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On a class of generalised Schmidt groups

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Abstract In this paper families of non-nilpotent subgroups covering the non-nilpotent part of a finite group are considered. An A_5-free group possessing one of these families is soluble, and soluble groups with this property have Fitting length at most three. A bound on the number of primes dividing the order of the group is also obtained.

Keywords Finite groups · Nilpotent groups · Maximal subgroups

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1 Introduction and statement of results

All groups considered in this paper are finite.

The results presented here spring from the classical results of Schmidt [15] about the structure of the minimal non-nilpotent groups and later developments from them [1–3,5,11–14,16]. Schmidt proved that if all the maximal subgroups of a group G are nilpotent, then G is soluble and that, in addition, if G is not nilpotent, $|G|$ has exactly two distinct prime factors, G has a normal Sylow subgroup and a cyclic non-normal Sylow subgroup. These groups are called *minimal non-nilpotent groups* or *Schmidt groups*.

Rose [14] studied the effects of replacing maximal subgroups by non-normal (or abnormal) maximal subgroups in the hypothesis of Schmidt's result, and the following fact is established:

Theorem A *If every non-normal maximal subgroup of a group G is nilpotent, then G has a normal Sylow subgroup P such that G/P is nilpotent.*

We shall say that a group G is a *Rose group* if every non-normal maximal subgroup of G is nilpotent.

In a recent paper [12], Li and Guo characterised Rose groups by means of certain families of normal non-nilpotent subgroups and obtained more detailed information about the number of primes dividing the order of the group. They also gave an alternative proof for solubility.

Theorem B *Let G be a Rose group. Then*

1. *G is soluble;*
2. *G is p-nilpotent for some prime p;*
3. *If G is non-nilpotent, then $2 \leq |\pi(G)| \leq k+2$, where k is the number of normal maximal subgroups of G which are not nilpotent.*

The present paper furnishes extensions of the main results of Rose, Li, and Guo and was motivated by some ideas of the paper [12]. We consider families of non-nilpotent subgroups covering the non-nilpotent part of the group and analyse how they determine the group structure.

It is abundantly clear that our results are not a mere exercise in generalisation. In fact, Theorem A and Theorem B cannot be extended directly: The alternating group of degree 5 is a fundamental obstruction to get solubility. We must seek to discover how nearly a non-nilpotent group with some of our coverings is soluble. With this purpose in view, we consider the solubility question (Theorem C) and give more detailed structural information in the soluble case.

Definition 1.1 Let G be a non-nilpotent group. A *Schmidt covering* of G is a, possibly empty, family of non-nilpotent proper subgroups $\mathscr{S} = \{K_1, \ldots, K_n\}$ of G satisfying the following two conditions:

1. If $i, j \in \{1, \ldots, n\}$ and $i \neq j$, then K_i is not contained in K_j.
2. If T is a proper subgroup of G supplementing the nilpotent residual $G^{\mathfrak{N}}$ of G, then at least one of the following statements holds:
   (a) T is nilpotent,
   (b) $T \in \mathscr{S}$, or
   (c) T is contained in K_t for all $t \in \{1, \ldots, n\}$.

Recall that the nilpotent residual of a group is the smallest normal subgroup with nilpotent quotient. We say that a group G is an *NNC-group* if G has a Schmidt covering. If G is a

Schmidt group, then G is an *NNC*-group with an empty Schmidt covering, and if G is a non-nilpotent Rose group, then the empty set and the set of all non-nilpotent normal maximal subgroups of G are both Schmidt coverings of G. Hence, G is an *NNC*-group. However, the symmetric group of degree 4 shows that the class of Rose groups is a proper subclass of the class of all *NNC*-groups.

Our first main theorem is the following.

Theorem C *Let G be an NNC-group. If G has no section isomorphic to* A_5*, then G is soluble.*

We note that the proof of the above result relies on the classification of finite simple groups.

For G, a non-trivial soluble group, we let $\mathrm{F}(G)$ denote the Fitting subgroup of G. The subgroups $\mathrm{F}_i(G)$ are defined inductively by $\mathrm{F}_0(G) = 1$ and $\mathrm{F}_{i+1}(G)/\mathrm{F}_i(G) = \mathrm{F}(G/\mathrm{F}_i(G))$. The smallest non-negative integer n such that $\mathrm{F}_n(G) = G$ is the Fitting length $\mathrm{l}(G)$ of G. The trivial group has Fitting length 0; a non-trivial nilpotent group has Fitting length 1, and if $G \neq 1$, then $\mathrm{l}(G/\mathrm{F}(G)) = \mathrm{l}(G) - 1$.

