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Estimates of invariant distances on "convex" domains

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Abstract Estimates for invariant distances of convexifiable, $\mathbb{C}$-convexifiable and planar domains are given.

Keywords Carathéodory · Kobayashi and Bergman distances · Bergman and Szegö kernels · Convex · Convexifiable and $\mathbb{C}$-convex domains

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1 Introduction and results

Diederich and Ohsawa [6, p. 182] asked if D is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, then the following lower bound for the Bergman distance b_D holds: For fixed z and w close to ∂D, one has that

$$b_D(z, w) \geq -c \log d_D(w),$$

where $d_D(w) = \text{dist}(w, \partial D)$ and $c > 0$ is a constant depending only on D. Błocki [4, Theorem 1.3] mentioned this fact for bounded convexifiable domains (not necessarily smooth).

We shall prove the estimate in the case of bounded $\mathbb{C}$-convex domains (or, more generally, $\mathbb{C}$-convexifiable). Recall that a set in $\mathbb{C}^n$ is called $\mathbb{C}$-convex if all its intersections with complex lines are contractible (cf. [2, p. 25]). Note that a C^1-smooth bounded domain is $\mathbb{C}$-convex if

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and only the complex tangent hyperplane through any boundary point does not intersect the domain (cf. [2, Theorem 2.5.2]).

Let D be a domain in $\mathbb{C}^n$. Denote by c_D and l_D the Carathéodory distance and the Lempert function of D, respectively:

$$c_D(z,w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D,\mathbb{D}) \text{ with } f(z)=0\},$$
$$l_D(z,w) = \inf\{\tanh^{-1}|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D},D) \text{ with } \varphi(0)=z, \varphi(\alpha)=w\},$$

where $\mathbb{D}$ is the unit disk (we refer to [10] for basic properties of the objects under consideration). The Kobayashi distance k_D is the largest pseudodistance not exceeding l_D. We have that

$$c_D \le k_D, \quad c_D \le b_D$$

(if b_D is well-defined). Note also that $k_D = l_D$ for any planar domain D (cf. [10, Remark 3.3.8(e)]). By Lempert's theorem [11, Theorem 1], combining with a result by Jacquet [9, Theorem 5], $c_D = l_D$ on any C^2-smooth bounded $\mathbb{C}$-convex domain D and hence on any convex domain. On the other hand, it follows by [14, Theorem 12] that there exists a constant $c_n > 0$, depending only on n, such that

$$k_D \le 4b_D \le c_n k_D \tag{1}$$

for any $\mathbb{C}$-convex domain D in $\mathbb{C}^n$, containing no complex lines (then b_D is well-defined). In other words, to estimate b_D, it is enough to find lower bounds for c_D and upper bounds for l_D.

Recall that b_D is the integrated form of Bergman metric

$$\beta_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}}, \quad z \in D, X \in \mathbb{C}^n,$$

where

$$M_D(z;X) = \sup\{|f'(z)X| : f \in L_h^2(D)\, ||f||_D \le 1,\ f(z)=0\}$$

and

$$K_D(z) = \sup\{|f(z)|^2 : f \in L_h^2(D),\ ||f||_D \le 1\}$$

is the Bergman kernel on the diagonal ($K_D(z) > 0$ is assumed). So,

$$b_D(z,w) = \inf_\gamma \int_0^1 \beta_D(\gamma(t);\gamma'(t)),$$

where the infimum is taken over all smooth curves $\gamma : [0,1] \to D$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Estimates for invariant distances of strictly pseudoconvex domains in $\mathbb{C}^n$ and pseudoconvex domains of finite type in $\mathbb{C}^2$ can be found in [3] (see also [1,10]) and [8], respectively.

