Semilinear elliptic systems with measure data

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Abstract We study the Dirichlet problem for systems of the form $-\Delta u^k = f^k(x, u) + \mu^k$, $x \in \Omega$, $k = 1, \ldots, n$, where $\Omega \subset \mathbb{R}^d$ is an open (possibly nonregular) bounded set, $\mu^1, \ldots, \mu^n$ are bounded diffuse measures on Ω, $f = (f^1, \ldots, f^n)$ satisfies some mild integrability condition and the so-called angle condition. Using the methods of probabilistic Dirichlet forms theory, we show that the system has a unique solution in the generalized Sobolev space $\dot{H}^1_{\text{loc}}(\Omega)$ of functions having fine gradient. We also provide a stochastic representation of the solution.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded set. In the present paper, we study the existence and uniqueness of solutions of systems of the form

$$\begin{cases} -\frac{1}{2}\Delta u^k = f^k(x, u) + \mu^k & \text{in } \Omega, \quad k = 1, \ldots, n, \\ u^k = 0 & \text{on } \partial\Omega, k = 1, \ldots, n, \end{cases} \tag{1.1}$$

where $f = (f^1, \ldots, f^n) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and $\mu^1, \ldots, \mu^n$ belong to the space $\mathcal{M}_{0,b}$ of bounded diffuse measures on Ω (see Sect. 2).

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Let $qL^1_{\text{loc}}(\Omega)$ denote the space of locally quasi-integrable functions (see Sect. 2). In the scalar case ($n = 1$), it is known that if

$$x \mapsto \sup_{|u| \le r} |f(x, u)| \in qL^1_{\text{loc}}(\Omega) \quad \text{for every} \quad r \ge 0, \tag{1.2}$$

$$y \mapsto f(x, y) \text{ is continuous on } \mathbb{R} \quad \text{for a.e.} \quad x \in \Omega \tag{1.3}$$

and

$$f(x, u) \cdot u \le 0 \quad \text{for a.e. } x \in \Omega \quad \text{and every } u \in \mathbb{R}, \tag{1.4}$$

then there exists a solution of (1.1) (see [13]; see also [1] for equations with general Leray-Lions type operators). One of the crucial ingredients in the proof of the existence result for (1.1) is the following Stampacchia estimate

$$\|f(\cdot, u)\|_{L^1(\Omega;m)} \le \|\mu\|_{\text{TV}} \tag{1.5}$$

(see [15]), which holds true under (1.4) for every solution of (1.1). An attempt to generalize the existence result to $n = 2$ has been made in Orsina and Ponce [13]. It is proved there that if Ω is smooth, f does not depend on x, is continuous on $\mathbb{R}^2$ and monotone componentwise, i.e., $f^1(\cdot, v)$, $f^2(u, \cdot)$ are nonincreasing and if $f^1(0, v) = f^2(u, 0) = 0$ for every $(u, v) \in \mathbb{R}^2$, then there exists a unique solution of (1.1). In Orsina and Ponce [13], as in the scalar case, the key step in proving the existence of solutions is Stampacchia's estimate, which is derived by using the componentwise character of monotonicity of f and by introducing the important notion of quasi-integrability of functions. Note also that in [13, Remark 7.1], the authors raise the question of existence of solutions to (1.1) for f satisfying weaker than monotonicity sign condition with respect to each coordinate, i.e., for f such that $f^1(u, \cdot) \cdot u \le 0$, $f^2(\cdot, v) \cdot v \le 0$ for every $(u, v) \in \mathbb{R}^2$. We answer positively the question raised in Orsina and Ponce [13]. Actually, using quite different than in Orsina and Ponce [13] methods of proof, we show existence and uniqueness results for more general systems.

In the present paper, we assume that μ, f satisfy the following assumptions analogous to assumptions (1.2)–(1.4) considered in the theory of scalar equations:

(A1) $\mu \in \mathcal{M}_{0,b}$,
(A2) For every $r \ge 0$, $x \mapsto \sup_{|y| \le r} |f(x, y)| \in qL^1_{\text{loc}}(\Omega)$,
(A3) For a.e. $x \in \Omega$, $y \to f(x, y)$ is continuous,
(A4) $\langle f(x, y), y \rangle \le 0$ for every $y \in \mathbb{R}^n$ and a.e. $x \in \Omega$ (Here $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^n$).

Following Landes [12], we will call (A4) the angle condition. In Landes [12], more general than (1.1) elliptic systems with the perturbed Leray-Lions type operator are considered. As a matter of fact, the assumptions in Landes [12] when adjusted to our problem say that the perturbation satisfies some strong growth conditions and stronger then (A4) condition

(A5) There exists $\alpha > 0$ such that for every $y \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^d$,

$$\langle f(x, y), y \rangle \le -\alpha |y| |f(x, y)|,$$

which we will call the uniform angle condition.

In the paper, we show that if the right-hand side of (1.1) satisfies (A5) then Stampacchia's estimate (1.5) holds true for any solution of (1.1), which immediately implies that any solution of (1.1) belongs to the Sobolev space $W^{1,q}_0(\Omega)$ with $1 \le q < d/(d-1)$. Under (A4), no analog of Stampacchia's estimate appears to be available. Consequently, it seems that in general $f(\cdot, u) \notin L^1_{\text{loc}}(\Omega)$. Therefore, the first problem we have to address is to describe

the regularity space for u and then formulate suitable definition of a solution of (1.1). We propose two equivalent definitions: the probabilistic and analytic one.

Let $\mathbb{X} = (X, P_x)$ denote the Wiener process killed upon leaving Ω and let ζ denote its life-time. In the probabilistic definition by a solution, we mean a quasi-continuous in the restricted sense function $u : \Omega \to \mathbb{R}$ such that the following stochastic equation

$$u(X_t) = u(X_\tau) + \int_t^\tau f(X_\theta, u(X_\theta)) \, d\theta$$

$$+ \int_t^\tau dA_\theta^\mu - \int_t^\tau dM_\theta, \quad 0 \le t \le \tau, \quad P_x\text{-a.s.} \tag{1.6}$$

is satisfied for quasi-every (with respect to the Newtonian capacity) $x \in \Omega$. Here, τ is an arbitrary stopping time such that $0 \le \tau < \zeta$, M is some local martingale additive functional of $\mathbb{X}$ (as a matter of fact M corresponds to the gradient of u), and A^μ is a continuous additive functional of $\mathbb{X}$ associated with the measure μ via Revuz duality (see Sect. 2).

In the analytic definition (see Sect. 4), a quasi-continuous function $u : \Omega \to \mathbb{R}$ is a solution of (1.1) if

$$\mathcal{E}(u^k, v) = (f^k(\cdot, u), v)_{L^2(\Omega;m)} + \langle \mu, v \rangle, \quad v \in H_0^1(G_l) \tag{1.7}$$

for every $k = 1, \ldots, n$ and $l \ge 1$, where

$$\mathcal{E}(u, v) = \frac{1}{2} \int_\Omega \nabla u(x) \nabla v(x) \, m(dx), \quad u, v \in D[\mathcal{E}] = H_0^1(\Omega) \tag{1.8}$$

is the Dirichlet form associated with the operator $(\frac{1}{2}\Delta, H_0^1(\Omega))$, $\{G_l, l \ge 1\}$ is a suitable family of finely open sets depending on u such $\bigcup_{l \ge 1} G_l = \Omega$ q.e. and (1.7) makes sense.

Since we are looking for solutions of (1.1) in Sobolev type spaces, our minimal regularity requirement for them is quasi-continuity or, equivalently, continuity in the fine topology (see [7]). Quasi-continuity provides some information on local, in terms of fine topology, regularity of functions, and allows one to control their behavior on finely open (closed) sets of the form $\{u < t\}$ ($\{u \le t\}$), $t \ge 0$. It is therefore natural to try to derive a priori estimates for solutions of (1.1) in Sobolev spaces on finely open sets to make sense of the analytic definition and then prove existence of solutions of (1.1). In the present paper, we prefer, however, a stochastic approach to the problem. The main reason for adopting the stochastic approach is that (1.6) is simpler to investigate than (1.7), because (1.7) is in fact a family of variational equations on finely open domains G_l which depend on the solution. Equations of the form (1.7) were considered for example in [6,9]. It seems that direct analysis of equations (1.7) would generate many technical difficulties in using the fine topology, while the stochastic approach avoids them, because in the latter approach, the fine topology is hidden in a very convenient way in probabilistic notions of the Dirichlet forms theory we are using in our proofs.

We prove that under (A1)–(A4), there exists a probabilistic solution u of (1.1), $f(\cdot, u) \in qL_{\text{loc}}^1(\Omega)$ and u belongs to the generalized Sobolev space $\dot{H}_{\text{loc}}^1(\Omega)$ of functions having fine gradient (see [9]). The space $\dot{H}_{\text{loc}}^1(\Omega)$ is wider than the space $\mathcal{T}^{1,2}$ of Borel functions whose truncations on every level belong to $H_0^1(\Omega)$, which was introduced in Bénilan et al. [1] to cope with elliptic equations with L^1 data. The solution u satisfies (1.7), because we show that in general, if $u \in \dot{H}_{\text{loc}}^1(\Omega)$, then u is a probabilistic solution of (1.1) iff it is a solution of (1.1) in the analytic sense. We also prove that if we replace (A4) by

(A4′) $\langle f(x, y) - f(x, y'), y - y' \rangle \leq 0$ for a.e. $x \in \Omega$ and every $y, y' \in \mathbb{R}^n$,

then the solution of (1.1) is unique. Let us note that besides proving our existence result under the general angle condition (A4), in contrast to Orsina and Ponce [13], we allow f to depend on x, the dimension of the system is arbitrary, and we impose no assumption on the regularity of Ω.

