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Composition operators acting on Besov spaces on the real line

Gérard Bourdaud · Madani Moussai · Winfried Sickel

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Abstract We study the composition operator $T_f(g) := f \circ g$ on Besov spaces $B^s_{p,q}(\mathbb{R})$. In case $1 < p < +\infty$, $0 < q \leq +\infty$ and $s > 1 + (1/p)$, we will prove that the operator T_f maps $B^s_{p,q}(\mathbb{R})$ to itself if, and only if, $f(0) = 0$ and f belongs locally to $B^s_{p,q}(\mathbb{R})$. For the case $p = q$, i.e., in case of Slobodeckij spaces, we can extend our results from the real line to $\mathbb{R}^n$.

Keywords Homogeneous and inhomogeneous Besov spaces on the real line · Slobodeckij spaces on $\mathbb{R}^n$ · Functions of bounded p-variation · Composition operators · Optimal inequalities

Mathematics Subject Classification (2000) 46E35 · 47H30

1 Introduction

The present paper is a continuation of our earlier investigations, see [13–15] as well as [20], on composition operators, i.e., mappings

$$T_f : g \mapsto f \circ g, \quad g \in E,$$

G. Bourdaud (✉)
IMJ-PRG (UMR 7586), Université Paris Diderot, Bâtiment Sophie Germain, Case 7012,
75205 Paris Cedex 13, France
e-mail: bourdaud@math.univ-paris-diderot.fr

M. Moussai
Department of Mathematics, LAFGE, University of M'Sila, P.O. Box 166, 28000 M'Sila, Algeria
e-mail: mmoussai@yahoo.fr

W. Sickel
Institute of Mathematics, Friedrich-Schiller-University Jena, Ernst-Abbe-Platz 1-2, 07743 Jena, Germany
e-mail: winfried.sickel@uni-jena.de

where f is a function of $\mathbb{R}$ to itself and E is a Besov or Lizorkin-Triebel space. Except the trivial case that f is linear, these operators are nonlinear. The theory of these mappings is rather incomplete, in particular in the framework of function spaces with fractional order of smoothness. A survey about the state of the art from our point of view has been given in [16].

Besov spaces represent one approach to fill in the gaps between Sobolev spaces $W^m_p(\mathbb{R}^n)$, with integer exponent m, by introducing spaces of fractional order of smoothness. Before turning to our main results with respect to Besov spaces, we recall what is known in case of Sobolev spaces.

Proposition 1 *Let n be a natural number ≥ 1, let m be a natural number ≥ 2, and $1 \leq p < +\infty$.*

(i) *Let $f \in \dot{W}^1_\infty \cap \dot{W}^m_p(\mathbb{R})$. For all $g \in \dot{W}^1_\infty \cap W^m_p(\mathbb{R}^n)$, it holds*

$$\|f \circ g\|_{\dot{W}^m_p} \leq c \left(\|f\|_{\dot{W}^1_\infty} + \|f\|_{\dot{W}^m_p}\right) \|g\|_{W^m_p} \left(1 + \|\nabla g\|_\infty\right)^{m-1-(1/p)} \tag{1}$$

with a constant c independent of f and g.

(ii) *Let $f \in \dot{W}^1_\infty \cap \dot{W}^m_p(\mathbb{R})$. For all $g \in L_\infty \cap W^m_p(\mathbb{R}^n)$, it holds*

$$\|f \circ g\|_{\dot{W}^m_p} \leq c \left(\|f\|_{\dot{W}^1_\infty} + \|f\|_{\dot{W}^m_p}\right) \|g\|_{W^m_p} \left(1 + \|g\|_\infty\right)^{m-1-(1/p)} \tag{2}$$

with a constant c independent of f and g.

(iii) *The operator T_f maps $W^m_p \cap L_\infty(\mathbb{R}^n)$ to itself if, and only if, $f \in W^{m,\ell oc}_p(\mathbb{R})$ and $f(0) = 0$.*

For a proof of the second statement, we refer to [8], see in particular Proposition 4, estimate (18)[1]. The first statement follows by a minor modification of the proof. Sufficiency in part (iii) is a consequence of (ii). Necessity is more or less obvious.

Based upon the last statement in Proposition 1, we believe on the following variant in case of Besov spaces on $\mathbb{R}^n$:

Conjecture 1 *If $s > 1+(1/p)$, $1 \leq p < \infty$ and $0 < q \leq \infty$, then T_f maps $B^s_{p,q} \cap L_\infty(\mathbb{R}^n)$ to itself if, and only if, $f \in B^{s,\ell oc}_{p,q}(\mathbb{R})$ and $f(0) = 0$.*

We refer to [9] for a more extended introduction into Conjecture 1. For $0 < s < 1$, the characterization of all f such that T_f takes $B^s_{p,q}(\mathbb{R}^n)$ to itself has been known for a longer time, see [5,16]. In case $1 < s < 1 + (1/p)$ not so much is known, even when we restrict us to $n = 1$. We refer to [16,19] and [23, 5.3] for some sufficient conditions, and to [9] for a reasonable conjecture.

In our earlier articles, we established Conjecture 1 in case $n = 1$, $p > 1$, with some restrictions on q, including the case $q \geq p$. In comparison with those works, we have obtained progresses in three different directions. First of all, we have been able to remove the restrictions on q. Second, we improved the inequalities reflecting the acting condition $T_f(B^s_{p,q}(\mathbb{R})) \subset B^s_{p,q}(\mathbb{R})$. Finally, based on these extensions and improvements, we can deal with E being the Slobodeckij spaces on $\mathbb{R}^n$.

Our main tools in the proofs are always a combination of appropriate characterizations by differences in the function spaces, including various embeddings between them. In principle, this is not complicated. However, the main difficulty in our proof consists in finding a

[1] This estimate must be corrected in [8]: the term $\|f\|_{\dot{W}^1_\infty} + \|f\|_{\dot{W}^m_p}$ is missing.

clever decomposition of the term $\|f \circ g\|_E$ to apply these tools. These rather sophisticated decompositions depend on s and p.

The homogeneous Besov spaces will play an important technical role in our work. We need their definition by the Littlewood–Paley decomposition as well as their characterizations by differences. However, the link between these two points of view is not completely referenced in the literature (in our opinion). For this reason, we will give a complete proof of the equivalence in the Appendix at the end of the paper. Furthermore, a number of basic properties of homogeneous Besov spaces is either recalled or proved there. In our opinion, Sect. 4 (the Appendix) is of self-contained interest.

1.1 Notation and plan of the paper

The paper is organized as follows. In Sect. 2, we state our main results. The next section is devoted to the proofs. In Sect. 4, we collect definitions and basic properties of the function spaces under consideration.

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the natural numbers including 0, $\mathbb{Z}$ the integers, and $\mathbb{R}$ the real numbers. The integer part of a real number x is denoted by $[x]$. If A is any finite set, we denote by Card A its cardinal number. All functions are assumed to be real-valued, except in Sect. 4.

If E is a quasi-Banach function space on $\mathbb{R}^n$, we denote by $E^{\ell oc}$ the collection of all functions f such that the product φf belongs to E, for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. If E and F are quasi-Banach spaces, then the symbol $E \hookrightarrow F$ indicates a continuous embedding. All the function spaces we consider are subspaces of $L_1^{\ell oc}(\mathbb{R}^n)$, i.e., spaces of equivalence classes w.r.t. almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function.

As usual, the symbol c denotes a positive constant which depends only on the fixed parameters n, s, p, q, unless otherwise stated; its value may vary from line to line.

If $p \in [1, +\infty]$ and $m \in \mathbb{N}$, we denote by $\|f\|_p$ the norm of a function f in $L_p(\mathbb{R}^n)$, by $W_p^m(\mathbb{R}^n)$ the usual Sobolev space, and $\dot{W}_p^m(\mathbb{R}^n)$ its homogeneous counterpart. For the definitions of the inhomogeneous as well as the homogeneous Besov spaces, we refer to Sect. 4. General information about these function spaces can be found, e.g., in [22,23,25–27].

The Fourier transform of a function $f \in L_1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-ix\cdot\xi}\, \mathrm{d}x .$$

It is extended to tempered distributions in the usual way.

We choose, once and for all, a cutoff function, i.e., a radial, positive, C^∞ function ρ such that $0 \le \rho \le 1$, $\rho(\xi) = 1$ for $|\xi| \le 1$, $\rho(\xi) = 0$ for $|\xi| \ge 3/2$. We associate with ρ the sequence of operators $(S_j)_{j\in\mathbb{Z}}$ defined by

$$\widehat{S_j f}(\xi) := \rho\left(2^{-j}\xi\right)\, \widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{3}$$

Clearly, the operator S_j is defined on $\mathcal{S}'(\mathbb{R}^n)$ and takes values in the space of analytical functions of exponential type.