Recall now in details two estimates. The proof of [4, Theorem 5.4] (cf. also [12, Proposition 2.4]) implies that if D is a proper convex domain in $\mathbb{C}^n$, then

$$c_D(z,w) \ge \frac{1}{2}\log\frac{d_D(z)}{d_D(w)} \tag{2}$$

(this proof uses only the existence of an appropriate supporting (real) hyperplane and the formula for the Poincaré distance of the upper half-plane). On the other hand, by [13, Theorem 1], for any $C^{1+\varepsilon}$-smooth bounded domain, there exists a constant $c > 0$ such that

$$l_D(z, w) \le -\frac{1}{2}\log(d_D(z)d_D(w)) + c \tag{3}$$

(see [7, Proposition 2.5] for a stronger estimate for k_D).

The smoothness is essential as an example of a C^1-smooth bounded $\mathbb{C}$-convex planar domain shows (see [13, Example 2]). Moreover, using [16, p. 146, Theorem 7], one may find a bounded $\mathbb{C}$-convex planar domain for which there is no similar estimate with any constant instead of $-1/2$.

So, it natural to find an upper bound for l_D in the convex case and a lower bound for c_D in the $\mathbb{C}$-convex case.

Proposition 1 *Let D be a proper convex domain in $\mathbb{C}^n$. Then*

$$l_D(z, w) \le \frac{||z-w||}{d(z)-d(w)}\log\frac{d(z)}{d(w)} \le \frac{||z-w||}{\min(d(z), d(w))}.$$

[1] *In particular, if, in addition, D is bounded, then for any compact subset K of D, there is a constant $c_K > 0$ such that*

$$b_D \le -c_K \log d_D(w) + 1/c_K, \quad z \in K, w \in D.$$

The last estimate for k_D instead of b_D (and K a singleton) is the content of [12, Proposition 2.3]. Similar estimates for the Kobayashi distance of pseudoconvex Reinhardt domains can be found in [19].

Proposition 2 *Let D be a proper $\mathbb{C}$-convex domain in $\mathbb{C}^n$. Then*

$$c_D(z, w) \ge \frac{1}{4}\log\frac{d_D(z)}{4d_D(w)}.$$

Hence, if, in addition, D is bounded, then for any compact subset K of D, there is a constant $c_K > 0$ such that

$$b_D(z, w) \ge -\frac{1}{4}\log d_D(w) - c_K, \quad z \in K, w \in D.$$

Note that by [5, p. 2381], the first estimate in Proposition 2 implies the following

Corollary 3 *The Bergman and Szegö kernels (on the diagonal) are comparable on any C^2-smooth bounded $\mathbb{C}$-convex domain.*

We point out that [5, Theorem 1.3] deals with the convex case.

Remark (a) The estimate for l_D is sharp when $z \to w$. Moreover, it is sharp up to a constant when z is fixed and $w \to \partial D$. Indeed, denote by $R_D(z, w)$, the right-hand side of the first inequality in Proposition 1. If $\theta \in (0, \pi)$ and $D_\theta = \{z \in \mathbb{C}_* : |\arg z| < \theta\}$, then

$$\lim_{\theta\to 0}\lim_{x\to 0+}\frac{l_{D_\theta}(1, x)}{R_{D_\theta}(1, x)} = \frac{\pi}{4}.$$

[1] If $d(z) = d(w)$, then $l_D(z, w) \le ||z-w||/d(w)$.

(b) The factor $1/4$ in the bound for c_D is optimal as $D = \mathbb{C}_* \setminus \mathbb{R}^+$ shows.
(c) Estimates for the infinitesimal forms of the distances under consideration, namely, the Carathéodory, Kobayashi and Bergman metrics, of convex and $\mathbb{C}$-convex domains can be found in [14]. The bounds there depend only on the distance to the boundary from the respective point in the respective direction.

Our main result is in the spirit of [4, Theorem 1.3], where a lower bound for the Bergman metric is mentioned in the locally convexifiable case (and a hint for a proof is given).

Proposition 4 *Let D be a bounded domain in $\mathbb{C}^n$ which is locally $\mathbb{C}$-convexifiable, i.e., for any point $a \in \partial D$, there exist a neighborhood U_a of a, an open set V_a in $\mathbb{C}^n$ and a biholomorphism $F_a : U_a \to V_a$ such that $F_a(D \cap U_a)$ is $\mathbb{C}$-convex. Then, there exists a constant $c > 0$ such that for any compact subset K of D one can find a constant $c_K > 0$ with*

$$s_D(z, w) \geq -c \log d_D(w) - c_K, \quad z \in K, w \in D,$$

where $s_D = k_D$ or $s_D = b_D$.