Moreover, we show that if u is a solution of (1.1) and $f(\cdot, u) \in L^1(\Omega; m)$, then $u \in W_0^{1,q}(\Omega)$ with $1 \leq q < d/(d-1)$ and u coincides with the distributional (renormalized, in the sense of duality) solution. Finally, we show that $f(\cdot, u) \in L^1(\Omega; m)$ if

(A4″) For every $k \in \{1, \ldots, n\}$, $y_i \in \mathbb{R}$, $i \in \{1, \ldots, n\} \setminus \{k\}$ and a.e. $x \in \Omega$,

$$f^k(x, y_1, \ldots, y_{k-1}, z, y_{k+1}, \ldots, y_n) \cdot z \leq 0 \quad \text{for every } z \in \mathbb{R},$$

i.e., if f satisfies the sign condition which respect to each coordinate.

2 Preliminary results

In the whole paper, we adopt the convention that for given class of real functions (measures) F and $\mathbb{R}^d$-valued function (measure) f we write $f \in F$, if each component f^k, $k = 1, \ldots, d$, of f belongs to F.

Let Ω be a nonempty bounded open subset in $\mathbb{R}^d$, $d \geq 2$. By m, we denote the Lebesgue measure on $\mathbb{R}^d$. Let $(\mathcal{E}, D[\mathcal{E}])$ be the Dirichlet form defined by (1.8), and let $\mathbb{X} = (\{X_t, t \geq 0\}, \{P_x, x \in \Omega\}, \{\mathcal{F}_t, t \geq 0\}, \zeta)$ be a diffusion process associated with $(\mathcal{E}, D[\mathcal{E}])$, i.e., for every $t \geq 0$,

$$(p_t f)(x) = E_x f(X_t) \tag{2.1}$$

for m-a.e. $x \in \Omega$, where $\{p_t, t \geq 0\}$ is the semigroup generated by $(\mathcal{E}, D[\mathcal{E}])$ and $\zeta = \inf\{t \geq 0, X_t = \Delta\}$, where Δ is a one-point compactification of the space Ω and E_x denote the expectation with respect to P_x (see [7]). We also adopt the convention that $u(\Delta) = 0$. It is well known that $\mathbb{X}$ is the Brownian motion killed upon leaving Ω so we sometimes use the notation B_t instead of X_t. From (2.1), it follows that $\{p_t, t \geq 0\}$ is a semigroup of contractions.

By $\{R_\alpha, \alpha > 0\}$, we denote the resolvent generated by $\{p_t, t \geq 0\}$. Since $\mathcal{E}$ is transient (see [7, Example 1.5.1]), R_0 is also well defined. Write $R = R_0$. From (2.1), it follows that

$$(Rf)(x) = E_x \int_0^\zeta f(X_t) \, dt$$

for m-a.e. $x \in \Omega$.

By cap, we denote the capacity associated with $\mathcal{E}$, i.e., cap $: 2^\Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is a subadditive set function defined as

$$\text{cap}(A) = \inf\{\mathcal{E}(u, u); u \in H_0^1(\Omega), u \geq \mathbf{1}_A, \ m\text{-a.e.}\}$$

for open set $A \subset \Omega$ (with the convention that $\inf \emptyset = +\infty$), and for arbitrary $A \subset \Omega$, $\text{cap}(A) = \inf\{\text{cap}(B) : A \subset B, B \text{ is an open subset of } \Omega\}$. We say that some property $P(x)$ holds q.e. if $\text{cap}(\{x; P(x) \text{ is not true}\}) = 0$.

A Borel measurable set $N \subset \Omega$ is called properly exceptional if for q.e. $x \in \Omega$,

$$P_x(\exists t > 0; X_t \in N) = 0. \tag{2.2}$$

It is known (see [7, Theorem 4.1.1 and page 140]) that every properly exceptional set is of capacity zero and if $N \subset \Omega$ is of capacity zero, then there exists a Borel properly exceptional set B such that $N \subset B$.

A nonnegative Borel measure μ on Ω is called smooth if it charges no set of zero capacity and there exists an ascending sequence $\{F_n\}$ of closed subsets of Ω such that $\mu(F_n) < \infty$ for $n \geq 1$ and for every compact set $K \subset \Omega$,

$$\text{cap}(K \setminus F_n) \to 0. \tag{2.3}$$

By S, we denote the set of all smooth measures on Ω. By $\mathcal{M}_{0,b}^+$, we denote the space of finite smooth measures, $\mathcal{M}_{0,b} = \mathcal{M}_{0,b}^+ - \mathcal{M}_{0,b}^+$. Elements of $\mathcal{M}_{0,b}$ are called diffuse or soft measures (see [3,4]).

Let $\mathcal{B}(\Omega) (\mathcal{B}^+(\Omega))$ denote the set of all real (nonnegative) Borel measurable functions on Ω. For $A \subset \Omega$, we write $A \in \mathcal{B}(\Omega)$ if $\mathbf{1}_A \in \mathcal{B}(\Omega)$. It is known (see [7, Section 5.1]) that for every $\mu \in S$, there exists a unique positive continuous additive functional A^μ (PCAF for short) such that for every $f, h \in \mathcal{B}^+(\Omega)$,

$$E_{h \cdot m} \int_0^t f(X_\theta) \, dA_\theta^\mu = \int_0^t \langle f \cdot \mu, p_\theta h \rangle \, d\theta, \quad t \geq 0, \tag{2.4}$$

where $P_\nu(B) = \int_\Omega P_x(B) \, \nu(dx)$ for any $B \in \mathcal{B}(\Omega)$ and nonnegative Borel measure ν, $(h \cdot \nu)(B) = \int_B h(x) \, \nu(dx)$ for any $B \in \mathcal{B}(\Omega)$ and $h \in \mathcal{B}^+(\Omega)$, and

$$\langle h, \nu \rangle = \int_\Omega h(x) \, \nu(dx).$$

On the other hand, for every PCAF A of $\mathbb{X}$, there exists a unique smooth measure μ such that (2.4) holds with A^μ replaced by A. The measure μ is called the Revuz measure associated with PCAF A.

Using (2.4), one can extend the resolvent R to S by putting

$$R\mu(x) = E_x \int_0^\zeta dA_t^\mu. \tag{2.5}$$

Similarly one can extend $R_\alpha, \alpha > 0$. The right-hand side of (2.5) is finite q.e. for every finite measure $\mu \in S$. This follows from (3.6) and Lemma 4.1 and Proposition 5.13 in Klimsiak and Rozkosz [11]. Moreover, by Theorem 2.2.2, Lemma 2.2.11 and Lemma 5.1.3 in Fukushima et al. [7], for every $\mu \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$,

$$\mathcal{E}(R\mu, \tilde{v}) = \langle \mu, \tilde{v} \rangle, \tag{2.6}$$

where $\tilde{v}$ is a quasi-continuous m-version of v.

By $\mathcal{C}(\Omega)$, we denote the space of all quasi-continuous functions on Ω. Let us recall that $u : \Omega \to \mathbb{R}$ is quasi-continuous if for every $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$ such that $\text{cap}(G_\varepsilon) < \varepsilon$ and $u_{|\Omega \setminus G_\varepsilon}$ is continuous. It is known that u is quasi-continuous iff the process $t \to u(X_t)$ is continuous on $[0, \zeta)$, P_x-a.s. for q.e. $x \in \Omega$ (see [7, Section 4.2]).

$\mathcal{C}_0(\Omega)$ is the space of quasi-continuous functions on Ω in the restricted sense. Let us note that a Borel measurable function u on Ω when considered as the function on $\Omega \cup \{\Delta\}$ (with the convention that $u(\Delta) = 0$) is continuous on $\Omega \cup \{\Delta\}$ if $u \in C_0(\Omega)$, where $C_0(\Omega)$ is the closure in $C(\overline{\Omega})$ of the space $C_c(\Omega)$ of all continuous functions with compact support in Ω.

A Borel function u on Ω is called quasi-continuous in the restricted sense if for every $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$ such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and $u_{|(\Omega \cup \{\Delta\}) \setminus G_\varepsilon}$ is continuous.

The following lemma shows that if $u \in \mathcal{C}_0(\Omega)$ then $u(x)$ tends to zero if x tends to the boundary of Ω along the trajectories of the process X.

Lemma 2.1 *Assume that $u \in \mathcal{C}_0(\Omega)$. Then for q.e. $x \in \Omega$,*

$$u(X_t) \to 0, \quad t \to \zeta^-, \quad P_x\text{-a.s.}$$

Proof Since $u \in \mathcal{C}_0(\Omega)$, there exists a sequence $\{U_n\}$ of open subsets of Ω such that $\operatorname{cap}(U_n) \to 0$ and $u_{|\Delta \cup \Omega \setminus U_n}$ is continuous. For every $T > 0$,

$$
\begin{aligned}
P_x(u(X_t) \nrightarrow 0, \ t \to \zeta^-) &\leq P_x(T \leq \zeta) + P_x(u(X_t) \nrightarrow 0, \ t \to \zeta^-, \zeta \leq T) \\
&\leq P_x(T \leq \zeta) + P_x(\exists_{t \leq T} \ X_t \in U_n) \\
&\leq P_x(T \leq \zeta) + P_x(\sigma_{U_n} \leq T) \\
&\leq P_x(T \leq \zeta) + e^T E_x e^{-\sigma_{U_n}},
\end{aligned}
$$

where

$$\sigma_{U_n} = \inf\{t > 0; \ X_t \in U_n\}.$$

Letting $n \to +\infty$ and applying [7, Theorem 4.2.1] we conclude that for q.e. $x \in \Omega$,

$$P_x(u(X_t) \nrightarrow 0, \ t \to \zeta^-) \leq P_x(T \leq \zeta).$$

Since $E_x \zeta < +\infty$ (see (3.5)) for every $x \in \Omega$, the desired result follows. $\square$

Remark 2.2 Let us note that by [7, Theorem 2.1.3], if $u \in H_0^1(\Omega)$ then $u \in \mathcal{C}_0(\Omega)$.

By $qL^1(\Omega)$ (resp. $qL_{\mathrm{loc}}^1(\Omega)$), we denote the class of all Borel measurable functions $f : \Omega \to \mathbb{R}$ such that for q.e. $x \in \Omega$,

$$
P_x \left(\int_0^{T \wedge \zeta} |f(X_t)| \, dt < +\infty, \ T \geq 0 \right) = 1
$$

$$
\left(\text{resp. } P_x \left(\int_0^T |f(X_t)| \, dt < +\infty, \ 0 \leq T < \zeta \right) = 1 \right). \tag{2.7}
$$

Elements of $qL^1(\Omega)$ ($qL_{\mathrm{loc}}^1(\Omega)$) will be called quasi-integrable functions (locally quasi-integrable functions).