2 Statement of the main results

We prefer to formulate the results for the one-dimensional case and the n-dimensional case separately.

2.1 Results in the one-dimensional case

Our main results consist in the following two theorems.

Theorem 1 *Let $1 < p < +\infty$, $0 < q \le +\infty$ and $s > 1 + (1/p)$. For a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, the composition operator T_f acts on $B^s_{p,q}(\mathbb{R})$ if, and only if, $f(0) = 0$ and $f \in B^{s,\ell oc}_{p,q}(\mathbb{R})$.*

The necessity part of Theorem 1 is almost immediate: it suffices to test T_f on a function $g \in \mathcal{D}(\mathbb{R})$ such that $g(x) = x$ on an arbitrary bounded interval of $\mathbb{R}$.

The sufficiency part of Theorem 1 relies upon a precise estimate of the quasi-norm of $f \circ g$. For the formulation of this result, it is convenient to introduce the space

$$\mathcal{B}^s_{p,q}(\mathbb{R}^n) := \{f \in L_\infty(\mathbb{R}^n) \ :\ \|f\|_{\dot{B}^s_{p,q}} < +\infty\}, \tag{4}$$

endowed with the quasi-norm

$$\|f\|_{\mathcal{B}^s_{p,q}} := \|f\|_\infty + \|f\|_{\dot{B}^s_{p,q}},$$

see Sect. 4.2 for the definition of the homogeneous quasi-seminorm $\|\cdot\|_{\dot{B}^s_{p,q}}$. The real number $\delta := s - 1 - (1/p)$ will play a central role in our investigations (as an exponent); this notation will be used all along the paper (except if several values of s are under consideration, see Proposition 2 and Sect. 3.1).

Theorem 2 *Let s, p, q be real numbers so as in Theorem 1. Then, there exists a constant $c > 0$ such that the inequality*

$$\|(f \circ g)'\|_{\mathcal{B}^{s-1}_{p,q}} \le c\, \|f'\|_{\mathcal{B}^{s-1}_{p,q}}\, \|g\|_{B^s_{p,q}}\, \left(1 + \|g'\|_\infty\right)^\delta \tag{5}$$

holds for all functions f such that $f' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$ and all $g \in B^s_{p,q}(\mathbb{R})$.

Let us add a few comments.

(i) The connection between both theorems is clear. From the embedding $B^s_{p,q}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R})$, a consequence of $s > 1/p$, we derive $f \circ g = f\varphi \circ g$, where $\varphi \in \mathcal{D}(\mathbb{R})$ satisfies $\varphi(x) = 1$ on the range of g. Hence, we can apply Theorem 2 to $f\varphi$ and deduce the sufficiency part of Theorem 1. Notice that the cutoff function φ depends only on $\|g\|_\infty$. Indeed, we can take $\varphi(t) := \rho\left(t\, \|g\|_\infty^{-1}\right)$. Under the assumptions of Theorem 1, it follows that *any composition operator acting on $B^s_{p,q}(\mathbb{R})$ maps bounded sets to bounded sets.*

(ii) Thus, Conjecture 1 turns out to be true for $n = 1$ and $p > 1$. Indeed, we have been able to prove it also for $n = p = 1$, but our proof has the following defaults:

- It does not cover the case of s being an integer, i.e., the Besov spaces $B^m_{1,q}(\mathbb{R})$ for $m = 3, 4, \ldots$.
- We did not succeed in obtaining the "good" estimate (5); hence, the extension to the general n-dimensional case is open.

(iii) The exponent δ is known to be sharp in estimate (5), see [15, prop. 16].

(iv) In our earlier publications, we always proved estimates of the type

$$\|(f \circ g)'\|_{\mathcal{B}^{s-1}_{p,q}} \leq c\, \|f'\|_{\mathcal{B}^{s-1}_{p,q}}\, \|g\|_{B^s_{p,q}} \, \left(1 + \|g\|_{B^s_{p,q}}\right)^{\delta} . \tag{6}$$

Of course, (5) implies (6) in view of the embedding $B^s_{p,q}(\mathbb{R}) \hookrightarrow W^1_\infty(\mathbb{R})$, since $s > 1 + (1/p)$. The difference between (5) and (6) does not look so important. However, (5) allows an extension to the n-dimensional case, at least partially, whereas we have been unable to do this using (6).

(v) Of course, inequality (5) represents the counterpart of (1) in case of Besov spaces. There should be also a counterpart of (2), reading as:

$$\|(f \circ g)'\|_{\mathcal{B}^{s-1}_{p,q}} \leq c\, \|f'\|_{\mathcal{B}^{s-1}_{p,q}}\, \|g\|_{B^s_{p,q}} \, (1 + \|g\|_\infty)^{\delta} .$$

Such an optimal estimate should open the door to an extension to the n-dimensional case for the natural range of parameters, see [7] for a significant partial result.

2.2 Results in the n-dimensional case

Now, we turn to the consequences of Theorems 1 and 2 for the n-dimensional situation.

Theorem 3 *Let* $1 < p < +\infty$ *and* $s > 1 + (1/p)$. *For a Borel measurable function* $f : \mathbb{R} \to \mathbb{R}$, *the composition operator* T_f *acts on* $B^s_{p,p} \cap W^1_\infty(\mathbb{R}^n)$ *if, and only if,* $f(0) = 0$ *and* $f \in B^{s,\ell oc}_{p,p}(\mathbb{R})$.

Remark 1 (i) For $s > 0$ not being a natural number, the spaces $B^s_{p,p}(\mathbb{R}^n)$ are usually called Slobodeckij spaces.

(ii) In case $1 < p < +\infty$, we have

$$B^s_{p,p} \cap W^1_\infty(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n) \quad \text{if} \quad s > \frac{n}{p} + 1.$$

Thus, the Conjecture 1 holds true for Slobodeckij spaces $B^s_{p,p}(\mathbb{R}^n)$ under the condition $s > (n/p) + 1$. However, the full Conjecture 1 remains open if $n > 1$, unlike in the case of the Sobolev spaces, see Proposition 1.

Theorem 4 *Let* $1 < p < +\infty$ *and* $s > 1 + (1/p)$. *Then, there exists a constant* $c > 0$ *such that the inequality*

$$\|f \circ g\|_{B^s_{p,p}} \leq c\, \|f'\|_{\mathcal{B}^{s-1}_{p,p}}\, \|g\|_{B^s_{p,p}} \, (1 + \|\nabla g\|_\infty)^{\delta}$$

holds for all functions f *such that* $f' \in \mathcal{B}^{s-1}_{p,p}(\mathbb{R})$ *and* $f(0) = 0$, *and all* $g \in B^s_{p,p} \cap \dot{W}^1_\infty(\mathbb{R}^n)$.

When turning to the situation on domains, not so much is changed.

Theorem 5 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain. Let* $1 < p < +\infty$ *and* $s > 1 + (1/p)$. *For a Borel measurable function* $f : \mathbb{R} \to \mathbb{R}$, *the composition operator* T_f *acts on* $B^s_{p,p} \cap \dot{W}^1_\infty(\Omega)$ *if, and only if,* $f \in B^{s,\ell oc}_{p,p}(\mathbb{R})$.

3 Proofs of the main theorems

Here, we collect the proofs of Theorems 2, 4 and 5 (recall that Theorems 1 and 3 follow easily).

In our preceding papers [13–15,20], we always used, as the first step of the proof of Theorem 2, some arguments to simplify the situation. We will do this here as well. We claim that Theorem 2 can be derived from the following statement:

Proposition 2 *Let* $1 < p < +\infty$, $0 < q \le +\infty$ *and* $1 + (1/p) < s \le 2 + (1/p)$. *There exists a constant* $c > 0$ *such that the estimate*

$$\| f \circ g \|_{B^s_{p,q}} \le c \, \| f' \|_{\mathcal{B}^{s-1}_{p,q}} \, \| g \|_{B^s_{p,q}} \, \left(1 + \| g' \|_\infty \right)^{s-1-(1/p)} \tag{7}$$

holds for all functions f *and* g *satisfying the following conditions:*

(i) f *is of class* C^2, $f' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$ *and* $f(0) = 0$,
(ii) g *is real analytic and* $g \in B^s_{p,q}(\mathbb{R})$.