Moreover, if D is locally convexifiable or $C^{1+\varepsilon}$-smooth and locally $\mathbb{C}$-confexifiable, then for any compact subset K of D, one can find a constant $c'_K > 0$ with

$$s_D(z, w) \leq -c'_K \log d_D(w) + 1/c'_K, \quad z \in K, w \in D.$$

Finally, we consider the planar case. We shall say that a boundary point p of a planar domain D is Dini-smooth if ∂D near p is a Dini-smooth curve $\gamma : [0, 1] \to \mathbb{C}$.[2] Call a planar domain Dini-smooth if it is Dini-smooth near any boundary point.

Proposition 5 *Let p be a Dini-smooth boundary point of a planar domain D. Then, for any neighborhood U of p and any compact subset K of D, there exist a neighborhood V of p and a constant $c > 0$ such that*

$$s_D(z, w) \geq -\frac{1}{2} \log d_D(w) - c, \quad z \in D \setminus U, w \in D \cap V,$$

$$|s_D(z, w) + \frac{1}{2} \log d_D(w)| \leq c, \quad z \in K, w \in D \cap V,$$

where $s_D = c_D$, $s_D = l_D (= k_D)$ or $s_D = b_D/\sqrt{2}$.

Since k_D and b_D are the integrated forms of κ_D and β_D, we get the following

Corollary 6 *Let p and q be different Dini-smooth boundary points of a planar domain D. If $s_D = l_D (= k_D)$ or $s_D = b_D/\sqrt{2}$, then the function*

$$2s_D(z, w) + \log d_D(z) + \log d_D(w)$$

is bounded for z near q and w near p.

In general, c_D is not an inner distance (even in the plane). So, the next proposition is not a direct consequence of Proposition 5.

Proposition 7 *Let p and q be different Dini-smooth boundary points of a planar domain D. Then, the function*

$$2c_D(z, w) + \log d_D(z) + \log d_D(w)$$

is bounded for z near q and w near p.

[2] This means that $\int_0^1 \frac{\omega(t)}{t} dt < \infty$, where ω is the modulus of continuity of γ'.

The next result is optimal for the boundary behavior of c_D and $l_D(= k_D)$ in the planar case. It is more general than the last results, but its proof uses these results. Similar (and slightly weaker) result for k_D on C^2-smooth strictly pseudoconvex bounded follows by [3, Theorem 1, Proposition 1.2].

Proposition 8 *Let D be a Dini-smooth bounded planar domain.*[3] *Then, there exists a constant $c \geq 1$ such that*

$$\log\left(1+\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}}+\frac{|z-w|^2}{cd_D(z)d_D(w)}\right)\leq 2c_D(z,w)\leq 2l_D(z,w)$$
$$\leq \log\left(1+\frac{c|z-w|}{\sqrt{d_D(z)d_D(w)}}+\frac{c|z-w|^2}{d_D(z)d_D(w)}\right).$$

In particular, the function $l_D - c_D$ is bounded on $D \times D$.

It is shown in [18, Theorem 1] that if D is strongly pseudoconvex domain in $\mathbb{C}^n$, then

$$\lim_{\substack{w\to\partial D\\ z\neq w}}\frac{c_D(z,w)}{k_D(z,w)}=1 \quad \text{uniformly in } z\in D.$$

We have the following planar extension of this result.

