Annali di Matematica (2015) 194:109–130
DOI 10.1007/s10231-013-0367-1

Curvature estimates for properly immersed ϕ_h-bounded submanifolds

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Received: 19 November 2012 / Accepted: 21 July 2013 / Published online: 8 August 2013

Abstract Jorge–Koutrofiotis (Am J Math 103:711–725, 1980) and Pigola et al. (Memoirs Am Math Soc 174(822), 2005) proved sharp sectional curvature estimates for extrinsically bounded submanifolds. In Alías et al. (Trans Am Math Soc 364(7):3513–3528, 2012), Alias et al. showed that these estimates hold on properly immersed cylindrically bounded submanifolds. On the other hand, Alias et al. (Math Ann 345(2):367–376, 2009) proved mean curvature estimates for properly immersed cylindrically bounded submanifolds. In this paper, we prove these sectional and mean curvature estimates for a larger class of submanifolds, the properly immersed ϕ-bounded submanifolds, see Theorems 5 and 6. With the ideas developed, we prove stronger forms of these estimates, see the results in Sect. 4.

Keywords Curvature estimates · ϕ-Bounded submanifolds · Omori–Yau pairs · Omori–Yau maximum principle

Mathematics Subject Classification (2000) 53C40 · 53C42 · 35B50

The authors were partially supported by CNPq-Brazil, PROCAD-PICME-CAPES-Brazil and The Abdus Salam International Centre for Theoretical Physics—ICTP, Italy.

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1 Introduction

The classical isometric immersion problem asks whether there exists an isometric immersion $\varphi\colon M \to N$ for given Riemannian manifolds M and N of dimension m and n, respectively, with $m < n$. The model result for this type of problem is the celebrated Efimov–Hilbert Theorem [11,14] that says that there is no isometric immersion of a geodesically complete surface M with sectional curvature $K_M \leq -\delta^2 < 0$ into $\mathbb{R}^3$, $\delta \in \mathbb{R}$. On the other hand, the Nash Embedding Theorem shows that there is always an isometric embedding into the Euclidean n-space $\mathbb{R}^n$ provided the codimension $n - m$ is sufficiently large, see [18]. For small codimension, meaning in this paper that $n - m \leq m - 1$, the answer in general depends on the geometries of M and N. For instance, a classical result of Tompkins [28] states that a compact, flat, m-dimensional Riemannian manifold cannot be isometrically immersed into $\mathbb{R}^{2m-1}$. Tompkin's result was extended in a series of papers, by Chern and Kuiper [9], Moore [17], O'Neill [20], Otsuki [21] and Stiel [26], whose results can be summarized in the following theorem.

Theorem 1 (Tompkins, Chern, Kuiper, Moore, O'Neil, Otsuki, Stiel) *Let $\varphi\colon M \to N$ be an isometric immersion of a compact Riemannian m-manifold M into a Cartan–Hadamard n-manifold N with $n - m \leq m - 1$. Then, the sectional curvatures of M and N satisfy*

$$\sup_M K_M > \inf_N K_N. \tag{1}$$

Jorge and Koutrofiotis [15] considered complete extrinsically bounded[1] submanifolds with scalar curvature bounded from below and proved the curvature estimates (3). Pigola et al. [24] proved an all general and abstract version of the Omori–Yau maximum principle [8,29], and in consequence, they were able to extend Jorge–Koutrofiotis' Theorem to complete m-submanifolds M immersed into regular balls of any Riemannian n-manifold N with scalar curvature bounded below as $s_M \geq -c \cdot \rho_M^2 \cdot \prod_{j=1}^{k} \left(\log^{(j)}(\rho_M)\right)^2$, $\rho_M \gg 1$.

Their version of Jorge–Koutrofiotis Theorem is the following:

Theorem 2 (Jorge–Koutrofiotis, Pigola–Rigoli–Setti) *Let $\varphi\colon M \to N$ be an isometric immersion of a complete Riemannian m-manifold M into a n-manifold N, $n - m \leq m - 1$, with $\varphi(M) \subset B_N(r)$, where $B_N(r)$ is a regular geodesic ball of N. If the scalar curvature of M satisfies*

$$s_M \geq -c \cdot \rho_M^2 \cdot \prod_{j=1}^{k} \left(\log^{(j)}(\rho_M)\right)^2, \ \ \rho_M \gg 1, \tag{2}$$

for some constant $c > 0$ and some integer $k \geq 1$, where ρ_M is the distance function on M to a fixed point and $\log^{(j)}$ is the j-th iterate of the logarithm. Then,

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N, \tag{3}$$

where $b = \sup_{B_N(r)} K_N^{\mathrm{rad}}$ and

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,t) & \text{if } b > 0 \text{ and } \ 0 < t < \pi/2\sqrt{b} \\ 1/t & \text{if } b = 0 \text{ and } \ t > 0 \\ \sqrt{-b}\coth(\sqrt{-b}\,t) & \text{if } b < 0 \text{ and } \ t > 0. \end{cases} \tag{4}$$

[1] Meaning: immersed into regular geodesic balls of a Riemannian manifold.

Remark 1 The curvature estimate (3) is sharp. For instance, if $B_{\mathbb{N}^n(b)}(r) \subset \mathbb{N}^n(b)$ is a geodesic ball of radius r in the simply connected space form of sectional curvature b, $\partial B_{\mathbb{N}^n(b)}(r)$ its boundary and $\varphi\colon \partial B_{\mathbb{N}^n(b)}(r-\epsilon) \to B_{\mathbb{N}^n(b)}(r)$ is the canonical immersion, where $\epsilon > 0$ is small, then we have

$$\sup_M K_M = K_{\partial B_{\mathbb{N}^n(b)}(r-\epsilon)} = \begin{cases} b/\sin^2(\sqrt{b}\,(r-\epsilon)) & \text{if } b > 0 \\ 1/(r-\epsilon)^2 & \text{if } b = 0 \\ -b/\sinh^2(\sqrt{-b}\,(r-\epsilon)) & \text{if } b < 0. \end{cases}$$

Therefore, $\sup_M K_M - [C_b^2(r) + \inf_{\mathbb{N}^n(b)} K_{\mathbb{N}^n(b)}] = [C_b^2(r-\epsilon) - C_b^2(r)] \searrow 0$ as $\epsilon \to 0$ showing that the inequality (3) is sharp.

Remark 2 One may assume without loss of generality that $\sup_M K_M < \infty$. This together with the scalar curvature bounds (2) implies that

$$K_M \geq -c^2 \cdot \rho_M^2 \cdot \prod_{j=1}^{k} \left(\log^{(j)}(\rho_M)\right)^2, \ \rho_M \gg 1$$

for some positive constant $c > 0$. This curvature lower bound implies that M is stochastically complete, which is equivalent to the fact that M satisfies the weak maximum principle—a weaker form of Omori–Yau maximum principle, see details in [23]—and that is enough to reproduce Jorge–Koutrofitis original proof of the curvature estimate (3).

Recently, Alias et al. [2] extended Theorem 2 to the class of cylindrically bounded, properly immersed submanifolds, where an isometric immersion $\varphi\colon M \hookrightarrow N \times \mathbb{R}^\ell$ is cylindrically bounded if $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$. One should also see [13] for sectional curvature estimates for cylindrically bounded submanifolds with scalar curvature bounded below. Here, $B_N(r)$ is a geodesic ball in N of radius $r > 0$. They proved the following theorem.

Theorem 3 (Alias–Bessa–Montenegro) *Let $\varphi\colon M \to N \times \mathbb{R}^\ell$ be a cylindrically bounded isometric immersion, $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, where $B_N(r)$ is a regular geodesic ball of N and $b = \sup K^{\mathrm{rad}}_{B_N(r)}$. Let $\dim(M) = m$, $\dim(N) = n - \ell$ and assume that $n - m \leq m - \ell - 1$. If either*

i. *the scalar curvature of M is bounded below as (2), or*
ii. *the immersion φ is proper and*

$$\sup_{\varphi^{-1}(B_N(r)\times\partial B_{\mathbb{R}^\ell}(t))} \|\alpha\| \leq \sigma(t), \tag{5}$$

where α is the second fundamental form of φ and $\sigma : [0, +\infty) \to \mathbb{R}$ is a positive function satisfying $\int_0^{+\infty} dt/\sigma(t) = +\infty$, then

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N. \tag{6}$$

Remark 3 The idea is to show that the hypotheses in both items i. & ii. imply that M is stochastically complete, then Remark 2 applies.

In the same spirit, Alias et al. [1] had proved the following mean curvature estimates for cylindrically bounded submanifolds properly immersed into $N \times \mathbb{R}^\ell$ immersed submanifolds.

Theorem 4 (Alias–Bessa–Dajczer) *Let $\varphi\colon M \to N \times \mathbb{R}^\ell$ be a cylindrically bounded isometric immersion, $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, where $B_N(r)$ is a regular geodesic ball of N and $b = \sup K^{\mathrm{rad}}_{B_N(r)}$. Here, M and N are complete Riemannian manifolds of dimension m and $n - \ell$, respectively, satisfying $m \geq \ell + 1$. If the immersion φ is proper, then*

$$\sup_M |\overrightarrow{H}| \geq (m - \ell) \cdot C_b(r). \tag{7}$$

2 Main results

The purpose of this paper is to extend these curvature estimates to a larger class of submanifolds, precisely, the properly immersed ϕ-bounded submanifolds. To describe this class, we need to introduce few preliminaries (Fig. 1).