3.1 From Proposition 2 to Theorem 2

Let us introduce the following intermediate property:

($\mathcal{Q}_s$) For some constant $c > 0$ depending only on s, p and q, the inequality

$$\| (f \circ g)' \|_{\mathcal{B}^{s-1}_{p,q}} \le c \, \| f' \|_{\mathcal{B}^{s-1}_{p,q}} \, \| g \|_{B^s_{p,q}} \, \left(1 + \| g' \|_\infty \right)^{s-1-(1/p)} \tag{8}$$

holds true for all functions f such that $f' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$ and all $g \in B^s_{p,q}(\mathbb{R})$.

3.1.1 From Proposition 2 to ($\mathcal{Q}_s$) for $1 + (1/p) < s \le 2 + (1/p)$.

We give here a sketchy proof and refer to our previous articles for details, in particular to [20, sect. 4.1].

Step 1. Assuming Proposition 2, let us prove the inequality (7) under the sole assumptions $f' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$, $f(0) = 0$ and $g \in B^s_{p,q}(\mathbb{R})$.

We use the cutoff function ρ and the operators S_j introduced in Notation, see (3). Then, we define

$$g_j := S_j g, \quad f_j := S_j f - S_j f(0) \rho.$$

The functions g_j and f_j are real analytic, and by standard estimations, it holds

$$\| g_j \|_{B^s_{p,q}} \le c \, \| g \|_{B^s_{p,q}}, \quad \| g_j' \|_\infty \le c \, \| g' \|_\infty,$$
$$\| f_j' \|_{\mathcal{B}^{s-1}_{p,q}} \le c \left(\| f' \|_{\mathcal{B}^{s-1}_{p,q}} + |S_j f(0)| \right),$$

where the constant c does not depend on j. Applying Proposition 2 to f_j and g_j, and using the above estimates, we obtain

$$\| f_j \circ g_j \|_{B^s_{p,q}} \le c \, (\| f' \|_{\mathcal{B}^{s-1}_{p,q}} + |S_j f(0)|) \, \| g \|_{B^s_{p,q}} \, \left(1 + \| g' \|_\infty \right)^{s-1-(1/p)} .$$

It is easily seen that $f_j \circ g_j$ tends to $f \circ g$ in $L_p(\mathbb{R})$, and that $S_j f(0)$ tends to 0, as $j \to +\infty$. By using the Fatou property of the Besov space, see [20, prop. 3.18], we complete the proof of (7).

Step 2. Now, assume $f' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$ and $g \in B^s_{p,q}(\mathbb{R})$.

Let us define $\tilde{f} := f - f(0)$. Then, we can apply Step 1 to the functions $\tilde{f}$ and g. By Proposition 8 and by embedding (55), we deduce

$$\begin{aligned}\|(f \circ g)'\|_{\mathcal{B}^{s-1}_{p,q}} &= \|(\tilde{f} \circ g)'\|_{\mathcal{B}^{s-1}_{p,q}}\\ &\le c_1 \, \|\tilde{f} \circ g\|_{B^s_{p,q}}\\ &\le c_2 \, \|\tilde{f}'\|_{\mathcal{B}^{s-1}_{p,q}} \, \|g\|_{B^s_{p,q}} \, \left(1 + \|g'\|_\infty\right)^{s-1-(1/p)}.\end{aligned}$$

This complete the proof of (8).

3.1.2 *End of the proof of Theorem 2*

By Sect. 3.1.1, the proof of Theorem 2 will be complete if we establish the following:

Claim: $(\mathcal{Q}_s)$ implies $(\mathcal{Q}_{s+1})$ for all $s > 1 + (1/p)$.

Proof of the Claim Let us assume $(\mathcal{Q}_s)$. Let f, g be such that $f' \in \mathcal{B}^s_{p,q}(\mathbb{R})$ and $g \in B^{s+1}_{p,q}(\mathbb{R})$. By Proposition 8 and by embedding (55), it holds

$$\|g'\|_{\mathcal{B}^s_{p,q}} \le c_1 \, \|g'\|_{B^s_{p,q}} \le c_2 \, \|g\|_{B^{s+1}_{p,q}}.$$

By Proposition 12 and by (39), it holds $f'' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$, $g \in B^s_{p,q}(\mathbb{R})$, and

$$\|f''\|_{\mathcal{B}^{s-1}_{p,q}} \le c \, \|f'\|_{\mathcal{B}^s_{p,q}}, \quad \|g\|_{B^s_{p,q}} \le c \, \|g\|_{B^{s+1}_{p,q}}. \tag{9}$$

Applying $(\mathcal{Q}_s)$ to f' and g, we deduce $(f' \circ g)' \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R})$ and

$$\|(f' \circ g)'\|_{\mathcal{B}^{s-1}_{p,q}} \le c \, \|f''\|_{\mathcal{B}^{s-1}_{p,q}} \, \|g\|_{B^s_{p,q}} \, \left(1 + \|g'\|_\infty\right)^{s-1-(1/p)}.$$

By Proposition 12 and by (9), it holds

$$\|f' \circ g\|_{\mathcal{B}^s_{p,q}} \le c \left(\|f'\|_{\mathcal{B}^s_{p,q}} \, \|g\|_{B^{s+1}_{p,q}} \, \left(1 + \|g'\|_\infty\right)^{s-1-(1/p)} + \|f'\|_\infty \right).$$

Applying Proposition 11 to $f' \circ g$ and g', we obtain

$$\begin{aligned}\|(f \circ g)'\|_{\mathcal{B}^s_{p,q}} &\le c \left(\|f' \circ g\|_{\mathcal{B}^s_{p,q}} \, \|g'\|_\infty + \|f' \circ g\|_\infty \, \|g'\|_{\mathcal{B}^s_{p,q}} \right)\\ &\le c \, \|f'\|_{\mathcal{B}^s_{p,q}} \, \|g\|_{B^{s+1}_{p,q}} \left(\left(1 + \|g'\|_\infty\right)^{s-1-(1/p)} \, \|g'\|_\infty + 1 \right).\end{aligned}$$

The estimate $(\mathcal{Q}_{s+1})$ follows at once. □

3.2 Proof of Proposition 2: a preparation

First, we begin with some notation. For all functions f on $\mathbb{R}^n$, we set

$$\Delta_h f(x) := f(x+h) - f(x).$$

The mth power of Δ_h is defined inductively as usual:

$$\Delta_h^1 := \Delta_h; \quad \Delta_h^{m+1} := \Delta_h \circ \Delta_h^m, \quad \forall m \in \mathbb{N}.$$

The following formulas allow the computation of Δ_h^2 for the product and the composition of functions f, g:

$$\Delta_h^2(fg)(x) = g(x)\,\Delta_h^2 f(x) + f(x+2h)\,\Delta_h^2 g(x) + 2\,\Delta_h f(x+h)\,\Delta_h g(x), \quad (10)$$

$$\begin{aligned} 2\,\Delta_h^2(f\circ g)(x) = {} & f(g(x+2h)) - f(2g(x+h)-g(x)) \\ & + f(g(x)) - f(2g(x+h)-g(x+2h)) \\ & + f(g(x+2h)) + f(2g(x+h)-g(x+2h)) - 2f(g(x+h)) \\ & + f(g(x)) + f(2g(x+h)-g(x)) - 2f(g(x+h)). \end{aligned}$$

We will use the following modified L_p-moduli of continuity:

$$\Omega_p^m(f;t) := \left(\int_{\mathbb{R}^n} \sup_{|h|\le t} |\Delta_h^m f(x)|^p \,\mathrm{d}x \right)^{1/p}.$$

The Hardy–Littlewood maximal function Mg of a locally integrable function g on $\mathbb{R}$ is defined by

$$Mg(x) := \sup_{x\in I} \frac{1}{|I|} \int_I |g(y)|\,\mathrm{d}y, \quad \forall x \in \mathbb{R}.$$

Here, the supremum is taken with respect to all intervals I containing x, and $|I|$ denotes the length of I. In our proof below the Wiener classes BV_p will play an important role. Let us recall its definition. For a function $g : \mathbb{R} \to \mathbb{R}$, we denote by $\|g\|_{BV_p}$ the supremum of numbers

$$\left(\sum_{k=1}^{N} |g(b_k) - g(a_k)|^p \right)^{1/p},$$

taken over all finite sets $\{]a_k, b_k[\,;\ k = 1,\ldots,N\}$ of pairwise disjoint open intervals. A function g is said to be of *bounded p-variation* if $\|g\|_{BV_p} < +\infty$. The collection of all such functions is called a Wiener class and denoted by BV_p. Their connection with Besov spaces is given by the Peetre embedding:

$$B^{1/p}_{p,1}(\mathbb{R}) \hookrightarrow BV_p(\mathbb{R}), \quad 1 \le p < +\infty, \quad (11)$$

see [22, thm. 7, p. 122] or [11]. We refer also to [12,28] for some further properties of these classes.