Proposition 9 *If D is finitely connected bounded planar domain without isolated boundary points, then*

$$\lim_{\substack{w\to\partial D\\ z\neq w}}\frac{c_D(z,w)}{l_D(z,w)}=1 \quad \text{uniformly in} \quad z\in D.$$

2 Proofs

Proof of Proposition 1 Denote by $C_{z,w}$ the convex hull of the union of the disks $\mathbb{D}(z, d_D(z))$ and $\mathbb{D}(w, d_D(w))$, lying in the complex line through z and w. Let $\gamma(t) = z + t(w - z)$. Since $C_{z,w} \subset D$ and $l_{C_{z,w}} = k_{C_{z,w}}$ is the integrated form of the Kobayashi metric $\kappa_{C_{z,w}}$,[4] then

$$l_D(z,w)\leq l_{C_{z,w}}(z,w)\leq \int_0^1 \kappa_{C_{z,w}}(\gamma(t);\gamma'(t))dt$$
$$\leq \int_0^1 \frac{|\gamma'(t)|}{d_{C_{z,w}}(\gamma(t))}dt=\frac{||z-w||}{d(z)-d(w)}\log\frac{d(z)}{d(w)}.$$

This inequality and (1) lead to the wanted result for b_D.

Proof of Proposition 2 Let $p(w) \in \partial D$ be such that $||w - p(w)|| = d_D(w)$. Since E is $\mathbb{C}$-convex, there exists a hyperplane $H_{p(w)}$ through $p(w)$ and disjoint from D (cf. [2, Theorem 2.3.9(ii)]). Denote by D_w and z_w the projections of D and z onto the complex line through w and $p(w)$ in direction $H_{(p(w)}$, respectively. By [2, Theorem 2.3.6], D_w is a simply connected

[3] This means that D is Dini-smooth near any boundary point.

[4] If $D \subset \mathbb{C}^n$, then $\kappa_D(z; X) = \inf\{|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \alpha\varphi'(0) = X\}$.

domain and $p(w) \in \partial D_w$. Denote by $\psi_w \in \mathcal{O}(\mathbb{D}, D_w)$ a Riemann map such that $\psi_w(0) = z_w$. If $\psi_w(\alpha_w) = w$, then

$$c_D(z, w) \geq c_{D_w}(z_w, w) = \tanh^{-1} |\alpha_w|.$$

By [16, p. 139, Corollary 6] (which is a consequence of the Köbe 1/4 and the Köbe distortion theorems),

$$\tanh^{-1} |\alpha_w| \geq \frac{1}{4} \log \frac{|\psi'_w(0)|}{4d_{D_w}(w)}.$$

Since $d_{D_w}(w) = d_D(w)$ and $|\psi'_w(0)| \geq d_{D_w}(z_w) \geq d_D(z)$, it follows that

$$c_D(z, w) \geq \frac{1}{4} \log \frac{d_D(z)}{4d_D(w)}.$$

This inequality and $b_D \geq c_D$ imply the desired result for b_D.

Proof of Proposition 4 [5] First, we shall prove the lower bound.

Note that

$$0 < c_a \leq \frac{d_{F_a(D\cap U_a)}(F_a(w))}{d_D(w)} \leq \frac{1}{c_a} \quad \text{near any } a \in \partial D. \tag{4}$$

Then, by Proposition 2, we may find a finite set $M \subset \partial D$ and a constant $c_1 > 0$ such that

$$s_{D\cap U_a}(z, w) \geq \frac{1}{4} \log \frac{d_D(z)}{d_D(w)} - c_1, \quad z, w \in D \cap V_a, a \in M,$$

where $V_a \subset U_a$ is a neighborhood of a such that $\partial D \subset \cup_{a\in M} V_a$.

Denote now by S_D the Kobayashi or Bergman metrics of D. By localization principles (cf. [10, Proposition 7.2.9 and Proposition 6.3.5], since D is pseudoconvex), there exists a constant $c_2 > 0$ such that

$$S_D \geq 4c_2 S_{D\cap U_a} \quad \text{on } (D \cap V_a) \times \mathbb{C}^n.$$

Let $W_a \Subset V_a$ be such that $W = \cup_{a\in M} W_a$ does not intersect K and contains ∂D. Set $r = \min_{a\in M} \operatorname{dist}(\partial W_a, \partial V_a)$.