According to Theorem A, a Rose group has Fitting length at most 2. The symmetric group of degree 4 is an *NNC*-group of Fitting length 3. Hence, Theorem A does not hold for soluble *NNC*-groups. However, we have:

Theorem D *Let G be a soluble NNC-group. Then the Fitting length of G is at most* 3.

In view of the third assertion of Theorem B, it is of interest to inquire whether there is a bound on the number of distinct prime factors of $|G|$, at least when G is a soluble *NNC*-group. Note that every non-empty Schmidt covering contains every conjugacy class of abnormal maximal subgroups, and a bound for the number of distinct primes dividing the order of a group is naturally related to the number of conjugacy classes of subgroups contained in the maximal covering.

Let G be an *NNC*-group. We say that the Schmidt covering $\mathscr{A}$ is the *maximal covering* of G if $\mathscr{A}$ contains every non-nilpotent maximal subgroup of G.

Theorem E *Let G be a soluble NNC-group. Let* $\mathscr{A}$ *be the maximal covering of G. Then* $2 \leq |\pi(G)| \leq l + 2$*, where l is the number of conjugacy classes of subgroups contained in* $\mathscr{A}$.

Note that the order of a Schmidt group is divisible by two different primes, and if A is a Schmidt group and p is a prime that does not divide its order, and B is a cyclic group of order p, then $G = A \times B$ is an *NNC*-group and $\{A\}$ is the maximal Schmidt covering of G. Hence, the bounds of the above theorem are best possible.

2 Preliminaries

Before taking up the proofs of our main results, we shall give in this section a few very useful results on *NNC*-groups.

Recall that a subgroup H of a group G is abnormal in G if $g \in \langle H, H^g \rangle$ for all $g \in G$. Our first result shows that every abnormal subgroup in a Schmidt covering of G should be a maximal subgroup of G.

Proposition 2.1 *Let* $\{K_1, \ldots, K_n\}$ *be a Schmidt covering of an NNC-group G. If, for some* $j \in \{1, \ldots, n\}$*,* K_j *is not maximal in G, then there exists a normal maximal subgroup L of G containing* K_j *such that* $\{L\} \cup \{K_t : K_t \nleq L\}$ *is a Schmidt covering of G.*

Proof Suppose that K_j is not maximal in G, for some $j \in \{1, \ldots, n\}$. Let L be a maximal subgroup of G containing K_j. Clearly, $L \notin \{K_1, \ldots, K_n\}$ and $L \nleq K_j$. If L were not normal in G, we would have $G = G^{\mathfrak{N}}L$, and since $\{K_1, \ldots, K_n\}$ is a Schmidt covering of G, this would imply the nilpotency of L. Hence, L is normal in G. Clearly, $\{L\} \cup \{K_t : K_t \nleq L\}$ is a Schmidt covering of G.

Corollary 2.2 *Let G be an NNC-group, which is not a Schmidt group. Then G has a Schmidt covering composed of maximal subgroups of G, and every Schmidt covering of G contains each non-nilpotent abnormal maximal subgroup of G.*

As we said in the 'Introduction', the symmetric group of degree 4 is a typical example of an *NNC*-group which is not a Rose group. However, *NNC*-groups with small Schmidt coverings are Rose groups.

Proposition 2.3 *Let* $\mathscr{A} = \{K_1, \ldots, K_n\}$ *be a Schmidt covering of an NNC-group G. If the number of maximal subgroups of G in* $\mathscr{A}$ *is at most* 2*, then G is a Rose group.*

Proof We may suppose that $\mathscr{A}$ is non-empty. Let k denote the number of maximal subgroups of G in $\mathscr{A}$.

Assume $k = 1$, and K_1 is the unique maximal subgroup of G in $\mathscr{A}$. Let $L \neq K_1$ be a maximal subgroup of G conjugate to K_1 in G. Since L is not nilpotent, it follows that L belongs to $\mathscr{A}$. Hence, K_1 is normal in G, and then every abnormal maximal subgroup of G is nilpotent. Then G is a Rose group.

