On existence and scattering theory for the Klein–Gordon–Schrödinger system in an infinite L^2 -norm setting

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Abstract This paper is concerned with the initial value problem for the nonlinear Klein–Gordon–Schrödinger (KGS) system in $\mathbb{R}^n \times \mathbb{R}$, $n \geq 1$. We consider general polynomial nonlinearities that include in particular the classical Yukawa–KGS model. We show existence of local and global mild solutions for the KGS system with initial data in weak L^r -spaces, which is an infinite L^2 -norm setting. Moreover, we obtain a persistence result in H^s when the initial data belong to this class, which shows that the constructed data-solution map in weak- L^r recovers H^s -regularity. We also prove results of scattering and wave operators in that singular framework.

Keywords Klein–Gordon–Schrödinger system · Local and global solutions · Scattering theory · Weak- L^p spaces

Mathematics Subject Classification 35Q55 · 35A01 · 35A02 · 35B40 · 35P25

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1 Introduction

The subject of this paper is the existence and scattering theory for the Klein–Gordon–Schrödinger (KGS) system with general nonlinearities, whose corresponding Cauchy problem reads as

$$\begin{cases} i\partial_t u + \Delta u = -u|u|^{\sigma-1}v|v|^{\lambda-1}, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ \partial_t^2 v - \Delta v + v = |u|^\rho, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{R}^n, \\ \partial_t v(x, 0) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $\sigma, \lambda \geq 1$, $\rho \geq 2$ and $n \geq 1$. This system generalizes the classical model of Yukawa's interaction ($\sigma = \lambda = 1, \rho = 2$) between scalar nucleons and neutral scalar mesons (see Fukuda and Tsutsumi [19], Yukawa [45]). Here, the nucleons are described by the complex scalar function $u(x, t)$ and the mesons by the real scalar function $v(x, t)$, defined for $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Yukawa interaction can be used to describe the strong nuclear force between nucleons, mediated by pions. Also, it is used in the standard model to describe the coupling between Higgs field and massless quark and electron fields (see [45]).

The classical Yukawa–KGS system has been analyzed by several authors and from various points of view (see for instance [1, 2, 5, 7, 11, 19–22, 26, 29, 34–37]). To our knowledge, the first studies of the KGS system were made by Fukuda and Tsutsumi in [19–22] in which existence of strong solutions was proved in bounded domains of $\mathbb{R}^3$ (with smooth boundary) by means of Galerkin's method. Latter, still in a L^2 -framework, the unique global existence and some conservation laws for the Yukawa–KGS system were established by Bachelot [1], Baillon and Chadam [2], and Hayashi and Von Wahl [26]. More recently, results of local well-posedness for initial data $(u_0, v_0, v_1) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)$ were obtained in Pecher [34]. These solutions exist uniquely in Bourgain-type spaces that are contained in the class $C([0, T]; L^2(\mathbb{R}^3) \times C([0, T]; L^2(\mathbb{R}^3) \times C([0, T]; H^{-1}(\mathbb{R}^3))$. Global well-posedness in $\mathbb{R}^3$ for initial data $(u_0, v_0, v_1) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)$ was obtained in [11] by using L^2 -mass conservation and Strichartz estimates (see also [42] for global well-posedness in $H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ with $s > \frac{7}{10}$). In the recent works [35, 36], Pecher obtained local and global well-posedness for data $u_0 \in H^s(\mathbb{R}^n)$, $v_0 \in H^\sigma(\mathbb{R}^n)$ and $v_1 \in H^{\sigma-1}(\mathbb{R}^n)$ for $n = 2, 3$ and adequate s, σ including negative values. By employing Strichartz estimates for local well-posedness and afterward using *a priori* estimate with finite energy and the local well-posedness result, existence of global finite-energy solutions for a related KGS system was established in [37]. [5] and [7] also considered a related KGS system in bounded domains, and they obtained (via Galerkin method) results about polynomial and exponential decay rates of global solutions, respectively.

Regarding the results of scattering for the coupled KGS system, [38–40] studied existence of wave operators under certain smallness conditions on the scattering states in a framework of Sobolev spaces. These results consider the 2D-case and quadratic interaction without assuming any restriction on the support of the Fourier transform of the scattering states (see [33] for conditions involving Fourier transform). In contrast to the coupled KGS system, there are many works about asymptotic behavior of solutions for the Klein–Gordon equation

$$\partial_t^2 v - \Delta v + v = \pm v|v|^\rho, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

which, to our knowledge, have been addressed in the framework of Sobolev spaces. For the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \pm u|u|^{\sigma-1}, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

and some related models, scattering results and existence of wave operators in weak- L^p spaces can be found in [9, 12, 13].

In this paper, we are interested in results of local and global existence for KGS system and scattering theory, outside the context of finite L^2 -mass. We consider initial data in a class of Lorentz spaces $L^{(r,d)}$, including the Marcinkiewicz space $L^{(r,\infty)}$, also called weak L^r -space (see Bergh and Löfström [3]). We recall that Lorentz spaces are defined as the set of measurable function f on $\mathbb{R}^n$ such that the quantity

$$\|f\|_{(q,d)} = \begin{cases} \left(\frac{q}{d} \int_0^\infty \left[t^{\frac{1}{q}} f^{**}(t) \right]^d \frac{dt}{t} \right)^{\frac{1}{d}}, & \text{if } 1 < q < \infty, 1 \leq d < \infty, \\ \sup_{t>0} t^{\frac{1}{q}} f^{**}(t), & \text{if } 1 < q \leq \infty, d = \infty, \end{cases}$$

is finite. Here

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds, \quad f^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : |f(x)| > s\}) \leq t\}, \quad t > 0,$$

with μ denoting the Lebesgue measure. In particular, $L^q(\Omega) = L^{(q,q)}(\Omega)$, and when $d = \infty$, $L^{(q,\infty)}(\Omega)$ are called weak- L^q spaces. Moreover, $L^{(q,d_1)} \subset L^q \subset L^{(q,d_2)} \subset L^{(q,\infty)}$ for $1 \leq d_1 \leq q \leq d_2 \leq \infty$. In particular, weak- L^r spaces contain singular functions with infinite L^2 -mass such as homogeneous functions of degree $-n/r$. Comparing to $H^{s,l}$ -spaces, the continuous inclusion $H^{s,l}(\mathbb{R}^n) \subset L^{(r,\infty)}(\mathbb{R}^n)$ holds true for $l \leq r$, $s \geq 0$ and $\frac{1}{l} - \frac{s}{n} \leq \frac{1}{r}$, and $H^{s,l}$ -spaces do not contain any weak- L^r spaces if $s \in \mathbb{R}$, $1 \leq l \leq 2$ and $l \leq r$. In particular, $L^{(r,\infty)}(\mathbb{R}^n) \not\subset H^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, when $r \geq 2$. In the case $n = 1$, $L^{(r,\infty)}(\mathbb{R}) \not\subset H^{-\frac{1}{4}}(\mathbb{R})$ for either $1 < r \leq \frac{4}{3}$ or $r \geq 2$.

Concerning the related Schrödinger equation, coupled Schrödinger equations and wave equations, some existence results of local and global solutions with initial data in classes of Lorentz spaces have been obtained for instance in [8, 16, 17, 41] by using the scaling principle and time-polynomial weighted norms (see also [10] for scaling techniques combined with Strichartz-type estimates in weak- L^p). This strategy for obtaining global solutions was initially introduced in order to study nonlinear parabolic equations and fluid mechanics models, for which there are results in several spaces (see, e.g., [14, 15, 18, 23, 27, 28, 44] and their references). However, the KGS system does not possess scaling which makes this system more awkward than related problems such as the Schrödinger or wave equations with nonlinearities of type $u|u|^\rho$. In this paper, we achieve suitable norms in which, despite having no scaling relation, it is possible to obtain the existence of global mild solutions. Having identified the adequate function spaces, we perform estimates for integral operators appearing in the mild formulation (see (1.2) below for definition) on Lorentz spaces and afterward employ Banach fixed point theorem in order to prove existence of local and global solutions. These can be physically interesting because although elements of $L^{(r,\infty)}$ have in general infinite L^2 -mass, they belong to L^2_{loc} for $r > 2$ (local finite L^2 -mass); that is, they can be realized in the physical space in any region with finite volume. In addition, for initial data in H^s , the corresponding solutions belong to H^s , which shows that the constructed data-solution map in $L^{(r,\infty)}$ recovers H^s -regularity and it is compatible with H^s -theory (see Sect. 3.2).

In order to state our results, we start by establishing the definition of mild solution for the Cauchy problem (1.1). A mild solution for system (1.1) is a pair (u, v) satisfying the following integral system:

$$\begin{cases} u(x, t) = e^{it\Delta} u_0(x) + i \int_0^t e^{i(t-s)\Delta} u(x, s) |u(x, s)|^{\sigma-1} v(x, s) |v(x, s)|^{\lambda-1} ds \\ v(x, t) = \partial_t W(t) v_0(x) + W(t) v_1(x) + \int_0^t W(t-s) |u(x, s)|^\rho ds \end{cases} \quad (1.2)$$

where $e^{it\Delta}$ and $W(t)$ are the free groups associated with the linear Schrödinger and Klein–Gordon equations, respectively, that is,

$$W(t)f(x) = (K_t * f)(x), \quad \text{with } K_t(x) = \left(\frac{\sin \left(t \sqrt{1 + |\xi|^2} \right)}{\sqrt{1 + |\xi|^2}} \right)^\vee(x)$$

and

$$e^{it\Delta} g = (4\pi t)^{-n/2} e^{-\frac{i|x|^2}{4t}} * g.$$

We consider the Banach space

$$X_{(q,d)} = L^{(q,d)}(\mathbb{R}^n) \times L^{(\frac{q}{q-\rho},d)}(\mathbb{R}^n),$$

endowed with the norm

$$\|[u, v]\|_{X_{(q,d)}} = \|u\|_{(q,d)} + \|v\|_{(\frac{q}{q-\rho},d)}.$$

For the existence of global solutions, we define the space E_{a_0, a_1}^∞ as the set of Bochner measurable pairs $[u, v] : (-\infty, \infty) \rightarrow X_{(q,\infty)}$ such that

$$\|[u, v]\|_{E_{a_0, a_1}^\infty} = \sup_{-\infty < t < \infty} |t|^{a_0} \|u(t)\|_{(q,\infty)} + \sup_{-\infty < t < \infty} |t|^{a_1} \|v(t)\|_{(\frac{q}{q-\rho}, \infty)} < \infty,$$

where (a_0, a_1) is the unique solution of the system

$$\begin{cases} 1 - b_0 - \lambda a_1 - (\sigma - 1)a_0 = 0, \\ 1 - \zeta\left(\frac{q}{\rho}\right) - \rho a_0 + a_1 = 0, \end{cases} \quad (1.3)$$

with $b_0 = \frac{n(q-2)}{2q}$, $2 \leq q < \infty$ and ζ the function defined by

$$\zeta(p) = \begin{cases} 2\left(1 - \frac{1}{p}\right), & \text{if } (1/p, 1/p') \in \overline{P_6 P_1} \\ \frac{n(2-p)}{2p}, & \text{if } (1/p, 1/p') \in \overline{P_0 P_6} \end{cases}, \quad (1.4)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Here, $\overline{PQ}$ denotes the closed segment joining P with Q ,

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}\right), \quad P_6 = \left(\frac{1}{2} + \frac{1}{n+2}, \frac{1}{2} - \frac{1}{n+2}\right) \quad \text{and} \quad P_1 = \left(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1}\right). \quad (1.5)$$

The definition of the function $\zeta(\cdot)$ appears from $L^{p,d} - L^{p',d}$ estimates of the linear Klein–Gordon group (see Lemma 2.4 below). In (1.5) we use the indexes 0, 6, 1 of P to keep the same notation of [29].