Let $\varepsilon > 0$. Since s_D is the integrated form of S_D, for any $z \in K$ and $w \in D \cap W$, there exists a smooth curve $\gamma : [0, 1] \to D$ with $\gamma(0) = z$, $\gamma(1) = w$ and

$$s_D(z, w) + \varepsilon > \int_0^1 S_D(\gamma(t); \gamma'(t))dt.$$

Let $t_1 = \max\{t \in (0, 1) : \gamma(t) \in G = D \setminus W\}$. Choose a point $a_1 \in M$ such that $\mathbb{B}_n(\gamma(t_1), r) \subset V_{a_1}$. Let $t_2 = \sup\{t \in (t_1, 1] : \gamma([t_1, t)) \in V_{a_1}\}$ and etc. In this way, we may find numbers $0 < t_1 < \cdots < t_{N+1} = 1$ and points $a_1, \ldots, a_{N+1} \in M$ such that

[5] Some difficulty arises from the fact that, in contrast to invariant metrics, general localization principles for invariant distances are not known. However, a strong localization principle holds for k_D and c_D if D is strongly pseudoconvex (see [18, Proposition 3, Theorem 1].

$\gamma[t_j, t_{j+1}) \subset D \cap V_{a_j}$ and $||\gamma(t_{j+1}) - \gamma(t_j)|| \geq r,\ 1 \leq j \leq N$. Then

$$\begin{aligned} s_D(z, w) + \varepsilon &> c_2 \sum_{j=1}^{N} s_{D\cap U_{a_j}}(\gamma(t_j), \gamma(t_{j+1})) \\ &\geq c_2 \sum_{j=1}^{N} \log \frac{d_D(\gamma(t_j))}{d_D(\gamma(t_{j+1}))} - c_3 N \\ &\geq c_2 \log \frac{\operatorname{dist}(G, \partial D)}{d_D(w)} - c_3 N, \end{aligned}$$

where $c_3 = 4c_1c_2$.

On the other hand, since D is a bounded domain, there exists a constant $c_4 > 0$ such that $s_D(z_1, z_2) \geq c_4||z_1 - z_2||$. Then

$$s_D(z, w) + \varepsilon > \sum_{j=1}^{N} s_D(\gamma(t_j), \gamma(t_{j+1})) \geq c_4 r N.$$

So,

$$\left(1 + \frac{c_3}{c_4 r}\right)(s_D(z, w) + \varepsilon) \geq c_2 \log \frac{\operatorname{dist}(G, \partial D)}{d_D(w)}.$$

The case when $w \in G$ is trivial which completes the proof of the lower bound.

The proof of the upper bound is easier. Fix a point $a \in \partial D$. It is enough to find a constant $c'_{a,K} > 0$ such that the estimate holds for w near a. Take a point $u \in U_a$ and a neighborhood $V_a \Subset U_a$ of a and a point $u \in D \cap U_a$. Proposition 1, (3) and (4) imply that

$$\begin{aligned} k_D(z, w) &\leq k_D(z, u) + k_D(u, w) \leq k_D(z, u) + k_{D\cap U}(u, w) \\ &\leq 1/c'_{a,K} - c'_{a,K} \log d_D(w),\ z \in K, w \in D \cap V_a. \end{aligned}$$

The upper bound for b_D follows similarly. It suffices to use that

$$b_D \leq \widetilde{c}_a b_{D\cap U_a} \leq \widetilde{c}_a c_n k_{D\cap U_a}$$

in view of [10, Proposition 6.3.5] and (1).

Proof of Proposition 5 for c_D *and* l_D We may find a Dini-smooth Jordan curve ζ such that $\zeta = \partial D$ near p and $D \subset G := \zeta_{\text{ext}}$. Take a point $a \notin \overline{G}$ and consider the union G_e of 0 and the image of G under the map $\varphi : z \to (z - a)^{-1}$. There exists a conformal map $\psi : G_e \to \mathbb{D}$. It extends to a C^1-diffeomorphism from $\overline{G_e}$ to $\overline{\mathbb{D}}$ (cf. [20, Theorems 3.5]). Setting $\eta = \psi \circ \varphi$, then

$$c_D(z, w) \geq c_{\mathbb{D}}(\eta(z), \eta(w)).$$

Now the lower bound for c_D follows by the same bound for $c_{\mathbb{D}}$ and an inequality of type (4).

