *if $\varphi : S^2 \to HP^n$ is an irreducible linearly full harmonic map with parallel second fundamental form, then, up to $Sp(n+1)$, it belongs to one of the following two* *cases: (5)* $\varphi = \left[(-2\bar{z},\ \sqrt{2}-\sqrt{2}z\bar{z},\ 2z)^T\right] : S^2 \to CP^2 \subset HP^2$, *with* $K = \frac{1}{2},\ B = 0$; *(6)* $\varphi = \left[(6\bar{z}^2,\ -6\bar{z}+6z\bar{z}^2,\ \sqrt{6}-4\sqrt{6}z\bar{z}+\sqrt{6}z^2\bar{z}^2,\ 6z-6z^2\bar{z},\ 6z^2)^T\right] : S^2 \to CP^4 \subset HP^4$, *with* $K = \frac{1}{6},\ \|B\|^2 = \frac{2}{3}$.

If $\varphi : S^2 \to HP^n$ is an irreducible linearly full harmonic map with parallel second fundamental form, then applying Lemma 3.2 and substituting (3.39) into the first equation of (3.33), we get

$$\lambda^2 = 4|X_1|^2, \quad 2K + ||B||^2 = 1, \quad \langle X_1, Y_1\rangle = 0, \quad |X_1|^2 = |Y_1|^2. \tag{3.41}$$

From (3.38) and (3.41) we have

$$\bar{\partial} E + \partial E^* + [E, E^*] = 0. \tag{3.42}$$

Let $\widetilde{X} \in \underline{\varphi}_0$ be another unit column vector such that $\underline{\varphi}_0 = \underline{\widetilde{X}} \oplus \mathbf{J}\underline{\widetilde{X}}$, then

$$[\widetilde{X}, \mathbf{J}\widetilde{X}] = [X, \mathbf{J}X]T, \tag{3.43}$$

where $T : S^2 \to SU(2)$ is to be determined such that $\widetilde{X}$ satisfies $[\widetilde{X}, \mathbf{J}\widetilde{X}]^* d[\widetilde{X}, \mathbf{J}\widetilde{X}] = 0$. Such T is a solution of the linear PDE

$$dT + (Edz - E^* d\bar{z})T = 0. \tag{3.44}$$

The integrability condition of (3.44) is just (3.42), so it has a unique solution locally on S^2 for any given initial value. Let T be a solution of (3.44) with the initial value $T(0) \in SU(2)$. From (3.44) we have $d(T^*T) = 0$ and $d|T| = 0$, so $T \in SU(2)$.

Now, we choose a unit column vector $X \in \underline{\varphi}_0$ such that $\underline{\varphi}_0 = \underline{X} \oplus \mathbf{J}\underline{X}$ and

$$[X, \mathbf{J}X]^* d[X, \mathbf{J}X] = 0. \tag{3.45}$$

It follows from (3.40) and (3.45) that

$$\partial\bar{\partial} X = -|X_1|^2 X. \tag{3.46}$$

Let $\underline{f} = [X] : S^2 \to CP^{2n+1}$ be a smooth immersion. Similarly, by calculating the equivalent condition of harmonic, we find $\underline{f}$ is harmonic by (3.45) and (3.46). Of course, $\mathbf{J}\underline{f} = [\mathbf{J}X] : S^2 \to CP^{2n+1}$ is also harmonic. So $\underline{\varphi}_0 = \underline{f} \oplus \mathbf{J}\underline{f}$, where $\underline{f}$ belongs to the following harmonic sequence

$$0 \longrightarrow \cdots \xrightarrow{A''_{p-1}} \underline{f}_{p-1} \xrightarrow{A''_p} \underline{f}_p = \underline{f} \xrightarrow{A'_p} \underline{f}_{p+1} \xrightarrow{A'_{p+1}} \cdots \longrightarrow 0. \tag{3.47}$$