For the existence of local solutions, fixed $0 < T < \infty$, we consider the space E_{b_0, b_1}^T of Bochner measurable pairs $[u, v] : (-T, T) \rightarrow X_{(q,\infty)}$ such that

$$\|[u, v]\|_{E_{b_0, b_1}^T} = \sup_{-T < t < T} |t|^{b_0} \|u(t)\|_{(q,\infty)} + \sup_{-T < t < T} |t|^{b_1} \|v(t)\|_{(\frac{q}{q-\rho}, \infty)} < \infty,$$

where

$$b_0 = \frac{n(q-2)}{2q}, \quad 2 \leq q < \infty \quad \text{and} \quad b_1 = \zeta\left(\frac{q}{\rho}\right).$$

Our local-in-time existence result reads as follows.

Theorem 1.1 (Local-in-time solutions) *Let $\lambda, \sigma \geq 1$, $\rho \geq 2$, and $\frac{2\rho(n+1)}{n+3} < q < 2\rho$ with $q \neq \frac{2\rho(n+2)}{(n+4)}$ be such that $\lambda\rho - \sigma - 1 = q(\lambda - 1)$ and $\max\{\rho b_0, (\lambda b_1 + \sigma b_0)\} < 1$. If $[u_0, v_0, v_1] \in [\mathcal{S}'(\mathbb{R}^n)]^3$ is such that the norm*

$$\sup_{-T < t < T} |t|^{b_0} \|e^{it\Delta} u_0\|_{(q,\infty)} + \sup_{-T < t < T} |t|^{b_1} \|\partial_t W(t)v_0 + W(t)v_1\|_{(\frac{q}{q-\rho},\infty)} \quad (1.6)$$

is finite, then there exists $0 < T^ \leq T < \infty$ such that the initial value problem (1.1) has a mild solution $[u, v] \in E_{b_0, b_1}^{T^*}$. The solution $[u, v]$ is unique in a given ball of $E_{b_0, b_1}^{T^*}$ and the data-solution map $[u_0, v_0, w_0] \mapsto [u, v]$ is Lipschitz.*

Remark 1.2

- (i) By using Lemmas 2.1 and 2.4 below, we get that (1.6) is finite if you take $u_0 \in L^{(q', \infty)}, v_0 = 0$ and $v_1 \in L^{(\frac{q}{\rho}, \infty)}$. More generally, for initial data $(u_0, v_0, v_1) \in L^{(q', \infty)} \times H^r \times H^{r-1}$, with $r > 0$ and $2 < \frac{q}{q-2} \leq \frac{2n}{n-2(r-1)}$ ($< \infty$ if $n < 2r - 2$), the quantity (1.6) is finite (see Sect. 3.2). So one can see that the class of initial data (u_0, v_0, v_1) satisfying (1.6) is large.
- (ii) If we assume that $[u_0, v_0, v_1]$ is such that

$$\sup_{-T < t < T} |t|^{b_0} \|e^{it\Delta} u_0\|_{(q,d)} + \sup_{-T < t < T} |t|^{b_1} \|\partial_t W(t)v_0 + W(t)v_1\|_{(\frac{q}{q-\rho}, d)}$$

is finite, for $1 \leq d < \infty$, with either $\sigma, \lambda \geq 2$ or $\lambda = \sigma = 1$, then the local-in-time solution verifies (reducing T if necessary)

$$\sup_{-T < t < T} |t|^{b_0} \|u(t)\|_{(q,d)} + \sup_{-T < t < T} |t|^{b_1} \|v(t)\|_{(\frac{q}{q-\rho}, d)} < \infty.$$

- (iii) Theorem 1.1 covers the relevant physical case of the classical model of Yukawa's interaction ($\sigma = \lambda = 1, \rho = 2$), at least for some values of n (see Remark 2.7 below).
- (iv) The proof of Theorem 1.1 is based on the contraction principle. We are able to prove that if the nonlinearity is smooth, then the data-solution map is also smooth (see Corollary 3.1). This general fact can be used to prove that, under certain conditions, it is not possible to carry out a fixed point argument to obtain local solutions for an equation (or system) (see for instance Molinet et al. [31, 32]).

For $1 \leq d \leq \infty$ and $h \geq 0$, let us define the initial data class $I_{0,d,h}$ of all $[u_0, v_0, v_1] \in [\mathcal{S}'(\mathbb{R}^n)]^3$ such that $\|[u_0, v_0, v_1]\|_{I_{0,d,h}} < \infty$, where

$$\begin{aligned} \|[u_0, v_0, v_1]\|_{I_{0,d,h}} := & \sup_{-\infty < t < \infty} |t|^{a_0+h} \|e^{it\Delta} u_0\|_{(q,d)} \\ & + \sup_{-\infty < t < \infty} |t|^{a_1+h} \|\partial_t W(t)v_0 + W(t)v_1\|_{(\frac{q}{q-\rho}, d)} \end{aligned} \quad (1.7)$$

In the case $h = 0$, we denote $I_{0,d,h} = I_{0,d,0}$ simply by $I_{0,d}$, and in particular $I_{0,\infty}$ stands for $I_{0,\infty,0}$.

Theorem 1.3 (Global-in-time solutions) *Let $\lambda, \sigma \geq 1$, $\rho \geq 2$, and $\frac{2\rho(n+1)}{n+3} < q < 2\rho$ with $q \neq \frac{2\rho(n+2)}{(n+4)}$ be such that $\lambda\rho - \sigma - 1 = q(\lambda - 1)$, and $\max\{b_0, b_1, \rho a_0, \lambda a_1 + \sigma a_0\} < 1$. There exists $\delta > 0$ such that if*

$$\|[u_0, v_0, v_1]\|_{I_{0,\infty}} \leq \frac{\delta}{2}, \quad (1.8)$$

then the initial value problem (1.1) has a unique global-in-time solution $[u, v] \in E_{a_0, a_1}^\infty$ with $\|[u, v]\|_{E_{a_0, a_1}^\infty} \leq \delta$. Moreover, the data-solution map $[u_0, v_0, v_1] \mapsto [u, v]$ from $I_{0, \infty}$ into E_{a_0, a_1}^∞ is locally Lipschitz at the origin.

Remark 1.4

- (i) Theorem 1.3 also covers the case of quadratic interaction (Yukawa system), at least for some values of n (see Remark 2.9 below).
- (ii) In addition to the hypotheses of Theorem 1.3, assume that $[u_0, v_0, v_1] \in I_{0, d, h}$ (see (1.7)) for $0 \leq h < 1 - \max\{\rho a_0, \lambda a_1 + \sigma a_0\}$ and some $1 \leq d < \infty$, with either $\sigma, \lambda \geq 2$ or $\lambda = \sigma = 1$. Then there exists δ_0 such that if $\|[u_0, v_0, w_0]\|_{I_{0, d}} < \delta_0$, then the solution given in Theorem 1.3 verifies

$$\sup_{-\infty < t < \infty} |t|^{a_0+h} \|u(t)\|_{(q, d)} + \sup_{-\infty < t < \infty} |t|^{a_1+h} \|v(t)\|_{(\frac{q}{q-\rho}, d)} < \infty. \quad (1.9)$$

Here, it is not necessary to assume a smallness hypothesis on the norm $\|\cdot\|_{I_{0, d, h}}$.

Now we establish our results about asymptotic behavior of solutions for the KGS system in the framework of weak- L^p spaces. Firstly, we find a initial data $[u_0^\pm, v_0^\pm, v_1^\pm]$ such that the solution $[u^\pm, v^\pm]$ of the linear problem associated with the KGS system, with initial data $[u_0^\pm, v_0^\pm, v_1^\pm]$, describes the asymptotic behavior of the global solution given in Theorem 1.3. This is the subject of the next theorem.

Theorem 1.5 (Scattering). *Suppose that $0 \leq h < 1 - \max\{\rho a_0, \lambda a_1 + \sigma a_0\}$, and let $[u, v]$ be the solution of (1.1) obtained in Theorem 1.3 with data $[u_0, v_0, v_1]$. If $[u_0, v_0, v_1]$ is as in item (ii) of Remark 1.4 with $d = \infty$, then there exists $[u_0^\pm, v_0^\pm, v_1^\pm] \in I_{0, \infty}$ such that*

$$\|[u(t) - u^\pm(t)]\|_{(q, \infty)} = O(|t|^{-a_0-h(\sigma+\lambda)}), \quad \text{as } t \rightarrow \pm\infty, \quad (1.10)$$

$$\|[v(t) - v^\pm(t)]\|_{(\frac{q}{q-\rho}, \infty)} = O(|t|^{-a_1-h\rho}), \quad \text{as } t \rightarrow \pm\infty, \quad (1.11)$$

where $[u^+(t), v^+(t)]$, $[u^-(t), v^-(t)]$ stand for the unique global mild solutions of the linear problem associated with (1.1) with initial data $[u_0^+, v_0^+, v_1^+]$ and $[u_0^-, v_0^-, v_1^-]$, respectively.

In this paper, we also are interested in the construction of a wave operator for the KGS system, that is, construction of solutions with prescribed scattering states. More precisely, for a given profile $[f_1, f_2, f_3]$, the problem consists in finding a global solution $[u, v]$ which converges as $|t| \rightarrow \infty$ to the solution of the associated linear problem with initial data $[f_1, f_2, f_3]$. This means that the interaction has no effect asymptotically at infinity.

In the following theorem, we construct a wave operator for the KGS system with arbitrary profiles $[f_1, f_2, f_3]$ in the framework of weak- L^p spaces.

Theorem 1.6 *Suppose that the parameters $\lambda, \sigma, \rho, a_0, b_0, q$ satisfy the hypotheses of Theorem 1.3. Let $\theta_0, \theta_1 \in [0, \infty)$ be such that $(1-\sigma)(\theta_0-a_0) < \lambda(\theta_1-a_1)$, $\theta_1-a_1 < \rho(\theta_0-a_0)$ and $\theta_0 \leq b_0$. For any $[f_1, f_2, f_3] \in [S'(\mathbb{R}^n)]^3$ such that $f_1 \in L^{(q', \infty)}$ and*

$$\sup_{t>0} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} < \infty,$$

there exist $T_0 = T_0(f_1, f_2, f_3) > 0$ and a solution $[u, v]$ of the integral equation (1.2) in $[T_0, \infty)$ with

$$\|[u, v]\|_{E_{T_0, \theta_0, \theta_1}} \equiv \sup_{t>T_0} t^{\theta_0} \|u(t)\|_{(q, \infty)} + \sup_{t>T_0} t^{\theta_1} \|v(t)\|_{(\frac{q}{q-\rho}, \infty)} < \infty$$

and

$$\lim_{t \rightarrow \infty} t^{\theta_0} \|u(t) - e^{it\Delta} f_1\|_{(q,\infty)} + \lim_{t \rightarrow \infty} t^{\theta_1} \|v(t) - (\partial_t W(t)f_2 + W(t)f_3)\|_{(\frac{q}{q-\rho},\infty)} = 0.$$

If $[\tilde{u}, \tilde{v}]$ is also a solution in $[T_0, \infty)$ satisfying

$$\sup_{t > T_0} t^{\theta_0} \|\tilde{u}(t)\|_{(q,\infty)} + \sup_{t > T_0} t^{\theta_1} \|\tilde{v}(t)\|_{(\frac{q}{q-\rho},\infty)} < \infty,$$

then there exists $T \geq T_0$ such that $[u, v] = [\tilde{u}, \tilde{v}]$ in $[T, \infty)$.