The estimate

$$l_D(z, w) \leq -\frac{1}{2} \log d_D(w) - c, \quad z \in K, w \in D \cap V$$

follows by (3). It can be also obtained in the following way. There exist a Dini-smooth domain simply connected domain $G_i \subset D$ and a neighborhood V of p such that $\partial G \cap V = \partial D \cap V$. Take a point $u \in V$. Since $l_D = k_D$, then

$$k_D(z, w) \leq k_D(z, u) + k_{G_i}(u, w).$$

It remains to repeat the final arguments from the first paragraph.

Proof of Proposition 5 for b_D [6] Choosing G as above, then

$$b_D(z,w) = b_{\eta(D)}(\eta(z),\eta(w)).$$

By the Dini-smoothness,

$$\lim_{z\to p}\frac{d_{\eta(D)}(\eta(z))}{d_D(z)} = |\eta'(p)|.$$

We may assume that $\eta(p)=1$. So, it is enough to get the estimates for $D\subset\mathbb{D}$ such that $F=\mathbb{D}\cap\mathbb{D}(1,r)\subset D$ for some $r\in(0,1)$.

First, we shall prove that if $0<r'<r$, then

$$\sqrt{2}b_D(z,w)\le -\log d_D(w)+c',\quad z\in K, w\in F'=\mathbb{D}\cap\mathbb{D}(1,r')$$

for some constant $c'>0$.

For a domain $\Omega\subset\mathbb{C}$ set $\beta_\Omega(z)=B_\Omega(z;1)$ and $\kappa_\Omega(z)=\kappa_\Omega(z;1)$. Let $\check F=\mathbb{D}\setminus F$ and

$$l_{\mathbb{D}}(u,\check F)=\inf_{w\in\check F} l_{\mathbb{D}}(u,w).$$

Then, for any $r''\in(r',r)$, we may find a constant $\tilde c>0$ such that

$$\begin{aligned}\beta_D(u) &\le \beta_F(u)\sqrt{\frac{K_F(u)}{K_{\mathbb{D}}(u)}} = \frac{\sqrt{2}\kappa_F^2(u)}{\kappa_{\mathbb{D}}(u)}\\ &\le \sqrt{2}\coth^2 l_{\mathbb{D}}(u,\check F)\kappa_{\mathbb{D}}(u)\le\frac{\sqrt{2}}{1-|u|^2}+\tilde c,\quad u\in F''=\mathbb{D}\cap\mathbb{D}(1,r'').\end{aligned}$$

(for the equality use that F is biholomorphic to $\mathbb{D}$ and for the inequality "between the lines" cf. [10, Proposition 7.2.9]).

Let $z\in K$, $w\in F'$ and $w'=[0,w]\cap\partial D(1,r'')$. Then

$$\begin{aligned}b_D(z,w)&\le b_D(z,w')+|w-w'|\left(\tilde c+\sqrt{2}\int_0^1\frac{dt}{1-|w'+t(w-w')|^2}\right)\\ &\le(-\log d_D(w)+c')/\sqrt{2}\end{aligned}$$

for some constant $c'>0$.

Now, shrinking r such that $\mathbb{D}(1,r)\subset U$, it remains to prove that

$$\sqrt{2}b_D(z,w)\ge -\log d_D(w)-c'',\quad z\in\check F, w\in F'$$

for some constant $c''>0$.

We have that

$$\begin{aligned}\beta_D(u)&\ge\beta_{\mathbb{D}}(u)\sqrt{\frac{K_{\mathbb{D}}(u)}{K_F(u)}}=\frac{\sqrt{2}\kappa_{\mathbb{D}}^2(u)}{\kappa_F(u)}\\ &\ge\sqrt{2}\tanh l_{\mathbb{D}}(u,\check F)\kappa_{\mathbb{D}}(u)\ge\frac{\sqrt{2}}{1-|u|^2}-\hat c,\quad u\in F''.\end{aligned}$$

[6] We have to modify the previous proof, since the Bergman distance is not monotone under inclusion of planar domains; to see this, use [15, Example 7].