2.1 ϕ-Bounded submanifolds

Consider $G \in C^\infty([0, \infty))$ satisfying

$$G_- \in L^1(\mathbb{R}^+), \quad t \int_t^{+\infty} G_-(s)\mathrm{d}s \leq \frac{1}{4} \quad \text{on } \mathbb{R}^+, \tag{8}$$

and h the solution of the following differential equation:

$$\begin{cases} h''(t) - G(t)h(t) = 0, \\ h(0) = 0, \quad h'(0) = 1. \end{cases} \tag{9}$$

In [6, Prop. 1.21], it is proved that the solution h and its derivative h' are positive in $\mathbb{R}^+ = (0, \infty)$, provided G satisfies (8), and furthermore, $h \to +\infty$ whenever the stronger condition

$$G(s) \geq -\frac{1}{4s^2} \quad \text{on } \mathbb{R}^+ \tag{10}$$

holds. Define $\phi_h \in C^\infty([0, \infty))$ by

$$\phi_h(t) = \int_0^t h(s)\mathrm{d}s. \tag{11}$$

Since h is positive and increasing in $\mathbb{R}^+$, we have that $\lim_{t\to\infty} \phi_h(t) = +\infty$. Moreover, ϕ_h satisfies the differential equation

$$\phi_h''(t) - \frac{h'}{h}(t)\phi_h'(t) = 0$$

for all $t \in [0, \infty)$.

Notation and basic assumptions on the ambient manifold $N \times L$. In this paper, N will always be a complete Riemannian manifold with a distinguished point z_0 and radial sectional curvatures along the minimal geodesic issuing from z_0 bounded above by $K_N^{rad}(z) \leq -G(\rho_N(z))$, where G satisfies the conditions (8), whereas L will be a complete Riemannian manifold with a distinguished point y_0 and sectional curvature bounded below by $-\Lambda^2$ for some $\Lambda > 0$. Let h be the solution of (9) associated with G and $\phi_h(t) = \int_0^t h(s)\mathrm{d}s$. Finally, $\rho_N(z) = \mathrm{dist}_N(z_0, z)$ will be the distance function on N, and $\rho_L(y) = \mathrm{dist}_L(y_0, y)$ will be

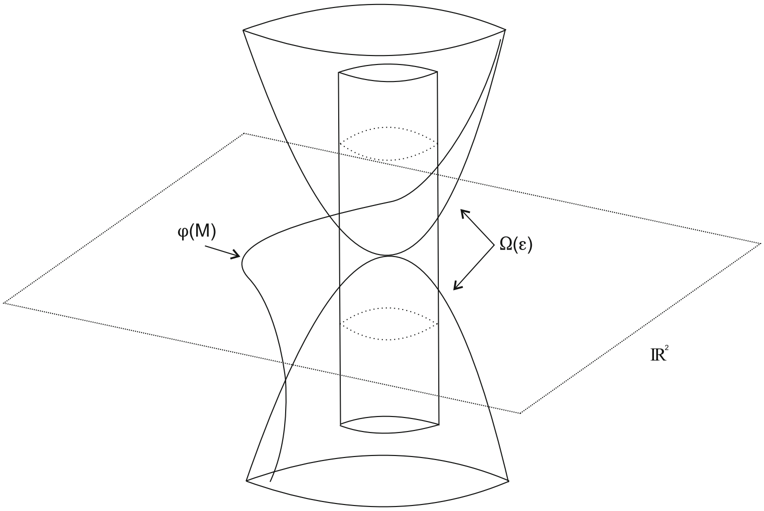

Fig. 1 ϕ_h-Bounded surface in $\mathbb{R}^3$

the distance function on L. For any given $\epsilon \in (0, 1)$, consider the subset $\Omega_{\phi_h}(\epsilon) \subset N \times L$ given by

$$\Omega_{\phi_h}(\epsilon) = \left\{(x, y) \in N \times L \colon \phi_h(\rho_N(x)) \leq \log(\rho_L(y) + 1)^{1-\epsilon}\right\}.$$

Definition 1 An isometric immersion $\varphi \colon M \to N \times L$ of a Riemannian manifold M into the product $N \times L$ is said to be ϕ_h-bounded if there exists a compact $K \subset M$ and $\epsilon \in (0, 1)$ such that $\varphi(M \setminus K) \subset \Omega_h(\epsilon)$.

Remark 4 The class of ϕ-bounded submanifolds contains the class of cylindrically bounded submanifolds. If $N \times L = \mathbb{R}^3$, then $\Omega_{\phi_h}(\epsilon) = \Omega(\epsilon)$ is given by

$$\Omega(\epsilon) = \left\{(x, y, z) \in \mathbb{R}^3 \colon z \geq e^{(x^2+y^2)^{1/(1-\epsilon)}} - 1\right\}.$$

2.2 Curvature estimates for ϕ-bounded submanifolds

In this section, we extend the cylindrically bounded version of Jorge–Koutrofiotis's Theorem, Theorem 3-ii., due to Alias et al., and the mean curvature estimates of Theorem 4 in [1], due to Alias et al., to the class of ϕ_h-bounded properly immersed submanifolds. These extensions are done in two ways. First, the class we consider is larger than the cylindrically bounded submanifolds. Second, there are no requirements on the growth on the second fundamental form as in Thm. 3. We also should observe that although ϕ-bounded properly immersed submanifold ($\varphi \colon M \to N \times_{\varrho} L$) is stochastically complete, provided L has an Omori–Yau pair, see Sect. 4, we do not need that to prove the following result.

Theorem 5 *Let $\varphi \colon M \to N^{n-\ell} \times L^{\ell}$ be a ϕ_h-bounded isometric immersion of a complete Riemannian m-manifold M with $n - m \leq m - \ell - 1$. If φ is proper and $-G \leq b \leq 0$, then*

$$sup_M K_M \geq |b| + \inf_N K_N. \tag{12}$$

With strict inequality, $\sup_M K_M > \inf_N K_N$ *if $b = 0$.*

Corollary 1 *Let $\varphi\colon M \to N^{n-\ell} \times L^{\ell}$ be a properly immersed, cylindrically bounded submanifold, $\varphi(M) \subset B_N(r) \times L^{\ell}$, where $B_N(r)$ is a regular geodesic ball of N. Suppose that $n - m \leq m - \ell - 1$. Then, the sectional curvature of M satisfies the following inequality:*

$$\sup_M K_M \geq C_b^2(r) + \inf_N K_N, \tag{13}$$

where $b = \sup_{B_N(r)} K_N^{rad}$ and C_b is defined in (4).

Our next main result extends the mean curvature estimates (7) to ϕ-bounded submanifolds.

Theorem 6 *Let $\varphi\colon M \to N^{n-\ell} \times L^{\ell}$ be a ϕ_h-bounded isometric immersion of a complete Riemannian m-manifold M with $m \geq \ell + 1$. If φ is proper, then the mean curvature vector $\vec{H} = \operatorname{tr}\alpha$ of φ satisfies*

$$\sup_M |\vec{H}| \geq (m - \ell) \cdot \inf_{r\in[0,\infty)} \frac{h'}{h}(r)\cdot \tag{14}$$

If $-G \leq b \leq 0$, then

$$\sup_M |\vec{H}| \geq (m - \ell) \cdot \sqrt{|b|}. \tag{15}$$

With strict inequality, $\sup_M |\vec{H}| > 0$ if $b = 0$.

3 Proof of the main results

3.1 Basic results

Let M and W be Riemannian manifolds of dimension m and n, respectively, and let $\varphi\colon M \to W$ be an isometric immersion. For a given function $g \in C^{\infty}(W)$, set $f = g \circ \varphi \in C^{\infty}(M)$. Since

$$\langle \operatorname{grad}_M f, X\rangle = \langle \operatorname{grad}_W g, X\rangle$$

for every vector field $X \in TM$, we obtain

$$\operatorname{grad}_W g = \operatorname{grad}_M f + (\operatorname{grad}_W g)^{\perp}$$

according to the decomposition $TW = TM \oplus T^{\perp}M$. An easy computation using the Gauss formula gives the well-known relation (see e.g. [15])

$$\operatorname{Hess}_M f(X, Y) = \operatorname{Hess}_W g(X, Y) + \langle \operatorname{grad}_W g, \alpha(X, Y)\rangle \tag{16}$$

for all vector fields $X, Y \in TM$, where α stands for the second fundamental form of φ. In particular, taking traces with respect to an orthonormal frame $\{e_1, \ldots, e_m\}$ in TM yields

$$\triangle_M f = \sum_{i=1}^{m} \operatorname{Hess}_W g(e_i, e_i) + \langle \operatorname{grad}_W g, \vec{H}\rangle, \tag{17}$$

where $\vec{H} = \sum_{i=1}^{m} \alpha(e_i, e_i)$.

In the sequel, we will need the following well-known results, see the classical Greene–Wu [12] for the Hessian Comparison Theorem and Pigola–Rigoli–Setti's "must looking at"book [25, Lemma 2.13], see also [27], [6, Thm. 1.9] for the Sturm Comparison Theorem.