We say that a Borel measurable function $f : \Omega \to \mathbb{R}$ is locally quasi-integrable in the analytic sense if for every compact $K \subset U$ and every $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \Omega$ such that $\operatorname{cap}(U_\varepsilon) < \varepsilon$ and $f_{|K \setminus U_\varepsilon} \in L^1(K \setminus U_\varepsilon)$. We say that a Borel measurable function $f : \Omega \to \mathbb{R}$ is quasi-integrable in the analytic sense if in the above definition one can replace K by Ω.

Remark 2.3 From Klimsiak and Rozkosz [11], it follows (see Remark 4.4) that f is quasi-integrable in the analytic sense iff $f \in qL^1(\Omega)$ (this is true for bounded domains). Moreover, if f is locally quasi-integrable in the analytic sense, then $f \in qL_{\mathrm{loc}}^1(\Omega)$. The reverse implication is also true. Indeed, let $f \in qL_{\mathrm{loc}}^1(\Omega)$ and $f \geq 0$. Then, by the very definition of the space $qL_{\mathrm{loc}}^1(\Omega)$, $A_t \equiv \int_0^t f(X_\theta) \, d\theta$ is a PCAF of $\mathbb{X}$ and its associated Revuz measure is

$f \cdot m$ (see [7, Section 5.1]). Since the associated Revuz measure is smooth, there exists an ascending sequence $\{F_n\}$ of closed subsets of Ω such that $f_{|F_n} \in L^1(F_n; m)$ for every $n \geq 1$ and (2.3) holds for every compact set $K \subset \Omega$. From this, one can easily deduce that f is locally quasi-integrable in the analytic sense.

The notion of quasi-integrability in the analytic sense was introduced in Orsina and Ponce [13]. In Orsina and Ponce [13], the authors do not distinguish between local quasi-integrability and quasi-integrability, and quasi-integrability in the sense of Orsina and Ponce [13] coincides with local quasi-integrability in the analytical sense defined in the present paper.

By $\mathcal{T}$, we denote the set of all finite stopping times with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ (see (2.1)). Let us recall (see [7, Section 5.2]) that M is called a martingale additive functional (MAF) of $\mathbb{X}$ if M is a finite continuous additive functional of $\mathbb{X}$ such that for every $t > 0$, $E_x M_t^2 < \infty$ and $E_x M_t = 0$ for q.e. $x \in \Omega$. By $\mathcal{M}$, we denote the space of all MAFs of $\mathbb{X}$. By $\mathcal{M}_{\mathrm{loc}}$, we denote the set of all local additive functionals of $\mathbb{X}$ (see [7, page 226]) for which there exists an ascending sequence $\{G_n, n \geq 1\}$ of finely open subsets of Ω such that $\bigcup_{n \geq 1} G_n = \Omega$ q.e., a sequence $\{M^n\} \subset \mathcal{M}$ and $N \subset \Omega$ such that cap(N)=0 and for every $n \geq 1$ and $x \in \Omega \setminus N$,

$$M_t = M_t^n, \quad t < \tau_{G_n}, \quad P_x\text{-a.s.}$$

Finally, by $\mathcal{M}^2$, we denote the space of all $M \in \mathcal{M}$ such that $\sup_{t \geq 0} E M_t^2 < \infty$ for q.e. $x \in \Omega$.

From now on,

$$f(u)(x) = f(x, u(x)), \quad x \in \Omega$$

for every measurable function $u : \Omega \to \mathbb{R}^n$.

Following [10,11], let us consider the class (FD) consisting of all functions $u \in \mathcal{B}(\Omega)$ with the property that the process $t \to u(X_t)$ is of Doob's class (D) under the measure P_x for q.e. $x \in \Omega$, i.e., for q.e. $x \in \Omega$ the family $\{u(X_\tau), \tau \in \mathcal{T}\}$ is uniformly integrable under P_x.

Definition We say that $u : \Omega \to \mathbb{R}^n$ is a probabilistic solution of (1.1) if

(a) u is of class (FD),
(b) $u \in \mathcal{C}_0(\Omega)$,
(c) $x \mapsto f(u)(x)$ belongs to $qL_{\mathrm{loc}}^1(\Omega)$,
(d) There exists $M \in \mathcal{M}_{\mathrm{loc}}$ such that for every stopping time $0 \leq \tau < \zeta$,

$$u(X_t) = u(X_\tau) + \int_t^\tau f(u)(X_\theta) \, d\theta + \int_t^\tau dA_\theta^\mu$$

$$- \int_t^\tau dM_\theta, \quad 0 \leq t \leq \tau, \quad P_x\text{-a.s.} \tag{2.8}$$

for q.e. $x \in \Omega$.

In the sequel, we adopt the convention that $\int_a^b = 0$ if $a \geq b$.

Remark 2.4 Under (b), if moreover $f(u) \in qL^1(\Omega)$, (2.8) is satisfied iff for every $T \geq 0$,

$$u(X_t) = u(X_{T \wedge \zeta}) + \int_t^{T \wedge \zeta} f(u)(X_\theta) \, d\theta + \int_t^{T \wedge \zeta} dA_\theta^\mu$$

$$- \int_t^{T \wedge \zeta} dM_\theta, \quad t \geq 0, \quad P_x\text{-a.s.} \tag{2.9}$$

for q.e. $x \in \Omega$. Indeed, by (2.5), $E_x A_\zeta < \infty$ for q.e. $x \in \Omega$. Since every stopping time with respect to Brownian filtration is predictable (see [14, Theorem 4, Chapter 3]), there exists a sequence $\{\tau_n\} \subset \mathcal{T}$ such that $0 \leq \tau_n < T \wedge \zeta$ and $\tau_n \nearrow T \wedge \zeta$. Taking τ_n in place of τ in (2.8) and letting $n \to +\infty$ we get (2.9) (the integral involving $f(u)$ is well defined since $f(u) \in qL^1_{\text{loc}}(\Omega)$). On the other hand, if (2.9) is satisfied, then replacing T by an arbitrary stopping time τ such that $0 \leq \tau < \zeta$ we get (2.8).

Remark 2.5 If f satisfies (A4) and u is a solution of (1.1) then u vanishes on the boundary of Ω in the sense of Sobolev spaces. Indeed, by the Itô-Tanaka formula (see [2]) and (A4), for any $\tau \in \mathcal{T}$ such that $0 \leq \tau < \zeta$,

$$E_x|u(X_t)| \leq E_x|u(X_{T \wedge \zeta})| + E_x \int_t^{T \wedge \zeta} \langle f(u)(X_\theta), \hat{u}(X_\theta) \rangle \, d\theta + E_x \int_t^{T \wedge \zeta} \langle \hat{u}(X_\theta), dA_\theta^\mu \rangle$$

$$\leq E_x \int_0^\zeta d|A^\mu|_t + E_x|u(X_{T \wedge \zeta})| \tag{2.10}$$

for q.e. $x \in \Omega$, where

$$\hat{y} = \frac{y}{|y|} \mathbf{1}_{\{y \neq 0\}}, \quad y \in \mathbb{R}^d.$$

Let $\{\tau_k\}$ be a sequence of stopping times such that $0 \leq \tau_k < \zeta$, $k \geq 1$, and $\tau_k \to \zeta$. Such a sequence exists since every stopping time with respect to Brownian filtration is predictable. It is clear that $u(X_{\tau_k}) \to 0$ as $k \to +\infty$, P_x-a.s. for q.e. $x \in \Omega$. This when combined with the fact that u is of class (FD) implies that

$$|u(x)| \leq E_x \int_0^\zeta d|A^\mu|_t \equiv v(x) \tag{2.11}$$

for q.e. $x \in \Omega$. By Klimsiak and Rozkosz [11], $T_k(v) \in H_0^1(\Omega)$ for every $k > 0$, which forces u to vanish on the boundary of Ω.

In Sect. 4, we give a different, analytic definition of a solution of (1.1), and we prove that actually it is equivalent to the probabilistic definition. Before doing this, we would like to present the motivation behind the two definitions. We begin with a concise presentation of famous Dynkin's formula.

Let $B \subset \Omega$ be a Borel set and let

$$H_0^1(B) = \{u \in H_0^1(\Omega); u = 0 \text{ q.e. on } B^c\}.$$

(see [5,6,9]). Here and in the sequel for a given function $u \in H_0^1(\Omega)$, we always consider its quasi-continuous version. It is clear that $H_0^1(B)$ is a closed subspace of the Hilbert space $H_0^1(\Omega)$. Therefore,

$$H_0^1(\Omega) = H_0^1(B) \oplus \mathcal{H}_{\Omega \setminus B},$$

where $\mathcal{H}_{\Omega \setminus B}$ is the orthogonal complement of $H_0^1(B)$. Let H_B denote the operator of the orthogonal projection on $\mathcal{H}_{\Omega \setminus B}$. By [7, Theorem 4.3.2], for every $u \in H_0^1(\Omega)$,

$$H_B(u)(x) = E_x u(X_{\tau_B}) \quad (= E_x u(X_{\tau_B \wedge \zeta})) \tag{2.12}$$

for q.e. $x \in \Omega$, where

$$\tau_B = \inf\{t > 0, X_t \notin B\}.$$

Let G be a finely open subset of Ω and let $(\mathcal{E}_G, D[\mathcal{E}_G])$ denote the restriction of the form defined by (1.8) to G, i.e.,

$$\mathcal{E}_G(u, v) = \mathcal{E}(u, v), \quad u, v \in D[\mathcal{E}_G] = H_0^1(G).$$

$(\mathcal{E}_G, D[\mathcal{E}_G])$ is again a Dirichlet form (it may be no longer regular). Let $\{p_t^G, t \geq 0\}$ denote the associated C_0-semigroup and $\{R_\alpha^G, \alpha \geq 0\}$ the associated resolvent. By Theorems 4.4.2 and 4.4.4 in [7], $(\mathcal{E}_G, D[\mathcal{E}_G])$ is transient and for q.e. $x \in \Omega$,

$$(R^G f)(x) = E_x \int_0^{\tau_G \wedge \zeta} f(X_t) \, dt, \tag{2.13}$$

where $R^G = R_0^G$, which is well defined due to the transiency of $(\mathcal{E}_G, D[\mathcal{E}_G])$. Dynkin's formula (see [7, page 153]) says that for every finely open set $G \subset \Omega$,

$$(Rf)(x) = (R^G f)(x) + [H_G(Rf)](x)$$

for q.e. $x \in \Omega$.