Notice that our function g belongs to $B^s_{p,q}(\mathbb{R})$ with $p < \infty$; hence, it cannot be a constant except if $g = 0$. Thus, all along the proof of Proposition 2, we will assume that $\|g'\|_\infty > 0$. Also, by assumption $s > 1 + (1/p)$, it holds $g \in C_0(\mathbb{R})$. Since g is assumed to be real analytic, this implies that the set of zeros of g' is a nonempty discrete set in $\mathbb{R}$.

Our proof will be divided into three parts, corresponding to the following cases:

$$1 + (1/p) < s < 2, \quad s = 2, \quad 2 < s \le 2 + (1/p).$$

Convention: In estimations of Δ_h and Δ_h^2, we often restrict ourselves to $h > 0$. Clearly, similar arguments can be applied for $h < 0$.

3.3 Proof of Proposition 2: the case $1 + (1/p) < s < 2$

This case was the first one which has been solved, with some restriction on q, see [6]. Then, the same basic ideas worked for $2 \le s \le 2 + (1/p)$, with more technicalities.

We apply Propositions 7, 8, and 9 in Sect. 4. This means we have to estimate

$$\| f \circ g \|_p + \left(\int_0^1 \left(\frac{\| \Delta_h ((f \circ g)') \|_p}{h^{s-1}} \right)^q \frac{\mathrm{d}h}{h} \right)^{1/q} .$$

Concerning the first term, we have

$$\| f \circ g \|_p = \| f \circ g - f(0) \|_p \le \| f' \|_\infty \, \| g \|_p .$$

(The above argument will work also for $s \ge 2$, so we will not refer anymore to it).

Now, we turn to the estimation of the second term. Let us define

$$U(h) := \Big(\int_{\mathbb{R}} |g'(x)|^p \; |\Delta_h (f' \circ g)(x)|^p \, \mathrm{d}x \Big)^{1/p} .$$

Since $|\Delta_h (f \circ g)'(x)| \le \|f'\|_\infty \, |\Delta_h(g')(x)| + |g'(x)| \, |\Delta_h(f' \circ g)(x)|$, we are reduced to prove the following estimate:

$$\left(\int_0^1 \Big(h^{1-s} U(h) \Big)^q \frac{\mathrm{d}h}{h} \right)^{1/q} \le c \, \|f'\|_{B^{s-1}_{p,q}} \, \|g\|_{B^s_{p,q}} \, \Big(1 + \|g'\|_\infty \Big)^\delta . \tag{12}$$

To prove (12), we observe that the set $\{x \in \mathbb{R} \; : \; g'(x) \ne 0\}$ is the union of a finite or countable family $(I_\ell)_{\ell \in \Lambda}$ of open disjoint intervals. For any $h > 0$, we denote by $I'_{\ell,h}$ the set of $x \in I_\ell$ whose distance to the boundary of I_ℓ is greater than $2h$, and we set

$$I''_{\ell,h} := I_\ell \setminus I'_{\ell,h}, \quad a_\ell := \sup_{I_\ell} |g'|, \quad g_\ell := g|_{I_\ell} .$$

Then, we have the following inequality:

$$\left(\sum_\ell a_\ell^p \right)^{1/p} \le \|g'\|_{BV_p} . \tag{13}$$

Proof of (13) Since $s - 1 > 1/p > 0$, g' is a member of $C_0(\mathbb{R})$. Then, there exists $\alpha_\ell \in \overline{I_\ell}$ such that $a_\ell = |g'(\alpha_\ell)|$. Moreover, since g' vanishes at the end points of I_ℓ, it holds $\alpha_\ell \in I_\ell$. As observed before, it holds $I_\ell \ne \mathbb{R}$: thus, we can consider one of the end points of I_ℓ, say β_ℓ. Let J_ℓ be the open interval with end points α_ℓ and β_ℓ. The intervals J_ℓ are pairwise disjoint, since $J_\ell \subset I_\ell$. As a consequence, it holds

$$\sum_\ell a_\ell^p = \sum_\ell |g'(\alpha_\ell) - g'(\beta_\ell)|^p \le \|g'\|^p_{BV_p} .$$

This completes the proof of (13). □

Let us notice that $I'_{\ell,h}$ is an open interval, possibly empty. In case it is not empty, we have

$$|\Delta_h g(g_\ell^{-1}(y))| \le a_\ell \, h, \quad \forall y \in g(I'_{\ell,h}) . \tag{14}$$

The set $I''_{\ell,h}$ is an interval of length $\leq 4h$, or the union of two intervals of length $2h$, and g' vanishes at one of the end points of this or those intervals. Now, we introduce

$$U_1(h) := \left(\sum_{\ell} \int_{I'_{\ell,h}} |g'(x)|^p \, |\Delta_h(f' \circ g)(x)|^p \, \mathrm{d}x \right)^{1/p}$$

and $U_2(h)$, defined in the same way, but replacing $I'_{\ell,h}$ by $I''_{\ell,h}$.

3.3.1 Estimation of U_1

By the change of variable $y := g_\ell(x)$ on $I'_{\ell,h}$ and by (14), it holds

$$U_1(h) \leq \left(\sum_{\ell} a_\ell^{p-1} \left(\Omega_p^1(f'; a_\ell h) \right)^p \right)^{1/p} . \tag{15}$$

We introduce the following notation:

- $\omega(t) := t^{1-s} \, \Omega_p^1(f'\,; t), \; t > 0\,;$
- $Z_m := \{\ell \in \Lambda \; : \; 2^{-m-1} \|g'\|_\infty < a_\ell \leq 2^{-m} \|g'\|_\infty\}, \; m \in \mathbb{N}_0 \,.$

By (13), it follows

$$\left(\sum_{m=0}^{\infty} 2^{-mp} \, (\mathrm{Card}\, Z_m) \right)^{1/p} \leq 2 \, \|g'\|_\infty^{-1} \left(\sum_{\ell} a_\ell^p \right)^{1/p} \leq 2 \, \|g'\|_\infty^{-1} \, \|g'\|_{BV_p}.$$

A fortiori the following estimate holds:

$$(\mathrm{Card}\, Z_m)^{1/p} \leq 2^{m+1} \|g'\|_\infty^{-1} \, \|g'\|_{BV_p}, \quad \forall m \in \mathbb{N}_0. \tag{16}$$

By the estimates (15,16) and by the monotonicity of Ω_p^1, we obtain

$$\begin{aligned} U_1(h) &\leq c_1 \, h^{s-1} \left(\sum_{\ell} a_\ell^{sp-1} \left(\omega(a_\ell h) \right)^p \right)^{1/p} \\ &\leq c_2 \, \|g'\|_\infty^{s-(1/p)} \, h^{s-1} \left(\sum_{m=0}^{\infty} 2^{-m(sp-1)} \mathrm{Card}\,(Z_m) \left(\omega(2^{-m} \, h \, \|g'\|_\infty) \right)^p \right)^{1/p} \\ &\leq c_3 \, \|g'\|_\infty^{\delta} \, \|g'\|_{BV_p} \, h^{s-1} \left(\sum_{m=0}^{\infty} 2^{-mp\delta} \left(\omega(2^{-m} \, h \, \|g'\|_\infty) \right)^p \right)^{1/p} . \end{aligned}$$

By condition $p \geq 1$, the above ℓ_p-norm is less than the corresponding ℓ_1-norm. Hence,

$$U_1(h) \leq c \, \|g'\|_\infty^{\delta} \, \|g'\|_{BV_p} \, h^{s-1} \sum_{m=0}^{\infty} 2^{-m\delta} \, \omega(2^{-m} \, h \, \|g'\|_\infty). \tag{17}$$

Then, we apply the following result:

Lemma 1 *For all* $\alpha > 0$ *and all* $q \in]0, +\infty]$, *there exists* $c = c(\alpha, q) > 0$ *such that the inequality*

$$\left(\int_0^\infty \Big(\sum_{m=0}^\infty 2^{-m\alpha}\, u(t\, 2^{-m}\, A) \Big)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le c \left(\int_0^\infty u(t)^q \, \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{18}$$

holds for all Borel measurable function $u :]0, +\infty[\to [0, +\infty[$ *and all* $A > 0$.

Proof of Lemma 1 By the change of variable $t' := tA$, we see that the left-hand side of (18) does not depend on A. Thus, we can assume $A = 1$.