Theorem 7 (Hessian Comparison Thm.) *Let W be a complete n-manifold and $\rho_W(x) = \mathrm{dist}_W(x_0, x)$, $x_0 \in W$ fixed. Let $D_{x_0} = W \setminus (\{x_0\} \cup \mathrm{cut}(x_0))$ be the domain of normal geodesic coordinates at x_0. Let $G \in C^0([0,\infty))$, and let h be the solution of (9). Let $[0, R)$ be the largest interval where $h > 0$. Then,*

i. *If the radial sectional curvatures along the geodesics issuing from x_0 satisfies*
$$K_W^{rad} \geq -G(\rho_W), \quad \text{in } B_W(R),$$
then,
$$\mathrm{Hess}_W \rho \leq \frac{h'}{h}(\rho_W)\,[\langle,\rangle - \mathrm{d}\rho \otimes \mathrm{d}\rho] \quad \text{on } D_{x_0} \cap B_W(R)$$

ii. *If the radial sectional curvatures along the geodesics issuing from x_0 satisfy*
$$K_W^{rad} \leq -G(\rho_W), \quad \text{in } B_W(R),$$
then,
$$\mathrm{Hess}_W \rho_W \geq \frac{h'}{h}(\rho)\,[\langle,\rangle - \mathrm{d}\rho \otimes \mathrm{d}\rho] \quad \text{on } D_{x_0} \cap B_W(R)$$

Lemma 1 (Sturm Comparison Thm.) *Let $G_1, G_2 \in L^\infty_{loc}(\mathbb{R})$, $G_1 \leq G_2$ and h_1 and h_2 solutions of the following problems:*
$$a.)\begin{cases} h_1''(t) - G_1(t)h_1(t) \leq 0 \\ h_1(0) = 0, \quad h_1'(0) > 0 \end{cases} \quad b.)\begin{cases} h_2''(t) - G_2(t)h_2(t) \geq 0 \\ h_2(0) = 0, \quad h_2'(0) > h_1'(0), \end{cases} \tag{18}$$
and let $I_1 = (0, S_1)$ and $I_2 = (0, S_2)$ be the largest intervals where $h_1 > 0$ and $h_2 > 0$, respectively. Then,

1. *$S_1 \leq S_2$. And on I_1, $\frac{h_1'}{h_1} \leq \frac{h_2'}{h_2}$ and $h_1 \leq h_2$.*
2. *If $h_1(t_o) = h_2(t_o)$, $t_o \in I_1$, then $h_1 \equiv h_2$ on $(0, t_o)$.*

For a more detailed Sturm Comparison Theorem, one should consult the beautiful book [25, Chapter 2.]. If $-G = b \in \mathbb{R}$, then the solution of $h_b''(t) - G \cdot h_b(t) = 0$ with $h_b(0) = 0$ and $h_b'(0) = 1$ is given by
$$h_b(t) = \begin{cases} \frac{1}{\sqrt{-b}} \cdot \sinh(\sqrt{-b}\,t) & \text{if } \; b < 0 \\ t & \text{if } \; b = 0 \\ \frac{1}{\sqrt{b}} \cdot \sin(\sqrt{b}\,t) & \text{if } \; b > 0. \end{cases}$$
In particular, if the radial sectional curvatures along the geodesics issuing from x_0 satisfy $K_W^{rad}(x) \leq -G(\rho_W(x)) \leq b$, $x \in B_W(R) = \{x, \mathrm{dist}_W(x_0, x) = \rho_W(x) < R\}$, then the solution h of (9) satisfies $(h'/h)(t) \geq (h_b'/h_b)(t) = C_b(t)$, $t \in (0, R)$, $R < \pi/2\sqrt{b}$, if $b > 0$. Therefore, $\mathrm{Hess}_W \rho_W \geq C_b(\rho_W)\left[\langle,\rangle - \mathrm{d}\rho_W \oplus \mathrm{d}\rho_W\right]$. Likewise, if $K_W^{rad}(x) \geq -G(\rho_W(x)) \geq b$, $x \in B_W(R)$, then $(h'/h)(t) \leq C_b(t)$, $t \in (0, R)$ and $\mathrm{Hess}_W \rho_W \leq C_b(\rho_W)\left[\langle,\rangle - \mathrm{d}\rho_W \oplus \mathrm{d}\rho_W\right]$.

3.2 Proof of Theorem 5

We may assume without loss of generality that there exists a $x_0 \in M$ such that $\varphi(x_0) = (z_0, y_0) \in N \times L$, z_0, y_0 the distinguished points of N and L. For each $x \in M$, let $\varphi(x) = (z(x), y(x))$. Define $g\colon N \times L \to \mathbb{R}$ by $g(z, y) = \phi_h(\rho_N(z)) + 1$, recalling that

$\phi_h(t) = \int_0^t h(s)\mathrm{d}s$, and define $f = g\circ\varphi\colon M \to \mathbb{R}$ by $f(x) = g(\varphi(x)) = \phi_h(\rho_N(z(x)))+1$. For each $k \in \mathbb{N}$, set $f_k(x) = f(x) - \frac{1}{k}\cdot\log(\rho_L(y(x)) + 1)$. Observe that $f_k(x_0) = 1$ for all k, since $\rho_N(z_0) = \rho_L(y_0) = 0$. First, let us prove the item i.

If $x \to \infty$ in M, then $\varphi(x) = (z(x), y(x)) \to \infty$ in $N \times L$ since φ is proper. If $\rho_N(z(x)) < \infty$ as $x \to \infty$ in M, then necessarily $\rho_L(y(x)) \to \infty$. If $\rho_N(z(x)) \to \infty$ and since $\varphi(M \setminus K) \subset \Omega_h(\epsilon)$ for some compact $K \subset M$ and $\epsilon \in (0, 1)$, we have that $\phi_h(\rho_N(z(x))) \leq \log(\rho_L(y(x)))^{(1-\epsilon)}$. This also imply that $\rho_L(y(x)) \to \infty$ as $x \to \infty$ in M. Thus,

$$\frac{f_k(x)}{\log(\rho_L(y(x)) + 1)} = \frac{f(x)}{\log(\rho_L(y(x)) + 1)} - \frac{1}{k} < \frac{1}{\log(\rho_L(y(x)) + 1)^{\epsilon}} - \frac{1}{k} < 0$$

for $\rho_M(x) \gg 1$. This implies that $f_k(x) < 0$ for $\rho_M(x) \gg 1$. Therefore, each f_k reaches a maximum at a point $x_k \in M$. This yields a sequence $\{x_k\} \subset M$ so that $\mathrm{Hess}_M f_k(x_k)(X, X) \leq 0$ for all $X \in T_{x_k}M$, that is, $\forall X \in T_{x_k}M$

$$\mathrm{Hess}_M f(x_k)(X, X) \leq \frac{1}{k}\cdot \mathrm{Hess}_M \log(\rho_L(y(x_k)) + 1)(X, X). \tag{19}$$

Observe that $\log(\rho_L(y(x_k))+1) = \log(\rho_L \circ\pi_L +1)\circ\varphi(x_k)$, $\pi_L\colon N\times L \to L$ the projection on the second factor; thus, the right-hand side of (19), using the formula (16), is given by

$$\begin{aligned}\mathrm{Hess}_M \log(\rho_L(y(x_k)) + 1)(X, X) &= \mathrm{Hess}_{N\times L} \log(\rho_L \circ \pi_L + 1)(\varphi(x_k))(X, X)\\ &\quad +\langle \mathrm{grad}_{N\times L} \log(\rho_L \circ \pi_L + 1), \alpha(X, X)\rangle\end{aligned} \tag{20}$$

where α is the second fundamental form of φ. For simplicity, set $\psi(t) = \log(t + 1)$, $z_k = z(x_k)$, $y_k = y(x_k)$, $s_k = \rho_N(z_k)$ and $t_k = \rho_L(y_k)$. Decomposing $X \in TM$ as $X = X^N + X^L \in TN \oplus TL$, we see that the first term of the right-hand side of (20) is

$$\begin{aligned}\mathrm{Hess}_{N\times L}\psi \circ \rho_L \circ y(x_k)(X, X) &= \psi''(t_k)|X^L|^2 + \psi'(t_k)\mathrm{Hess}_L\rho_L(y_k)(X, X)\\ &\leq \psi''(t_k)|X^L|^2 + C_{-\Lambda^2}(t_k)\frac{|X^N|^2}{(t_k + 1)}\\ &\leq C_{-\Lambda^2}(t_k)\frac{|X^N|^2}{(t_k + 1)},\end{aligned} \tag{21}$$

since $\mathrm{Hess}_L\rho_L(y_k)(X, X) \leq C_{-\Lambda^2}(t_k)|X^N|^2$ (by Theorem 7) and $\psi'' \leq 0$. Recall that $C_{-\Lambda^2}$ is defined in (4), and it comes from the curvature assumption on L.