For $z \in \check{F}$, $w \in F'$, and $\varepsilon > 0$, there exists a smooth curve $\gamma : [0, 1] \to D$ with

$$b_D(z, w) + \varepsilon > \int_0^1 \beta_D(\gamma(t))|\gamma'(t)|dt.$$

Let $t_0 = \sup\{t \in (0, 1) : \gamma(t) \notin F''\}$. Then,

$$b_D(z, w) + \varepsilon > \int_{t_0}^1 b_D(\gamma(t))|\gamma'(t)|dt \geq \hat{b}_{\mathbb{D}}(w, \check{F}),$$

where $\hat{b}_{\mathbb{D}}$ is the integrated form of the Finsler pseudometric

$$\hat{\beta}_{\mathbb{D}}(u; X) = |X|\left(\frac{\sqrt{2}}{1 - |u|^2} - \hat{c}\right)^+.$$

It remains to use that, shrinking r' (if necessary),

$$\hat{b}_{\mathbb{D}}(w, \check{F}) \geq (-\log d_D(w) - c'')/\sqrt{2}$$

for some constant $c'' > 0$ (cf. [3, Theorem 1.1]).

Proof of Corollary 6 Since k_D and b_D are the integrated forms of κ_D and β_D, the boundedness from below follows by the first inequality in Proposition 5 (cf. the proof of [10, Proposition 10.2.6]). Choosing a point $a \in D$, the boundedness from above is a consequence of the inequality $s_D(z, w) \leq s_D(z, a) + s_D(a, w)$ and the second inequality in Proposition 5.

Proof of Proposition 7 In virtue of the inequality $c_D \leq k_D$ and Corollary 6, we have to prove only the boundedness from below. For this, take disjoint Dini-smooth Jordan curves ζ' and ζ'' such that $\zeta' = \partial D$ near p, $\zeta'' = \partial D$ near q and $D \subset G := \zeta'_{\text{ext}} \cap \zeta''_{\text{ext}}$. Note that any Dini-smooth bounded double-connected planar $\tilde{G}$ domain can be conformally map to some annulus $A_r = \{z \in \mathbb{C} : 1/r < |z| < r\} (r > 1)$, and the respective mapping extends to a C^1-diffeomorphism from $\overline{\tilde{G}}$ to $\overline{A_r}$.[7]

Then, proceeding similarly to the proof of Proposition 5 for c_D, it is enough to show that

$$2c_{A_r}(z, w) + \log d_{A_r}(z) + \log d_{A_r}(w)$$

is bounded from below for $z \in \mathbb{R}$ near r and w near p, where $|p| = 1/r$; this is equivalent to

$$m_{A_r}(z, w) := \tanh c_{A_r}(z, w) \geq 1 - c d_{A_r}(z) d_{A_r}(w)$$

for some constant $c > 0$.

Recall that (cf. [10, Proposition 5.5])

$$m_{A_r}(z, w) = \frac{f(z, w) f(1/z, -|w|)}{r|w|},$$

[7] To see this, we can proceed as follows (S. R. Bell, private communication). First, take a conformal mapping φ_1 from the domain bounded by the outer boundary of $\tilde{G}$ to $\mathbb{D}$. Next, choose a point a in the interior of the inner boundary Γ of $\psi_1(\tilde{G})$ and set $\psi_2 : z \to (z - a)^{-1}$. Consider now a conformal mapping ψ_3 from the domain bounded by $\psi_2(\Gamma)$ to $\mathbb{D}$. Then, $\psi = \psi_3 \circ \psi_2 \circ \psi_1$ maps conformally $\tilde{G}$ to a bounded double-connected planar domain G' with real-analytic boundary. It remains to apply the reflection principle to a conformal mapping from G' to A_r.

where f is a holomorphic function on $\overline{A_r \times A_r} \setminus \{u = v \in \partial A_r\}$ and $|f(u, v)| = 1$ if $|u| = r$, $v \in \overline{A_r}$ or $u \in \overline{A_r}$, $|v| = 1/r (u \neq v)$.