Suppose $k = 2$ and that K_1 is one of the maximal subgroups of G in $\mathscr{A}$. If K_1 were not normal in G, then $\mathscr{A}$ would contain any conjugate of K_1 in G. This would mean that $|G : K_1| = 2$, and $K_1 \trianglelefteq G$, contrary to our assumption. Hence, the maximal subgroups of G in $\mathscr{A}$ are normal in G, and G is a Rose group. □

Lemma 2.4 *Let G be an NNC-group, and let* $\{K_1, \ldots, K_n\}$ *be a Schmidt covering of G. Assume that, for some* $i \in \{1, \ldots, n\}$, K_i *is not contained in any normal maximal subgroup of G. Then* $n > 2$*, and* K_i *is a Rose group.*

Proof According to Proposition 2.1, K_i is an abnormal maximal subgroup of G. Hence, $G = G^{\mathfrak{N}}K_i$. Since K_i is not nilpotent, it follows that $n > 2$. Let T be a maximal subgroup of K_i such that $K_i = K_i^{\mathfrak{N}}T$. Then $G = G^{\mathfrak{N}}T$. Since $T \notin \{K_1, \ldots, K_n\}$, we conclude that T is nilpotent. Consequently, K_i is a Rose group,

The next result is particularly useful when an inductive argument involving quotient groups is applied.

Lemma 2.5 *Let G be an NNC-group, and let* $\mathscr{S} = \{K_1, \ldots, K_n\}$ *be a Schmidt covering of G. If N is a normal subgroup of G, then one of the following statements holds:*

1. G/N *is nilpotent;*
2. $G = NK_i$ *for all* $i \in \{1, \ldots, n\}$ *and* G/N *is a Rose group;*
3. $\{K_i/N : N \leq K_i, K_i/N$ *is non-nilpotent*$\}$ *is a Schmidt covering of* G/N.

In other words, G/N *is either nilpotent or an NNC-group.*

Proof Suppose that G is a Schmidt group. If N is contained in the Frattini subgroup of G, it follows that G/N is a Schmidt group, and if N is supplemented in G by a maximal subgroup of G, then G/N is nilpotent. Hence, the lemma holds in this case. Therefore, by

Corollary 2.2, we may assume that $\mathscr{S}$ is a non-empty Schmidt covering of G composed of maximal subgroups. Suppose, now, that G/N is not nilpotent. If N is not contained in any K_i, then $G = NK_i$ for all $i \in \{1, \ldots, n\}$. Let T/N be an arbitrary maximal subgroup of G/N. Then $T \notin \{K_1, \ldots, K_n\}$, and so either $T \trianglelefteq G$ or T is nilpotent. Consequently, G/N is a Rose group, and Statement 2 holds.

Assume that $\mathscr{A} := \{K_1, \ldots, K_r\}$ is the non-empty set of subgroups of $\{K_1, \ldots, K_n\}$ containing N, $1 \leq r \leq n$, and write

$$\mathscr{C} := \{K_i/N : \ K_i/N \text{ is non-nilpotent}, 1 \leq i \leq r\}.$$

We show that $\mathscr{C}$ is a Schmidt covering of G/N. If $\mathscr{C}$ is empty, then every abnormal maximal subgroup of G containing N is nilpotent. Hence, G/N is a Rose group, and the lemma holds in this case. Suppose that $\mathscr{C}$ is non-empty. It is clear that $\mathscr{C}$ satisfies Condition 1 of Definition 1.1. Let T/N be a proper subgroup of G/N such that $T/N \notin \mathscr{C}$, $T/N \nleq K_t/N \in \mathscr{C}$, and $G/N = (G/N)^{\mathfrak{N}} \cdot (T/N)$. Then $G = G^{\mathfrak{N}}T$. If $T \leq K_j$ for some $1 \leq j \leq r$ such that K_j/N is nilpotent, then T/N is also nilpotent. Otherwise, $T \notin \{K_1, \ldots, K_n\}$. Since $\mathscr{S}$ is a Schmidt covering of G, T is nilpotent. In both cases, we conclude that T/N is nilpotent. Therefore, $\mathscr{C}$ is a Schmidt covering of G/N, and Statement 3 holds. □

The above lemma allows us to obtain an amenable characterisation of the Schmidt coverings.

Proposition 2.6 *Let $\mathscr{A} = \{K_1, \ldots, K_n\}$ be a family of non-nilpotent subgroups of a group G such that K_i is not contained in K_j, if $i \neq j$, $i, j \in \{1, \ldots, n\}$. Then $\mathscr{A}$ is a Schmidt covering of G if and only if every proper supplement of the nilpotent residual of G not belonging to $\mathscr{A}$ is nilpotent.*

Proof It is clear that only the necessity of the condition is in doubt. Assume that $\mathscr{A}$ is a Schmidt covering of G, and let $1 \neq T$ be a proper subgroup of G such that $G = G^{\mathfrak{N}}T$ and T does not belong to $\mathscr{A}$. We prove that T is nilpotent by induction on the order of G. If T is not contained in some K_i, then T is nilpotent. Hence, we may assume that $T \leq \bigcap\{K_i : 1 \leq i \leq n\}$ and $1 \neq G^{\mathfrak{N}}$. In particular, $\mathscr{A}$ is non-empty and so $\mathscr{A}$ contains every abnormal maximal subgroup of G. Let X be the intersection of the abnormal non-nilpotent maximal subgroups of G. Then X is normal in G and $XG^{\mathfrak{N}} = G$.