The plan of this paper is as follows. In the next section, we give some preliminaries and state linear and nonlinear estimates that will be used in order to employ a contraction argument. In Sects. 3 and 4, we prove the existence of local solutions (Theorem 1.1) and global solutions (Theorem 1.3). Finally, in Sect. 5 we prove the scattering results (Theorems 1.5 and 1.6).

2 Linear and nonlinear estimates

We start by recalling the Hölder inequality in Lorentz spaces which reads

$$\|h\|_{(r,s)} \leq C(r) \|f\|_{(q_1,d_1)} \|g\|_{(q_2,d_2)},$$

for $1 < q_1 \leq \infty$, $1 < q_2$, $r < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} < 1$, $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$, and $s \geq 1$ satisfies $\frac{1}{d_1} + \frac{1}{d_2} \geq \frac{1}{s}$.

Lemma 2.1 (see [17]) Let $1 \leq d \leq \infty$ and $1 < r \leq 2$, and consider r' be such that $\frac{1}{r} + \frac{1}{r'} = 1$. There exists a positive constant $C_1 = C_1(n, r)$ such that

$$\|e^{it\Delta} f\|_{(r',d)} \leq C_1 |t|^{-\frac{n}{2}(\frac{2}{r}-1)} \|f\|_{(r,d)},$$

for all $f \in L^{(r,d)}$ and $t \neq 0$.

Now we recall an estimate that comes from Theorems B and C in Marshall, Strauss and Wainger [29], when $(1/p, 1/q)$ belongs to the line of duality, i.e., $q = p' = \frac{p}{p-1}$.

Proposition 2.2 (see [29]) Assume that $\frac{2(n+1)}{n+3} \leq p \leq 2$ and let (p, p') and $\zeta(p)$ be as in (1.4). There exists a positive constant C such that

$$\|W(t)f\|_{L^{p'}} \leq C |t|^{-\zeta(p)} \|f\|_{L^p},$$

for all $t \in \mathbb{R}$ and all $f \in L^p(\mathbb{R}^n)$.

Remark 2.3 We observe that in the application of Theorems B and C in Marshall, Strauss and Wainger [29], with $(1/p, 1/q)$ on the so-called line of duality, the exponents a and b (in the notation of [29]) verify that $a \leq b$ and then $t^b \leq t^a$ on the interval $0 < t < 1$.

Lemma 2.4 Assume that $\frac{2(n+1)}{n+3} < p < 2$ and $1 \leq d \leq \infty$. Let (p, p') and $\zeta(p)$ be as in (1.4), but in the open segments, that is, $(1/p, 1/p')$ is neither P_0 , P_1 nor P_6 which are defined in (1.5). There exists a positive constant C_2 such that

$$\|W(t)f\|_{(p',d)} \leq C_2 |t|^{-\zeta(p)} \|f\|_{(p,d)}, \quad (2.1)$$

for all $t \in \mathbb{R}$ and all $f \in L^{(p,d)}$.

Proof We only give the proof in the case $(1/p, 1/p') \in \overline{P_0 P_6}$; the other case is similar. Consider $p_0 = \frac{2(n+2)}{n+4}$, $p_1 = 2$ and $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, with $0 < \theta < 1$. Note that in this case $\zeta(p_i) = n(\frac{1}{p_i} - \frac{1}{2})$, for $i = 0, 1$. From Proposition 2.2 we have that $W(t) : L^{p_0} \rightarrow L^{p'_0}$ and $W(t) : L^{p_1} \rightarrow L^{p'_1}$, with operators norms bounded by

$$\|W(t)\|_{p_0 \rightarrow p'_0} \leq C |t|^{-\zeta(p_0)} \quad \text{and} \quad \|W(t)\|_{p_1 \rightarrow p'_1} \leq C |t|^{-\zeta(p_1)}.$$

Since $L^p = L^{(p,p)}$, using real interpolation we get,

$$\|W(t)\|_{(p,d) \rightarrow (p',d)} \leq C |t|^{-\zeta(p_0)\lambda} |t|^{-\zeta(p_1)(1-\lambda)} = C |t|^{-[\zeta(p_0)\lambda + \zeta(p_1)(1-\lambda)]}.$$

But

$$\zeta(p_0)\lambda + \zeta(p_1)(1-\lambda) = n \left(\frac{1}{p_0} - \frac{1}{2} \right) \lambda + n \left(\frac{1}{p_1} - \frac{1}{2} \right) (1-\lambda) = n \left(\frac{1}{p} - \frac{1}{2} \right) = \zeta(p).$$

Thus we obtain that

$$\|W(t)\|_{(p,d) \rightarrow (p',d)} \leq C |t|^{-\zeta(p)},$$

which finishes the proof of the lemma. $\square$

In the following, we recall a lemma about the Beta function and give its proof for the sake of completeness.

Lemma 2.5 *Suppose that $1-x$, $1-y$ and t are positive numbers. Then*

$$\int_0^t (t-s)^{-x} s^{-y} ds = t^{1-x-y} B(1-x, 1-y), \quad (2.2)$$

where $B(x, y) = \int_0^1 (1-s)^{x-1} s^{y-1} ds$ is the Beta function.

Proof Using the change $s \rightarrow st$, it follows that

$$\int_0^t (t-s)^{-x} s^{-y} ds = t^{1-x-y} \int_0^1 (1-s)^{-x} s^{-y} ds,$$

which implies (2.2). $\square$

Throughout this paper, we denote by

$$\mathcal{F}(u, v) = i \int_0^t e^{i(t-s)\Delta} u(s) |u(s)|^{\sigma-1} v(s) |v(s)|^{\lambda-1} ds \quad \text{and} \quad \mathcal{G}(u) = \int_0^t W(t-s) |u(s)|^\rho ds$$

the nonlinear parts of the integral system (1.2). We also define

$$\mathcal{N}(u, v) = [\mathcal{F}(u, v), \mathcal{G}(u)]. \quad (2.3)$$

Lemma 2.6 *Let $\lambda, \sigma \geq 1$, $\rho \geq 2$, and $\frac{2\rho(n+1)}{n+3} < q < 2\rho$ with $q \neq \frac{2\rho(n+2)}{(n+4)}$ be such that $\lambda\rho - \sigma - 1 = q(\lambda - 1)$ and $\max\{\rho b_0, (\lambda b_1 + \sigma b_0)\} < 1$. There exists a constant $C_3 > 0$ (independent of T) such that*

$$\begin{aligned}
& \|\mathcal{N}(u, v) - \mathcal{N}(\tilde{u}, \tilde{v})\|_{E_{b_0, b_1}^T} \\
& \leq C_3 T^{1-\lambda b_1 - \sigma b_0} \sup_{-T < t < T} |t|^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{-T < t < T} |t|^{\lambda b_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \\
& \quad \cdot \sup_{-T < t < T} |t|^{(\sigma-1)b_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \\
& \quad + C_3 T^{1-\lambda b_1 - \sigma b_0} \sup_{-T < t < T} |t|^{b_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{-T < t < T} |t|^{\sigma b_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \\
& \quad \cdot \sup_{-T < t < T} |t|^{(\lambda-1)b_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \\
& \quad + C_3 T^{1-\rho b_0} \sup_{-T < t < T} |t|^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{-T < t < T} |t|^{(\rho-1)b_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right),
\end{aligned} \tag{2.4}$$

for all $u, v, \tilde{u}, \tilde{v}$ such that the right-hand side of (2.4) is finite.

Proof Without loss of generality suppose that $t > 0$. Then from Lemma 2.1 and Hölder inequality, we obtain

$$\begin{aligned}
& \|\mathcal{F}(u, v) - \mathcal{F}(\tilde{u}, \tilde{v})\|_{(q, \infty)} \leq \int_0^t (t-s)^{-\frac{n(q-2)}{2q}} \|u|u|^{\sigma-1}v|v|^{\lambda-1} - \tilde{u}|\tilde{u}|^{\sigma-1}\tilde{v}|\tilde{v}|^{\lambda-1}\|_{(q', \infty)} ds \\
& \leq C \int_0^t (t-s)^{-b_0} \|u - \tilde{u}\|_{(q, \infty)} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] ds \\
& \quad + C \int_0^t (t-s)^{-b_0} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \|\tilde{u}\|_{(q, \infty)}^\sigma \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] ds \\
& \leq C \sup_{0 < t < T} t^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{0 < t < T} t^{\lambda b_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \\
& \quad \cdot \left(\sup_{0 < t < T} t^{(\sigma-1)b_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \right) \int_0^t (t-s)^{-b_0} s^{-\lambda b_1 - \sigma b_0} ds \\
& \quad + C \sup_{0 < t < T} t^{b_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{0 < t < T} t^{\sigma b_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \\
& \quad \cdot \left(\sup_{0 < t < T} t^{(\lambda-1)b_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \right) \int_0^t (t-s)^{-b_0} s^{-\lambda b_1 - \sigma b_0} ds \\
& = C \sup_{0 < t < T} t^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{0 < t < T} t^{\lambda b_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \\
& \quad \cdot \left(\sup_{0 < t < T} t^{(\sigma-1)b_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \right) t^{-b_0} t^{1-\lambda b_1 - \sigma b_0} \\
& \quad + C \sup_{0 < t < T} t^{b_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{0 < t < T} t^{\sigma b_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \\
& \quad \cdot \left(\sup_{0 < t < T} t^{(\lambda-1)b_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \right) t^{-b_0} t^{1-\lambda b_1 - \sigma b_0}.
\end{aligned}$$

Therefore

$$\begin{aligned}
 & \sup_{0 < t < T} t^{b_0} \|\mathcal{F}(u, v) - \mathcal{F}(\tilde{u}, \tilde{v})\|_{(q, \infty)} \\
 & \leq C \sup_{0 < t < T} t^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{0 < t < T} t^{\lambda b_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \\
 & \quad \cdot \sup_{0 < t < T} t^{(\sigma-1)b_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] T^{1-\lambda b_1-\sigma b_0} \\
 & \quad + C \sup_{0 < t < T} t^{b_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{0 < t < T} t^{\sigma b_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \\
 & \quad \cdot \sup_{0 < t < T} t^{(\lambda-1)b_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] T^{1-\lambda b_1-\sigma b_0}. \quad (2.5)
 \end{aligned}$$

Notice that the condition $\lambda\rho - \sigma - 1 = q(\lambda - 1)$ leads us to $\frac{\lambda q}{q-\sigma-1} = \frac{q}{q-\rho}$ in (2.5). Here we have two cases: If $\lambda = 1$, we obtain $\rho = \sigma + 1$, and if $\lambda \neq 1$, we get $q = \frac{\lambda\rho - \sigma - 1}{\lambda - 1}$.