Suppose now that $\mu \in H^{-1}(\Omega)$ and there exists a weak solution $u \in H_0^1(\Omega)$ of (1.1) such that $f(u) \in L^2(\Omega; m)$. Then, for every $v \in H_0^1(\Omega)$,

$$\mathcal{E}(u^k, v) = (f^k(u), v)_{L^2(\Omega;m)} + \langle \mu^k, v \rangle, \quad k = 1, \ldots, n \tag{2.14}$$

that is

$$u^k = Rf^k(u) + R\mu^k, \quad k = 1, \ldots, n.$$

Applying the operator H_G to both sides of the above equation and using Dynkin's formula, we get

$$H_G(u^k) = Rf^k(u) - R^G f^k(u) + R\mu^k - R^G \mu^k = u^k - R^G f^k(u) - R^G \mu^k.$$

As a consequence,

$$u^k = H_G(u^k) + R^G f^k(u) + R^G \mu^k, \quad k = 1, \ldots, n. \tag{2.15}$$

This equation expresses the property that if u is a solution of (1.1) on Ω, then for every finely open set G, $u_{|G}$ is a solution of (1.1) on G with the boundary condition $u_{|G} = u$ on ∂G.

In general, if $\mu \in \mathcal{M}_{0,b}$, it is natural to look for solutions of (1.1) in the class of quasi-continuous functions vanishing at the boundary of Ω and, roughly speaking, such that they coincide with functions from the space $H_0^1(\Omega)$ on each set G from some family of finely

open set which covers Ω. Therefore, it is natural to require the solution u of (1.1) to satisfy (2.15) or (2.14) for each set G from some family of suitably chosen (depending on u in general) finely open sets. It is not easy to deal with such families of equations. Fortunately, we can obtain (2.15) from (2.9), and, in view of Remark 2.4, from (2.8) if we know that $f(u) \in qL^1(\Omega)$. Indeed, by standard arguments, one can replace T in (2.9) by τ_G with arbitrary finely open set $G \subset \Omega$ and then putting $t = 0$ and taking expectation one can get

$$u(x) = E_x u(X_{\tau_G \wedge \zeta}) + E_x \int_0^{\tau_G \wedge \zeta} f(u)(X_t)\, d\theta + E_x \int_0^{\tau_G \wedge \zeta} dA_t^\mu,$$

which in view of (2.12) and (2.13) gives (2.15). The stochastic equations (2.8), (2.9) are much more convenient to work with than systems of the form (2.15). One of the major advantage of (2.8) (resp. (2.9)) lies in the fact that it is well defined whenever $f \in qL^1_{loc}(\Omega)$ (resp. $f(u) \in qL^1(\Omega)$) and $\mu \in \mathcal{M}_{0,b}$. Moreover, (2.8), (2.9) allow one to apply stochastic analysis methods to study partial differential equations and are well suited for dealing with the fine topology.

3 Existence and uniqueness of probabilistic solutions

We begin with the uniqueness result.

Proposition 3.1 *Assume that* (A4$'$) *is satisfied. Then there exists at most one probabilistic solution of* (1.1).

Proof Assume that u_1, u_2 are solutions of (1.1) and M_1, M_2 are local MAFs associated with u_1, u_2, respectively. Then, denoting $u = u_1 - u_2$ and $M = M_1 - M_2$, we have

$$u(X_t) = u(X_\tau) + \int_t^\tau (f(u_1) - f(u_2))(X_\theta)\, d\theta - \int_t^\tau dM_\theta \quad 0 \le t \le \tau, \quad P_x\text{-a.s.}$$

for q.e. $x \in \Omega$. By the Itô-Tanaka formula and (A4$'$),

$$|u(X_t)| \le |u(X_\tau)| + \int_t^\tau \langle f(u_1) - f(u_2)(X_\theta), \hat{u}(X_\theta) \rangle\, d\theta - \int_t^\tau \hat{u}(X_\theta)\, dM_\theta$$

$$\le |u(X_\tau)| - \int_t^\tau \hat{u}(X_\theta)\, dM_\theta, \quad 0 \le t \le \tau, \quad P_x\text{-a.s.}$$

for q.e. $x \in \Omega$. Without loss of generality, we may assume that $\int_0^\cdot \hat{u}(X_\theta)\, dM_\theta$ is a true martingale (otherwise, one can apply the standard localization procedure). Therefore, putting $t = 0$ and taking the expectation with respect to P_x, we conclude that

$$|u(x)| \le E_x |u(X_\tau)| \tag{3.1}$$

for q.e. $x \in \Omega$. Let $\{\tau_k\}$ be a sequence of stopping times such that $0 \le \tau_k < \zeta$, $k \ge 1$, and $\tau_k \to \zeta$. Since u is of class (FD) and $u \in \mathcal{C}_0(\Omega)$, replacing τ by τ_k in (3.1) and then letting $k \to +\infty$ we conclude that $|u| = 0$ q.e. $\square$

Remark 3.2 In general, the class $\mathcal{C}_0(\Omega)$ is too large to ensure uniqueness of a solution of (1.1) under (A4′). To see this, let us set $n = 1$, $\Omega = B(0, 1) \equiv \{x \in \mathbb{R}^d; |x| < 1\}$ and $u(x) = \frac{1}{d-2}(\frac{1}{|x|^{d-2}} - 1)$, $d \geq 3$. Then $u \in \mathcal{C}_0(\Omega)$ and from the Fukushima decomposition (see [7, Theorem 5.5.2]), it follows that

$$u(X_t) = u(X_{T \wedge \zeta}) + \int_t^{T \wedge \zeta} \mathrm{d}M_\theta, \quad t \geq 0$$

for some $M \in \mathcal{M}_{\mathrm{loc}}$. Thus, u is a solution of (1.1) with $f \equiv 0$, $\mu \equiv 0$. Obviously, the other solution of the above equation is $v \equiv 0$. In fact, it is known (see [3]) that u is a renormalized solution of (1.1) with $f \equiv 0$ and $\mu = \sigma_{d-1}\delta_0$ (which is not a smooth measure for $d \geq 2$), where σ_{d-1} is the measure of $\partial B(0, 1)$.

Let us recall that for a given additive functional A of $\mathbb{X}$, its energy is given by

$$e(A) = \lim_{t \searrow 0} \frac{1}{2t} E_m A_t^2$$

whenever the limit exists. It is known that for fixed regular Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ and $u \in D[\mathcal{E}]$, the additive functional $A_t^{[u]} \equiv u(X_t) - u(X_0)$ admits the so-called Fukushima decomposition, i.e., for q.e. $x \in \Omega$,

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-a.s.},$$

where $M^{[u]}$ is a martingale additive functional of $\mathbb{X}$ of finite energy and $N^{[u]}$ is a continuous additive functional of $\mathbb{X}$ of zero energy. This decomposition is unique (see [7, Theorem 5.2.2]).

Lemma 3.3 *Assume that $u \in H_0^1(\Omega)$. Then for q.e. $x \in \Omega$,*

$$P_x \left(\int_0^T |\nabla u(X_t)|^2 \, \mathrm{d}t < +\infty, \, T \geq 0 \right) = 1 \tag{3.2}$$

and

$$M_t^{[u]} = \int_0^t \nabla u(X_\theta) \, \mathrm{d}B_\theta, \quad t \geq 0, \quad P_x\text{-a.s.}$$

Proof That ∇u satisfies (3.2) follows immediately from the fact that $\nabla u \in L^2(\Omega; m)$ (see (5.2.21) in [7]). Let $\{u_n\} \subset C_0^\infty(\Omega)$ be such that $u_n \to u$ strongly in $H_0^1(\Omega)$. By [7, Corollary 5.6.2],

$$M_t^{[u_n]} = \int_0^t \nabla u_n(X_\theta) \, \mathrm{d}B_\theta, \quad t \geq 0, \quad P_x\text{-a.s.}$$

for q.e. $x \in \Omega$. By [7, Theorem 5.2.2],

$$e(M^{[u_n]} - M^{[u]}) = \int_\Omega |\nabla(u_n - u)(x)|^2 \, m(\mathrm{d}x) \to 0.$$

On the other hand, by (5.2.8) in [7],

$$e(M^{[u_n]} - \int_0^\cdot \nabla u(X_\theta) \, dB_\theta) = \int_\Omega |\nabla (u_n - u)(x)|^2 \, m(dx) \to 0.$$

Therefore, the desired result follows from [7, Theorem 5.2.1].   □

Lemma 3.4 *Let $B \subset \Omega$ be a Borel set such that $m(\Omega \setminus B) = 0$. Then for q.e. $x \in \Omega$,*

$$P_x(X_t \in B \text{ for a.e. } 0 \le t < \zeta) = 1.$$

Proof Let $A_t^1 = \int_0^t \mathbf{1}_B(X_\theta) \, d\theta$, $A_t^2 = \int_0^t \mathbf{1}_\Omega(X_\theta) \, d\theta$. Then A^1, A^2 are PCAFs of $\mathbb{X}$ and their associated Revuz measures are $\mathbf{1}_B \cdot m$ and m, respectively. Since $\mathbf{1}_B \cdot m = m$ and the Revuz correspondence is ont-to-one, $A_t^1 = A_t^2$, $t \ge 0$, P_x-a.s. for q.e. $x \in \Omega$, which leads to the desired result.   □

Let $\mathcal{FS}^q$, $q > 0$, denote the set of all functions $u \in \mathcal{C}(\Omega)$ such that for q.e. $x \in \Omega$,

$$E_x \sup_{t \ge 0} |u(X_t)|^q < +\infty,$$

and let $\dot{H}_{\mathrm{loc}}^1(\Omega)$ denote the space of all Borel measurable functions on Ω for which there exists a quasi-total family $\{U_\alpha, \alpha \in I\}$ (i.e., $\mathrm{cap}(\Omega \setminus (\bigcup_{\alpha \in I} U_\alpha)) = 0$) of finely open subsets of Ω such that for every $\alpha \in I$, there exists a function $u_\alpha \in H_0^1(\Omega)$ such that $u = u_\alpha$ q.e. on U_α. For any function $u \in \dot{H}_{\mathrm{loc}}^1(\Omega)$, one can define its gradient as

$$\nabla u = \nabla u_\alpha \text{ on } U_\alpha.$$

For an arbitrary sequence $\{x_n\}$ of elements of some linear space, we set

$$\sigma_k(\{x_n\}) = \frac{x_1 + x_2 + \cdots + x_k}{k}, \quad k \ge 0.$$

Theorem 3.5 *Assume that* (A1)–(A4) *are satisfied. Then there exists a solution u of* (1.1) *such that $u \in \mathcal{FS}^q$ for $q \in (0, 1)$, $u \in \dot{H}_{\mathrm{loc}}^1(\Omega)$ and*

$$M_t = \int_0^t \nabla u(X_\theta) \, dB_\theta, \quad t \ge 0, \quad P_x\text{-a.s.}$$

for q.e. $x \in \Omega$.