The second term of the right-hand side of (20) is

$$\begin{aligned}\langle \mathrm{grad}_{N\times L}\psi \circ \rho_L \circ y(x_k), \alpha(X, X)\rangle &= \psi'(t_k)\langle \mathrm{grad}_L\rho_L(y_k), \alpha(X, X)\rangle\\ &\leq \frac{1}{(t_k + 1)}\|\alpha\| \cdot |X|^2\end{aligned} \tag{22}$$

From (21) and (22), we have the following:

$$\mathrm{Hess}_M\psi \circ \rho_L \circ y(x_k)(X, X) \leq \frac{C_{-\Lambda^2}(t_k) + \|\alpha\|}{(t_k + 1)} \cdot |X|^2, \tag{23}$$

and from (19) and (23), we have that

$$\mathrm{Hess}_M f(x_k)(X, X) \leq \frac{1}{k}\frac{(C_{-\Lambda^2}(t_k) + \|\alpha\|)}{(t_k + 1)}|X|^2 \tag{24}$$

We will compute the left-hand side of (19). Using the formula (16) again, recalling that $f = g \circ \varphi$ and g is given by $g(z, y) = \phi_h(\rho_N(z))$, where ϕ_h is defined in (11) and $\rho_N(z) = dist_N(z_0, z)$, we have

$$\operatorname{Hess}_M f(x_k) = \operatorname{Hess}_{N\times L} g(\varphi(x_k)) + \langle \operatorname{grad}_{N\times L} g, \alpha\rangle. \tag{25}$$

Let us consider an orthonormal basis (26)

$$\{\overbrace{\operatorname{grad}\rho_N, \partial/\partial\theta_1, \ldots, \partial/\partial\theta_{n-\ell-1}}^{\in TN}, \overbrace{\partial/\partial\gamma_1, \ldots, \partial/\partial\gamma_\ell}^{\in TL}\} \tag{26}$$

for $T_{\varphi(x_k)}(N\times L)$. Thus, if $X \in T_{x_k}M$, $|X| = 1$, we can decompose

$$X = a\cdot \operatorname{grad}\rho_N + \sum_{j=1}^{n-\ell-1} b_j\cdot \partial/\partial\theta_j + \sum_{i=1}^{\ell} c_i\cdot\partial/\partial\gamma_i$$

with $a^2 + \sum_{j=1}^{n-\ell-1} b_j^2 + \sum_{i=1}^{\ell} c_i^2 = 1$. Recalling that $s_k = \rho_N(z(x_k))$, we can see that the first term of the right-hand side of (25)

$$\begin{aligned}
\operatorname{Hess}_{N\times L} g(\varphi(x))(X,X) &= \phi_h''(s_k)\cdot a^2 + \phi_h'(s_k)\sum_{j=1}^{n-\ell-1} b_j^2\cdot \operatorname{Hess}\rho_N(z_k)\left(\frac{\partial}{\partial\theta_j}, \frac{\partial}{\partial\theta_j}\right)\\
&\geq \phi_h''(s_k)\cdot a^2 + \phi_h'(s_k)\sum_{j=1}^{n-\ell-1} b_j^2\cdot\frac{h'}{h}(s_k)\\
&= \phi_h''(s_k)\cdot a^2 + \left(1 - a^2 - \sum_{i=1}^{\ell} c_i^2\right)\cdot\phi_h'(s_k)\cdot\frac{h'}{h}(s_k)\\
&= \left[\overbrace{\left(\phi_h'' - \frac{h'}{h}\cdot\phi_h'\right)}^{\equiv 0} a^2 + \left(1 - \sum_{i=1}^{\ell} c_i^2\right)\cdot\phi_h'\cdot\frac{h'}{h}\right](s_k)\\
&= \left(1 - \sum_{i=1}^{\ell} c_i^2\right)\cdot\phi_h'(s_k)\cdot\frac{h'}{h}(s_k)
\end{aligned}$$

Thus,

$$\operatorname{Hess}_{N\times L} g(\varphi(x))(X,X) \geq \left(1 - \sum_{i=1}^{\ell} c_i^2\right)\cdot\phi_h'(s_k)\cdot\frac{h'}{h}(s_k). \tag{27}$$

The second term of the right-hand side of (25) is the following:

$$\begin{aligned}
\langle \operatorname{grad}_{N\times L} g, \alpha(X,X)\rangle &= \phi_h'(s_k)\langle \operatorname{grad}_N \rho_N(z_k), \alpha(X,X)\rangle\\
&\geq -\phi_h'(s_k)|\alpha(X,X)|
\end{aligned} \tag{28}$$

From (25), (27) and (28), we have that

$$\operatorname{Hess}_M f(x_k)(X,X) \geq \left[\left(1 - \sum_{i=1}^{\ell} c_i^2\right)\cdot\frac{h'}{h}(s_k) - |\alpha(X,X)|\right]\phi_b'(s_k) \tag{29}$$

Recall that $n + \ell \leq 2m - 1$. This dimensional restriction implies that $m \geq \ell + 2$, since $n \geq m + 1$. Therefore, for every $x \in M$, there exists a subspace $V_x \subset T_x M$ with $\dim(V_x) \geq (m - \ell) \geq 2$ such that $V_x \perp TL$; this is equivalent to $c_i = 0$ for $X \in V_x$. If we take any $X \in V_{x_k} \subset T_{x_k} M$, $|X| = 1$, we have by (24) and (29) that

$$\frac{(C_{-\Lambda^2}(t_k) + |\alpha(X,X)|)}{k(t_k+1)} \geq \mathrm{Hess}_M f(x_k)(X,X) \geq \left[\frac{h'}{h}(s_k) - |\alpha(X,X)|\right]\phi_h'(s_k)$$

Thus, reminding that $\phi_h' = h$,

$$\left[\frac{1}{k(t_k+1)} + h(s_k)\right]|\alpha(X,X)| \geq h'(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k+1)} \tag{30}$$

Since $-G \leq b \leq 0$, we have by Lemma 1 (Sturm's argument) that the solution h of (9) satisfies $(h'/h)(t) \geq C_b(t) > \sqrt{|b|}$ and that $h(t) \to +\infty$ as $t \to +\infty$, where C_b is defined in (4). Let us assume that x_k is unbounded in M, so that passing to a subsequence if necessary $x_k \to \infty$ (the case $\rho_M(x_k) \leq C^2 < \infty$ will be considered later), then $s_k \to \infty$ as well as $t_k \to \infty$. Thus, from (30), for sufficiently large k, we have at $\varphi(x_k)$ that

$$\begin{aligned}\left[\frac{1}{k(t_k+1)h(s_k)} + 1\right]|\alpha(X,X)| &\geq \frac{h'(s_k)}{h(s_k)} - \frac{C_{-\Lambda^2}(t_k)}{k(t_k+1)h(s_k)} \\ &\geq C_b(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k+1)h(s_k)} \\ &> 0 \end{aligned} \tag{31}$$

Thus, at x_k and $X \in T_{x_k} M$ with $|X| = 1$, we have

$$|\alpha(X,X)| \geq \left[C_b(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k+1)h(s_k)}\right]\left[\frac{1}{k(t_k+1)h(s_k)} + 1\right]^{-1} > 0. \tag{32}$$

We will need the following lemma known as Otsuki's Lemma [16, p. 28].

Lemma 2 (Otsuki) *Let $\beta\colon \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^d$, $d \leq q - 1$, be a symmetric bilinear form satisfying $\beta(X,X) \neq 0$ for $X \neq 0$. Then, there exists linearly independent vectors X, Y such that $\beta(X,X) = \beta(Y,Y)$ and $\beta(X,Y) = 0$.*

The *horizontal* subspace V_{x_k} has dimension $\dim(V_{x_k}) \geq m - \ell \geq 2$. Thus, by the inequality (32) and $n - m \leq m - \ell - 1 \leq \dim(V_{x_k}) - 1$, we may apply Otsuki's Lemma to $\alpha(x_k)\colon V_{x_k} \times V_{x_k} \to T_{x_k} M^{\perp} \simeq \mathbb{R}^{n-m}$ to obtain $X, Y \in V_{x_k}$, $|X| \geq |Y| \geq 1$ such that $\alpha(x_k)(X,X) = \alpha(x_k)(Y,Y)$ and $\alpha(x_k)(X,Y) = 0$.

By the Gauss equation, we have that

$$\begin{aligned}
K_M(x_k)(X,Y)-K_N(\varphi(x_k))(X,Y) &= \frac{\langle \alpha(x_k)(X,X), \alpha(x_k)(Y,Y)\rangle}{|X|^2|Y|^2-\langle X,Y\rangle^2}\\
&= \frac{|\alpha(x_k)(X,X)|^2}{|X|^2|Y|^2}\\
&\geq \left(\frac{|\alpha(x_k)(X,X)|}{|X|^2}\right)^2\\
&= \left|\alpha(x_k)\left(\frac{X}{|X|},\frac{X}{|X|}\right)\right|^2\\
&\geq \left[C_b(s_k)-\frac{C_{-\Lambda^2}(t_k)}{k(t_k+1)h(s_k)}\right]\left[\frac{1}{k(t_k+1)h(s_k)}+1\right]^{-1}\\
&> 0 \qquad (33)
\end{aligned}$$

where this last inequality is implied by (32). Since

$$\sup K_M - \inf K_N \geq K_M(x)(X,Y)-K_N(\varphi(x))(X,Y),$$

for all $X, Y \in T_xM \subset T_xN$, we have by (33) that

$$\sup K_M - \inf K_N > \left(\left[\frac{h'(s_k)}{h(s_k)}-\frac{C_{-\Lambda^2}(t_k)}{k(t_k+1)h(s_k)}\right]\left[\frac{1}{k(t_k+1)h(s_k)}+1\right]^{-1}\right)^2 > 0.$$

Therefore, $\sup K_M - \inf K_N > 0$ regardless of $b = 0$ or $b < 0$. If $b < 0$, we let $k \to +\infty$, and then, we have

$$\sup K_M - \inf K_N \geq \lim_{s_k\to\infty}\left[\frac{h'}{h}(s_k)\right]^2 = |b| \qquad (34)$$

The case where the sequence $\{x_k\} \subset M$ remains in a compact set, we proceed as follows. Passing to a subsequence, we have that $x_k \to x_\infty \in M$. Thus, $t_k \to t_\infty < \infty$ and $s_k \to s_\infty < \infty$. By (24)

$$\text{Hess}_M f(x_\infty)(X,X) \leq \lim_{k\to\infty}\frac{(C_{-\Lambda^2}(t_\infty)+|\alpha(x_\infty)(X,X)|)}{k(t_\infty+1)} = 0, \qquad (35)$$

for all $X \in T_{x_0}M$. Using the expression on the right-hand side of (29), we obtain for every $X \in V_{x_\infty}$

$$0 \geq \text{Hess}\, f(x_\infty)(X,X) \geq \left[\left(1-\sum_{i=1}^{\ell}c_i^2\right)\cdot\frac{h'}{h}(s_\infty)-|\alpha(X,X)|\right]\phi_b'(s_\infty).$$