We put $r := \min(1, q)$ and we use both the embedding of ℓ_r into ℓ_1 and the Minkowski inequality w.r.t. $q/r \ge 1$. We obtain

$$\left(\int_0^\infty \left(\sum_{m=0}^\infty 2^{-m\alpha}\, u(t\, 2^{-m}) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le \left(\int_0^\infty \left(\sum_{m=0}^\infty 2^{-rm\alpha}\, u(t\, 2^{-m})^r \right)^{q/r} \frac{\mathrm{d}t}{t} \right)^{1/q}$$
$$\le \left(\sum_{m=0}^\infty 2^{-rm\alpha} \left(\int_0^\infty u(t\, 2^{-m})^q \, \frac{\mathrm{d}t}{t} \right)^{r/q} \right)^{1/r} = \left(\sum_{m=0}^\infty 2^{-rm\alpha} \left(\int_0^\infty u(t)^q \, \frac{\mathrm{d}t}{t} \right)^{r/q} \right)^{1/r}.$$

We conclude the proof by using condition $r\alpha > 0$. □

Now applying (17) and Lemma 1, we deduce

$$\left(\int_0^\infty \left(h^{1-s} U_1(h) \right)^q \frac{\mathrm{d}h}{h} \right)^{1/q} \le c_1 \, \|g'\|_\infty^\delta \, \|g'\|_{BV_p} \left(\int_0^\infty \omega(t)^q \, \frac{\mathrm{d}t}{t} \right)^{1/q}$$
$$\le c_2 \, \|g'\|_\infty^\delta \, \|g'\|_{BV_p} \, \|f'\|_{\mathcal{B}^{s-1}_{p,q}},$$

see Proposition 13 in the Appendix. By the Peetre embedding (11), in combination with Proposition 8 in the Appendix, we conclude that (12) holds with U_1 instead of U.

3.3.2 Estimation of U_2

By the inequality $|\Delta_h(f' \circ g)(x)| \le \Omega^1_\infty(f'; h\, \|g'\|_\infty)$ and the properties of $I''_{\ell,h}$, it holds

$$U_2(h) \le c\ \Omega^1_\infty(f'; h\, \|g'\|_\infty)\, h^{1/p} \left(\sum_\ell a_\ell^p \right)^{1/p}.$$

By condition $0 < \delta < 1$ and by (13), we deduce

$$\left(\int_0^1 \left(h^{1-s} U_2(h) \right)^q \frac{\mathrm{d}h}{h} \right)^{1/q} \le c_1 \, \|g'\|_{BV_p} \left(\int_0^1 \left(h^{-\delta} \Omega^1_\infty(f'; h\, \|g'\|_\infty) \right)^q \frac{\mathrm{d}h}{h} \right)^{1/q}$$
$$\le c_2 \, \|f'\|_{B^\delta_{\infty,q}} \, \|g'\|_\infty^\delta \, \|g'\|_{BV_p}.$$

With the help of the embedding $\mathcal{B}^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow B^\delta_{\infty,q}(\mathbb{R})$, see (57) in Sect. 4.6, we conclude that (12) holds with U_2 instead of U.

3.4 Proof of Proposition 2: the case $s = 2$

Because of $s = 2$, we have $\delta = 1-(1/p)$, hence $0 < \delta < 1$. The use of first-order differences is not longer possible. Instead, we can work with the second-order differences operator Δ_h^2 defined in Sect. 3.2, see Propositions 7, 8, and 9 in Sect. 4. By (10), we can write

$$\Delta_h^2((f' \circ g)\, g')(x) = A_1(x,h) + A_2(x,h) + \frac{1}{2}\sum_{j=3}^{6} A_j(x,h),$$

where the A_j's are defined by:

$$\begin{aligned}
A_1(x,h) &:= f'(g(x+2h))\, \Delta_h^2 g'(x),\\
A_2(x,h) &:= 2\, \Delta_h g'(x)\, \Delta_h (f' \circ g)(x+h),\\
A_3(x,h) &:= g'(x)\, \big(f'(g(x)) + f'(2g(x+h) - g(x)) - 2f'(g(x+h))\big)\,,\\
A_4(x,h) &:= g'(x)\, \big(f'(g(x+2h)) + f'(2g(x+h) - g(x+2h)) - 2f'(g(x+h))\big)\,,\\
A_5(x,h) &:= g'(x)\, \big(f'(g(x)) - f'(2g(x+h) - g(x+2h))\big)\,,\\
A_6(x,h) &:= g'(x)\, \big(f'(g(x+2h)) - f'(2g(x+h) - g(x))\big)\,.
\end{aligned}$$

We introduce the notation

$$V_j(h) := \left(\int_{\mathbb{R}} |A_j(x,h)|^p\, \mathrm{d}x\right)^{1/p}. \tag{19}$$

Then, it suffices to prove

$$\left(\int_0^1 \left(h^{-1} V_j(h)\right)^q \frac{\mathrm{d}h}{h}\right)^{1/q} \le c\, \|f'\|_{\mathcal{B}^1_{p,q}}\, \|g\|_{B^2_{p,q}}\, \left(1 + \|g'\|_\infty\right)^\delta. \tag{20}$$

In some cases, the above estimate will follow by the stronger estimate:

$$V_j(h) \le c\, h^\alpha\, \|f'\|_{\mathcal{B}^1_{p,q}}\, \|g\|_{B^2_{p,q}}\, \left(1 + \|g'\|_\infty\right)^\delta, \quad \forall h \in]0,1], \quad \text{for some } \alpha > 1. \tag{21}$$

3.4.1 Estimation of V_1

We obtain immediately

$$\left(\int_0^1 \left(h^{-1} V_1(h)\right)^q \frac{\mathrm{d}h}{h}\right)^{1/q} \le \|f'\|_\infty \left(\int_0^1 \left(h^{-1} \|\Delta_h^2 g'\|_p\right)^q \frac{\mathrm{d}h}{h}\right)^{1/q} \le c\, \|f'\|_\infty\, \|g'\|_{B^1_{p,q}}.$$

Combined with Propositions 8 and 9 in the Appendix, this yields (20) in case $j = 1$.

3.4.2 Estimation of V_2

Using the embedding

$$B^1_{p,q}(\mathbb{R}) \hookrightarrow B^\gamma_{p,\infty}(\mathbb{R}),$$

where γ is any number < 1, see (38) and (39) in Sect. 4.3.2, we derive

$$V_2(h) \le c\, \|f'\|_{B^\delta_{\infty,\infty}}\, \|g'\|_\infty^\delta\, \|g'\|_{B^1_{p,q}}\, h^{\delta+\gamma}.$$

Choosing $1/p < \gamma < 1$, we obtain $\delta + \gamma > 1$. By embeddings $\mathcal{B}^1_{p,q}(\mathbb{R}) \hookrightarrow B^\delta_{\infty,\infty}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R})$, see (57) and (39) in Appendix, and Proposition 8, we conclude that (21) holds for $j = 2$.

3.4.3 Estimation of V_3

Notice that

$$\left| f'(g(x)) + f'(2g(x+h) - g(x)) - 2f'(g(x+h)) \right| = \left| \left(\Delta^2_{\Delta_h g(x)} f' \right)(g(x)) \right| \leq \sup_{|\theta| \leq h \|g'\|_\infty} \left| \left(\Delta^2_\theta f' \right)(g(x)) \right|. \tag{22}$$

With the same notation as in Sect. 3.3, it holds $V_3(h) \leq V_7(h) + V_8(h)$, where

$$V_7(h) := \left(\sum_\ell \int_{I'_{\ell,h}} |A_3(x,h)|^p \,\mathrm{d}x \right)^{1/p} \quad \text{and} \quad V_8(h) := \left(\sum_\ell \int_{I''_{\ell,h}} |A_3(x,h)|^p \,\mathrm{d}x \right)^{1/p}.$$

Estimation of V_7

On $I'_{\ell,h}$, the estimate (22) can be improved as follows:

$$\left| \left(\Delta^2_{\Delta_h g(x)} f' \right)(g(x)) \right| \leq \sup_{|\theta| \leq a_\ell h} \left| \left(\Delta^2_\theta f' \right)(g(x)) \right|.$$

Hence,

$$V_7(h) \leq \left(\sum_\ell a_\ell^{p-1} \left(\Omega^2_p(f'; a_\ell h) \right)^p \right)^{1/p}.$$

Then, we proceed exactly as in Sect. 3.3.1 to obtain

$$\left(\int_0^1 \left(h^{-1} V_7(h) \right)^q \frac{\mathrm{d}h}{h} \right)^{1/q} \leq c_1 \|g'\|^\delta_\infty \|g'\|_{BV_p} \|f'\|_{\mathcal{B}^1_{p,q}}$$
$$\leq c_2 \|g'\|^\delta_\infty \|g\|_{B^s_{p,q}} \|f'\|_{\mathcal{B}^1_{p,q}}.$$

This yields (20) in case of $j = 7$.