Proof Let T_r, $r > 0$, denote the truncature operator, i.e.

$$T_r(y) = \frac{ry}{|y| \vee r}, \quad y \in \mathbb{R}^d.$$

Put $f_n = T_n(f)$, $n \in \mathbb{N}$. Then f_n is bounded and satisfies (A4). Let $\{F_n\}$ be a generalized nest such that $\mu_n = \mathbf{1}_{F_n} \cdot \mu \in H^{-1}(\Omega)$, $n \in \mathbb{N}$ (for the existence of such family see [7, Theorem 2.2.4]). It is well known that there exists a solution $u_n \in H_0^1(\Omega)$ of (1.1) with f_n in place of f and μ_n in place of μ. By Klimsiak and Rozkosz [11], $u_n \in \mathcal{FS}^2$ and there exists

$M^n \in \mathcal{M}^2$ such that for q.e. $x \in \Omega$,

$$u_n(X_t) = u_n(X_{T \wedge \zeta}) + \int_t^{T \wedge \zeta} f_n^k(u_n)(X_\theta) \, d\theta + \int_t^{T \wedge \zeta} dA_\theta^{\mu_n}$$

$$- \int_t^{T \wedge \zeta} dM_\theta^n, \quad 0 \le t \le T < \infty, \quad P_x\text{-a.s.}$$

As in [7, page 201]) one can check that the CAFs $\int_0^\cdot f_n^k(u_n)(X_\theta) \, d\theta$, $A_t^{\mu_n}$ are of zero energy. It follows from uniqueness of the Fukushima decomposition and Lemma 3.3 that for q.e. $x \in \Omega$,

$$M_t^n = \int_0^t \nabla u_n(X_\theta) \, dB_\theta, \quad t \ge 0, \quad P_x\text{-a.s.}$$

Since u_n is of class (FD) and $u_n \in \mathcal{C}_0(\Omega)$, in much the same way as in the proof of (2.11), we show that

$$|u_n(x)| \le E_x \int_0^\zeta d|A^\mu|_t \qquad (3.3)$$

for q.e. $x \in \Omega$. Write

$$v(x) = E_x \int_0^\zeta d|A^\mu|_t.$$

By Klimsiak and Rozkosz [11], $v \in \mathcal{FS}^q$, $q \in (0, 1)$, for every $k > 0$, $T_k(v) \in H_0^1(\Omega)$ and v is of class (FD). Let $G_k = \{|v| < k\}$. Since v is quasi-continuous, G_k is finely open. Moreover, the family $\{G_k\}$ is quasi-total. Let us put

$$\tau_k = \inf\{t > 0; X_t \in G_k\}.$$

By Itô's formula, for q.e. $x \in \Omega$, we have

$$|u_n(x)|^2 + E_x \int_0^{\zeta \wedge \tau_k} |\nabla u_n|^2(X_t) \, dt$$

$$= E_x |u_n(X_{\tau_k \wedge \zeta})|^2 + 2E_x \int_0^{\zeta \wedge \tau_k} \langle f^n(u_n)(X_t), u_n(X_t) \rangle \, dt$$

$$+ 2E_x \int_0^{\zeta \wedge \tau_k} \langle u_n(X_t), dA_t^\mu \rangle.$$

By the definition of τ_k, (3.3) and (A4), for q.e. $x \in \Omega$,

$$E_x \int_0^{\zeta \wedge \tau_k} |\nabla u_n(X_t)|^2 \, dt \le 2k + 2kE_x \int_0^{\zeta \wedge \tau_k} d|A^\mu|_t. \qquad (3.4)$$

Since Ω is bounded, there exists $R > 0$ such that $\Omega \subset B(0, R)$. Hence,

$$E_x \zeta \leq E_x \tau_{B(0,R)} \leq C(d)(R^2 - |x|^2) \tag{3.5}$$

for every $x \in \Omega$ (for the last inequality see, e.g., [8, page 253]). From (2.4) and (3.5), it follows in particular that for every $f \in \mathcal{B}^+(\Omega)$ and $\mu \in S$,

$$E_m \int_0^\zeta f(X_t) \, \mathrm{d}A_t^\mu \leq C(\Omega, d)\|f \cdot \mu\|_{\mathrm{TV}}, \tag{3.6}$$

where $\| \cdot \|_{\mathrm{TV}}$ denotes the total variation norm. By (3.4) and (3.6),

$$E_m \int_0^{\zeta \wedge \tau_k} |\nabla u_n(X_t)|^2 \, \mathrm{d}t \leq 2km(\Omega) + 2kc|\mu|(\Omega). \tag{3.7}$$

Since G_k is finely open, it follows from [7, Theorem 4.2.2] that $\mathcal{E}_{G_k}$ is a Dirichlet form on $L^2(G_k; m)$ and the semigroup $\{p_t^{G_k}, t \geq 0\}$ is determined by the process $\mathbb{X}^{G_k}$ (see (2.13)). Therefore, for every $f \in L^2(G_k; m)$,

$$R^{G_k} f(x) = E_x \int_0^{\tau_k \wedge \zeta} f(X_t) \, \mathrm{d}t$$

for m-a.e. $x \in G_k$. Moreover, by [7, Theorem 4.4.4], $\mathcal{E}_{G_k}$ is transient and $D[\mathcal{E}_{G_k}] = H_0^1(G_k)$. Therefore, from (3.7), it follows that

$$\sup_{n \geq 1} \|R^{G_k}(|\nabla u_n|^2)\|_{L^1(\Omega; m)} < +\infty. \tag{3.8}$$

On the other hand, by [7, Lemma 5.1.10],

$$\|R^{G_k}(|\nabla u_n|^2)\|_{L^1(\Omega; m)} = \int_{G_k} |\nabla u_n|^2(y) R^{G_k} 1(y) \, m(\mathrm{d}y).$$

Since $R^{G_k} 1 \in D[\mathcal{E}_{G_k}] = H_0^1(G_k) \subset H_0^1(\Omega)$ and $|u_n(x)| \leq k$ on G_k for q.e. $x \in \Omega$, we conclude from the above estimate that

$$\sup_{n \geq 1} \int_\Omega |\nabla (u_n \cdot R^{G_k} 1)|^2(y) \, \mathrm{d}y < +\infty. \tag{3.9}$$

Since $L^2(\Omega; m)$ has the Banach-Saks property, it follows from (3.9) that one can choose a subsequence (still denoted by $\{n\}$) such that $\sigma_n(\{\nabla(u_n \cdot R^{G_k} 1)\})$ is convergent in $L^2(\Omega; m)$ for every $k \geq 1$. By [7, Theorem 2.1.4], one can find a further subsequence (still denoted by $\{n\}$) such that $\{\sigma_n(\{u_n R^{G_k} 1\})\}$ is convergent q.e. for every $k \geq 1$. Since G_k is finely open, $R^{G_k} 1(x) = E_x(\tau_k \wedge \zeta) > 0$ q.e. on G_k. Therefore, $\{\sigma_n(\{u_n\})\}$ is convergent q.e. on Ω. Set $u(s, x) = \limsup_{n \to +\infty} \sigma_n(\{u_n\})(s, x)$ for $(s, x) \in \Omega$. Since $R^{G_k} 1 \in H_0^1(\Omega)$, u is quasi-continuous. Using this and the fact that $R^{G_k} 1 > 0$ q.e. on G_k we see that one can find a quasi-total family $\{\tilde{G}_k\}$ such that for every $k \geq 1$,

$$\int_{\tilde{G}_k} |\sigma_n(\{\nabla u_n\}) - \sigma_m(\{\nabla u_m\})|^2(y) \, \mathrm{d}y \to 0$$

as $n \to +\infty$. Therefore, we may define a measurable function $w : \Omega \to \mathbb{R}^n \times \mathbb{R}^d$ such that $w_{|\tilde{G}_k} = \lim_{n \to +\infty} \sigma_n(\{\nabla u_n\})$ in $L^2(\tilde{G}_k; m)$ for every $k \geq 1$. Let us fix $\alpha > 0$ and $\nu \in S_{00}^{(0)}$ (see [7, Section 2]). Then

$$
P_\nu \left(\int_0^{\tilde{\tau}_k} |w(X_t)|^2 \, \mathrm{d}t > \alpha \right) \leq \alpha^{-1} E_\nu \int_0^{\tilde{\tau}_k} \mathbf{1}_{\tilde{G}_k} |w(X_t)|^2 \, \mathrm{d}t
$$

$$
= \alpha^{-1} \langle \nu, R^{\tilde{G}_k}(\mathbf{1}_{\tilde{G}_k} w^2) \rangle
$$

$$
= \alpha^{-1} \langle R^{\tilde{G}_k} \nu, \mathbf{1}_{\tilde{G}_k} w^2 \rangle
$$

$$
\leq \alpha^{-1} \|R\nu\|_\infty \int_{\tilde{G}_k} |w(y)|^2 \, \mathrm{d}y.
$$