There exists a subspace $V_x \subset T_xM$ with $\dim(V_x) \geq (m-\ell) \geq 2$ such that $V_x \perp T\mathbb{R}^\ell$; this is equivalent to $c_i = 0$, for $X \in V_x$. If we take any $X \in V_{x_\infty} \subset T_{x_\infty}M$, $|X| = 1$, we have

$$|\alpha_{x_\infty}(X,X)| \geq \frac{h'}{h}(s_\infty)|X|^2.$$

Again, using Otsuki's Lemma and Gauss equation, we conclude that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq \left[\frac{h'}{h}(s_\infty)\right]^2 > |b|. \qquad (36)$$

3.3 Proof of Theorem 6

We will follow the proof of Theorem 5 closely. Recall that f_k reaches a maximum at $x_k \in M$, $k = 1, 2, \ldots$, so that $\triangle_M f_k(x_k) \leq 0$. Thus,

$$\triangle_M f(x_k) \leq \frac{1}{k} \cdot \triangle_M (\log(\rho_L \circ \pi_L + 1) \circ \varphi(x_k)). \tag{37}$$

Using the formula (17),

$$\begin{aligned}\triangle_M (\log(\rho_L \circ \pi_L + 1) \circ \varphi(x_k)) &= \sum_{i=1}^{m} \mathrm{Hess}_{N\times L} \log(\rho_{\mathbb{R}^\ell} \circ \pi_L + 1)(\varphi(x_k))(X_i, X_i)\\ &+ \langle \mathrm{grad}_{N\times L} \log(\rho_L \circ \pi_L + 1), \vec{H}\rangle\end{aligned} \tag{38}$$

where $\vec{H} = \sum_{i=1}^m \alpha(X_i, X_i)$ is the mean curvature vector, while α is the second fundamental form of the immersion φ, and $\{X_i\}$ is an orthonormal basis of $T_{x_k}M$.

As before, decomposing $X \in TM$ as $X = X^N + X^L \in TN \oplus TL$ and setting $\psi(t) = \log(t+1)$, $y_k = y(x_k)$ and $t_k = \rho_L(y_k)$ we have that the right-hand side of (38)

$$\begin{aligned}\sum_{i=1}^{m} \mathrm{Hess}_{N\times L} \psi \circ \rho_L \circ y(x_k)(X_i, X_i) &= \psi''(t_k) \sum_{i=1}^{m} |X_i^L|^2\\ &\quad + \psi'(t_k) \sum_{i=1}^{m} \mathrm{Hess}_L \rho_L(y_k)(X_i, X_i)\\ &\leq \frac{C_{-\Lambda^2}(t_k)}{(t_k+1)} \sum_{i=1}^{m} |X_i^N|^2,\\ &\leq \frac{m \cdot C_{-\Lambda^2}(t_k)}{(t_k+1)}\end{aligned} \tag{39}$$

since $\psi'' \leq 0$ and

$$\begin{aligned}\langle \mathrm{grad}_{N\times L} \psi \circ \rho_L \circ y(x_k), \vec{H}\rangle &= \psi'(t_k)\langle \mathrm{grad}\, \rho_L(y_k), \vec{H}\rangle\\ &\leq \frac{1}{(t_k+1)} |\vec{H}|\end{aligned} \tag{40}$$

From (38), (39) and (40), we have

$$\triangle_M \log(\rho_L(y(x_k)) + 1) \leq \frac{m \cdot C_{-\Lambda^2}(t_k) + |\vec{H}|}{(t_k+1)} \tag{41}$$

And from (37) and (41), we have that

$$\triangle_M f(x_k) \leq \frac{m \cdot C_{-\Lambda^2}(t_k) + |\vec{H}|}{k(t_k+1)} \tag{42}$$

We will compute the left-hand side of (37). Recall that $f = g \circ \varphi$ and g is given by $g(z, y) = \phi_h(\rho_N(z))$, where ϕ is defined in (11). Using the formula (17) again, we have

$$\triangle_M f(x_k) = \sum_{i=1}^{m} \mathrm{Hess}_{N\times L}\, g(\varphi(x_k))(X_i, X_i) + \langle \mathrm{grad}\, g, \vec{H}\rangle \tag{43}$$

Consider the orthonormal basis (26) for $T_{\varphi(x_k)}(N \times L)$. Thus, if $X_i \in T_{x_k}M$, $|X_i| = 1$, we can decompose

$$X_i = a_i \cdot \operatorname{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_{ij} \cdot \partial/\partial\theta_j + \sum_{l=1}^{\ell} c_{il} \cdot \partial/\partial\gamma_l$$

with $a_i^2 + \sum_{j=1}^{n-\ell-1} b_{ij}^2 + \sum_{l=1}^{\ell} c_{il}^2 = 1$. Set $z_k = z(x_k)$ and $s_k = \rho_N(z_k)$. We have as in (27)

$$\operatorname{Hess}_{N\times L}\, g(\varphi(x))(X_i, X_i) \geq \left(1 - \sum_{l=1}^{\ell} c_{il}^2\right) \cdot \phi_h'(s_k) \cdot \frac{h'}{h}(s_k) \tag{44}$$

The second term of the right-hand side of (43) is the following, if $|X| = 1$,

$$\begin{aligned}\langle \operatorname{grad} g, \vec{H}\rangle &= \phi_h'(s_k)\langle \operatorname{grad} \rho_N(z_k), \vec{H}\rangle \\ &\geq -\phi_h'(s_k)|\,\vec{H}\,|\end{aligned} \tag{45}$$

Therefore, from (43), (44) and (45), we have that

$$\triangle_M f(x_k) \geq \left[\left(m - \sum_{i=1}^{m}\sum_{l=1}^{\ell} c_{il}^2\right) \cdot \frac{h'}{h}(s_k) - |\,\vec{H}\,|\right]\phi_b'(s_k) \tag{46}$$

From (42) and (46), we have

$$\frac{m \cdot C_{-\Lambda^2}(t_k) + |\,\vec{H}\,|}{k(t_k+1)} \geq \triangle_M f(x_k) \geq \left[(m-\ell) \cdot \frac{h'}{h}(s_k) - |\,\vec{H}\,|\right]\phi_h'(s_k) \tag{47}$$

Therefore,

$$\sup_M |\,\vec{H}\,| \left[\frac{1}{h(s_k)\cdot k \cdot (t_k+1)} + 1\right] \geq (m-\ell) \cdot \frac{h'}{h}(s_k) - \frac{m \cdot C_{-\Lambda^2}(t_k)}{h(s_k)\cdot k\cdot(t_k+1)}$$

Letting $k \to \infty$, we have

$$\sup_M |\,\vec{H}\,| \geq (m-\ell) \cdot \lim_{k\to\infty} \frac{h'}{h}(s_k)\cdot$$

If in addition, we have that $-G \leq b \leq 0$ then $(h'/h)(s) \geq C_b(s)$. The case that $b = 0$, we have $(h'/h)(s_k) \geq 1/s_k$ and $h(s_k) \geq s_k$. Since the immersion is ϕ-bounded, we have $s_k^2 \leq 2\log(t_k+1)^{(1-\epsilon)}$. Thus, for sufficient large k,

$$\sup_M |\,\vec{H}\,| \left[\frac{1}{s_k \cdot k \cdot (t_k+1)} + 1\right] \geq \frac{m-\ell}{s_k} - \frac{m \cdot C_{-\Lambda^2}(t_k)}{s_k \cdot k \cdot (t_k+1)} > 0.$$

This shows that $\sup_M |\,\vec{H}\,| > 0$.

In the case $b < 0$, we have $(h'/h)(s_k) \geq C_b(s_k) \geq \sqrt{|b|}$ and

$$\sup_M |\,\vec{H}\,| \geq (m-\ell) \cdot \lim_{k\to\infty} \frac{h'}{h}(s_k) \geq \sqrt{|b|}.$$

Remark 5 The statements of Theorems 5 and 6 are also true in a slightly more general situation. This is, if, instead a proper ϕ-bounded immersion, one asks a proper immersion $\varphi\colon M \to N \times L$ with the property

$$\lim_{x\to\infty_{in\,M}} \frac{\phi_h(\rho_N(z(x)))}{\log(\rho_L(y(x))+1)} = 0,$$

where $\varphi(x) = (z(x), y(x)) \in N \times L$.

4 Omori–Yau pairs

Omori [19], in discovered an important global maximum principle for complete Riemannian manifolds with sectional curvature bounded below. Omori's maximum principle was refined and extended by Cheng and Yau [8,29,30], to Riemannian manifolds with Ricci curvature bounded below and applied to find elegant solutions to various analytic-geometric problems on Riemannian manifolds.

There were other generalizations of the Omori–Yau maximum principle under more relaxed curvature requirements in [7,10] and an extension to an all general setting by Pigola et al. [24] in their beautiful book. There, they introduced the following terminology.