Estimation of V_8

The same arguments as in Sect. 3.3.2 can be applied. We find

$$V_8(h) \leq \Omega^2_\infty(f'; h\|g'\|_\infty) \left(\sum_\ell \int_{I''_{\ell,h}} |g'(x)|^p \,\mathrm{d}x \right)^{1/p}$$
$$\leq c\, h^{1/p} \|g'\|_{BV_p} \Omega^2_\infty(f'; h\|g'\|_\infty).$$

Since $h^{-1}\, h^{1/p} = h^{-\delta}$, we deduce

$$\left(\int_0^1 \left(h^{-1} V_8(h)\right)^q \frac{\mathrm{d}h}{h}\right)^{1/q} \leq c_1\, \|g'\|_{BV_p} \left(\int_0^1 \left(h^{-\delta}\, \Omega^2_\infty(f'; h\, \|g'\|_\infty)\right)^q \frac{\mathrm{d}h}{h}\right)^{1/q}$$
$$\leq c_2\, \|g'\|_{BV_p} \|f'\|_{B^\delta_{\infty,q}} \|g'\|^\delta_\infty,$$

the latter term following from Proposition 9, and the fact that $0 < \delta < 2$. We conclude with the help of the embedding $\mathcal{B}^1_{p,q}(\mathbb{R}) \hookrightarrow B^\delta_{\infty,q}(\mathbb{R})$. This yields (20) also in case $j = 8$.

3.4.4 Estimation of V_4

We need a further splitting $A_4 = -A_9 + A_{10}$, where

$$A_9(x,h) := \Delta_{2h}(g')(x) \left(f'(g(x+2h)) + f'(2g(x+h) - g(x+2h)) - 2f'(g(x+h))\right),$$
$$A_{10}(x,h) := g'(x+2h) \left(f'(g(x+2h)) + f'(2g(x+h) - g(x+2h)) - 2f'(g(x+h))\right).$$

Then, we define V_9 and V_{10} according to (19).

Estimation of V_9

It holds:

$$\left|f'(g(x+2h)) + f'(2g(x+h) - g(x+2h)) - 2f'(g(x+h))\right|$$
$$\leq \left|f'(g(x+2h)) - f'(g(x+h))\right| + \left|f'(2g(x+h) - g(x+2h)) - f'(g(x+h))\right|$$
$$\leq c\, h^\delta\, \|f'\|_{B^\delta_{\infty,\infty}}\, \|g'\|^\delta_\infty.$$

Thus, the estimation of V_9 is similar to that of V_2.

Estimation of V_{10}

By a change of variable, it holds

$$V_{10}(h) = \left(\int_{\mathbb{R}} |g'(x)|^p\, |f'(g(x)) + f'(2g(x-h) - g(x)) - 2f'(g(x-h))|^p\, \mathrm{d}x\right)^{1/p}.$$

Thus, the estimation of V_{10} is similar to that of V_3.

3.4.5 Estimation of V_5

Taking in account the inequality $|\Delta^2_h g(x)| \leq 2h\|g'\|_\infty$, it makes sense to compare $|\Delta^2_h g(x)|$ with $h^r\|g'\|_\infty$, for some $r > 0$, to be chosen later on. Then, we introduce the following notation:

$$C(h) := \left\{x \in \mathbb{R} : |\Delta^2_h g(x)| \leq h^r\, \|g'\|_\infty\right\}.$$

We split V_5 w.r.t. to $C(h)$ by setting

$$V_{11}(h) := \left(\int_{C(h)} |A_5(x,h)|^p\, \mathrm{d}x\right)^{1/p}$$

and V_{12} defined similarly, with $\mathbb{R} \setminus C(h)$ instead of $C(h)$.

Estimation of V_{12}

For all $x \in \mathbb{R}$ and all $h > 0$, we have

$$|f'(g(x)) - f'(2g(x+h) - g(x+2h))| \leq c \, \|f'\|_{B^{\delta}_{\infty,\infty}} \, \|g'\|_{\infty}^{\delta} \, h^{\delta}$$

and

$$\left(\int\limits_{\mathbb{R}\setminus C(h)} |g'(x)|^p \, \mathrm{d}x \right)^{1/p} \leq h^{-r} \left(\int\limits_{\mathbb{R}} |\Delta_h^2 g(x)|^p \, \mathrm{d}x \right)^{1/p}$$

$$\leq c \, h^{2-\varepsilon-r} \, \|g\|_{B^{2-\varepsilon}_{p,\infty}},$$

for an arbitrary $\varepsilon \in]0, 2[$. Hence,

$$V_{12}(h) \leq c \, h^{\delta+2-r-\varepsilon} \, \|f'\|_{B^{\delta}_{\infty,\infty}} \, \|g'\|_{\infty}^{\delta} \, \|g\|_{B^2_{p,q}}. \tag{23}$$

V_{12} will satisfy (21) if $\delta + 2 - r - \varepsilon > 1$, for a sufficiently small ε. Thus we need the condition

$$r < 1 + \delta. \tag{24}$$

Estimation of V_{11}

Consider a number $v > p$, to be fixed later on. By Hölder inequality with exponents v/p and $v/(v-p)$, we have

$$V_{11}(h) \leq \|g'\|_p^{1-(p/v)} \left(\int\limits_{C(h)} |g'(x)|^p \left| f'(g(x)) - f'(2g(x+h) - g(x+2h)) \right|^v \mathrm{d}x \right)^{1/v}.$$

Using the notations $I'_{\ell,h}$, $I''_{\ell,h}$ and a_ℓ of Sect. 3.3, we have

$$V_{11}(h) \leq \|g'\|_p^{1-(p/v)} \, (V_{13}(h) + V_{14}(h)), \tag{25}$$

where

$$V_{13}(h) := \left(\sum_{\ell} \int\limits_{I'_{\ell,h} \cap C(h)} |g'(x)|^p \left| f'(g(x)) - f'(2g(x+h) - g(x+2h)) \right|^v \mathrm{d}x \right)^{1/v},$$

and $V_{14}(h)$ is defined similarly, with $I''_{\ell,h}$ instead of $I'_{\ell,h}$.

Estimation of V_{13}

Clearly, for every $x \in I'_{\ell,h} \cap C(h)$, it holds

$$|\Delta_h^2 g(x)| \leq c \, \min\Big(h \, a_\ell, \, h^r \, \|g'\|_\infty\Big).$$

Then, by the change of variable $y := g_\ell(x)$, we have

$$V_{13}(h) \leq c \left(\sum_{\ell} a_\ell^{p-1} \Big(\Omega_v^1 \Big(f'; c \, \min\big(h \, a_\ell, \, h^r \, \|g'\|_\infty\big) \Big) \Big)^v \right)^{1/v}. \tag{26}$$

Next, we will use the embedding $\mathcal{B}^1_{p,q}(\mathbb{R}) \hookrightarrow \mathcal{B}^{\delta+(1/v)}_{v,\infty}(\mathbb{R})$, see (56) in Sect. 4.6, a consequence of

$$\delta + \frac{1}{v} = 1 - \frac{1}{p} + \frac{1}{v} < 1.$$

This yields

$$\Omega^1_v(f';t) \le c\, \|f'\|_{\mathcal{B}^1_{p,q}}\, t^{\delta+(1/v)}, \quad \forall t > 0.$$

By (26), we obtain

$$\begin{aligned} V_{13}(h) &\le c_1\, \|f'\|_{\mathcal{B}^1_{p,q}} \Big(\sum_{a_\ell \le h^{r-1}\|g'\|_\infty} a_\ell^{p-1}\, (h\, a_\ell)^{\delta v+1} + \sum_{a_\ell > h^{r-1}\|g'\|_\infty} a_\ell^{p-1}\, (h^r\, \|g'\|_\infty)^{\delta v+1} \Big)^{1/v} \\ &\le c_2\, \|f'\|_{\mathcal{B}^1_{p,q}} \Big(\sum_\ell a_\ell^p\Big)^{1/v} \Big(h^{\delta v+1} (h^{r-1}\|g'\|_\infty)^{\delta v} + h^{r(\delta v+1)} \|g'\|_\infty^{\delta v+1} (h^{-r+1}\|g'\|_\infty^{-1}) \Big)^{1/v}. \end{aligned}$$

By (11) and (13), this implies

$$V_{13}(h) \le c\, \|f'\|_{\mathcal{B}^1_{p,q}}\, \|g\|^{p/v}_{B^2_{p,q}}\, \|g'\|^\delta_\infty\, h^{r\delta+(1/v)}. \tag{27}$$

In view of condition (21), we need

$$r\delta + \frac{1}{v} > 1. \tag{28}$$

Estimation of V_{14}

In this situation, for $x \in I''_{\ell,h} \cap C(h)$, it follows

$$|f'(g(x)) - f'(2g(x+h) - g(x+2h))| \le c\, \|f'\|_{B^\delta_{\infty,\infty}}\, \|g'\|^\delta_\infty\, h^{r\delta},$$

resulting in

$$V_{14}(h) \le c\, \|f'\|_{B^\delta_{\infty,\infty}} \Big(\sum_\ell a_\ell^p\Big)^{1/v}\, \|g'\|^\delta_\infty\, h^{r\delta+(1/v)}.$$

Hence, V_{14} satisfies the same estimate (27) as V_{13}.