Similarly,

$$
P_\nu \left(\int_0^{\tilde{\tau}_k} |(\sigma_n(\{\nabla u_n\}) - w)(X_t)|^2 \, \mathrm{d}t > \alpha \right) \leq \alpha^{-1} \|R\nu\|_\infty \int_{\tilde{G}_k} |\sigma_n(\{\nabla u_n\})(y) - w(y)|^2 \, \mathrm{d}y,
$$

which converges to 0 as $n \to \infty$. Using the above two inequalities and the Borel–Cantelli lemma, one can show that $P_x(\int_0^T |w(X_t)|^2 \, \mathrm{d}t < +\infty, \ T \geq 0) = 1$ for q.e. $x \in \Omega$ and there exists a subsequence (still denoted by $\{n\}$) such that for every $T > 0$,

$$
\int_0^{\cdot} \sigma_n(\{\nabla u_n\})(X_t) \, \mathrm{d}B_t \to \int_0^{\cdot} w(X_t) \, \mathrm{d}B_t \tag{3.10}
$$

uniformly on $[0, T]$ in probability P_x for q.e. $x \in \Omega$ (see the proof of [7, Theorem 5.2.1]). Furthermore, by (3.9), the Rellich–Kondrachov theorem and the fact that $R^{G_k} 1 > 0$ q.e. on G_k and $\{G_k\}$ is a quasi-total family, we conclude that there exists a subsequence (still denoted by $\{n\}$) such that $u_n \to u$ m-a.e. Hence, by Lemma 3.4, for $n \geq k$ and $0 \leq \tau < \zeta$, we have

$$
P_x \left(\int_0^{\tau_k \wedge \tau} |(f^n(u_n) - f(u))(X_t)| \, \mathrm{d}t > \varepsilon \right)
$$

$$
= P_x \left(\int_0^{\tau_k \wedge \tau} |(f(u_n) - f(u))(X_t)| \, \mathrm{d}t > \varepsilon \right)
$$

$$
= P_x \left(\int_0^{\tau_k \wedge \tau} \mathbf{1}_{\{|u_n| \leq k\}} \mathbf{1}_{\{|u| \leq k\}} |(f(u_n) - f(u))(X_t)| \, \mathrm{d}t > \varepsilon \right),
$$

which for q.e. $x \in \Omega$ converges to 0 as $n \to +\infty$. From this, we conclude that for q.e. $x \in \Omega$,

$$
\int_0^{\cdot \wedge \tau} |(f_n(u_n) - f(u))(X_t)| \, \mathrm{d}t \to 0 \tag{3.11}
$$

uniformly on $[0, T]$ in probability P_x, for every $0 \leq \tau < \zeta$. Moreover,

$$\sigma_n(\{u_n\})(X_t) = \sigma_n(\{u_n\})(X_{T \wedge \tau}) + \int_t^{T \wedge \tau} \sigma_n(\{f^n(u_n)\})(X_\theta) \, \mathrm{d}\theta$$

$$+ \int_t^{T \wedge \tau} \sigma_n(\{\mathrm{d}A_\theta^{\mu_n}\}) - \int_t^{T \wedge \tau} \sigma_n(\{\nabla u_n\})(X_\theta) \, \mathrm{d}B_\theta \qquad (3.12)$$

and without loss of generality (see [7, Theorem 4.1.1]), we may assume that $\sigma_n(\{u_n\})$ is convergent on Ω except for some properly exceptional set $N \subset \Omega$ (see (2.2)). Therefore letting $n \to \infty$ in (3.12) and using (3.10), (3.11) we see that for every $T > 0$,

$$u(X_t) = u(X_{T \wedge \tau}) + \int_t^{T \wedge \tau} f(u)(X_\theta) \, \mathrm{d}\theta + \int_t^{T \wedge \tau} \mathrm{d}A_\theta^\mu$$

$$- \int_t^{T \wedge \tau} w(X_\theta) \, \mathrm{d}B_\theta, \quad 0 \leq t \leq T \wedge \tau, \quad P_x\text{-a.s.}$$

for q.e. $x \in \Omega$. From this, we get (2.8). From (2.8), the fact that $|u(X_t)| \leq |v(X_t)|$, $t \geq 0$, P_x-a.s. for q.e. $x \in \Omega$ and Remark 2.2 it follows that u is a solution of (1.1), $u \in \mathcal{F}\mathcal{S}^q$ for $q \in (0, 1)$ and u is of class (FD). By (3.8), $\{|\nabla u_n|^2 T_r(R^{G_k}1)\}_n$ is bounded in $L^1(\Omega; m)$ for every $r > 0$. Since $R^{G_k}1 \in D[\mathcal{E}_{G_k}]$, $T_r(R^{G_k}1) \in D[\mathcal{E}_{G_k}]$. From this, we conclude that

$$\sup_{n \geq 1} \int_\Omega |\nabla(u_n T_r(R^{G_k}1))|^2(y) \, \mathrm{d}y < +\infty,$$

which implies that $u T_r(R^{G_k}1) \in H_0^1(\Omega)$ for every $r > 0$ and $k \geq 0$. Since $R^{G_k}1$ is quasi-continuous and positive q.e. on G_k, $\{R^{G_k}1 > r, k \geq 1, r > 0\}$ forms a finely open quasi-total family. This shows that $u \in \dot{H}_{\mathrm{loc}}^1(\Omega)$ and $w = \nabla u$ since $r^{-1} u T_r(R^{G_k}1) = u$ q.e. on $\{R^{G_k} > r\}$. $\square$

4 Analytic solutions

To formulate the definition of a solution of (1.1) in the analytic sense, we will need the following lemma.

Lemma 4.1 *Let μ be a smooth measure on Ω. Then there exists a finely open quasi-total family $\{G_k\}$ such that $\mathbf{1}_{G_k}\mu \in H^{-1}(\Omega)$ for every $k \geq 0$.*

Proof The proof is a slight modification of the proof of [7, Lemma 5.1.7]. Let f be a Borel bounded positive function and let

$$\phi(x) = E_x \int_0^\infty e^{-t-A_t} f(X_t) \, \mathrm{d}t,$$

where $A = A^\mu$. Then $\phi = R_1^A f$, where $\{R_\alpha^A\}$ is the resolvent associated with the perturbed form (see [7]). It follows in particular that ϕ is quasi-continuous. Put

$$G_k = \{x \in \Omega; \phi(x) > k^{-1}\}, \quad \mu_k = \mathbf{1}_{G_k} \cdot \mu.$$

Since f is positive and ϕ is quasi-continuous, $\{G_k\}$ is a finely open quasi-total family. We have

$$R_1\mu_k(x) = E_x \int_0^\zeta e^{-t} \mathbf{1}_{G_k}(X_t)\,dA_t \le kE_x \int_0^\zeta e^{-t}\phi(x)\,dA_t \le kR^1 f(x),$$

the last inequality being a consequence of [7, Lemma 5.1.5]. Since $R^1 f \in H_0^1(\Omega)$, it follows from Theorem 2.2.1 and Lemma 2.3.2 in [7] that $R_1\mu_k \in H_0^1(\Omega)$, hence that $\mu_k \in H^{-1}(\Omega)$. □

Remark 4.2 Suppose that $\{U_\alpha,\ \alpha\ \in\ T_1\}$, $\{V_\beta,\ \beta\ \in\ T_2\}$ are two finely open quasi-total families. Then there exists a finely open quasi-total family $\{W_\gamma,\ \gamma \in T_3\}$ such that for every $\gamma \in T_3$ there exists $\alpha \in T_1, \beta \in T_2$ such that $W_\gamma \subset U_\alpha \cap V_\beta$. To see this, it suffices to take $T_3 = T_1 \times T_2$ and $\{W_{(\alpha,\beta)} = U_\alpha \cap V_\beta,\ \alpha \in T_1,\ \beta_1 \in T_2\}$.

From the definition of the space $\dot{H}_{\mathrm{loc}}^1(\Omega)$, Lemma 4.1 and Remark 4.2, it follows that for given $u \in \dot{H}_{\mathrm{loc}}^1(\Omega)$, $f \in qL_{\mathrm{loc}}^1(\Omega)$ and $\mu \in S$, there exists a finely open quasi-total family $\{U_\alpha, \alpha \in T\}$ such that for every $\alpha \in T$,

$$u = u_\alpha \text{ q.e. on } U_\alpha \text{ for some } u_\alpha \in H_0^1(\Omega), \quad \mathbf{1}_{U_\alpha} f \in L^2(\Omega; m), \quad \mathbf{1}_{U_\alpha} \cdot \mu \in H^{-1}. \quad (4.1)$$

In what follows we say that $\{U_\alpha,\ \alpha\ \in\ T\}$ is a finely open quasi-total family for the triple $(u, f, \mu) \in \dot{H}_{\mathrm{loc}}^1(\Omega) \times qL_{\mathrm{loc}}^1(\Omega) \times S$ if (4.1) is satisfied for every $\alpha \in T$.

Proposition 3.1 and Remark 3.2 suggest that the class $\dot{H}_{\mathrm{loc}}^1(\Omega) \cap \mathcal{C}_0(\Omega)$ is too large to get uniqueness of analytic solutions of (1.1) under (A4′), and secondly that the uniqueness holds if we restrict the class of solutions to functions which additionally are of class (FD). Unfortunately, we do not know how to define the class (FD) analytically. Therefore, to state the definition of a solution of (1.1) in a purely analytic way, we introduce a class $\mathcal{U}$, which is a bit narrower than (FD).

$\mathcal{U} = \{u \in \mathcal{C}_0(\Omega) : |u| \le v \text{ q.e., where } v \text{ is a solution in the sense of duality (see [15]) of the problem}$

$$-\frac{1}{2}\Delta v = \nu \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega \quad (4.2)$$

for some $\nu \in \mathcal{M}_{0,b}\}$.