Suppose first that G is soluble. Since X is a proper normal subgroup of G, there exists a maximal normal subgroup Y of G containing X. Then G/Y is nilpotent, and so $G^{\mathfrak{N}}$ is contained in Y. Therefore, $G = XG^{\mathfrak{N}} \leq Y$. This contradiction shows that G cannot be soluble.

Suppose now that G is a monolithic primitive group. Let N be the minimal normal subgroup of G, and let M be a maximal subgroup of G such that $G = NM$. Note that M cannot be normal in G, because otherwise $N \leq M$ and this would imply $G = M$. Hence, M is abnormal. If M were non-nilpotent, then $N \leq X \leq M$ and so $N \leq M$ again. Hence, M is nilpotent. Suppose first that N is non-abelian. By a result of Rose [14, Lemma 1], G has even order and $M = \mathrm{N}_G(G_2)$ for some Sylow subgroup G_2 of G. In particular, all maximal subgroups supplementing N are conjugate. By [4, 1.1.11 (4)], it follows that G cannot be a primitive group of type 2, that is, N is abelian and G is a primitive group of type 1. Since M is nilpotent, G is soluble, but we have seen that this cannot happen.

Now we note that an epimorphic image of an *NNC*-group is an *NNC*-group or a nilpotent group by Lemma 2.5. Let M be a maximal subgroup of G. Suppose that G/M_G is an *NNC*-group. Suppose that

$$\{K_i/M_G : M_G \le K_i,\ \ K_i/M_G \text{ is non-nilpotent}\}$$

is non-empty, by Lemma 2.5 it is a Schmidt covering of G/M_G. Furthermore, XM_G/M_G is contained in the intersection of all maximal abnormal subgroups of the Schmidt covering $\mathscr{A}$ containing M_G. By the arguments of the previous paragraph, if G/M_G is a primitive group of type 1 or a primitive group of type 2, XM_G/M_G is a nilpotent group. It follows that $X/(X \cap M_G)$ is nilpotent. Obviously, the same conclusion holds if G/M_G is nilpotent. Suppose that G/M_G is a Rose group. Then M/M_G is nilpotent, and G/M_G is soluble by Theorem B. Let Z/M_G be the unique minimal normal subgroup of G/M_G. Suppose that X is not contained in M_G. Then XM_G/M_G is a normal subgroup of G/M_G and so $Z \le XM_G$. Since M/M_G is nilpotent, XM_G/M_G is contained in a normal maximal subgroup Y/M_G of G/M_G. But G/Z is isomorphic to the nilpotent group M/M_G. Hence, $G^{\mathfrak{N}}$ is contained in Z and so $G = XG^{\mathfrak{N}} \le XZ \le XM_G \le Y$. This contradiction shows that $X \le M_G$ and so XM_G/M_G is trivial. In particular, XM_G/M_G is a nilpotent group. Now assume that G/M_G is a primitive group of type 3. Let W_1/M_G and W_2/M_G be the minimal normal subgroups of G/M_G. By [4, 1.1.13], G/W_1 and G/W_2 are primitive groups of type 2. By the previous arguments, $X/(X \cap W_1)$ and $X/(X \cap W_2)$ are nilpotent. Hence, $X/(X \cap W_1 \cap W_2) = X/(X \cap M_G)$ is nilpotent. Since $X/(X \cap M_G)$ is nilpotent for all maximal subgroups M of G and the class of all nilpotent groups is a formation, we will get that $X/(X \cap \Phi(G))$ is nilpotent. By a result of Gaschütz [7, III, 3.5], we obtain that X is nilpotent. It follows that T is nilpotent. □

3 Proofs of the main results

Proof of Theorem C Suppose that the result is false, and let the group G provide a counterexample of least possible order. Then, by Proposition 2.3 and Theorem A, the number of abnormal maximal subgroups in every Schmidt covering of G is greater than 2. Since the properties of G, as enunciated in the statement of the theorem, are inherited by non-nilpotent quotients of G, the minimality of G implies that G has a unique minimal normal subgroup, say N. Then N must be insoluble, $\mathrm{C}_G(N) = 1$, and N is a direct power of a simple non-abelian group. Assume that G/N is not nilpotent, and let H/N be a non-normal maximal subgroup of G/N. Then H is a non-normal maximal subgroup of G supplementing the nilpotent residual of G. If H is not in any Schmidt covering of G, then H is nilpotent; otherwise, H is a Rose group by Lemma 2.4. In both cases, H should be soluble, contrary to assumption. Therefore, G/N is nilpotent and $N = G^{\mathfrak{N}}$ is the nilpotent residual of G.