Conclusion.

We have to justify that the choice of v and r is possible. First, we observe that

$$\frac{1}{\delta}\Big(1 - \frac{1}{v}\Big) \to 1+, \quad \text{for} \quad v \to p+ .$$

Thus, we can chose $v > p$ such that

$$1 < \frac{1}{\delta}\Big(1 - \frac{1}{v}\Big) < 1 + \delta.$$

Then, we chose r such that

$$\frac{1}{\delta}\Big(1 - \frac{1}{v}\Big) < r < 1 + \delta.$$

The last condition implies (24) and (28). This completes the estimation of V_5.

3.4.6 Estimation of V_6

We write $A_6 = -A_{15} + A_{16}$, where

$$A_{15}(x,h) := \Delta_{2h}(g')(x)\,\big(f'(g(x+2h)) - f'(2g(x+h) - g(x))\big)$$
$$A_{16}(x,h) := g'(x+2h)\,\big(f'(g(x+2h)) - f'(2g(x+h) - g(x))\big)\,.$$

Then, we define V_{15} and V_{16} according to (19). The estimations of V_{15} and V_{16} are similar to that of V_2 and V_5, respectively.

3.5 Proof of Proposition 2: the case $2 < s \le 2 + (1/p)$

Since f and g are functions of class C^2, it holds

$$(f \circ g)'' = (f'' \circ g)\, g'^2 + (f' \circ g)\, g''.$$

Step 1 : Estimation of $(f' \circ g)\, g''$.
Let β be a parameter such that $\beta \le \delta$ and $s - 2 < \beta < 1$ (recall that $p > 1$, hence $s < 3$). Then, $\mathcal{B}^{s-1}_{p,q}(\mathbb{R})$ is embedded into $B^{\beta}_{\infty,\infty}(\mathbb{R})$, see (39, 57).
A straightforward computation leads to

$$\| f' \circ g \|_{B^{\beta}_{\infty,\infty}} \le c\, \| f' \|_{B^{\beta}_{\infty,\infty}} \big(1 + \| g' \|_\infty\big)^{\beta}\,.$$

By a classical result on multipliers, see [23, thm. 4.7.1], and by assumption $\beta > s - 2$, we deduce

$$\begin{aligned}\| (f' \circ g)\, g'' \|_{B^{s-2}_{p,q}} &\le c_1\, \| f' \|_{B^{\beta}_{\infty,\infty}} \big(1 + \| g' \|_\infty\big)^{\beta}\ \| g'' \|_{B^{s-2}_{p,q}} \\ &\le c_2\, \| f' \|_{\mathcal{B}^{s-1}_{p,q}} \big(1 + \| g' \|_\infty\big)^{\beta}\ \| g \|_{B^{s}_{p,q}}\,,\end{aligned}$$

see Proposition 8 in the Appendix.

Step 2 : Estimation of $(f'' \circ g)\, g'^2$.
We employ Proposition 9(ii). Since $0 < s - 2 < 1$, we have to estimate

$$W(t) := \Big(\int_{\mathbb{R}} \Big(t^{-1} \int_{-t}^{t} |\Delta_h((f'' \circ g)\, g'^2)(x)|\, \mathrm{d}h \Big)^p \mathrm{d}x \Big)^{1/p}.$$

Similarly to [15, p. 1118], we split the area of integration with respect to h. For $x \in \mathbb{R}$, we define

$$Q(x) := \Big\{ h \in \mathbb{R} :\ |g'(x+h)| \le |g'(x)| \Big\},$$
$$P(x) := \Big\{ h \in \mathbb{R} :\ |g'(x)| < |g'(x+h)| \Big\},$$

and

$$Q(x;t) := Q(x) \cap [-t,t], \quad P(x;t) := P(x) \cap [-t,t].$$

On $Q(x;t)$ we will use the elementary identity

$$\Delta_h((f'' \circ g)\, g'^2)(x) = g'(x+h)^2\, \Delta_h(f'' \circ g)(x) + f''(g(x))\, \Delta_h(g'^2)(x),$$

whereas on $P(x;t)$ we will use

$$\Delta_h((f''\circ g)\,g'^2)(x) = g'(x)^2\,\Delta_h(f''\circ g)(x) + f''(g(x+h))\,\Delta_h(g'^2)(x)$$

instead. Hence, $W(t) \le \sum_{j=1}^4 W_j(t)$, where

$$W_1(t) := \Big(\int_{\mathbb{R}} \Big(t^{-1} \int_{Q(x;t)} |f''(g(x))|\,|\Delta_h(g'^2)(x)|\,\mathrm{d}h\Big)^p \mathrm{d}x\Big)^{1/p},$$

$$W_2(t) := \Big(\int_{\mathbb{R}} \Big(t^{-1} \int_{P(x;t)} |f''(g(x+h))|\,|\Delta_h(g'^2)(x)|\,\mathrm{d}h\Big)^p \mathrm{d}x\Big)^{1/p},$$

$$W_3(t) := \Big(\int_{\mathbb{R}} \Big(t^{-1} \int_{Q(x;t)} |\Delta_h(f''\circ g)(x)|\,g'(x+h)^2\,\mathrm{d}h\Big)^p \mathrm{d}x\Big)^{1/p},$$

$$W_4(t) := \Big(\int_{\mathbb{R}} \Big(t^{-1} \int_{P(x;t)} |\Delta_h(f''\circ g)(x)|\,g'(x)^2\,\mathrm{d}h\Big)^p \mathrm{d}x\Big)^{1/p}.$$

3.5.1 Estimations of W_3 and W_4

We concentrate on W_3. The estimation of W_4 can be done in a similar way. Using the notation I_ℓ, $I'_{\ell,t}$, $I''_{\ell,t}$, a_ℓ and g_ℓ, as in Sect. 3.3, we can write $W_3(t) \le W_5(t) + W_6(t)$ where

$$W_5(t) := \Big(\sum_\ell \int_{I'_{\ell,t}} \Big(t^{-1} \int_{Q(x;t)} |\Delta_h(f''\circ g)(x)|\,g'(x+h)^2\,\mathrm{d}h\Big)^p \mathrm{d}x\Big)^{1/p},$$

and W_6 is defined in the same way, but with $I''_{\ell,t}$ instead of $I'_{\ell,t}$.

Estimation of W_5

We begin with the elementary inequality

$$g'(x+h)^2 \le |g'(x)|\,|g'(x+h)|, \quad \forall h \in Q(x).$$

Then, we perform the following changes of variables:

$$y := g_\ell(x) \quad \text{and} \quad \Theta := \Theta(h) = g(g_\ell^{-1}(y)+h) - y.$$

Since $|\Theta| \le a_l\,t$ for all $h \in [-t,t]$, see (14), we obtain

$$W_5(t) \le \left(\sum_\ell a_\ell^{p-1} \int_{\mathbb{R}} \Big(t^{-1} \int_{|\Theta|\le a_\ell t} |\Delta_\Theta f''(y)|\,\mathrm{d}\Theta\Big)^p \mathrm{d}y\right)^{1/p}.$$

With the abbreviation

$$\omega(t) := t^{2-s} \left(\int_{\mathbb{R}} \Big(t^{-1} \int_{|\Theta|\le t} |\Delta_\Theta f''(y)|\,\mathrm{d}\Theta\Big)^p \mathrm{d}y\right)^{1/p},$$

it follows

$$W_5(t) \le c\, t^{s-2} \left(\sum_{\ell} a_\ell^{sp-1} \, \big(\omega(a_\ell t)\big)^p \right)^{1/p} .$$

Now, arguing so as in Sect. 3.3.1, we conclude

$$\left(\int_0^1 \left(t^{2-s} W_5(t)\right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le c\, \|g'\|_\infty^\delta \, \|g'\|_{BV_p} \, \|f'\|_{\mathcal{B}^{s-1}_{p,q}},$$

where we have used

$$M^{s-2,1,1}_{p,q}(f'') \le c\, \|f'\|_{\mathcal{B}^{s-1}_{p,q}},$$

see Proposition 14.