On the other hand, for $\frac{2(n+1)}{n+3} < z < 2$ with $z \neq \frac{2(n+2)}{(n+4)}$ such that $\frac{1}{z} = \frac{1}{q} + \frac{\rho-1}{q}$, again Lemma 2.4 and Hölder inequality yield

$$\begin{aligned}
 \|\mathcal{G}(u) - \mathcal{G}(\tilde{u})\|_{(z', \infty)} & \leq C \int_0^t (t-s)^{-\zeta(z)} \| |u|^\rho - |\tilde{u}|^\rho \|_{(z, \infty)} ds \\
 & \leq C \int_0^t (t-s)^{-\zeta(z)} \|u - \tilde{u}\|_{(q, \infty)} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right) ds, \quad (2.6)
 \end{aligned}$$

where $z' = \frac{q}{q-\rho}$. From (2.6) and Lemma 2.5, we obtain

$$\begin{aligned}
 \|\mathcal{G}(u) - \mathcal{G}(\tilde{u})\|_{\left(\frac{q}{q-\rho}, \infty\right)} & \leq C \int_0^t (t-s)^{-\zeta\left(\frac{q}{q-\rho}\right)} \|u - \tilde{u}\|_{(q, \infty)} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right) ds \\
 & \leq C \int_0^t (t-s)^{-b_1} s^{-\rho b_0} ds \sup_{0 < t < T} t^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{0 < t < T} t^{(\rho-1)b_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right) \\
 & \leq C t^{-b_1} t^{1-\rho b_0} \sup_{0 < t < T} t^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{0 < t < T} t^{(\rho-1)b_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right).
 \end{aligned}$$

Consequently

$$\begin{aligned}
 & \sup_{0 < t < T} t^{b_1} \|\mathcal{G}(u) - \mathcal{G}(\tilde{u})\|_{\left(\frac{q}{q-\rho}, \infty\right)} \\
 & \leq C T^{1-\rho b_0} \sup_{0 < t < T} t^{b_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{0 < t < T} t^{(\rho-1)b_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right). \quad (2.7)
 \end{aligned}$$

The estimate (2.4) follows from (2.5) and (2.7). $\square$

Remark 2.7 Let us consider the relevant physical case of the classical model of Yukawa's interaction ($\sigma = \lambda = 1$ and $\rho = 2$). Then, the conditions in Lemma 2.6 can be rewritten as follows:

$$q \neq \frac{4(n+2)}{(n+4)}, \quad q \in \left(\frac{4(n+1)}{(n+3)}, 4 \right), \text{ and}$$

$$2b_0 < 1 \Rightarrow q \in \left\{ \left(\frac{4(n+1)}{(n+3)}, \frac{4(n+2)}{(n+4)} \right) \cup \left(\frac{4(n+2)}{(n+4)}, 4 \right), \quad \text{if } n = 1 \right. \\ \left. \left(\frac{4(n+1)}{n+3}, \frac{4(n+2)}{(n+4)} \right) \cup \left(\frac{4(n+2)}{(n+4)}, \frac{2n}{n-1} \right), \quad \text{if } n = 2, 3 \right\}. \quad (2.8)$$

In order to describe the conditions on b_1 , we have two cases.

Case 1: $(2/q, 1 - 2/q) \in \overline{P_0 P_1}$. In this case $b_1 = 2 - 4/q$, and therefore

$$b_1 < 1 \Leftrightarrow q < 4, \quad (2.9)$$

and

$$b_1 + b_0 < 1 \Leftrightarrow q < \frac{2(n+4)}{n+2}. \quad (2.10)$$

Combining (2.8)–(2.10), we have that

$$q \in \left\{ \left(2, \frac{12}{5} \right) \cup \left(\frac{12}{5}, \frac{10}{3} \right), \quad \text{if } n = 1 \right. \\ \left. \left(\frac{12}{5}, \frac{8}{3} \right) \cup \left(\frac{8}{3}, 3 \right), \quad \text{if } n = 2 \right. \\ \left. \left(\frac{8}{3}, \frac{14}{5} \right), \quad \text{if } n = 3 \right\}.$$

Case 2: $(2/q, 1 - 2/q) \in \overline{P_0 P_6}$. Then $b_1 = \frac{n(4-q)}{2q}$. Therefore

$$b_1 < 1 \Leftrightarrow q > \frac{4n}{n+2}, \quad (2.11)$$

and

$$b_1 + b_0 < 1 \Leftrightarrow q > n. \quad (2.12)$$

As $b_1 + b_0 < 1$ implies $b_1 < 1$, then from (2.11) and (2.12) $n > \frac{4n}{n+2}$ and thus $n \geq 3$. Therefore, combining with (2.8), we have that in Case 2, there is not admissible values for q .

Lemma 2.8 *Let $\lambda, \sigma \geq 1$, $\rho \geq 2$, and $\frac{2\rho(n+1)}{n+3} < q < 2\rho$ with $q \neq \frac{2\rho(n+2)}{(n+4)}$ be such that $\lambda\rho - \sigma - 1 = q(\lambda - 1)$, and $\max\{b_0, b_1, \rho a_0, \lambda a_1 + \sigma a_0\} < 1$. There exists a constant $C_4 > 0$ such that*

$$\begin{aligned} & \|\mathcal{N}(u, v) - \mathcal{N}(\tilde{u}, \tilde{v})\|_{E_{a_0, a_1}^\infty} \\ & \leq C_4 \sup_{-\infty < t < \infty} |t|^{a_0} \|u - \tilde{u}\|_{(q, \infty)}^{\lambda a_1} \sup_{-\infty < t < \infty} |t|^{\lambda a_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \\ & \quad \cdot \sup_{-\infty < t < \infty} |t|^{(\sigma-1)a_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \\ & + C_4 \sup_{-\infty < t < \infty} |t|^{a_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{-\infty < t < \infty} |t|^{\sigma a_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \\ & \quad \cdot \sup_{-\infty < t < \infty} |t|^{(\lambda-1)a_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \\ & + C_4 \sup_{-\infty < t < \infty} |t|^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{-\infty < t < \infty} |t|^{(\rho-1)a_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right), \end{aligned} \quad (2.13)$$

for all $u, v, \tilde{u}, \tilde{v}$ such that the right-hand side of (2.13) is finite.

Proof Again, we can consider $t > 0$, without loss of generality. Using Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
\|\mathcal{F}(u, v) - \mathcal{F}(\tilde{u}, \tilde{v})\|_{(q, \infty)} &\leq \int_0^t (t-s)^{-\frac{n(q-2)}{2q}} \|u|u|^{\sigma-1}v|v|^{\lambda-1} - \tilde{u}|\tilde{u}|^{\sigma-1}\tilde{v}|\tilde{v}|^{\lambda-1}\|_{(q', \infty)} \, ds \\
&\leq C \int_0^t (t-s)^{-b_0} \|u - \tilde{u}\|_{(q, \infty)} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \, ds \\
&\quad + C \int_0^t (t-s)^{-b_0} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \|\tilde{u}\|_{(q, \infty)}^\sigma \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \, ds \\
&\leq C \sup_{t>0} t^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{t>0} t^{\lambda a_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \sup_{t>0} t^{(\sigma-1)a_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \\
&\quad \cdot \int_0^t (t-s)^{-b_0} s^{-\lambda a_1 - \sigma a_0} \, ds \\
&\quad + C \sup_{t>0} t^{a_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{t>0} t^{\sigma a_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \sup_{t>0} t^{(\lambda-1)a_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \\
&\quad \cdot \int_0^t (t-s)^{-b_0} s^{-\lambda a_1 - \sigma a_0} \, ds \\
&= C \sup_{t>0} t^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{t>0} t^{\lambda a_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \sup_{t>0} t^{(\sigma-1)a_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \\
&\quad \cdot t^{-a_0} t^{1-b_0-\lambda a_1-(\sigma-1)a_0} \\
&\quad + C \sup_{t>0} t^{a_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{t>0} t^{\sigma a_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \sup_{t>0} t^{(\lambda-1)a_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right] \\
&\quad \cdot t^{-a_0} t^{1-b_0-\lambda a_1-(\sigma-1)a_0}.
\end{aligned}$$

Since $1 - b_0 - \lambda a_1 - (\sigma - 1)a_0 = 0$, we conclude that

$$\begin{aligned}
&\sup_{t>0} t^{a_0} \|\mathcal{F}(u, v) - \mathcal{F}(\tilde{u}, \tilde{v})\|_{(q, \infty)} \\
&\leq C \sup_{t>0} t^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{t>0} t^{\lambda a_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^\lambda \sup_{t>0} t^{(\sigma-1)a_0} \left[\|u\|_{(q, \infty)}^{\sigma-1} + \|\tilde{u}\|_{(q, \infty)}^{\sigma-1} \right] \\
&\quad + C \sup_{t>0} t^{a_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{t>0} t^{\sigma a_0} \|\tilde{u}\|_{(q, \infty)}^\sigma \\
&\quad \cdot \sup_{t>0} t^{(\lambda-1)a_1} \left[\|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} + \|\tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \right]. \tag{2.14}
\end{aligned}$$

Now, from Lemma 2.4, Lemma 2.5 and Hölder inequality we also obtain

$$\begin{aligned}
\|\mathcal{G}(u) - \mathcal{G}(\tilde{u})\|_{\left(\frac{q}{q-\rho}, \infty\right)} &\leq C \int_0^t (t-s)^{-\zeta(\frac{q}{\rho})} \|u - \tilde{u}\|_{(q, \infty)} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right) \, ds \\
&\leq C \int_0^t (t-s)^{-\zeta(\frac{q}{\rho})} s^{-\rho a_0} \, ds \sup_{t>0} t^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{t>0} t^{(\rho-1)a_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right) \\
&\leq C t^{-a_1} t^{1-\zeta(\frac{q}{\rho})-\rho a_0+a_1} \sup_{t>0} t^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{t>0} t^{(\rho-1)a_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right).
\end{aligned}$$

Since $1 - \zeta(\frac{q}{\rho}) - \rho a_0 + a_1 = 0$, we arrive at

$$\begin{aligned} & \sup_{t>0} t^{a_1} \|\mathcal{G}(u) - \mathcal{G}(\tilde{u})\|_{(\frac{q}{q-\rho}, \infty)} \\ & \leq C \sup_{t>0} t^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \sup_{t>0} t^{(\rho-1)a_0} \left(\|u\|_{(q, \infty)}^{\rho-1} + \|\tilde{u}\|_{(q, \infty)}^{\rho-1} \right). \end{aligned} \quad (2.15)$$

The estimate (2.13) follows from (2.14) and (2.15). $\square$

Remark 2.9 Let us consider again the classical Yukawa-KGS system, that is, take $\sigma = \lambda = 1$ and $\rho = 2$. Using (1.3) we have that

$$q \neq \frac{4(n+2)}{(n+4)}, \quad q \in \left(\frac{4(n+1)}{(n+3)}, 4 \right), \quad \text{and} \quad (2.16)$$

$$b_0 < 1 \Rightarrow q \in \left\{ \left(\frac{4(n+1)}{(n+3)}, \frac{4(n+2)}{(n+4)} \right) \cup \left(\frac{4(n+2)}{(n+4)}, 4 \right), \quad \text{if } n = 1, 2, 3, 4 \right\}.$$

$$\left(\frac{4(n+1)}{(n+3)}, \frac{4(n+2)}{(n+4)} \right) \cup \left(\frac{4(n+2)}{(n+4)}, \frac{2n}{n-2} \right), \quad \text{if } n = 5 \quad (2.17)$$

$$2a_0 < 1 \Leftrightarrow b_0 + b_1 > 1,$$

and

$$a_0 + a_1 < 1 \Leftrightarrow 3b_0 + b_1 > 2. \quad (2.18)$$

Now we have two cases.