Definition 2 (Pigola–Rigoli–Setti) The Omori–Yau maximum principle holds on a Riemannian manifolds W if for any $u \in C^2(W)$ with $u^*\colon = \sup_W u < \infty$, there exists a sequence of points $x_k \in W$, depending on u and on W, such that

$$\lim_{k\to\infty} u(x_k) = u^*, |\mathrm{grad}\, u|(x_k) < \frac{1}{k}, \triangle u(x_k) < \frac{1}{k}. \tag{48}$$

Likewise, the Omori–Yau maximum principle *for the Hessian* holds on W if

$$\lim_{k\to\infty} u(x_k) = u^*, |\mathrm{grad}\, u|(x_k) < \frac{1}{k}, \mathrm{Hess}_W u(x_k)(X, X) < \frac{1}{k}\cdot |X|^2, \tag{49}$$

for every $X \in T_{x_k}W$.

A natural and important question is, what are the Riemannian geometries that the Omori–Yau maximum principle holds on? It does hold on complete Riemannian manifolds with sectional curvature bounded below [19], and it also holds on complete Riemannian manifolds with Ricci curvature bounded below [8,29,30]. It follows from the work of Pigola et al. [24] that the Omori–Yau maximum principle holds on complete Riemannian manifolds W with Ricci curvature with strong quadratic decay,

$$Ric_W \geq -c^2\cdot \rho_W^2 \cdot \Pi_{i=1}^k(\log^{(i)}(\rho_W+1), \quad \rho_W \gg 1.$$

The notion of the Omori–Yau pair was formalized in [3], after the work of Pigola-Rigoli-Setti. The Omori–Yau pair is described for the Laplacian and for the Hessian; however, it certainly can be extended to other operators or bilinear forms.

Definition 3 A pair $(\mathcal{G}, \gamma)$ of functions $\mathcal{G}\colon [0, +\infty) \to (0, +\infty)$, $\gamma\colon W \to [0, +\infty)$, $\mathcal{G} \in C^1([0,\infty))$, $\gamma \in C^2([0,\infty))$, forms an Omori–Yau pair for the Laplacian in a Riemannian manifold W if they satisfy the following conditions:

h.1) $\gamma(x) \to +\infty$ as $x \to \infty$in W.
h.2) $\mathcal{G}(0) > 0$, $\mathcal{G}'(t) \geq 0$ and $\int_0^{+\infty} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} = +\infty$.

h.3) $\exists A > 0$ constant such that $|\text{grad}_W \gamma| \leq A\sqrt{\mathcal{G}(\gamma)}\left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right)$ off a compact set.

h.4) $\exists B > 0$ constant such that $\triangle_W \gamma \leq B\sqrt{\mathcal{G}(\gamma)}\left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right)$ off a compact set.

The pair $(\mathcal{G}, \gamma)$ forms an Omori–Yau pair for the Hessian if instead h.4) one has

h.5) $\exists C > 0$ constant such that $\text{Hess}\,\gamma \leq C\sqrt{\mathcal{G}(\gamma)}\left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right)$ off a compact set, in the sense of quadratic forms.

The result [24, Thm. 1.9] captured the essence of the Omori–Yau maximum principle, and it can be stated as follows.

Theorem 8 *If a Riemannian manifold M has an Omori–Yau pair $(\mathcal{G}, \gamma)$, then the Omori–Yau maximum principle holds on it.*

The main step in the proof of Alias–Bessa–Montenegro's Theorem (Thm. 3) and Alias–Bessa–Dajczer's Theorem (Thm. 4) is to show that a cylindrically bounded submanifold, properly immersed into $N \times L$, with *controlled* second fundamental form or *controlled* mean curvature vector, has an Omori–Yau pair, provided L has an Omori–Yau pair. Thus, the Omori–Yau maximum principle holds on those submanifolds and their proof follows the steps of Jorge–Koutrofiotis's Theorem. On the other hand, the idea behind the proof of Theorems 5 and 6 is that: The factor L has bounded sectional curvature; hence, it has a natural Omori–Yau pair $(\mathcal{G}, \gamma)$. This Omori–Yau pair together with the geometry of the factor N allows us to consider an unbounded region Ω_ϕ such that if $\varphi\colon M \to \Omega_\phi \subset N \times L$ is an isometric immersion, then there exists a function $f \in C^2(M)$, not necessarily bounded, and a sequence $x_k \in M$ satisfying $\triangle f(x_k) \leq 1/k$. We show that a properly immersed ϕ-bounded submanifold has an Omori–Yau pair for the Laplacian, provided the fiber L has an Omori–Yau pair for the Hessian. We show in Theorem 10 that an Omori–Yau pair for the Hessian guarantees the Omori–Yau sequence for certain unbounded functions, as this unbounded function f we are working. This leads to stronger forms of Theorems 5 and 6.

Let M, N, L be complete Riemannian manifolds of dimension $m, n - \ell$ and ℓ, with distinguished points x_0, z_0 and y_0, respectively. Let $\rho_N(z) = \text{dist}_N(z_0, z)$ and suppose that $K_N^{\text{rad}} \leq -G(\rho_N)$, G satisfying (8). Let h be the solution of (9) and ϕ_h as in (11). Suppose in addition that L has an Omori–Yau pair for the Hessian $(\gamma, \mathcal{G})$. Let $\Omega_{h,\gamma,\mathcal{G}}(\epsilon) \subset N \times L$ be the region defined by

$$\Omega_{h,\gamma,\mathcal{G}}(\epsilon) = \{(z, y) \in N \times L\colon \phi_h \circ \rho_N(z(x)) \leq \left[\psi \circ \gamma(y(x))\right]^{1-\epsilon}\},$$

where $\psi(t) = \log\left(\int_0^t \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right)$. In this setting, we have the following result.

Theorem 9 *Let $\varphi\colon M \to N \times L$ be a properly immersed submanifold such that $\varphi(M \setminus K) \subset \Omega_{h,\gamma,\mathcal{G}}(\epsilon)$ for some compact $K \subset M$ and positive $\epsilon \in (0, 1)$. Suppose that $K_N^{\text{rad}} \leq -G(\rho_N) \leq b \leq 0$.*

1. *If the codimension satisfies $n - m \leq m - \ell - 1$, then*

$$\sup_M K_M \geq |b| + \inf_N K_N. \tag{50}$$

   With strict inequality, $\sup_M K_M > \inf_N K_N$ *if $b = 0$.*

2. *If $m \geq \ell + 1$, then*

$$\sup_M |\vec{H}| \geq (m - \ell) \cdot \sqrt{|b|}. \tag{51}$$

   With strict inequality, $\sup_M |\vec{H}| > 0$ *if $b = 0$.*

The curvature assumption can be relaxed to admit some positive curvature; that is, if $K_N^{\mathrm{rad}} \leq -G(\rho_N)$, G satisfying (8) and if $m \geq \ell + 1$, then

$$\sup_M |\overrightarrow{H}| \geq (m-\ell) \cdot \inf_{r\in[0,\infty)} \frac{h'}{h}(r)\cdot \tag{52}$$

Proof Assume without loss of generality that there exists $x_0 \in M$ such that $\varphi(x_0) = (z_0, y_0) \in N \times L$. As before, $\varphi(x) = (z(x), y(x))$ and $g, p\colon N \times L \to \mathbb{R}$ given by $g(z, y) = \phi_h(\rho_N(z)) + 1$, $p(z, y) = \psi(\gamma(y))$. For each $k \in \mathbb{N}$, let $g_k\colon M \to \mathbb{R}$ given by $g_k(x) = g\circ\varphi(x) - p\circ\varphi(x)/k$. Observe that $g_k(x_0) = 1 - \psi(\gamma(y_0))/k > 0$ if $k \gg 1$. We have that $g_k(x) < 0$ for $\rho_M(x) \gg 1$. This implies that g_k has a maximum at a point x_k, yielding in this way a sequence $\{x_k\} \subset M$ such that $\mathrm{Hess}_M g_k(x_k) \leq 0$ in the sense of quadratic forms. Proceeding as in the proof of Theorem 5, we have that for $X \in T_{x_k}M$,

$$\mathrm{Hess}_M g\circ\varphi(x_k)(X,X) \leq \frac{1}{k}\mathrm{Hess}_M p\circ\varphi(x_k)(X,X). \tag{53}$$

We have to compute both terms of this inequality. Considering once more the orthonormal basis (26) for $T_{\varphi(x_k)}(N\times L)$, we can decompose, $X \in T_{x_k}M$, $|X| = 1$ (after identifying X with $d\varphi X$), as

$$X = a\cdot \mathrm{grad}\,\rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial/\partial\theta_j + \sum_{i=1}^{\ell} c_i \cdot \partial/\partial\gamma_i$$

with $a^2 + \sum_{j=1}^{n-\ell-1} b_j^2 + \sum_{i=1}^{\ell} c_i^2 = 1$. Setting $s_k = \rho_N(z(x_k))$, $t_k = \gamma(y(x_k))$, we have as in (29),