In particular,

$$\frac{\partial^n f}{\partial u^n} = \frac{\partial^n f}{\partial v^n} = 0, \quad n \in \mathbb{N},$$

at any point (u, v) with $|u| = r$ and $|v| = 1/r$. Then, by the Taylor expansion,

$$|f(z, w) - f(r, w(r|w|)^{-1})| \leq c_1 d_{A_r}(z) d_{A_r}(w).$$

This implies that

$$|f(z, |w|) - f(r, 1/r)| \leq c_1 d_{A_r}(z) d_{A_r}(|w|)$$

(the constant can be chosen the same for z near r and w away from r). Since $f(r, \cdot)$ is a unimodular constant and $d_{A_r}(w) = d_{A_r}(|w|)$, it follows that

$$|m_{A_r}(z, w) - m_{A_r}(z, |w|)| \leq c_2 d_{A_r}(z) d_{A_r}(|w|).$$

Further, $c_{A_r}(z, |w|) = c_{A_r}(z, t) + c_{A_r}(t, |w|)$ for $t \in [|w|, z]$ (cf. [10, Lemma 5.11(b)]). Then, Proposition 5 implies that

$$m_{A_r}(z, |w|) \geq 1 - c_3 d_{A_r}(z) d_{A_r}(|w|).$$

Hence we may choose $c = c_2 + c_3$ which completes the proof.

Proof of Proposition 8 Using Corollary 6 and Proposition 7, it is enough to prove the inequalities for z and w near a fixed point $p \in \partial D$. Moreover, it is easy to see that these inequalities are equivalent to

$$\frac{|z - w|}{\sqrt{c d_D(z) d_D(w) + |z - w|^2}} \leq \tanh c_D(z, w) \leq \tanh l_D(z, w)$$
$$\leq \frac{|z - w|}{\sqrt{c^{-1} d_D(z) d_D(w) + |z - w|^2}}$$

for some constant $c \geq 1$.[8]

To prove the lower bound for $\tanh c_D(z, w)$, let η be as in the proof of Proposition 5 for c_D and l_D. Then, it is not difficult to find a constant $c_1 > 0$ such that

$$\tanh c_D(z, w) \geq \tanh c_{\mathbb{D}}(z_1, w_1) \geq \frac{|z_1 - w_1|}{\sqrt{c_1 d_{\mathbb{D}}(z_1) d_{\mathbb{D}}(w_1) + |z_1 - w_1|^2}},$$

where $z_1 = \eta(z)$ and $w_1 = \eta(w)$. It remains to use that, similarly to (4), $d_D \geq c_2 d_{\mathbb{D}}$ and $|z_1 - w_1| \geq c_2 |z - w|$ for some constant $c_2 > 0$.

The proof of the upper bound for $\tanh l_D(z, w)$ is similar (by using G_i from the second part of the proof mentioned above) and we skip it.

Proof of Proposition 9 By the Köbe uniformization theorem, we may assume that ∂D consists of disjoint circles. Using Proposition 8 and compactness, it is enough to prove that for any point $p \in \partial D$,

$$\lim_{z \neq w \to p} \frac{c_D(z, w)}{l_D(z, w)} = 1.$$

[8] These estimates imply the bounds for the Green function g_D from the crucial Lemma 4.2 in [17], since $\tanh c_D \leq \exp(-2\pi g_D) \leq \tanh l_D$.

Applying an inversion, we may suppose that the outer boundary of D is the unit circle Γ and $p \in \Gamma$. Let U be a disk centered at p such that $\mathbb{D} \cap U \subset D$. Then,

$$1 \geq \frac{c_D(z,w)}{l_D(z,w)} \geq \frac{c_{\mathbb{D}}(z,w)}{l_{\mathbb{D}\cap U}(z,w)} = \frac{k_{\mathbb{D}}(z,w)}{k_{\mathbb{D}\cap U}(z,w)}.$$

Considering $\mathbb{D}$ as a part of the unit ball in $\mathbb{C}^2$, it follows that the last ratio tends to 1 as a particular case of the same result for strongly pseudoconvex domains (see [18, Proposition 3]).

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