As in the case (Ia), let f_0 be a holomorphic section of $\underline{f}_0$, i.e. $\bar{\partial} f_0 = 0$, and f_p satisfy the corresponding formulas. From (3.41), we know $l_{p-1} = l_p$, which implies that $\underline{f}_p$ is totally real by ([2], Theorem 7.3), i.e. $\underline{f}_p = \underline{f}_m^{(2m)} : S^2 \to RP^{2m} \subset CP^{2m} \subset CP^{2n+1}$, where $2 \le 2m \le 2n+1$. Let $f_p = f_m^{(2m)}$ satisfy the corresponding formulas, then in harmonic sequence (3.34) by (3.40) and (3.45) we have

$$\underline{\varphi}_0 = \underline{f}_m^{(2m)} \oplus \mathbf{J}\underline{f}_m^{(2m)}, \quad \underline{\varphi}_1 = \underline{f}_{m+1}^{(2m)} \oplus \mathbf{J}\underline{f}_{m-1}^{(2m)}, \quad \underline{\varphi}_2 = \underline{f}_{m+2}^{(2m)} \oplus \mathbf{J}\underline{f}_{m-2}^{(2m)}, \tag{3.48}$$

where $l_i^{(2m)} = l_{2m-1-i}^{(2m)}$ $(i = 0, \ldots, m-1)$ and $\underline{\varphi}_0,\ \underline{\varphi}_1,\ \underline{\varphi}_2$ are mutually orthogonal.

At this time, from (3.48), we have $\varphi_0 = \frac{f_m^{(2m)}(f_m^{(2m)})^*}{|f_m^{(2m)}|^2} + \frac{(\mathbf{J}f_m^{(2m)})(\mathbf{J}f_m^{(2m)})^*}{|f_m^{(2m)}|^2}$. Then by (2.1) and a series of calculations, we obtain

$$\begin{cases} \lambda^2 = 4l_m^{(2m)}, \\ K = \frac{1}{2} - \frac{l_{m+1}^{(2m)}}{2l_m^{(2m)}}, \\ A_z = \frac{(\mathbf{J}f_m^{(2m)})(\mathbf{J}f_{m+1}^{(2m)})^*}{|f_m^{(2m)}|^2} + \frac{(\mathbf{J}f_{m-1}^{(2m)})(\mathbf{J}f_m^{(2m)})^*}{|f_{m-1}^{(2m)}|^2} - \frac{f_{m+1}^{(2m)}f_m^{(2m)*}}{|f_m^{(2m)}|^2} - \frac{f_m^{(2m)}f_{m-1}^{(2m)*}}{|f_{m-1}^{(2m)}|^2}, \\ P = \frac{1}{4}\left[\frac{l_{m-2}^{(2m)}}{|f_m^{(2m)}|^2} f_m^{(2m)}f_{m-2}^{(2m)*} + \frac{(\mathbf{J}f_m^{(2m)})(\mathbf{J}f_{m+2}^{(2m)})^*}{|f_{m+1}^{(2m)}|^2} - \frac{f_{m+2}^{(2m)}f_m^{(2m)*}}{|f_{m+1}^{(2m)}|^2} - \frac{l_{m-2}^{(2m)}}{|f_m^{(2m)}|^2}(\mathbf{J}f_{m-2}^{(2m)})(\mathbf{J}f_m^{(2m)})^*\right], \\ ||B||^2 = \frac{l_{m+1}^{(2m)}}{l_m^{(2m)}}. \end{cases} \tag{3.49}$$

Then applying Lemma 3.2 and substituting (3.49) into the second equation of (3.33), we get $m = 1$ or

$$\left\langle \mathbf{J}f_{m-2}^{(2m)}, f_{m-1}^{(2m)}\right\rangle = \left\langle f_{m+2}^{(2m)}, \mathbf{J}f_{m+1}^{(2m)}\right\rangle = 0, \ 3l_{m+1}^{(2m)} = 2l_m^{(2m)}. \tag{3.50}$$

In the latter case, since $l_{m+1}^{(2m)} = \frac{(m+2)(m-1)}{(1+z\bar{z})^2}$ and $l_m^{(2m)} = \frac{(m+1)m}{(1+z\bar{z})^2}$ by ([2], §5), we have $m = 2$ by (3.50). Hence, in the following, we discuss the above two cases of $m = 1$ and $m = 2$ respectively.

Case II1, $m = 1$.

In this case, by (3.50) we have

$$\underline{\varphi}_0 = U\underline{V}_1^{(2)} \oplus \mathbf{J}U\underline{V}_1^{(2)}, \tag{3.51}$$

where $\underline{V}_1^{(2)}$ is a Veronese surface in $CP^2 \subset CP^{2n+1}$ with the standard expression given in ([2], Section 5) and $U \in U(2n+2)$ satisfies $tr\left(V_2^{(2)}{V_0^{(2)}}^T U^T J_{n+1}U\right) = 0$, as this expresses the orthogonality of $Jf_0^{(2)}$ and $f_2^{(2)}$.