$$\begin{aligned}
\mathrm{Hess}_M g\circ\varphi(x_k)(X,X) &= \mathrm{Hess}_{N\times L}\, g(\varphi(x_k))(X,X) + \langle \mathrm{grad}_{N\times L}\, g, \alpha(X,X)\rangle \\
&\geq \left[\left(1 - \sum_{i=1}^{\ell} c_i^2\right)\cdot \frac{h'}{h}(s_k) - |\alpha(X,X)|\right]\phi_b'(s_k) \qquad (54)\\
\mathrm{Hess}_M p\circ\varphi(x_k)(X,X) &= \mathrm{Hess}_{N\times L}\, p(\varphi(x_k))(X,X) + \langle \mathrm{grad}_{N\times L}\, p, \alpha(X,X)\rangle \\
&= \psi''(t_k)\langle X, \mathrm{grad}_L \gamma\rangle^2 + \psi'(t_k)\mathrm{Hess}_L\gamma(X,X) \\
&\quad + \psi'(t_k)\langle \mathrm{grad}_L\gamma, \alpha(X,X)\rangle \\
&\leq \psi'(t_k)\left(\mathrm{Hess}_L\gamma(X,X) + |\mathrm{grad}_L\gamma|\cdot|\alpha(X,X)|\right) \\
&\leq \frac{\left[\sqrt{\mathcal{G}(\gamma(t_k))}\left(\int_0^{t_k}\frac{\mathrm{d}s}{\sqrt{\mathcal{G}(\gamma(s))}} + 1\right)\right](C + A\cdot|\alpha(X,X)|)}{\sqrt{\mathcal{G}(\gamma(t_k))}\left(\int_0^{t_k}\frac{\mathrm{d}s}{\sqrt{\mathcal{G}(\gamma(s))}} + 1\right)} \\
&= C + A\cdot|\alpha(X,X)|, \qquad (55)
\end{aligned}$$

since $\psi'' \leq 0$. Taking in consideration the bounds (54) and (55), the inequality (53) yields, $(\phi'(s) = h(s))$,

$$\left[\frac{A}{k\cdot h(s_k)} + 1\right]|\alpha(X,X)| \geq \left(1 - \sum_{i=1}^{\ell} c_i^2\right)\frac{h'}{h}(s_k) - \frac{C}{k\cdot h(s_k)}. \tag{56}$$

Under the hypotheses of item 1., we have that $(h'/h)(s) \geq C_b(s) > \sqrt{|b|}$ and $h(s) \to \infty$ as $s \to \infty$. Moreover, there exists a subspace $V_{x_k} \subset T_{x_k}M$, $\dim V_{x_k} \geq 2$, such that if $X \in V_{x_k}$, then $X = a \cdot \operatorname{grad} \rho_{_N} + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial/\partial\theta_j$. Therefore, for $X \in V_{x_k}$, $|X| = 1$, we have for $k \gg 1$.

$$\left[\frac{A}{k \cdot h(s_k)} + 1\right] |\alpha(X, X)| \geq \frac{h'}{h}(s_k) - \frac{C}{k \cdot h(s_k)} > |b| - \frac{C}{k \cdot h(s_k)} > 0. \tag{57}$$

The proof follows exactly the steps of the proof of Theorem 5, and we obtain that $\sup_{_M} K_M \geq |b| + \inf_{_N} K_N$ if $b < 0$ and $\sup_{_M} K_M > \inf_{_N} K_N$ if $b = 0$.

To prove item 2., take an orthonormal basis $X_1, \ldots, X_q, \ldots, X_m \in T_{x_k}M$,

$$X_q = a_q \cdot \operatorname{grad} \rho_{_N} + \sum_{j=1}^{n-\ell-1} b_{jq} \cdot \partial/\partial\theta_j + \sum_{i=1}^{\ell} c_{iq} \cdot \partial/\partial\gamma_i$$

with $a_q^2 + \sum_{j=1}^{n-\ell-1} b_{jq}^2 + \sum_{i=1}^{\ell} c_{iq}^2 = 1$. Tracing the inequality (56) to obtain

$$\left[\frac{A}{k \cdot h(s_k)} + 1\right] |H| \geq (m - \sum_{q=1}^{m}\sum_{i=1}^{\ell} c_{iq}^2)\frac{h'}{h}(s_k) - \frac{C}{k \cdot h(s_k)} \geq (m-\ell)C_b(s_k) - \frac{C}{k \cdot h(s_k)} > 0 \tag{58}$$

for $k \gg 1$. If $b = 0$, then $C_b(s) = 1/s$; then, coupled with the estimate $h(s) \geq s\sqrt{s}$, see [6], we deduce that $\sup_{_M} |\vec{H}| > 0$. And if $b < 0$, then $C_b(s) \geq \sqrt{|b|} > 0$; then, letting $k \to \infty$, we have $\sup_{_M} |\vec{H}| \geq (m-\ell)\sqrt{|b|} > 0$ if $b < 0$. □

These curvature estimates can be seen as geometric applications of the Theorem 10 below, which is an extension of the Pigola et al. [24, Thm. 1.9]. In [22, Cor. A1.], Pigola–Rigoli–Setti proved an Omori–Yau type result for quite general operators on Riemannian manifolds M, applicable to unbounded functions $u \in C^2(M)$ with growth rate to infinity faster than ours. However, their result require a decay condition on the Ricci curvature of M instead a general Omori–Yau pair $(\mathcal{G}, \gamma)$, although the Ricci curvature decay $Ric_M \geq -B^2 G(\rho_M)$, ($B > 0$ constant) yields an Omori–Yau pair, i.e., (G, ρ_M). Another difference between our result and theirs is that we have a condition on the gradient of the unbounded function u.

Theorem 10 *Let W be a complete Riemannian manifold with an Omori–Yau pair $(\mathcal{G}, \gamma)$ for the Hessian (Laplacian). If $u \in C^2(W)$ satisfies $\lim_{x\to\infty} \frac{u(x)}{\psi(\gamma(x))} = 0$, $\psi(t) = \log\left(\int_0^t \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right)$,, then there exist a sequence $x_k \in M$, $k \in \mathbb{N}$ such that*

$$|\operatorname{grad}_{_W} u|(x_k) \leq \frac{A}{k}, \operatorname{Hess}_{_W} u(x_k) \leq \frac{C}{k}\left(\triangle_{_W} u(x_k) \leq \frac{B}{k}\right) \tag{59}$$

If $u^* = \sup_M u < \infty$, then $u(x_k) \to u^*$. The constants A, B and C come from the Omori–Yau pair $(\mathcal{G}, \gamma)$, see Definition 3.

Proof Assume that the Omori–Yau pair $(\mathcal{G}, \gamma)$ is for the Hessian. The case of the Laplacian is similar. Fix a point $x_0 \in M$ such that $\gamma(x_0) > 0$ and define for each $k \in \mathbb{N}$, $u_k \colon M \to \mathbb{R}$ by $u_k(x) = u(x) - \frac{1}{k}\psi(\gamma(x)) + 1 - u(x_0) - \frac{1}{k}\psi(\gamma(x_0))$. We have that $u_k(x_0) = 1$ and

$u_k(x) \le 0$ for $\rho_W(x) = \mathrm{dist}_W(x_0, x) \gg 1$. Thus, there is a point x_k such that u_k reaches a maximum. In this way, we find a sequence $x_k \in M$ such that for all $X \in T_{x_k} W$

$$\begin{aligned}
\mathrm{Hess}_W u(X,X) &\le \frac{1}{k}\mathrm{Hess}_W \psi(\gamma)(X,X) \\
&= \frac{1}{k}\left[\psi''(\gamma)\langle \mathrm{grad}_W \gamma, X\rangle^2 + \psi'(\gamma)\mathrm{Hess}_W \gamma(X,X)\right] \\
&\le \frac{1}{k}\left[\frac{1}{\sqrt{\mathcal{G}(\gamma)}}\frac{1}{\left(\int_0^\gamma \frac{\mathrm{d}s}{\mathcal{G}(s)} + 1\right)} C\sqrt{\mathcal{G}(\gamma)}\left(\int_0^\gamma \frac{\mathrm{d}s}{\mathcal{G}(s)} + 1\right)\right]|X|^2 \\
&= \frac{C}{k}|X|^2.
\end{aligned}$$

We used that $\psi'' \le 0$ and $\mathrm{Hess}_W \gamma(X,X) \le C\sqrt{\mathcal{G}(\gamma)}\left(\int_0^\gamma \frac{\mathrm{d}s}{\mathcal{G}(s)} + 1\right)$.

$$\begin{aligned}
|\mathrm{grad}_W u| &= \frac{1}{k}|\mathrm{grad}_W \psi(\gamma)| \\
&\le \frac{1}{k}\left[\frac{1}{\sqrt{\mathcal{G}(\gamma)}}\frac{1}{\left(\int_0^\gamma \frac{\mathrm{d}s}{\mathcal{G}(s)} + 1\right)} A\sqrt{\mathcal{G}(\gamma)}\left(\int_0^\gamma \frac{\mathrm{d}s}{\mathcal{G}(s)} + 1\right)\right] \le \frac{A}{k}.
\end{aligned}$$

□

4.1 Omori–Yau pairs and warped products

Let (N, g_N) and (L, g_L) be complete Riemannian manifolds of dimension $n - \ell$ and ℓ, respectively, and $\varrho\colon L \to \mathbb{R}_+$ be a smooth function. Let $\varphi\colon M \to L \times_\varrho N$ be an isometric immersion into the warped product $L \times_\varrho N = (L \times N, \mathrm{d}s^2 = g_L + \varrho^2 g_N)$. The immersed submanifold $\varphi(M)$ is cylindrically bounded if $\pi_N(\varphi(M)) \subset B_N(r)$, where $\pi_N\colon L \times N \to N$ is the canonical projection in the N-factor and $B_N(r)$ is a regular geodesic ball of radius r of N. Alías and Dajczer in the proof of [4, Thm. 1] showed that if φ is proper in $L \times N$, then the existence of an Omori–Yau pair for the Hessian in L induces an Omori–Yau pair for the Laplacian on M provided the mean curvature $|H|$ is bounded. We can prove a slight extension of this result.