Definition We say that $u : \Omega \to \mathbb{R}^n$ is a solution of (1.1) in the analytic sense if

(a) $u \in \mathcal{U}$,
(b) $u \in \dot{H}_{\mathrm{loc}}^1(\Omega) \cap \mathcal{C}_0(\Omega)$, $f(u) \in qL_{\mathrm{loc}}^1(\Omega)$,
(c) For some finely open quasi-total family $\{G_l, l \ge 1\}$ for $(u, f(u), \mu)$,

$$\mathcal{E}(u^k, v) = (f^k(u), v)_{L^2(\Omega;m)} + \langle \mu, v \rangle, \quad v \in H_0^1(G_l) \quad (4.3)$$

for every $k = 1, \ldots, n$ and $l \ge 1$.

Example 4.3 Let Ω, u be as in Remark 3.2 and let $G_l = \{x \in B(0, 1); l^{-1} < |x| < 1\}$. Then, $\{G_l,\ l \ge 1\}$ is a finely open quasi-total family for u, $u \in \dot{H}_{\mathrm{loc}}^1(\Omega) \cap \mathcal{C}_0(\Omega)$ and $\mathcal{E}(u^k, v) = 0$ for $v \in H_0^1(G_l)$, i.e., u satisfies (b), (c) of the above definition with $f \equiv 0$, $\mu \equiv 0$. Of course, $v \equiv 0$ satisfies the same conditions, too.

Lemma 4.4 *If $u \in H_0^1(\Omega)$ then u is of class* (FD).

Proof If $u \in H_0^1(\Omega)$, then $u^+ \in H_0^1(\Omega)$, so we may assume that $u \geq 0$. Let $v \in H_0^1(\Omega)$ be such that $v \geq u$ q.e. and $-\Delta v \geq 0$ in the distributional sense (one can, for example, take v to be a solution of the obstacle problem with barrier u). Then, there exists a nonnegative diffuse measure $\mu \in H^{-1}(\Omega)$ such that $-\frac{1}{2}\Delta v = \mu$ in $H^{-1}(\Omega)$. By Klimsiak and Rozkosz [11], v is of class (FD), and hence, so is u. $\square$

Proposition 4.5 *Assume that $u \in \dot{H}_{\mathrm{loc}}^1(\Omega)$ and $f(u) \in qL_{\mathrm{loc}}^1$. Then u satisfies* (2.8) *iff u satisfies* (4.3).

Proof Assume that u satisfies (2.8). Let $\{G_l, \ l \geq 1\}$ be a finely open quasi-total family for $(u, f(u), \mu)$. Fix $l \in \mathbb{N}$, $v \in H_0^1(G_l)$ and let $\tau_l = \tau_{G_l}$. By (2.8), for q.e. $x \in \Omega$ and every $0 \leq \tau < \zeta$, we have

$$u(X_t) = u(X_{\tau_l \wedge \tau}) + \int_t^{\tau_l \wedge \tau} f(u)(X_\theta)\,\mathrm{d}\theta + \int_t^{\tau_l \wedge \tau} \mathrm{d}A_\theta^\mu$$

$$- \int_t^{\tau_l \wedge \tau} \mathrm{d}M_\theta, \quad 0 \leq t \leq \tau_l \wedge \tau, \quad P_x\text{-a.s.} \tag{4.4}$$

Let $u_l \in H_0^1(\Omega)$ be such that $u = u_l$ q.e. on G_l. Then by (4.4),

$$u_l(x) = u_l(X_{\tau_l \wedge \tau}) + \int_0^{\tau_l \wedge \tau} f(u)(X_t)\,\mathrm{d}t + \int_0^{\tau_l \wedge \tau} \mathrm{d}A_t^\mu - \int_0^{\tau_l \wedge \tau} \mathrm{d}M_t. \tag{4.5}$$

By the definition of τ_l and the remark following (2.5),

$$E_x \int_0^{\tau_l \wedge \tau} |f(u)(X_t)|\,\mathrm{d}t = E_x \int_0^{\tau_l \wedge \tau} \mathbf{1}_{G_l}(X_t)|f(u)(X_t)|\,\mathrm{d}t$$

$$\leq E_x \int_0^\zeta \mathbf{1}_{G_l}|f(u)(X_t)|\,\mathrm{d}t < +\infty,$$

for q.e. $x \in \Omega$. By Fatou's lemma, $E_x \int_0^{\tau_l \wedge \zeta} |f(u)(X_t)|\,\mathrm{d}t < +\infty$ for q.e. $x \in \Omega$. Likewise, by the remark following (2.5), $E_x \int_0^\zeta \mathrm{d}|A^\mu|_t < +\infty$ and, by [7, Theorem 4.3.2], $E_x|u(X_{\tau_l \wedge \zeta})| < +\infty$ for q.e. $x \in \Omega$. Therefore, from (4.5), the fact that $u_l \in \mathcal{C}_0(\Omega)$ (see Remark 2.2) and u_l is of class (FD) (see Lemma 4.4) it follows that for q.e. $x \in \Omega$,

$$u_l(x) = E_x u_l(X_{\tau_l \wedge \zeta}) + E_x \int_0^{\tau_l \wedge \zeta} f(u)(X_t)\,\mathrm{d}t + E_x \int_0^{\tau_l \wedge \zeta} \mathrm{d}A_t^\mu. \tag{4.6}$$

By (2.12) and (2.13), one can rewrite the above equation in the form

$$u_l(x) = (H_{G_l} u_l)(x) + (R^{G_l} f(u))(x) + (R^{G_l} \mu_l)(x). \tag{4.7}$$

Hence,

$$(u_l, v)_{L^2(\Omega;m)} = (H_{G_l}(u_l), v)_{L^2(\Omega;m)} + (R^{G_l} f, v)_{L^2(\Omega;m)} + (R^{G_l} \mu_l, v)_{L^2(\Omega;m)}, \tag{4.8}$$

that is

$$(u_l - H_{G_l}(u_l), v)_{L^2(\Omega;m)} = (R^{G_l} f, v)_{L^2(\Omega;m)} + (R^{G_l} \mu_l, v)_{L^2(\Omega;m)} \tag{4.9}$$

for every $v \in H_0^1(G_l)$. Since $v, u_l^k - H_{G_l}(u_l^k) \in H_0^1(G_l) \subset H_0^1(\Omega)$,

$$(u_l^k - H_{G_l}(u_l^k), v)_{L^2(\Omega;m)} = \mathcal{E}(u_l^k - H_{G_l}(u_l^k), R^{G_l} v), \quad k = 1, \ldots, n.$$

Moreover,

$$\mathcal{E}(u_l^k - H_{G_l}(u_l^k), R^{G_l} v) = \mathcal{E}(u_l^k, R^{G_l} v), \quad k = 1, \ldots, n,$$

because $H_0^1(G_l)$ and $\mathcal{H}_{\Omega \setminus G_l}$ are orthogonal and $R^{G_l} v \in H_0^1(G_l)$. Using this and symmetry of R^{G_l}, we conclude from (4.9) that

$$\mathcal{E}(u_l^k, R^{G_l} v) = (f^k(u), R^{G_l} v)_{L^2(\Omega;m)} + (\mu_l^k, R^{G_l} v)_{L^2(\Omega;m)}, \quad k = 1, \ldots, n,$$

hence that

$$\mathcal{E}(u^k, R^{G_l} v) = (f^k(u), R^{G_l} v)_{L^2(\Omega;m)} + (\mu^k, R^{G_l} v)_{L^2(\Omega;m)}, \quad k = 1, \ldots, n \tag{4.10}$$

since $R^{G_l} v \in H_0^1(G_l)$. Finally, since (4.10) holds true for every $v \in H_0^1(G_l)$, $l \in \mathbb{N}$, and $R^{G_l}(H_0^1(G_l))$ is a dense subset of $H_0^1(G_l)$ in the norm induced by the form $\mathcal{E}_{G_l}$, i.e., the norm defined as $\|v\|_{\mathcal{E}_{G_l}} = \mathcal{E}_{G_l}(v, v) + (v, v)_{L^2(\Omega,m)}$ for $v \in H_0^1(G_l)$, we conclude from (4.10) that for every $v \in H_0^1(G_l)$,

$$\mathcal{E}(u^k, v) = (f^k(u), v)_{L^2(\Omega;m)} + (\mu^k, v)_{L^2(\Omega;m)}, \quad k = 1, \ldots, n,$$

which shows that u satisfies (2.8).

Now, let us assume that u satisfies (4.3). Let $\{G_l, l \geq 1\}$ be a finely open quasi-total family for $(u, f(u), \mu)$. Then, taking $v = R^{G_l} \nu$ with $\nu \in \mathcal{M}_{0,b} \cap H^{-1}(\Omega)$ as a test function in (4.3) and reversing the steps of the proof of (4.10), we get (4.8) with v replaced by ν and with the duality $\langle \cdot, \cdot \rangle$ in place of $(\cdot, \cdot)_{L^2(\Omega;m)}$, i.e.

$$\langle u_l, \nu \rangle = \langle H_{G_l}(u_l), \nu \rangle + \langle R^{G_l} f, \nu \rangle + \langle R^{G_l} \mu_l, \nu \rangle.$$

By [7, Theorem 2.2.3], this implies (4.7), which in turn gives (4.6). Now, we show that (4.6) implies (4.4). To this end, let us first note that by [7, Theorem 4.1.1] one can assume that (4.6) holds except for some properly exceptional set. Let $\tau \in \mathcal{T}$ be such that $0 \leq \tau < \tau_l \wedge \zeta$. Then

$$u_l(X_\tau) = E_{X_\tau} u_l(X_{\tau_l \wedge \zeta}) + E_{X_\tau} \int_0^{\tau_l \wedge \zeta} f(u)(X_t) \, dt + E_{X_\tau} \int_0^{\tau_l \wedge \zeta} dA_t^\mu, \quad P_x\text{-a.s.}$$

Let $\{\theta_t, t \geq 0\}$ denote the family of shift operators. By the strong Markov property and the fact that $(\tau_l \wedge \zeta) \circ \theta_\tau + \tau = \tau_l \wedge \zeta$,

$$u_l(X_\tau) = E_x \left(u_l(X_{\tau_l \wedge \zeta}) + \int_\tau^{\tau_l \wedge \zeta} f(u)(X_t) \, dt + \int_\tau^{\tau_l \wedge \zeta} dA_t^\mu \Big| \mathcal{F}_\tau \right), \quad P_x\text{-a.s.}$$