Let $\mathscr{A}$ be a Schmidt covering of G composed of maximal subgroups. Write $N = S_1 \times \cdots \times S_m$, where S_j are pairwise isomorphic non-abelian simple groups, $j \in \{1, \dots, m\}$. Let us denote $C = \mathrm{C}_G(S_1)$, $M = \mathrm{N}_G(S_1)$, and $K = S_2 \times \cdots \times S_m$. Let H be a core-free maximal subgroup of G. Then H is soluble. Write $V = M \cap H$.

Applying [4, 1.1.52], one of the following statements holds:

1. G is an almost simple group;
2. (G, H) is equivalent to a primitive pair with simple diagonal action; in this case, $H \cap N$ is a full diagonal subgroup of N;
3. (G, H) is equivalent to a primitive pair with product action such that $H \cap N \cong D_1 \times \cdots \times D_l$, a direct product of $l > 1$ subgroups such that, for each $j = 1, \dots, l$, the subgroup D_j is a full diagonal subgroup of a direct product $\prod_{i \in \mathcal{I}_j} S_i$.
4. (G, H) is equivalent to a primitive pair with product action such that the projection $R_1 = (H \cap N)^{\pi_1}$ is a non-trivial proper subgroup of S_1; in this case $R_1 = VC \cap S_1$ and

VC/C is a maximal subgroup of the almost simple group M/C, where π_1 is a projection from N to S_1;

5. $H \cap N = 1$.

The solubility of H implies that $H \cap N$ cannot contain any copy of the composition factor of N. If $H \cap N = 1$, we know, by a result of Lafuente (see [4, 1.1.51 (2)]), that H would be a primitive group with non-abelian socle. Since H is soluble, it follows that G is an almost simple group or (G, H) is equivalent to a primitive pair with product action such that the projection $R_1 = (H \cap N)^{\pi_1}$ is a non-trivial proper subgroup of S_1. Assume that N is not simple. Since M/C is almost simple, we have that M/K is neither nilpotent nor a Rose group. In addition, N/K is the nilpotent residual of M/K. Let L/K a non-nilpotent maximal subgroup of M/K supplementing N/K in M/K. Applying [4, 1.1.35], there exists a maximal subgroup A of $G = AN$ such that $L = (A \cap M)K$. Then A belongs to $\mathscr{A}$. Since every supplement U/K of N/K in M/K is of the form $U/K = (B \cap M)K/K$ for some supplement B of N in G by [4, 1.1.35], we conclude that the set of all non-nilpotent maximal subgroups of M/K supplementing N/K is a Schmidt covering of M/K. Therefore, M/K is an NNC-group with no sections isomorphic to A_5. The minimal choice of G implies that M/K is soluble, contradicting the fact that N/K is non-abelian simple. Thus, G must be an almost simple group.

Let $p \geq 5$ be a prime dividing $|N|$, and let P be a Sylow p-subgroup of N. Then $G = N\mathrm{N}_G(P) = G^{\mathfrak{N}}\mathrm{N}_G(P)$, and so either $\mathrm{N}_G(P)$ is contained in some maximal abnormal subgroups of $\mathscr{A}$ or $\mathrm{N}_G(P)$ is nilpotent. If the latter were true, then we would have $\mathrm{O}^p(N) < N$ by [8, X, 8.13], contrary to the hypothesis. Therefore, $Z = \mathrm{N}_G(P)$ is contained in some abnormal maximal subgroups E belonging to $\mathscr{A}$. By Lemma 2.4, E is a Rose group. Applying Theorem A, $Z^{\mathfrak{N}}$ is a q-group for some prime q. Suppose that $q \neq p$, and let Q be a Sylow q-subgroup of Z. Then Q is normal in Z and $Z = QZ_{q'}$, where $Z_{q'}$ is a Hall q'-subgroup of Z. In particular, $Z_{q'}$ is nilpotent. Moreover, $Z_{q'}$ cannot be a normal subgroup of Z; otherwise, we should have Z nilpotent, contradicting our hypothesis. Hence, $\mathrm{N}_Z(Z_{q'})$ is contained in a non-normal maximal subgroup Y of Z. Let W be a maximal subgroup of E containing Y. Then W is not normal in E and so W is nilpotent. Thus, Y is nilpotent and $Z_{q'}$ centralises P. Since Q also centralises P, we have that $Z/\mathrm{C}_G(P)$ is a p-group. Applying [8, X, 8.13], $\mathrm{O}^p(N) < N$, contradicting the prescribed minimality of N. Consequently $q = p$. Then the nilpotent residual of E is a p-group and so a Sylow p-subgroup of E is normal in E. It implies that $E = Z$. In this case, P is a Sylow p-subgroup of G.