Case 1: $(2/q, 1 - 2/q) \in \overline{P_6 P_1}$. Then $b_1 = 2 - 4/q$ and therefore, from (2.17) we obtain

$$2a_0 < 1 \Leftrightarrow q > \frac{2(n+4)}{n+2}, \quad (2.19)$$

$$b_1 < 1 \Leftrightarrow q < 4. \quad (2.20)$$

It follows from (2.18) that

$$a_0 + a_1 < 1 \Leftrightarrow q > \frac{2(3n+4)}{3n}. \quad (2.21)$$

Now, combining (2.16), (2.19), (2.20) and (2.21), we have that

$$q \in \begin{cases} \left(\frac{10}{3}, 4 \right), & \text{if } n = 2 \\ \left(\frac{26}{9}, 4 \right), & \text{if } n = 3 \\ \left(\frac{8}{3}, 3 \right) \cup (3, 4), & \text{if } n = 4 \\ \left(3, \frac{28}{9} \right) \cup \left(\frac{28}{9}, \frac{10}{3} \right), & \text{if } n = 5 \end{cases}.$$

Case 2: $(2/q, 1 - 2/q) \in \overline{P_0 P_6}$. Then $b_1 = \frac{n(4-q)}{2q}$. Therefore

$$2a_0 < 1 \Leftrightarrow q < n, \quad (2.22)$$

$$b_1 < 1 \Leftrightarrow q > \frac{4n}{n+2}. \quad (2.23)$$

and

$$a_0 + a_1 < 1 \Leftrightarrow 3b_0 + b_1 > 2 \Rightarrow q > \frac{n-2}{n}. \quad (2.24)$$

Combining (2.16), (2.22) and (2.23), we have that

$$q \in \begin{cases} \left(\frac{8}{3}, \frac{20}{7} \right) \cup \left(\frac{20}{7}, 3 \right), & \text{if } n = 3 \\ \left(\frac{20}{7}, 3 \right) \cup (3, 4), & \text{if } n = 4 \\ \left(3, \frac{28}{9} \right) \cup \left(\frac{28}{9}, \frac{10}{3} \right), & \text{if } n = 5 \end{cases}.$$

3 Local solutions

3.1 Proof of Theorem 1.1

The proof is based on the Banach fixed point theorem. From the assumption on $[u_0, v_0, v_1]$, we have

$$\begin{aligned} \| [e^{it\Delta} u_0, \partial_t W(t) v_0 + W(t) v_1] \|_{E_{b_0, b_1}^T} &= \sup_{-T < t < T} |t|^{b_0} \| e^{it\Delta} u_0 \|_{(q, \infty)} \\ &\quad + \sup_{-T < t < T} |t|^{b_1} \| \partial_t W(t) v_0 + W(t) v_1 \|_{(\frac{q}{q-\rho}, \infty)} \\ &= \frac{\epsilon}{2} < \infty. \end{aligned}$$

Note that ϵ can be large. Recall the nonlinear operator $\mathcal{N}$ defined in (2.3). We wish to show that the map

$$\Upsilon([u, v]) = [e^{it\Delta} u_0, \partial_t W(t) v_0 + W(t) v_1] + \mathcal{N}(u, v) \quad (3.1)$$

is a contraction on (B_ϵ, d) , where B_ϵ is the closed ball $\left\{ [u, v] \in E_{b_0, b_1}^T : \|[u, v]\|_{E_{b_0, b_1}^T} \leq \epsilon \right\}$ endowed with the complete metric $\mathcal{M}(\cdot, \cdot)$ defined by

$$\mathcal{M}([u, v], [\tilde{u}, \tilde{v}]) = \| [u - \tilde{u}, v - \tilde{v}] \|_{E_{b_0, b_1}^T}.$$

Now take T such that

$$C_3 \left(\epsilon^{\lambda+\sigma-1} T^{1-\sigma b_0-\lambda b_1} + \epsilon^{\rho-1} T^{1-\rho b_0} \right) < \frac{1}{2}, \quad (3.2)$$

where C_3 is as in Lemma 2.6. From Lemma 2.6 with $[\tilde{u}, \tilde{v}] = [0, 0]$, we obtain

$$\begin{aligned} \|\Upsilon([u, v])\|_{E_{b_0, b_1}^T} &\leq \| [e^{it\Delta} u_0, \partial_t W(t) v_0 + W(t) v_1] \|_{E_{b_0, b_1}^T} + \|\mathcal{N}(u, v)\|_{E_{b_0, b_1}^T} \\ &\leq \frac{\epsilon}{2} + C_3 \left(\epsilon^{\lambda+\sigma} T^{1-\sigma b_0-\lambda b_1} + \epsilon^{\rho} T^{1-\rho b_0} \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for all $[u, v] \in B_\epsilon$ and consequently $\Upsilon(B_\epsilon) \subset B_\epsilon$. Using again Lemma 2.6, for all $[u, v], [\tilde{u}, \tilde{v}] \in B_\epsilon$, we also obtain

$$\begin{aligned} \|\Upsilon([u, v]) - \Upsilon([\tilde{u}, \tilde{v}])\|_{E_{b_0, b_1}^T} &= \|\mathcal{N}(u, v) - \mathcal{N}(\tilde{u}, \tilde{v})\|_{E_{b_0, b_1}^T} \\ &\leq [2C_3 \epsilon^{\sigma+\lambda-1} T^{1-\sigma b_0-\lambda b_1} + 2C_3 \epsilon^{\rho-1} T^{1-\rho b_0}] \|[u, v] - [\tilde{u}, \tilde{v}]\|_{E_{b_0, b_1}^T}. \end{aligned} \quad (3.3)$$

From (3.2) and (3.3), we get that the map Υ is a contraction on B_ϵ and then the Banach fixed point theorem assures the existence of a unique solution $[u, v] \in E_{b_0, b_1}^T$ for (1.2).

Now, assume that $[v, v], [\tilde{u}, \tilde{v}] \in B_\epsilon$ are two mild solutions with respective initial data $[u_0, v_0, v_1], [\tilde{u}_0, \tilde{v}_0, \tilde{v}_1]$. Let us denote

$$\|[u_0, v_0, v_1]\|_{\mathcal{I}_{0,T}} = \sup_{-T < t < T} |t|^{b_0} \| e^{it\Delta} u_0 \|_{(q, \infty)} + \sup_{-T < t < T} |t|^{b_1} \| \partial_t W(t) v_0 + W(t) v_1 \|_{(\frac{q}{q-\rho}, \infty)}.$$

Working as in (3.3), we arrive at

$$\begin{aligned}
 \| [u, v] - [\tilde{u}, \tilde{v}] \|_{E_{b_0, b_1}^T} &= \| [e^{it\Delta}(u_0 - \tilde{u}_0), \partial_t W(t)(v_0 - \tilde{v}_0) + W(t)(v_1 - \tilde{v}_1)] \|_{E_{b_0, b_1}^T} \\
 &\quad + \| \mathcal{N}(u, v) - \mathcal{N}(\tilde{u}, \tilde{v}) \|_{E_{b_0, b_1}^T} \\
 &\leq \| [(u_0 - \tilde{u}_0), (v_0 - \tilde{v}_0), (v_1 - \tilde{v}_1)] \|_{\mathcal{I}_{0, T}} \\
 &\quad + 2C_3 \left(\epsilon^{\sigma+\lambda-1} T^{1-\sigma b_0-\lambda b_1} + \epsilon^{\rho-1} T^{1-\rho b_0} \right) \| [u, v] - [\tilde{u}, \tilde{v}] \|_{E_{b_0, b_1}^T}.
 \end{aligned} \tag{3.4}$$

Since $2C_3 (\epsilon^{\sigma+\lambda-1} T^{1-\sigma b_0-\lambda b_1} + \epsilon^{\rho-1} T^{1-\rho b_0}) < 1$, from (3.4), we obtain the local Lipschitz continuity of the data-solution map. $\square$

3.2 Comparisons with H^s -results

In this subsection, we show that some local existence results for the KGS system with data in Sobolev spaces (when $\sigma = \lambda = 1, \rho = 2$) can be recovered (in a certain sense) from Theorem 1.1. Let $s, r > 0$ and $q > 2$ be such that $q \leq \frac{2n}{n-2s}$ if $n > 2s$ ($q \leq \infty$ if $n < 2s$), and $2 < \frac{q}{q-2} \leq \frac{2n}{n-2r}$ if $n > 2r$ ($2 < \frac{q}{q-2} < \infty$ if $n < 2r$). Then we have the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{(q, \infty)}$ and $H^r(\mathbb{R}^n) \hookrightarrow L^{(\frac{q}{q-2}, \infty)}$. Therefore,

$$\begin{aligned}
 &\sup_{-T < t < T} |t|^{b_0} \| e^{it\Delta} u_0 \|_{(q, \infty)} + \sup_{-T < t < T} |t|^{b_1} \| \partial_t W(t) v_0 + W(t) v_1 \|_{(\frac{q}{q-2}, \infty)} \\
 &\leq C \sup_{-T < t < T} |t|^{b_0} \| e^{it\Delta} u_0 \|_{H^s} + C \sup_{-T < t < T} |t|^{b_1} \| \partial_t W(t) v_0 + W(t) v_1 \|_{H^r} \\
 &\leq C T^{b_0} \| u_0 \|_{H^s} + C T^{b_1} (\| v_0 \|_{H^r} + \| v_1 \|_{H^{r-1}}) \\
 &\leq C \max\{T^{b_0}, T^{b_1}\} \| [u_0, v_0, v_1] \|_{H^s \times H^r \times H^{r-1}}.
 \end{aligned}$$

Thus, for $[u_0, v_0, v_1] \in H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n) \times H^{r-1}(\mathbb{R}^n)$ we arrived at

$$\sup_{-T < t < T} |t|^{b_0} \| e^{it\Delta} u_0 \|_{(q, \infty)} + \sup_{-T < t < T} |t|^{b_1} \| \partial_t W(t) v_0 + W(t) v_1 \|_{(\frac{q}{q-2}, \infty)} < \infty.$$

Hence, Theorem 1.1 gives us a solution $[u(t), v(t)]$ on $(-T, T)$ (with small $T > 0$) satisfying

$$\sup_{-T < t < T} |t|^{b_0} \| u(t) \|_{(q, \infty)} + \sup_{-T < t < T} |t|^{b_1} \| v(t) \|_{(\frac{q}{q-2}, \infty)} < \infty.$$

Now, we consider finite-energy solutions for (1.1) on $\mathbb{R}^n$, $n = 2, 3$. For this, let $r - 2 < s < r + 1$, $r - 2s < 3/2$ and $[u_0, v_0, v_1] \in H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n) \times H^{r-1}(\mathbb{R}^n)$. Then, from [34–36], there exists a unique mild solution $[\tilde{u}, \tilde{v}] \in C([-T_0, T_0]; H^s \times H^r)$ for the KGS system, with small $T_0 > 0$. Since $H^s \times H^r \hookrightarrow L^{(q, \infty)} \times L^{(\frac{q}{q-2}, \infty)}$, we get that $[\tilde{u}, \tilde{v}] \in E_{b_0, b_1}^{T_0}$ and $\| [\tilde{u}, \tilde{v}] \|_{E_{b_0, b_1}^{T_0}} \rightarrow 0$ as $T_0 \rightarrow 0^+$. Taking T_0 small enough, the uniqueness of solutions in $E_{b_0, b_1}^{T_0}$ given by Theorem 1.1 implies that the mild solution $[u, v]$ coincides with $[\tilde{u}, \tilde{v}]$ on $[-T_0, T_0]$ and therefore $[u, v] \in C([-T_0, T_0]; H^s \times H^r)$.