Write

$$M_t^{l,x} = E_x \left(\int_0^{\tau_l \wedge \zeta} f(u)(X_\theta) \, dt + \int_0^{\tau_l \wedge \zeta} dA_\theta^\mu \Big| \mathcal{F}_t \right) - u_l(X_0), \quad t \geq 0.$$

By [7, Lemma A.3.5], $M^{l,x}$ has a continuous version M^l which does not depend on x. Observe that M^l is a MAF of $\mathbb{X}$ and

$$u_l(X_\tau) = u(X_0) - \int_0^\tau f(u)(X_t)\, dt - \int_0^\tau dA_t^\mu + \int_0^\tau dM_t^l, \quad P_x\text{-a.s.}$$

Since the above equality is satisfied for every $\tau \in \mathcal{T}$ such that $0 \le \tau < \tau_l \wedge \zeta$, we can assert that

$$u_l(X_t) = u(X_0) - \int_0^t f(u)(X_\theta)\, d\theta - \int_0^t dA_\theta^\mu + \int_0^t dM_\theta^l, \quad 0 \le t \le \tau_l \wedge \zeta, \quad P_x\text{-a.s.}$$

Without loss of generality, we may assume that the family $\{G_l,\ l \ge 1\}$ is ascending. Then, by [7, Lemma 5.5.1], $M_t^l = M_t^{l+1}$, $0 \le t \le \tau_l$. Therefore, the process M defined as $M_t = \lim_{l \to +\infty} M_t^l$, $t \ge 0$, is well defined, and M is a local MAF of $\mathbb{X}$ and

$$u_l(X_t) = u(X_0) - \int_0^t f(u)(X_\theta)\, d\theta - \int_0^t dA_\theta^\mu + \int_0^t dM_\theta, \quad 0 \le t \le \tau_l \wedge \zeta, \quad P_x\text{-a.s.},$$

which implies (4.4). Finally, letting $l \to +\infty$ in (4.4), we get (2.8). $\qquad\square$

Proposition 4.6 *If $u \in \mathcal{U}$ then u is of class* (FD).

Proof From Klimsiak and Rozkosz [11], it follows that if $\nu \in \mathcal{M}_{0,b}$, then the solution of (4.2) in the sense of duality is of class (FD). This implies the desired result. $\qquad\square$

Corollary 4.7 *Assume that f satisfies* (A4′). *Then there exists at most one analytic solution of* (1.1).

Proof Follows from Propositions 3.1, 4.5, and 4.6. $\qquad\square$

Remark 4.8 From Lemma 3.3, it follows that the martingale M from Proposition 4.5 has the representation

$$M_t = \int_0^t \nabla u(X_\theta)\, dB_\theta, \quad t \ge 0, \quad P_x\text{-a.s.}$$

for q.e. $x \in \Omega$.

In the following, three propositions by a solution of (1.1) we mean either probabilistic solution or analytic solution.

Let us recall that by T_r, we denote the truncature operator defined in the proof of Theorem 3.5.

Proposition 4.9 *Assume that u is a solution of* (1.1) *such that $f(u) \in L^1(\Omega; m)$. Then $T_r(u) \in H_0^1(\Omega)$ for every $r > 0$, $u \in W_0^{1,q}(\Omega)$ for $q \in [1, \frac{d}{d-1})$ and*

$$\mathcal{E}(u^k, v) = (f^k(u), v)_{L^2(\Omega;m)} + \langle v, \mu^k\rangle, \quad v \in C_0^2(\overline{\Omega}), \quad k = 1, \ldots, n. \tag{4.11}$$

Proof Since $f(u) \in L^1(\Omega; m)$ and $\mu \in \mathcal{M}_{0,b}$, $E_x \int_0^\zeta |f(u)(X_t)| \, dt < +\infty$ and $E_x \int_0^\zeta d|A^\mu|_t < +\infty$ for q.e. $x \in \Omega$ (see the remark following (2.5)). Using this and the fact that u is of class (FD), we conclude from (2.9) that

$$u(x) = E_x \int_0^\zeta f(u)(X_t) \, dt + E_x \int_0^\zeta dA_t^\mu$$

for q.e. $x \in \Omega$. From the results proved in Klimsiak and Rozkosz [11], it follows now that u is a solution of (1.1) in the sense of duality (see [15]). Consequently, u satisfies (4.11). Moreover, by Stampacchia [15], $u \in W_0^{1,q}(\Omega)$ with $1 \leq q < d/(d-1)$, while by Dal Maso et al. [3], $T_r(u) \in H_0^1(\Omega)$ for every $r > 0$. $\square$

We do not know whether under (A4) the function $f(u)$ is integrable. We can show, however, that it is integrable under stronger than (A4) condition (A4'').

Proposition 4.10 *Let assumption* (A4'') *hold and let u be a solution of* (1.1). *Then $f(u) \in L^1(\Omega; m)$,* (4.11) *is satisfied and*

$$\|f(u)\|_{L^1(\Omega;m)} \leq \|\mu\|_{\mathrm{TV}}.$$

Proof Since u is a solution of (1.1), u^k is a solution of the scalar equation

$$\begin{cases} -\frac{1}{2} \Delta u^k = g^k(x, u^k) + \mu^k & \text{in } \Omega, \\ u^k = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g^k : \Omega \times \mathbb{R} \to \mathbb{R}$, $g^k(x, t) = f^k(x, u^1(x), \ldots, u^{k-1}(x), t, u^{k+1}(x), \ldots, u^n(x))$. Observe that by (A4''), $g^k(x, t) \cdot t \leq 0$ for a.e. $x \in \Omega$ and every $t \in \mathbb{R}$. Therefore, by [11, Lemma 2.3] (in [11, Lemma 2.3], it is assumed that $g^k(x, \cdot)$ is monotone but as a matter of fact in the proof only the sign condition formulated above is used) and Proposition 4.5, for every $\tau \in \mathcal{T}$ such that $0 \leq \tau < \zeta$, we have

$$E_x \int_0^\tau |g^k(X_t, u^k(X_t))| \, dt \leq E_x |u^k(X_\tau)| + E_x \int_0^\tau d|A^{\mu^k}|_t$$

for q.e. $x \in \Omega$. Let $\{\tau_k\} \subset \mathcal{T}$ be such that $0 \leq \tau_k < \zeta$ and $\tau_k \to \zeta$. Replacing τ by τ_k in the above inequality, passing to the limit and using the fact that u is of class (FD) and $u \in \mathcal{C}_0(\Omega)$, we get

$$E_x \int_0^\zeta |g^k(X_t, u^k(X_t))| \, dt \leq E_x \int_0^\zeta d|A^{\mu^k}|_t.$$

By the above and [11, Lemma 5.4],

$$\|f^k(u)\|_{L^1(\Omega;m)} = \|g^k(u_k)\|_{L^1(\Omega;m)} \leq \|\mu^k\|_{\mathrm{TV}},$$

which implies the desired inequality. That (4.11) is satisfied now follows from Proposition 4.9. $\square$

Finally, we show that if the right-hand side of (1.1) satisfies the uniform angle condition (A5), then Stampacchia's estimate (1.5) holds for every solution of (1.1).

Proposition 4.11 *Assume* (A5). *If u is a solution of* (1.1) *then*

$$\|f(u)\|_{L^1(\Omega;m)} \leq \alpha^{-1}\|\mu\|_{\text{TV}}.$$

Proof Using the Itô-Tanaka formula, we get the first inequality in (2.10). From this inequality, the fact that u is of class (FD), $u \in \mathcal{C}_0(\Omega)$ and (A5) one can conclude that for every $\tau \in \mathcal{T}$ such that $0 \leq \tau < \zeta$,

$$E_x \int\limits_0^\tau |f(u)(X_t)|\,\mathrm{d}t \leq \alpha^{-1} E_x \int\limits_0^\zeta d|A^\mu|_r$$

for q.e. $x \in \Omega$. The desired inequality now follows from Fatou's lemma and [11, Lemma 5.4]. □

References

1. Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vazquez, J.-L.: L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22**, 241–273 (1995)
2. Briand, Ph, Delyon, B., Hu, Y., Pardoux, E., Stoica, L.: L^p solutions of backward stochastic differential equations. Stoch. Process. Appl. **108**, 109–129 (2003)
3. Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Scuola Norm. Sup. Pisa. Cl. Sci. **4**, 741–808 (1999)
4. Droniou, J., Poretta, A., Prignet, A.: Parabolic capacity and soft measures for nonlinear equations. Potential Anal. **19**, 99–161 (2003)
5. Fuglede, B.: Integral representation of fine green potentials. Math. Ann. **262**, 191–214 (1983)
6. Fuglede, B.: The Dirichlet Laplacian on finely open sets. Potential Anal. **10**, 91–101 (1999)
7. Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet Forms and Symmetric Markov Processes. Walter de Gruyter, Berlin (1994)
8. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Springer, Berlin (1991)
9. Kilpeläinen, T., Malý, J.: Supersolutions to degenerate elliptic equation on quasi open sets. Comm. Partial Differ. Equ. **17**, 371–405 (1992)
10. Klimsiak, T.: Reflected BSDEs and the obstacle problem for semilinear PDEs in divergence form. Stoch. Process. Appl. **122**, 134–169 (2012)
11. Klimsiak, T., Rozkosz, A.: Dirichlet forms and semilinear elliptic equations with measure data. J. Funct. Anal. (2013). doi:10.1016/j.jfa.2013.05.028
12. Landes, R.: On the angle condition for the perturbation of elliptic systems. Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **29**, 253–268 (2000)
13. Orsina, L., Ponce, A.C.: Semilinear elliptic equations and systems with diffuse measures. J. Evol. Equ. **8**, 781–812 (2008)
14. Protter, P.: Stochastic Integration and Differential Equations, 2nd edn. Springer, Berlin (2004)
15. Stampacchia, G.: Équations elliptiques du second ordre à coefficients discontinus. Séminaire de Mathémtiques Supérieures 16 (1966)