Assume that G is not simple. Applying [10, 1.1], there exists a normal subgroup G_0 of G which is minimal such that $E_0 := E \cap G_0$ is maximal in G_0, and E is isomorphic to E_0G/G_0. Moreover, with the exceptions for which the numbers c of conjugacy classes of such subgroups E_0 are listed in [10, Table 1], all of such subgroups E_0 are conjugate. Note that the subgroups E_0 contain a Sylow subgroup of G for a prime $p \geq 5$. Therefore, N cannot be isomorphic to any simple group in [10, Table 1]. Hence, all of such subgroups E_0 are conjugate, and they are the normalisers of a Sylow p-subgroup of G for a fixed prime $p \geq 5$ dividing the order of N. This implies that $|N|$ is divisible by exactly three different primes. According to [9, Table 1], N is isomorphic to A_5, A_6, $\mathrm{PSU}(4, 2)$, $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 8)$, $\mathrm{PSU}(3, 3)$, $\mathrm{PSL}(3, 3)$, or $\mathrm{PSL}(2, 17)$. Clearly, N cannot be isomorphic to A_5, A_6, or $\mathrm{PSU}(4, 2)$ since G is A_5-free. Since the nilpotent residual of G is isomorphic to N, $G \cong \mathrm{Aut}(N)$ and G is an almost simple group, it follows by Atlas [6] that $|G : N| = 2$ if $N \in \{\mathrm{PSL}(2, 7), \mathrm{PSU}(3, 3), \mathrm{PSL}(3, 3), \mathrm{PSL}(2, 17)\}$, and $|G : N| = 3$ if $N = \mathrm{PSL}(2, 8)$. Let p be the largest prime divisor of $|N|$ and P a Sylow p-subgroup of N. Then $|P| = p$ and $\mathrm{N}_G(P)$ belongs to $\mathscr{A}$. By [6], $\mathrm{N}_G(P)$ is isomorphic to $7 : 6$ if N is isomorphic to one of the groups PSL(2, 7), PSL(2, 8), or PSU(3, 3); it is isomorphic to 13 : 6 if $N \cong$ PSL(3, 3), or isomorphic to 17 : 16 if $N \cong$ PSL(2, 17). Hence, $\mathrm{N}_G(P)$ has a subgroup K isomorphic to 7 : 2, 7 : 3, 7 : 2, 13 : 2, 17 : 8, if N is isomorphic to PSL(2, 7), PSL(2, 8), PSU(3, 3), PSL(3, 3), or PSL(2, 17), respectively. Since $\mathrm{N}_G(P)$ is a Rose group, we have that K is nilpotent. This contradiction implies that G cannot be an almost simple group.

Consequently, G is a simple group. Then every maximal subgroup of G either is nilpotent or belongs to $\mathscr{A}$. Therefore, every maximal subgroup is either nilpotent or a Schmidt group. By [7, II, 7.5], G is isomorphic to one of the following groups:

1. PSL(2, p), where p is a prime, $p > 3$, and $p^2 - 1 \not\equiv 0 \pmod 5$;
2. PSL(2, 2^q), where q is a prime,
3. PSL(2, 3^q), where q is an odd prime,
4. PSL(3, 3), or
5. Sz(2^q), where q is an odd prime.

In the following, we analyse each one of these cases and derive a contradiction.

1. $G \cong$ PSL(2, p), where $p > 3$ is a prime and $p^2 - 1 \not\equiv 0 \pmod 5$.