Lemma 3 *Let $\varphi\colon M \to L \times_\varrho N$ be an isometric immersion, proper in the first entry, where L carries an Omori–Yau pair $(\mathcal{G}, \gamma)$ for the Hessian, $\varrho \in C^\infty(L)$ is a positive function satisfying*

$$|\mathrm{grad}\,\log\varrho(y)| \le \ln\left(\int_0^{\gamma(y)} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right). \tag{60}$$

Letting $\varphi(x) = (y(x), z(x))$ and if

$$|\vec{H}(\varphi(x))| \le \ln\left(\int_0^{\gamma(y(x))} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1\right), \tag{61}$$

then M has an Omori–Yau pair for the Laplacian. In particular, M holds the Omori–Yau maximum principle for the Laplacian.

Proof The idea of the proof is presented in [4] and therefore will try to follow the same notation to simplify the demonstration. Let $(\mathcal{G}, \gamma)$ be the Omori–Yau pair for the Hessian of L. Assume w.l.o.g. that M is non-compact and denote $\varphi(x) = (y(x), z(x))$. Define $\Gamma(y, z) = \gamma(y)$ and define $\vartheta(x) = \Gamma \circ \varphi = \gamma(y(x))$. We will show that $(\mathcal{G}, \vartheta)$ is an Omori–Yau pair for the Laplacian in M. Indeed, let $q_k \in M$ a sequence such that $q_k \to \infty$ in M as $k \to +\infty$. Since φ is proper in the first entry, we have that $y(q_k) \to \infty$ in L. Since $\vartheta(q_k) = \gamma(y(q_k))$, we have $\vartheta(q_k) \to \infty$ as $q_k \to \infty$ in M.

We have that

$$\operatorname{grad}_{L\times_\varrho N} \Gamma(z, y) = \operatorname{grad}_L \gamma(z). \tag{62}$$

Since $\varrho = \Gamma \circ \varphi$, we obtain at $\varphi(q)$

$$\begin{aligned}\operatorname{grad}_{L\times_\varrho N} \Gamma &= (\operatorname{grad}_{L\times_\varrho N} \Gamma)^T + (\operatorname{grad}_{L\times_\varrho N} \Gamma)^\perp \\ &= \operatorname{grad}_M \varrho + (\operatorname{grad}_{L\times_\varrho N} \Gamma)^\perp.\end{aligned}$$

By hypothesis, we have

$$\begin{aligned}|\operatorname{grad}_M \varrho|(q) \le |\operatorname{grad}_{L\times_\varrho N} \Gamma|(\varphi(q)) &= |\operatorname{grad}_L \gamma|(y(q)) \\ &\le \sqrt{\mathcal{G}(\gamma(y(q)))}\left(\int_0^{\gamma(y(q))} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1\right)\end{aligned}$$

out of a compact subset of M.

Let $T, S \in TL$, $X, Y \in TN$ and $\nabla^{L\times_\varrho N}$, ∇^L and ∇^N be the Levi-Civita connections of the metrics $\mathrm{d}s^2 = g_L + \varrho^2 g_N$, g_L and g_N, respectively. It is easy to show that $\nabla_S^{L\times_\varrho N} T = \nabla_S^L T$ and $\nabla_X^{L\times_\varrho N} T = \nabla_T^{L\times_\varrho N} X = T(\eta)X$ where $\eta = \log \varrho$. Therefore,

$$\begin{aligned}\nabla_T^{L\times_\varrho N} \operatorname{grad}_{L\times_\varrho N} \Gamma &= \nabla_T^L \operatorname{grad}_L \gamma \\ \nabla_X^{L\times_\varrho N} \operatorname{grad}_{L\times_\varrho N} \Gamma &= \operatorname{grad}_L \gamma(\eta)\, X.\end{aligned}$$

Hence,

$$\begin{aligned}&\operatorname{Hess}_{L\times_\varrho N} \Gamma(T, S) = \operatorname{Hess}_L \gamma(T, S), \quad \operatorname{Hess}_{L\times_\varrho N} \Gamma(T, X) = 0 \\ &\operatorname{Hess}_{L\times_\varrho N} \Gamma(X, Y) = \langle \operatorname{grad}_L \eta, \operatorname{grad}_L \gamma\rangle\langle X, Y\rangle.\end{aligned}$$

For any unit vector $e \in T_q M$, decompose $e = e^L + e^N$, where $e^L \in T_{y(q)}L$ and $e^N \in T_{z(q)}N$. Then, we have at $\varphi(q)$

$$\operatorname{Hess}_{L\times_\varrho N} \Gamma(e, e) = \operatorname{Hess}_L \gamma(y(q))(e^L, e^L) + \langle \operatorname{grad}_L \gamma, \operatorname{grad}_L \eta\rangle(y(q))|e^N|^2.$$

On the other hand, $\operatorname{Hess}_M \varrho(q)(e, e) = \operatorname{Hess}_{L\times_\varrho N} \Gamma(e, e) + \langle \operatorname{grad}_{L\times_\varrho N} \Gamma, \alpha(e, e)\rangle$. Therefore,

$$\begin{aligned}\operatorname{Hess}_M \varrho(q)(e, e) = \operatorname{Hess}_L \gamma(e^L, e^L) &+ \langle \operatorname{grad}_L \gamma, \operatorname{grad}_L \eta\rangle(z(q))|e^P|^2 \\ &+ \langle \operatorname{grad}_L \gamma, \alpha(e, e)\rangle.\end{aligned} \tag{63}$$

However,

$$\operatorname{Hess}_L \gamma \leq \sqrt{\mathcal{G}(\gamma)} \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right), \tag{64}$$

out of a compact subset of L. By hypothesis, see (60),

$$\begin{aligned} \langle \operatorname{grad}_L \gamma, \operatorname{grad}_L \eta \rangle (y(q)) &\leq |\operatorname{grad}_L \gamma| \cdot |\operatorname{grad}_L \eta| \\ &\leq \sqrt{\mathcal{G}(\gamma)} \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right) \ln \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right). \end{aligned} \tag{65}$$

Considering (64), (65) and (63), we have that (off a compact set)

$$\begin{aligned} \operatorname{Hess}_M \varrho(q)(e,e) \leq C \cdot \sqrt{\mathcal{G}(\gamma)} \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right) \ln \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right) \\ + \langle \operatorname{grad}_L \gamma, \alpha(e,e) \rangle, \end{aligned}$$

for some positive constant C. Thus, by (61), it follows that

$$\triangle \gamma \leq B \sqrt{G(\gamma)} \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right) \ln \left(\int_0^{\gamma} \frac{\mathrm{d}s}{\sqrt{\mathcal{G}(s)}} + 1 \right)$$

for some positive constant B. Concluding that $(\mathcal{G}, \varrho)$ is an Omori–Yau pair for the Laplacian in M. □

The proof of [4, Thm. 1] coupled with Lemma 3 allows us to state the following technical extension of Alias–Dajczer's Theorem [4, Thm. 1].

Theorem 11 (Alias–Dajczer) *Let $\varphi\colon M \to L \times_\varrho N$ be an isometric immersion, proper in the first entry, where L carries an Omori–Yau pair $(\mathcal{G}, \gamma)$ for the Hessian, $\varrho \in C^\infty(L)$ is a positive function satisfying*

$$|\operatorname{grad} \log \varrho(y)| \leq \ln \left(\int_0^{\gamma(y)} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right). \tag{66}$$

Letting $\varphi(x) = (y(x), z(x))$ and if

$$|\vec{H}(\varphi(x))| \leq \ln \left(\int_0^{\gamma(y(x))} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right). \tag{67}$$

Suppose that $\varphi(M) \subset \{(y,z) : y \in L,\ z \in B_N(r)\}$; then,

$$\sup_M \varrho |\vec{H}| \geq (m - \ell) C_b(r),$$

where $b = \sup_{B_N(r)} K_N^{\mathrm{rad}}$.

Remark 6 The Theorems 5 and 6 should have versions for ϕ-bounded submanifold of warped product $L \times_{\varrho} N$. Specially interesting should be the Jorge–Koutrofiotis Theorem in this setting. We leave to the interested reader to pursue it.

As a last application of Theorem 10, let $N^{n+1} = I \times_{\varrho} P^n$ the product manifold endowed with the warped product metric, $I \subset \mathbb{R}$ is a open interval, P^n is a complete Riemannian manifold, and $\varrho\colon I \to \mathbb{R}_+$ is a smooth function. Given an isometrically immersed hypersurface $\varphi\colon M^n \to N^{n+1}$, define $h : M^n \to I$ the $C^\infty(M^n)$ height function by setting $h = \pi_I \circ \varphi$, where $\pi_I\colon I \times P \to I$ is a projection. This result below is a technical extension of [5, Thm. 7]. The key point is to prove the existence of an Omori–Yau sequence. Here, we prove it under hypotheses weaker than of those of [5].

Theorem 12 *Let $\varphi\colon M^n \to N^{n+1}$ be an isometrically immersed hypersurface. If M^n has an Omori–Yau pair $(\mathcal{G}, \gamma)$ for the Laplacian and the height function h satisfies $\lim_{x\to\infty} \frac{h(x)}{\psi(\gamma(x))} = 0$, then*

$$\sup_{M^n} |\overrightarrow{H}| \geq \inf_{M^n} \mathcal{H}(h), \tag{68}$$

with $\overrightarrow{H}$ being the mean curvature and $\mathcal{H}(t) = \frac{\rho'(t)}{\rho(t)}$.

Acknowledgments We want to express our gratitude to our friend Newton Santos for their suggestions along the preparation on this paper. We also would like to heartily thank a generous and anonymous referee that thoroughly checked every detail of this paper.

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