By calculating, we find in this case $\overline{W} = U^T J_{n+1}U \in U(2n+2)$ is the same type as (3.24). Then, the type of the corresponding $U \in U(2n+2)$ is the same as (3.25). Thus, we have

$$UV_1^{(2)} = \frac{1}{1+z\bar{z}}(-2\bar{z},\ 0,\ \sqrt{2}-\sqrt{2}z\bar{z},\ 0,\ 2z,\ 0,\ 0,\ldots,0)^T,$$
$$\mathbf{J}UV_1^{(2)} = \frac{1}{1+z\bar{z}}(0,\ -2z,\ 0,\ \sqrt{2}-\sqrt{2}z\bar{z},\ 0,\ 2\bar{z},\ 0,\ldots,0)^T.$$

In this case, it is easy to check that the corresponding map φ is totally geodesic. Obviously, it is congruent to the case (5) with $K = \frac{1}{2},\ B = 0$.

Case II2, $m = 2$.

In this case, by (3.50) we have

$$\underline{\varphi}_0 = U\underline{V}_2^{(4)} \oplus \mathbf{J}U\underline{V}_2^{(4)}, \tag{3.52}$$

where $\underline{V}_2^{(4)}$ is a Veronese surface in $CP^4 \subset CP^{2n+1}$ with the standard expression given in ([2], Section 5) and $U \in U(2n+2)$ satisfies $tr\left(V_4^{(4)}{V_0^{(4)}}^T U^T J_{n+1}U\right) = 0$, as this expresses the orthogonality of $Jf_0^{(4)}$ and $f_4^{(4)}$.

Similarly, we get the type of $\overline{W} = U^T J_{n+1} U \in U(2n+2)$ as follows:

$$\overline{W} = \begin{bmatrix} & & 0 & & & a_{16} & \cdots & a_{1,2n+2} \\ & & & & & a_{26} & \cdots & a_{2,2n+2} \\ & & & & & a_{36} & \cdots & a_{3,2n+2} \\ & & & & & a_{46} & \cdots & a_{4,2n+2} \\ & & & & & a_{56} & \cdots & a_{5,2n+2} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & \cdots & a_{6,2n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1,2n+2} & -a_{2,2n+2} & -a_{3,2n+2} & -a_{4,2n+2} & -a_{5,2n+2} & -a_{6,2n+2} & \cdots & 0 \end{bmatrix}.$$

And the type of the corresponding $U \in U(2n+2)$ is as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{16} & \cdots & -a_{1,2n+2} \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{26} & \cdots & -a_{2,2n+2} \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{36} & \cdots & -a_{3,2n+2} \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{46} & \cdots & -a_{4,2n+2} \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \mathbf{0} & & & & * & \end{bmatrix}. \tag{3.53}$$

Thus, we have

$$\begin{aligned} UV_2^{(4)} &= \frac{2}{(1+z\overline{z})^2}(6\overline{z}^2, 0, -6\overline{z}+6z\overline{z}^2, 0, \sqrt{6}-4\sqrt{6}z\overline{z} \\ &\quad + \sqrt{6}z^2\overline{z}^2, 0, 6z-6z^2\overline{z}, 0, 6z^2, 0, \ldots, 0)^T, \\ \mathbf{J}UV_2^{(4)} &= \frac{2}{(1+z\overline{z})^2}(0,\ 6z^2, 0, -6z+6z^2\overline{z}, 0, \sqrt{6}-4\sqrt{6}z\overline{z} \\ &\quad + \sqrt{6}z^2\overline{z}^2, 0, 6\overline{z}-6z\overline{z}^2, 0, 6\overline{z}^2, \ldots, 0)^T. \end{aligned}$$

In this case, it is easy to check that the Eq. (3.1) holds, which shows the second fundamental form of the corresponding map φ is parallel. Obviously, it is congruent to the case (6) with $K = \frac{1}{6}$, $||B||^2 = \frac{2}{3}$.

It is easy to check that no two of the above six cases are congruent, i.e., we cannot transform any one into any other by left multiplication by $Sp(n+1)$. To sum up, we get Theorem 1.1.

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