3.3 Smooth data-solution mapping

The proof of Theorem 1.1 is based on the contraction principle. This method has the advantage that if the nonlinearity is smooth enough (e.g., C^∞), then the data-solution mapping is also smooth. This general fact can be used to prove that, under certain conditions, it is not possible

to implement a fixed point argument to obtain local solutions for an equation or system (see for instance Molinet, Saut and Tzvetkov [31, 32]). Here we prove that, in our case, this consequence of the fixed point argument is valid. We prove the following corollary.

Corollary 3.1 *Under the hypotheses of Theorem 1.1, suppose that $R(u, v) = u|u|^{\sigma-1}v|v|^{\lambda-1}$ and $K(u) = |u|^\rho$ are smooth (i.e., consider σ, λ odd positive integers and ρ an even positive integer). Then, there exists a neighborhood V of $[u_0, v_0, v_1]$ such the data-solution mapping $V \rightarrow B_\epsilon = \{[u, v] \in E_{b_0, b_1}^T : \|[u, v]\|_{E_{b_0, b_1}^T} \leq \epsilon\}$ is smooth.*

Proof Define $\mathbb{M} : V \times B_\epsilon \rightarrow B_\epsilon$ as

$$\begin{aligned} \mathbb{M}([a_0, b_0, c_0, f(t), g(t)]) &= [f(t), g(t)] - \Upsilon_{[a_0, b_0, c_0]}([f(t), g(t)]) \\ &= \left[f(t) - e^{it\Delta}a_0 - \mathcal{F}([f(t), g(t)]), g(t) - \partial_t W(t)b_0 - W(t)c_0 - \mathcal{G}(f(t)) \right], \end{aligned}$$

where $\mathcal{F}$ and $\mathcal{G}$ are defined in (2.3). Hence, $\mathbb{M}$ is smooth and $\mathbb{M}([u_0, v_0, v_1, u(t), v(t)]) = [0, 0]$. Additionally,

$$\begin{aligned} D_{u,v}\mathbb{M}([u_0, v_0, v_1, u(t), v(t)])([f(t), g(t)]) &= \\ &= \left[f(t) + \int_0^t e^{i(t-s)\Delta} [\partial_f R(u, v)f + \partial_{\bar{f}} R(u, v)\bar{f} + \partial_g R(u, v)g] ds, g(t) \right. \\ &\quad \left. - \int_0^t W(t-s) [\partial_f K(u)f + \partial_{\bar{f}} K(u)\bar{f}] ds \right]. \end{aligned}$$

Following the proof of Proposition 2.6, we get

$$\begin{aligned} &\sup_{-T < t < T} |t|^{b_0} \left\| \int_0^t e^{i(t-s)\Delta} [\partial_f R(u, v)f + \partial_{\bar{f}} R(u, v)\bar{f} + \partial_g R(u, v)g] ds \right\|_{(q, \infty)} \\ &\leq CT^{1-\lambda b_1 - \sigma b_0} \sup_{-T < t < T} |t|^{b_0} \|f\|_{(q, \infty)} \sup_{-T < t < T} |t|^{\lambda b_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{-T < t < T} |t|^{(\sigma-1)b_0} \|u\|_{(q, \infty)}^{\sigma-1} \\ &\quad + CT^{1-\lambda b_1 - \sigma b_0} \sup_{-T < t < T} |t|^{b_1} \|g\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \sup_{-T < t < T} |t|^{(\lambda-1)b_1} \|v\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)}^{\lambda-1} \sup_{-T < t < T} |t|^{\sigma b_0} \|u\|_{(q, \infty)}^\sigma \end{aligned}$$

and

$$\begin{aligned} &\sup_{-T < t < T} |t|^{b_1} \left\| \int_0^t W(t-s) [\partial_f K(u)f + \partial_{\bar{f}} K(u)\bar{f}] ds \right\|_{\left(\frac{q}{q-\rho}, \infty\right)} \\ &\leq CT^{1-\rho b_0} \sup_{-T < t < T} |t|^{b_0} \|f\|_{(q, \infty)} \sup_{-T < t < T} |t|^{(\rho-1)b_0} \|u\|_{(q, \infty)}^{\rho-1}. \end{aligned}$$

Using the last two inequalities and the ideas established on Theorem 1.1, we arrive at

$$\begin{aligned} &\|D_{u,v}\mathbb{M}([u_0, v_0, v_1, u(t), v(t)])([f, g])\|_{E_{b_0, b_1}^T} \\ &\leq \left(1 + \epsilon^{\lambda+\sigma-1} T^{1-\sigma b_0 - \lambda b_1} + \epsilon^{\rho-1} T^{1-\rho b_0}\right) \|[f, g]\|_{E_{b_0, b_1}^T} \end{aligned}$$

and

$$\begin{aligned} &\|[f, g]\|_{E_{b_0, b_1}^T} \left(1 - \epsilon^{\lambda+\sigma-1} T^{1-\sigma b_0 - \lambda b_1} - \epsilon^{\rho-1} T^{1-\rho b_0}\right) \\ &\leq \|D_{u,v}\mathbb{M}([u_0, v_0, v_1, u(t), v(t)])([f, g])\|_{E_{b_0, b_1}^T}. \end{aligned}$$

Therefore $D_{u,v}\mathbb{M}([u_0, v_0, v_1, u(t), v(t)]) : B_\epsilon \rightarrow B_\epsilon$ is one to one and onto. Then by the implicit function theorem, there exist a neighborhood V of $[u_0, v_0, v_1]$ and a smooth function $m : V \rightarrow B_\epsilon$ such that

$$\mathbb{M}(a_0, b_0, c_0, m(a_0, b_0, c_0)) = [0, 0], \quad \text{for all } [a_0, b_0, c_0] \in V.$$

Hence, if $m(x, y, z) = [m_1(x, y, z), m_2(x, y, z)]$, we have that

$$m(a_0, b_0, c_0) = [e^{it\Delta}a_0 + \mathcal{F}([m(a_0, b_0, c_0)]), \partial_t W(t)b_0 + W(t)c_0 + \mathcal{G}(m_1(a_0, b_0, c_0))]$$

is a solution of (1.2) with data $[a_0, b_0, c_0]$ (instead of $[u_0, v_0, v_1]$) and this finishes the proof of the corollary. $\square$

4 Global solutions

4.1 Proof of Theorem 1.3

Let $\delta > 0$ be such that $2C_4(\delta^{\sigma+\lambda-1} + \delta^{\rho-1}) < 1$, with C_4 the constant given in Lemma 2.8. Define the ball $B_\delta = \{[u, v] \in E_{a_0, a_1}^\infty : \|[u, v]\|_{E_{a_0, a_1}^\infty} \leq \delta\}$ and consider the map Υ defined in (3.1). From (1.8) and Lemma 2.8 with $\tilde{u}, \tilde{v} = 0$, we obtain (for all $[u, v] \in B_\delta$)

$$\begin{aligned} \|\Psi([u, v])\|_{E_{a_0, a_1}^\infty} &\leq \| [e^{it\Delta}u_0, \partial_t W(t)v_0 + W(t)v_1] \|_{E_{a_0, a_1}^\infty} + \|\mathcal{N}(u, v)\|_{E_{a_0, a_1}^\infty} \\ &\leq \|[u_0, v_0, v_1]\|_{I_{0, \infty}} + C_4(\|[u, v]\|_{E_{a_0, a_1}^\infty}^\sigma \|[u, v]\|_{E_{a_0, a_1}^\infty}^\lambda + \|[u, v]\|_{E_{a_0, a_1}^\infty}^\rho) \\ &\leq \frac{\delta}{2} + C_4(\delta^{\sigma+\lambda} + \delta^\rho) \leq \delta. \end{aligned}$$

Consequently $\Phi(B_\delta) \subset B_\delta$. Taking $[u, v], [\tilde{u}, \tilde{v}] \in B_\delta$, we get from Lemma 2.8 that

$$\begin{aligned} \|\Upsilon([u, v]) - \Upsilon([\tilde{u}, \tilde{v}])\|_{E_{a_0, a_1}^\infty} &= \|\mathcal{N}(u, v) - \mathcal{N}(\tilde{u}, \tilde{v})\|_{E_{a_0, a_1}^\infty} \\ &\leq 2C_4\delta^\lambda\delta^{\sigma-1} \sup_{-\infty < t < \infty} |t|^{a_0} \|u - \tilde{u}\|_{(q, \infty)} + 2C_4\delta^\sigma\delta^{\lambda-1} \sup_{-\infty < t < \infty} |t|^{\alpha_1} \|v - \tilde{v}\|_{\left(\frac{\lambda q}{q-\sigma-1}, \infty\right)} \\ &\quad + 2C_4\delta^{\rho-1} \sup_{-\infty < t < \infty} |t|^{a_0} \|u - \tilde{u}\|_{(q, \infty)} \leq 2C_4(\delta^{\sigma+\lambda-1} + \delta^{\rho-1}) \|[u, v] - [\tilde{u}, \tilde{v}]\|_{E_{a_0, a_1}^\infty}. \end{aligned}$$

Since $2C_4(\delta^{\sigma+\lambda-1} + \delta^{\rho-1}) < 1$, the map Υ is a contraction on B_δ . Thus, we have a unique fixed point in B_δ , which is the unique solution $[u, v]$ of the integral Eq. (1.2) satisfying $\|[u, v]\|_{E_{a_0, a_1}^\infty} \leq \delta$.

Now, we prove the local Lipschitz continuity of the data-solution map. For that, consider $[u, v], [\tilde{u}, \tilde{v}] \in B_\delta$ satisfying (1.2) with respective data $[u_0, v_0, v_1], [\tilde{u}_0, \tilde{v}_0, \tilde{v}_1] \in I_{0, \infty}$ (see (1.8)). Since $2C_4(\delta^{\sigma+\lambda-1} + \delta^{\rho-1}) < 1$, we get

$$\begin{aligned} \|[u, v] - [\tilde{u}, \tilde{v}]\|_{E_{a_0, a_1}^\infty} &= \| [e^{it\Delta}(u_0 - \tilde{u}_0), \partial_t W(t)(v_0 - \tilde{v}_0) + W(t)(v_1 - \tilde{v}_1)] \|_{E_{a_0, a_1}^\infty} \\ &\quad + \|\mathcal{N}(u, v) - \mathcal{N}(\tilde{u}, \tilde{v})\|_{E_{a_0, a_1}^\infty} \\ &\leq \|u_0 - \tilde{u}_0, v_0 - \tilde{v}_0, v_1 - \tilde{v}_1\|_{I_{0, \infty}} \\ &\quad + 2C_4(\delta^{\sigma+\lambda-1} + \delta^{\rho-1}) \|[u, v] - [\tilde{u}, \tilde{v}]\|_{E_{a_0, a_1}^\infty}, \end{aligned}$$

which gives the desired continuity. $\square$

5 Scattering theory

5.1 Scattering: Proof of Theorem 1.5.

We only analyze the case $t \rightarrow \infty$. The case $t \rightarrow -\infty$ can be proved in an analogous way. Consider $[u_0^+, v_0^+, v_1^+] \in [S'(\mathbb{R}^n)]^3$ defined by

$$[u_0^+, v_0^+, v_1^+] = [u_0, v_0, v_1] + \int_0^\infty \left[i e^{i(-s)\Delta} u(s) |u(s)|^{\sigma-1} v(s) |v(s)|^{\lambda-1}, 0, W(-s) |u(s)|^{\rho-1} \right] ds,$$

where $[u, v]$ is the solution obtained in Theorem 1.3. Let $[u^+, v^+]$ be the solution of the linear problem associated with (1.1) with data $[u_0^+, v_0^+, v_1^+]$, that is,

$$[u^+, v^+] = [e^{it\Delta} u_0^+, \partial_t W(t) v_0^+ + W(t) v_1^+].$$

Notice that $[u^+, v^+]$ can be expressed as

$$\begin{cases} u^+(t) = e^{it\Delta} u_0 + i \int_0^\infty e^{i(t-s)\Delta} u(s) |u(s)|^{\sigma-1} v(s) |v(s)|^{\lambda-1} ds, \\ v^+(t) = \partial_t W(t) v_0 + W(t) v_1 + \int_0^\infty W(t-s) |u(s)|^{\rho-1} ds. \end{cases} \quad (5.1)$$