Applying [7, II, 8.27], G has subgroups $G_1 \cong D_r$ and $G_2 \cong D_s$, where $r = 2 \cdot \frac{p+1}{(2,p-1)} = p+1$ and $s = 2 \cdot \frac{p-1}{(2,p-1)} = p-1$. Since G_1 and G_2 are either nilpotent or Schmidt groups, it follows that $\frac{r}{2} = \frac{p+1}{2}$ and $\frac{s}{2} = \frac{p-1}{2}$ are either a power of 2 or a prime. Note that $p-1$, p, and $p-1$ are consecutive integers, and so 3 must divide one of them. As $p > 3$ is a prime, we have that $3 \nmid p$. Assume that $3 \mid p+1$. Then $6 \mid p+1$, which implies $3 \mid \frac{p+1}{2}$, and thus $\frac{p+1}{2} = 3$ since $\frac{p+1}{2}$ is a prime. Hence, $p = 5$. In this case, $G \cong$ PSL(2, 5) $\cong A_5$, contradicting our assumption. Analogously, $3 \mid p - 1$ yields $p = 7$. Then G has a maximal subgroup isomorphic to S_4, which is neither nilpotent nor a Schmidt group. This is a contradiction.

2. $G \cong$ PSL(2, 2^q), where q is a prime.

With similar arguments to those used above, we have that $3 \mid 2^q - 1$ and $q = 2$. Then $G \cong$ PSL(2, 4) $\cong A_5$, a contradiction.

3. $G \cong$ PSL(2, 3^q), where q is an odd prime. We again argue as in the above cases. Since p is odd and $\frac{3^q+1}{2}$, $\frac{3^q-1}{2}$ are consecutive integers, it implies that only one of them is a power of 2.

Assume first $\frac{3^q+1}{2} = 2^{\alpha_1}$ for some integer α_1. Then

$$3^q + 1 = 2^\alpha \tag{1}$$

where $\alpha = \alpha_1 + 1$. Write $q = 2k + 1$. If $\alpha \geq 3$, we can take classes module 8 to conclude that $\bar{3}^{2k+1} + \bar{1} = \bar{0}$, and so $\bar{4} = \bar{0}$. This contradiction yields $\alpha = 1$ or 2. As a result, $q = 0$ or 1. This is impossible.

Hence,

$$3^q - 1 = 2^{\beta_1} \tag{2}$$

for some integer β_1. We can argue as above to get a contradiction.

4. $G \cong$ PSL(3, 3).

In this case, G has a subgroup isomorphic to GL(2, 3), which is neither nilpotent nor a Schmidt group.

5. $G \cong$ Sz(2^q), where q is an odd prime.

It is known that G has a subgroup isomorphic to Sz(2), which is neither nilpotent nor a Schmidt group.

Consequently, G cannot be a non-abelian simple group, and this contradiction establishes the theorem. □

Proof of Theorem D We use induction on the order of G. We may assume that G is not a Rose group by Theorem A. Then, by Lemma 2.5, quotient groups of G that are non-nilpotent also are *NNC*-groups. Let $\mathscr{A}$ be a Schmidt covering of G, and let N be a minimal normal subgroup of G contained in $G^{\mathfrak{N}}$. Then G/N is either nilpotent or an *NNC*-group. Therefore, the Fitting length of G/N is at most three. If $N \leq \Phi(G)$, then G has length at most three since the class of groups of Fitting length at most three is a saturated formation. Assume that $N \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = NM$. If M does not belong to $\mathscr{A}$, then M is nilpotent, and hence, the Fitting length of G is two. Suppose that $M \in \mathscr{A}$. Since $G = G^{\mathfrak{N}}M$, we conclude that M is abnormal in G. Then M is a Rose group by Proposition 2.4. Applying Theorem A, M has Fitting length at most two, and so the Fitting length of G is at most three. The proof of the theorem is now complete. □

Proof of Theorem E Assume that the result is false, and let the *NNC*-group G provide a counterexample of minimal order. Put $\mathscr{A} = \{K_1, \ldots, K_n\}$.

Let N be a minimal normal subgroup of G contained in $G^{\mathfrak{N}}$. Then Lemma 2.5 implies that G/N is either nilpotent or an *NNC*-group.

Assume that G/N is an *NNC*-group. There are two possibilities:

1. There exists $j \in \{1, \ldots, n\}$ such that $\mathrm{Core}_G(K_j) = 1$. In this case, G is a primitive group and $G = NK_j$. Write $A = K_j$. Then A is a Rose group, and $A^{\mathfrak{N}}$ is a q-group for some prime $q \neq p \in \pi(N)$. Moreover, all core-free maximal subgroups of G are conjugate (see [4, 1.1.10]). Let C be a Carter subgroup of A. Then $A = A^{\mathfrak{N}}C$ and NC is a proper subgroup of G supplementing the nilpotent residual of G.
   Assume that NC does not belong to $\mathscr{A}$. Since $\mathscr{A}$ is a Schmidt covering of G, NC is nilpotent. Suppose that we have two distinct primes dividing the order of the Hall q'-subgroup of A. Then there exists a Sylow subgroup of NC centralising N. This contradicts the fact that $\mathrm{C}_G(N) = N$ (see [4, 1.1.7]). This contradiction implies $|\pi(G)| \leq 3 \leq l+2$, against our choice of G.
   Consequently, we may assume that NC is a maximal subgroup of G belonging to $\mathscr{A}$. All these maximal subgroups are conjugate in G because C is a Carter subgroup of G and all Carter subgroups are conjugate (see [4, 2.3.2]). Put $\pi(C) = \{p_1, \ldots, p_t\}$. Then $|\pi(G)| \leq t + 2$. If $t = 1$, then $|\pi(G)| \leq 3 \leq l + 2$. Hence, we may assume that $t > 1$. Let C_i be a maximal subgroup of C such that $|C : C_i| = p_i$ for all $i \in \{1, \ldots, t\}$. We have that $S_i = NA^{\mathfrak{N}}C_i$ is a non-nilpotent maximal normal subgroup of G for all $i \in \{1, \ldots, t\}$, and hence, $\{S_1, \ldots, S_t\}$ is contained in $\mathscr{A}$. This means that $l \geq t + 2$. This contradicts our supposition.
2. $\mathrm{Core}_G(K_j) \neq 1$ for all $j \in \{1, \ldots, n\}$. Assume that there exists $i \in \{1, \ldots, n\}$ such that $C = K_i$ is an abnormal maximal subgroup of G. Let L be a minimal normal subgroup of G contained in C. Then G/L is not nilpotent, and by Lemma 2.5,

   $$\mathscr{C} = \{K_j/L : \ L \leq K_j, \ K_j/L \text{ is non-nilpotent}\}$$

   is a Schmidt covering of G/L. Assume that $\mathscr{C}$ is empty. Then every maximal subgroup of G/L is nilpotent, and so G/L is a Schmidt group. In this case $|\pi(G/L)| = 2$ and so $|\pi(G)| \leq 3$, contrary to supposition. Assume that $\mathscr{C}$ is non-empty. If L is not contained in some K_t for some $t \in \{1, \ldots, n\}$, then the minimal choice of G forces $2 \leq |\pi(G/L)| \leq (l-1)+2$. Therefore, $2 \leq |\pi(G)| \leq l+2$. Hence, we can assume that L is contained in every element of $\mathscr{A}$. If G/L were a Rose group, then C/L would be nilpotent and the number of conjugacy classes of maximal subgroups in $\mathscr{C}$ is less than or equal to $l-1$. The minimal choice of G would imply $2 \leq |\pi(G/L)| \leq (l-1)+2$ and then $2 \leq |\pi(G)| \leq l+2$. This would be a contradiction. Therefore, we may assume that

C/L is not nilpotent. Since C is a Rose group by Lemma 2.4, it follows that L is not a Sylow subgroup of C. This means that $|\pi(G/L)| = |\pi(G)|$. The minimality of G and Lemma 2.5 yields $2 \leq |\pi(G)| \leq l + 2$, contrary to assumption.

Then we may assume that every maximal subgroup of G belonging to $\mathscr{A}$ is normal in G and so $N \leq \bigcap\{K_i : 1 \leq i \leq n\}$. If N were not a Sylow subgroup of G, then the theorem would be applied to the group G/N, and we would get $2 \leq |\pi(G)| \leq n + 2$. Thus, we may assume that N is a Sylow subgroup of G, and so N is a complement of an abnormal maximal subgroup of G. This means that $N = G^{\mathfrak{N}}$, and G/N is nilpotent. This impossibility clearly shows that G/N must not be an *NNC*-group.

If G/N is nilpotent, then $N = G^{\mathfrak{N}}$, and $N \leq \bigcap\{K_i : 1 \leq i \leq n\}$, because otherwise some of the subgroups in $\mathscr{A}$ would be nilpotent, and G is a Rose group. The minimal choice of G implies that N is a Sylow subgroup of G, and so N is complemented by a nilpotent Hall subgroup C of G. Write $\pi(C) = \{p_1, \ldots, p_t\}$. Again we may assume that $t > 2$. Let C_i be a maximal subgroup of C such that $|C : C_i| = p_i$ for all $i \in \{1, \ldots, t\}$. Then $S_i = NC_i$ belongs to $\mathscr{A}$ for all i. We conclude that $t \leq n$ and so $|\pi(G)| = t + 1 \leq n + 2$. This final contradiction proves the result. □

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