From Remark 1.4, we have that $[u, v]$ satisfies (1.9). Taking the difference between (1.2) and (5.1), and computing the $X_{(q,\infty)}$ -norm, we have

$$\begin{aligned} & \| [u - u^+, v - v^+] \|_{X_{(q,\infty)}} \\ &= \left\| \int_t^\infty [e^{i(t-s)\Delta} u(s) |u(s)|^{\sigma-1} v(s) |v(s)|^{\lambda-1}, W(t-s) |u(s)|^{\rho-1}] ds \right\|_{X_{(q,\infty)}} \\ &\leq \left\| \int_t^\infty e^{i(t-s)\Delta} u(s) |u(s)|^{\sigma-1} v(s) |v(s)|^{\lambda-1} ds \right\|_{(q,\infty)} \\ &\quad + \left\| \int_t^\infty W(t-s) |u(s)|^{\rho-1} ds \right\|_{\left(\frac{q}{q-\rho}, \infty\right)} \equiv J_1 + J_2. \end{aligned} \quad (5.2)$$

Working as in the proof of Lemma 2.8, we can bound J_1 and J_2 as follows.

$$\begin{aligned} J_1 &\leq C \int_t^\infty (s-t)^{-b_0} s^{-\lambda a_1 - \sigma a_0 - h(\lambda+\sigma)} ds \sup_{t>0} t^{\lambda(a_1+h)} \|v\|_{\left(\frac{q}{q-\rho}, \infty\right)}^\lambda \sup_{t>0} t^{\sigma(a_0+h)} \|u\|_{(q,\infty)}^\sigma \\ &\leq C t^{-a_0-h(\lambda+\sigma)} \int_1^\infty (s-1)^{-b_0} s^{-\lambda a_1 - \sigma a_0 - h(\lambda+\sigma)} ds \leq C t^{-a_0-h(\lambda+\sigma)}, \end{aligned} \quad (5.3)$$

because $1 - \lambda a_1 - \sigma a_0 - b_0 = -a_0$ and

$$\begin{aligned} & \int_1^\infty (s-1)^{-b_0} s^{-\lambda a_1 - \sigma a_0 - h(\lambda+\sigma)} ds \leq \int_1^2 (s-1)^{-b_0} ds \\ & \quad + \int_2^\infty (s-1)^{-b_0} s^{-\lambda a_1 - \sigma a_0 - h(\lambda+\sigma)} ds < \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} J_2 &\leq C \left(\sup_{t>0} t^{a_0+h} \|u\|_{(q,\infty)} \right)^\rho \int_t^\infty (s-t)^{-\zeta(\frac{q}{\rho})} s^{-\rho a_0-\rho h} ds \\ &\leq C \int_t^\infty (s-t)^{-\zeta(\frac{q}{\rho})} s^{-\rho a_0-\rho h} ds \leq C t^{-a_1} \int_1^\infty (s-1)^{-\zeta(\frac{q}{\rho})} s^{-\rho a_0-\rho h} ds \leq C t^{-a_1-\rho h}, \end{aligned} \quad (5.4)$$

in view of $1 - \zeta(\frac{q}{\rho}) - \rho a_0 + a_1 = 0$ and

$$\int_1^\infty (s-1)^{-\zeta(\frac{q}{\rho})} s^{-\rho a_0-\rho h} ds < \infty.$$

From (5.2), (5.3) and (5.4), we obtain (1.10) and (1.11) as $t \rightarrow \infty$.

We finish the proof by observing that, in view of $[u, v] \in E_{a_0, a_1}^\infty$ and $[u_0, v_0, v_1] \in I_{0,\infty}$ (see (1.7)), one can proceed as in (5.3) and (5.4) in order to show $[u_0^+, v_0^+, v_1^+] \in I_{0,\infty}$. $\square$

5.2 Wave operator: Proof of Theorem 1.6.

The aim of this section is to prove Theorem 1.6. The proof is divided into four steps.

First step: Construction of the solution.

For $T > 0$ we consider the set $E_{T, \theta_0, \theta_1}$ of all Bochner measurable pairs $[w, \omega] : [T, \infty) \rightarrow X_{(q,\infty)}$ such that

$$\|[w, \omega]\|_{E_{T, \theta_0, \theta_1}} \equiv \sup_{t>T} t^{\theta_0} \|w\|_{(q,\infty)} + \sup_{t>T} t^{\theta_1} \|\omega\|_{(\frac{q}{q-\rho}, \infty)} < \infty.$$

Let $R > 0$ and denote by

$$K_T(0, R) = \left\{ [w, \omega] \in E_{T, \theta_0, \theta_1} : \|[w, \omega]\|_{E_{T, \theta_0, \theta_1}} \leq R \right\}.$$

Define the mapping $\Pi : K_T(0, R) \rightarrow K_T(0, R)$ by

$$\Pi([w, \omega]) = [\Pi_1([w, \omega]), \Pi_2([w, \omega])]$$

with

$$\begin{aligned} \Pi_1([w, \omega]) &= -i \int_t^\infty e^{i(t-s)\Delta} (M_1([w(s), \omega(s)] + [e^{is\Delta} f_1, \partial_t W(s) f_2 + W(s) f_3])) ds, \\ \Pi_2([w, \omega]) &= - \int_t^\infty W(t-s) (M_2([w(s), \omega(s)] + [e^{is\Delta} f_1, \partial_t W(s) f_2 + W(s) f_3])) ds, \end{aligned}$$

and M_1, M_2 denoting

$$M_1([z_1, z_2]) = z_1 |z_1|^{\sigma-1} z_2 |z_2|^{\lambda-1}, \quad M_2([z_1, z_2]) = |z_1|^\rho.$$

We have that if $[w, \omega]$ is a fixed point of the operator $\Pi : K_{T_0}(0, R) \rightarrow K_{T_0}(0, R)$, for some $T_0 > 0$ which will be chosen later, then the pair $[u(t), v(t)]$ defined by

$$[u(t), v(t)] \equiv [e^{it\Delta} f_1 + w(t), \partial_t W(t) f_2 + W(t) f_3 + \omega(t)]$$

satisfies $[u, v] \in E_{T_0, \theta_0, \theta_1}$, $u \in C([T_0, \infty); S'(\mathbb{R}^n))$ and $v \in C^1([T_0, \infty); S'(\mathbb{R}^n))$, and it makes sense to take $u(T_0)$, $v(T_0)$ and $\partial_t v(T_0) = \lim_{t \rightarrow T_0^+} \partial_t v(t)$. Using the group properties of $e^{it\Delta}$ and $W(t)$, one can show that

$$[u, v] = [e^{i(t-T_0)\Delta} u(T_0), \partial_t W(t-T_0)v(T_0) + W(t-T_0)\partial_t v(T_0)] + \mathcal{N}_{T_0}(u, v), \quad (5.5)$$

where $\mathcal{N}_{T_0}(u, v) = [\mathcal{F}_{T_0}(u, v), \mathcal{G}_{T_0}(u)]$ and

$$\mathcal{F}_{T_0}(u, v) = i \int_{T_0}^t e^{i(t-s)\Delta} u(s) |u(s)|^{\sigma-1} v(s) |v(s)|^{\lambda-1} ds \quad \text{and} \quad \mathcal{G}_{T_0}(u) = \int_{T_0}^t W(t-s) |u(s)|^\rho ds.$$

Also, $\mathcal{N}_{T_0}(u, v) \in [S'(\mathbb{R}^n)]^2$ and Eq. (5.5) holds in $[S'(\mathbb{R}^n)]^2$, for a.e. $t > T_0$. In other words, $[u, v]$ is a solution for (1.2) in the time interval $[T_0, \infty)$. In fact, we should take (5.5) as the concept of solution in the interval $[T_0, \infty)$.

Second step: Π is a contraction.

It follows from Lemma 2.1, Hölder and triangle inequalities that

$$\begin{aligned} & \|\Pi_1([w(t), \omega(t)])\|_{(q, \infty)} \\ & \leq C \int_t^\infty |t-s|^{-b_0} \|w + e^{is\Delta} f_1\|_{(q, \infty)}^\sigma \|\omega + \partial_s W(s) f_2 + W(s) f_3\|_{(\frac{q}{q-\rho}, \infty)}^\lambda ds \\ & \leq C \left(\sup_{t>T} t^{\theta_0} \|w + e^{is\Delta} f_1\|_{(q, \infty)} \right)^\sigma \left(\sup_{t>T} t^{\theta_1} \|\omega + \partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} \right)^\lambda \\ & \quad \cdot \int_t^\infty (s-t)^{-b_0} s^{-\sigma\theta_0-\lambda\theta_1} ds \\ & \leq C \left(\sup_{t>T} t^{\theta_0} \|w\|_{(q, \infty)} + \sup_{t>T} t^{\theta_0-b_0} \|f_1\|_{(q', \infty)} \right)^\sigma \\ & \quad \cdot \left(\sup_{t>T} t^{\theta_1} \|\omega\|_{(\frac{q}{q-\rho}, \infty)} + \sup_{t>T} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} \right)^\lambda \\ & \quad \cdot \int_t^\infty (s-t)^{-b_0} s^{-\sigma\theta_0-\lambda\theta_1} ds. \end{aligned} \quad (5.6)$$

Notice that $\theta_0 + 1 - b_0 - \sigma\theta_0 - \lambda\theta_1 < a_0 + 1 - b_0 - \sigma a_0 - \lambda a_1 = 0$, $b_0 < 1$, and $b_0 + \sigma\theta_0 + \lambda\theta_1 > \theta_0 + 1 > 1$. Then the integral in (5.6) can be bounded as follows:

$$\begin{aligned} & t^{\theta_0} \int_t^\infty (s-t)^{-b_0} s^{-\sigma\theta_0-\lambda\theta_1} ds = t^{\theta_0+1-b_0-\sigma\theta_0-\lambda\theta_1} \int_1^\infty (s-1)^{-b_0} s^{-\sigma\theta_0-\lambda\theta_1} ds \\ & \leq C T^{\theta_0+1-b_0-\sigma\theta_0-\lambda\theta_1} \left(\int_1^2 (s-1)^{-b_0} ds + \sup_{s>2} \left(\frac{s}{s-1} \right)^{b_0} \left(\int_2^\infty s^{-b_0-\sigma\theta_0-\lambda\theta_1} ds \right) \right) \\ & \leq C_1(T) \rightarrow 0 \text{ as } T \rightarrow \infty. \end{aligned} \quad (5.7)$$

Therefore, from (5.6)–(5.7) and using that $\sup_{t \geq T} t^{\theta_0 - b_0} \leq 1$ for $T \geq 1$, we get

$$\begin{aligned} \sup_{t > T} t^{\theta_0} \|\Pi_1([w(t), \omega(t)])\|_{(q, \infty)} &\leq C_1(T) (\sup_{t > T} t^{\theta_0} \|w\|_{(q, \infty)} + \|f_1\|_{(q', \infty)})^\sigma \\ &\cdot \left(\sup_{t > T} t^{\theta_1} \|\omega\|_{(\frac{q}{q-\rho}, \infty)} + \sup_{t > T} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} \right)^\lambda. \end{aligned} \quad (5.8)$$

On the other hand, using Lemma 2.4 and $T \geq 1$, we estimate

$$\begin{aligned} \|\Pi_2([w(t), \omega(t)])\|_{(\frac{q}{q-\rho}, \infty)} &\leq C \int_t^\infty |t-s|^{-\zeta(\frac{q}{\rho})} \left(\|w + e^{is\Delta} f_1\|_{(q, \infty)} \right)^\rho ds \\ &\leq C \left(\sup_{t > T} t^{\theta_0} \|w\|_{(q, \infty)} + \sup_{t > T} t^{\theta_0 - b_0} \|f_1\|_{(q', \infty)} \right)^\rho \int_t^\infty (s-t)^{-\zeta(\frac{q}{\rho})} s^{-\rho\theta_0} ds \\ &\leq C \left(\sup_{t > T} t^{\theta_0} \|w\|_{(q, \infty)} + \|f_1\|_{(q', \infty)} \right)^\rho \int_t^\infty (s-t)^{-\zeta(\frac{q}{\rho})} s^{-\rho\theta_0} ds. \end{aligned} \quad (5.9)$$

Since $\theta_1 + 1 - \zeta(\frac{q}{\rho}) - \rho\theta_0 < a_1 + 1 - \zeta(\frac{q}{\rho}) - \rho a_0 = 0$, and $\zeta(\frac{q}{\rho}) + \rho\theta_0 > \theta_1 + 1 > 1$, then

$$\begin{aligned} t^{\theta_1} \int_t^\infty (s-t)^{-\zeta(\frac{q}{\rho})} s^{-\rho\theta_0} ds &\leq t^{\theta_1} \int_t^\infty (s-t)^{-\zeta(\frac{q}{\rho})} s^{-\rho\theta_0} ds \\ &= t^{\theta_1 + 1 - \zeta(\frac{q}{\rho}) - \rho\theta_0} \int_1^\infty (s-1)^{-\zeta(\frac{q}{\rho})} s^{-\rho\theta_0} ds \\ &\leq C T^{\theta_1 + 1 - \zeta(\frac{q}{\rho}) - \rho\theta_0} \left(\int_1^2 (s-1)^{-\zeta(\frac{q}{\rho})} ds + \left(\sup_{s \geq 2} \left(\frac{s}{s-1} \right)^{\zeta(\frac{q}{\rho})} \right) \int_2^\infty s^{-\zeta(\frac{q}{\rho}) - \rho\theta_0} ds \right) \\ &\leq C_2(T) \rightarrow 0 \text{ as } T \rightarrow \infty, \end{aligned} \quad (5.10)$$

which implies that

$$\sup_{t > T} t^{\theta_1} \|\Pi_2([w(t), \omega(t)])\|_{(q, \infty)} \leq C_2(T) (\sup_{t > T} t^{\theta_0} \|w\|_{(q, \infty)} + \|f_1\|_{(q', \infty)})^\rho, \quad (5.11)$$

for $T \geq 1$. Therefore, from (5.8) and (5.11) we obtain

$$\begin{aligned} \|\Pi([w(t), \omega(t)])\|_{E_{T, \theta_0, \theta_1}} &\leq C_1(T) (\sup_{t > T} t^{\theta_0} \|w\|_{(q, \infty)} + \|f_1\|_{(q', \infty)})^\sigma \\ &\cdot \left(\sup_{t > T} t^{\theta_1} \|\omega\|_{(\frac{q}{q-\rho}, \infty)} + \sup_{t > T} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} \right)^\lambda \\ &+ C_2(T) (\sup_{t > T} t^{\theta_0} \|w\|_{(q, \infty)} + \|f_1\|_{(q', \infty)})^\rho \\ &\leq C_1(T) \left(\|[w, \omega]\|_{E_{T, \theta_0, \theta_1}} + \|f_1\|_{(q', \infty)} + \sup_{t > T} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} \right)^{\sigma + \lambda} \\ &+ C_2(T) (\|[w, \omega]\|_{E_{T, \theta_0, \theta_1}} + \|f_1\|_{(q', \infty)})^\rho. \end{aligned} \quad (5.12)$$

Now we estimate the norm $\|\Pi([w_1(t), \omega_1(t)] - \Pi([w_2(t), \omega_2(t)])\|_{E_{T,\theta_1,\theta_1}}$. For that, notice that

$$\begin{aligned}
 & \|\Pi_1([w - \tilde{w}, \omega - \tilde{\omega}])\|_{(q,\infty)} \\
 &= \left\| \int_t^\infty e^{i(t-s)\Delta} (M_1([w + e^{is\Delta} f_1, \omega + \partial_s W(s) f_2 \right. \\
 & \quad \left. + W(s) f_3]) - M_1([\tilde{w} + e^{is\Delta} f_1, \tilde{\omega} + \partial_s W(s) f_2 + W(s) f_3])) ds \right\|_{(q,\infty)} \\
 &\leq C \sup_{t>T} t^{\theta_0} \|w - \tilde{w}\|_{(q,\infty)} \left(\sup_{t>T} t^{\lambda\theta_1} \|\omega + \partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)}^\lambda \right. \\
 & \quad \cdot \sup_{t>T} t^{(\sigma-1)\theta_0} \left[\|w + e^{it\Delta} f_1\|_{(q,\infty)}^{\sigma-1} + \|\tilde{w} + e^{it\Delta} f_1\|_{(q,\infty)}^{\sigma-1} \right] \\
 & \quad \cdot \int_0^t (t-s)^{-b_0} s^{-\lambda\theta_1 - \sigma\theta_0} ds \\
 & \quad + C \sup_{t>T} t^{\theta_1} \|\omega - \tilde{\omega}\|_{(\frac{q}{q-\rho}, \infty)} \sup_{t>T} t^{\sigma\theta_0} \|\tilde{w} + e^{it\Delta} f_1\|_{(q,\infty)}^\sigma \\
 & \quad \cdot \sup_{t>T} t^{(\lambda-1)\theta_1} \left[\|\omega + \partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)}^{\lambda-1} + \|\tilde{\omega} + \partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)}^{\lambda-1} \right] \\
 & \quad \cdot \int_0^t (t-s)^{-b_0} s^{-\lambda\theta_1 - \sigma\theta_0} ds.
 \end{aligned}$$

Thus, using that $[w, \omega], [\tilde{w}, \tilde{\omega}] \in K_T(0, R)$, Lemma 2.1 and (5.7), the last inequality yields

$$\begin{aligned}
 & \sup_{t>T} t^{\theta_0} \|\Pi_1([w - \tilde{w}, \omega - \tilde{\omega}])\|_{(q,\infty)} \leq C \| [w - \tilde{w}, \omega - \tilde{\omega}] \|_{E_{T,\theta_0,\theta_1}} \\
 & \quad \cdot C_1(T) \left[2 \left(R + \|f_1\|_{(q',\infty)} + \sup_{t>T} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho}, \infty)} \right)^{\sigma+\lambda-1} \right] \quad (5.13)
 \end{aligned}$$

On the other hand, using Lemma 2.4 we obtain

$$\begin{aligned}
 & \|\Pi_2([w - \tilde{w}, \omega - \tilde{\omega}])\|_{(q,\infty)} \\
 &= \left\| \int_t^\infty W(t-s) (|w + e^{is\Delta} f_1|^\rho - |\tilde{w} + e^{is\Delta} f_1|^\rho) ds \right\|_{(\frac{q}{q-\rho}, \infty)} \\
 &\leq C \sup_{t>T} t^{\theta_0} \|w - \tilde{w}\|_{(q,\infty)} \cdot \sup_{t>T} t^{(\rho-1)\theta_0} \left[\|w + e^{it\Delta} f_1\|_{(q,\infty)}^{\rho-1} + \|\tilde{w} + e^{it\Delta} f_1\|_{(q,\infty)}^{\rho-1} \right] \\
 & \quad \cdot \int_0^t (t-s)^{-\zeta(\frac{q}{\rho})} s^{-\rho\theta_0} ds.
 \end{aligned}$$

Since $[w, \omega], [\tilde{w}, \tilde{\omega}] \in K_T(0, R)$, using Lemma 2.1 and (5.10), we obtain

$$\begin{aligned}
 & \sup_{t>T} t^{\theta_1} \|\Pi_2([w - \tilde{w}, \omega - \tilde{\omega}])\|_{(\frac{q}{q-\rho}, \infty)} \leq C \| [w - \tilde{w}, \omega - \tilde{\omega}] \|_{E_{T,\theta_0,\theta_1}} \\
 & \quad \cdot C_2(T) [2(R + \|f_1\|_{(q',\infty)})^{\rho-1}]. \quad (5.14)
 \end{aligned}$$

As $C_1(T), C_2(T) \rightarrow 0$ as $T \rightarrow \infty$, from (5.12), (5.13) and (5.14), we can choose $T_0 > 0$ such that Π is a contraction on $K_{T_0}(0, R)$, which implies the existence of a unique fixed point $[w, \omega]$.

Third step: Proof of the convergence.

Let $\eta < T_0$ be fixed and $\tau > 2T_0$. Denote $\tilde{\tau} = \tau - \eta > T_0$. From inequality (5.12), we have

$$\begin{aligned} & \tau^{\theta_0} \|u(\tau) - e^{i\tau\Delta} f_1\|_{(q,\infty)} + \tau^{\theta_1} \|v(\tau) - (\partial_t W(\tau) f_2 + W(\tau) f_3)\|_{(\frac{q}{q-\rho},\infty)} \\ &= \tau^{\theta_0} \|w(\tau)\|_{(q,\infty)} + \tau^{\theta_1} \|\omega(\tau)\|_{(\frac{q}{q-\rho},\infty)} \\ &\leq \sup_{t>\tilde{\tau}} t^{\theta_0} \|w(t)\|_{(q,\infty)} + \sup_{t>\tilde{\tau}} t^{\theta_1} \|\omega(t)\|_{(\frac{q}{q-\rho},\infty)} \\ &= \|[w, \omega]\|_{E_{\tilde{\tau},\theta_0,\theta_1}} = \|\Pi[w, \omega]\|_{E_{\tilde{\tau},\theta_0,\theta_1}} \\ &\leq C_1(\tilde{\tau}) \left(R + \|f_1\|_{(q',\infty)} + \sup_{t>T_0} t^{\theta_1} \|\partial_t W(t) f_2 + W(t) f_3\|_{(\frac{q}{q-\rho},\infty)} \right)^{\sigma+\lambda} \\ &\quad + C_2(\tilde{\tau}) (R + \|f_1\|_{(q',\infty)})^\rho \rightarrow 0 \text{ as } \tau \rightarrow \infty. \end{aligned}$$

Fourth step: Uniqueness. The uniqueness assertion follows from standard arguments (see, e.g., [12]). $\square$

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