Estimation of W_6

By definition of $Q(x)$, we find

$$W_6(t) \le W_7(t) + 2^{1/p}\, W_8(t), \quad 0 < t \le 1,$$

where

$$W_7(t) := \Bigg(\sum_{\ell} \int_{I''_{\ell,t}} \Big(t^{-1} \int_{Q(x;t)} |f''(g(x+h))|\, g'(x+h)^2 \,\mathrm{d}h \Big)^p \mathrm{d}x \Bigg)^{1/p},$$

$$W_8(t) := \Bigg(\sum_{\ell} \int_{I''_{\ell,t}} |f''(g(x))|^p \, |g'(x)|^{2p} \,\mathrm{d}x \Bigg)^{1/p}.$$

Estimation of W_7

The main difficulty consists in the fact that f'' need not be bounded. Instead, we use the embedding $\mathcal{B}^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow \dot{W}^1_v(\mathbb{R})$ for all v satisfying

$$1 - \delta < \frac{1}{v} < \frac{1}{(s-1)p}, \tag{29}$$

see (59) in Sect. 4.6 (notice that $1 - \delta < ((s-1)p)^{-1}$ follows by $s > 2$). The value of v will be chosen later. The restrictions in (29) imply $v > p$. Hence, the following definitions make sense:

$$\frac{1}{w} := \frac{1}{p} - \frac{1}{v} \quad \text{and} \quad \alpha := \frac{p+1}{v}. \tag{30}$$

Observe that (29) and $2 < s \le 2 + (1/p)$ imply $\alpha < 2$. By definition of the set $Q(x;t)$, we find

$$\int_{I''_{\ell,t}} \Big(t^{-1} \int_{Q(x;t)} |f''(g(x+h))|\, g'(x+h)^2 \,\mathrm{d}h \Big)^p \,\mathrm{d}x$$
$$\le \int_{I''_{\ell,t}} |g'(x)|^{p(2-\alpha)} \Big(t^{-1} \int_{-t}^{t} |f''(g(x+h))|\, |g'(x+h)|^{\alpha} \,\mathrm{d}h \Big)^p \,\mathrm{d}x$$
$$\le c \int_{I''_{\ell,t}} |g'(x)|^{p(2-\alpha)} \Big(M((f''\circ g)\,|g'|^{\alpha})(x) \Big)^p \,\mathrm{d}x.$$

Hölder inequality in $\Lambda \times \mathbb{R}$ (Λ has been defined in Sect. 3.3) yields

$$W_7(t) \le c_1 \Bigg(\sum_\ell \int_{I''_{\ell,t}} |g'(x)|^{(2-\alpha)w} \,\mathrm{d}x \Bigg)^{1/w} \Bigg(\sum_\ell \int_{I''_{\ell,t}} \Big(M((f''\circ g)\,|g'|^{\alpha})(x) \Big)^v \,\mathrm{d}x \Bigg)^{1/v}$$
$$\le c_1 \Bigg(\sum_\ell \int_{I''_{\ell,t}} |g'(x)|^{(2-\alpha)w} \,\mathrm{d}x \Bigg)^{1/w} \Bigg(\int_{\mathbb{R}} \Big(M((f''\circ g)\,|g'|^{\alpha})(x) \Big)^v \,\mathrm{d}x \Bigg)^{1/v}$$
$$\le c_2 \Bigg(\sum_\ell \int_{I''_{\ell,t}} |g'(x)|^{(2-\alpha)w} \,\mathrm{d}x \Bigg)^{1/w} \quad \|(f''\circ g)|g'|^{\alpha}\|_v,$$

where the last estimate follows by the Hardy–Littlewood maximal inequality in L_v. By using the identity $\alpha v = p+1$, we conclude

$$\|\,(f''\circ g)\,|g'|^{\alpha}\,\|_v \le \Big(\sum_\ell a_\ell^p \int_{g(I_\ell)} |f''(y)|^v \,\mathrm{d}y \Big)^{1/v}$$
$$\le \|f'\|_{\dot{W}^1_v} \;\|g'\|_{BV_p}^{p/v}. \tag{31}$$

Since $w(2-\alpha) = p + w(1-(1/v))$ and g' vanishes at one of the end points of $I''_{\ell,t}$, we obtain

$$\Bigg(\sum_\ell \int_{I''_{\ell,t}} |g'(x)|^{(2-\alpha)w} \,\mathrm{d}x \Bigg)^{1/w} \le \|g'\|_\infty^{1-(1/v)} \Bigg(\int_{\mathbb{R}} \sup_{|h|\le 2t} |\Delta_h g'(x)|^p \,\mathrm{d}x \Bigg)^{1/w}$$
$$\le c\, t^{rp/w}\, \|g'\|_\infty^{1-(1/v)}\, \|g'\|_{B^r_{p,\infty}}^{p/w}$$

as long as

$$\frac{1}{p} < r < 1, \tag{32}$$

see Proposition 9 in the Appendix. As used many times before, we know $B^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow BV_p(\mathbb{R})$. Furthermore, since $s > 2$, we have $s-1 > 1 > r$ and hence $B^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow B^r_{p,\infty}(\mathbb{R})$.

Summarizing, we proved up to now the inequality

$$\begin{aligned} W_7(t) &\le c\,\|f'\|_{\dot W^1_v}\;\|g\|^{p/v}_{B^s_{p,q}}\,t^{rp/w}\,\|g'\|^{1-(1/v)}_\infty\;\|g'\|^{p/w}_{B^{s-1}_{p,q}} \\ &\le c\,\|f'\|_{\dot W^1_v}\;\|g\|_{B^s_{p,q}}\,\|g'\|^{1-(1/v)}_\infty\,t^{rp/w}. \end{aligned}$$

For the desired estimate of W_7, we wish to have also

$$\frac{rp}{w} > s-2. \tag{33}$$

Looking at this inequality, it becomes clear that we should choose r and v as large as possible. Obviously,

$$\lim_{r\uparrow 1}\;\lim_{v\uparrow 1/(1-\delta)}\frac{rp}{w} = p\,(s-2).$$

Since $p>1$, we can always find appropriate parameters r and v with (33). This leads to

$$\begin{aligned} \left(\int_0^1 \left(t^{2-s}\,W_7(t)\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q} &\le c\,\|f'\|_{\mathcal{B}^{s-1}_{p,q}}\;\|g\|_{B^s_{p,q}}\;\|g'\|^{1-(1/v)}_\infty \\ &\le c\,\|f'\|_{\mathcal{B}^{s-1}_{p,q}}\;\|g\|_{B^s_{p,q}}\;(1+\|g'\|_\infty)^\delta. \end{aligned} \tag{34}$$

Estimation of W_8

It is similar to that of W_7, indeed a little simpler since the maximal inequality is no more needed. We omit the details. The estimate (34) holds with W_7 replaced by W_8.

3.5.2 Estimations of W_1 and W_2

We concentrate on W_2. The estimation of W_1 is similar. Let us take v and w as in (29,30). First, observe the elementary inequality

$$|\Delta_h(g'^2)(x)| \le c\,\|g'\|^{1-(1/v)}_\infty |g'(x+h)|^{(p+1)/v}\,|\Delta_h g'(x)|^{1-(p/v)},\quad \forall h\in P(x).$$

Now, we argue so as in the estimation of W_7 and obtain

$$\begin{aligned} W_2(t) &\le c_1\,\|g'\|^{1-(1/v)}_\infty\left(\int_{\mathbb{R}}\left(M((f''\circ g)\,|g'|^{(p+1)/v})(x)\right)^p(\sup_{|h|\le t}|\Delta_h g'(x)|)^{p(1-(p/v))}\,\mathrm{d}x\right)^{1/p} \\ &\le c_2\,\|g'\|^{1-(1/v)}_\infty\|(f''\circ g)\,|g'|^{(p+1)/v}\|_v\left(\int_{\mathbb{R}}(\sup_{|h|\le t}|\Delta_h g'(x)|)^p\,\mathrm{d}x\right)^{1/w} \\ &\le c_3\,t^{rp/w}\|g'\|^{1-(1/v)}_\infty\|f''\|_v\,\|g'\|^{p/v}_{BV_p}\;\|g'\|^{p/w}_{B^r_{p,\infty}}. \end{aligned}$$

From this, again as above, we deduce

$$\left(\int_0^1\left(t^{2-s}\,W_2(t)\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q} \le c\,\|f'\|_{\mathcal{B}^{s-1}_{p,q}}\,\|g\|_{B^s_{p,q}}(1+\|g'\|_\infty)^\delta.$$

This completes the proof of Proposition 2. □

3.6 Proof of Theorem 4

Our main ingredient is the Fubini-type characterization of $B^s_{p,p}(\mathbb{R}^n)$. For $n \geq 2,\ 1 \leq p \leq \infty$ and $s > 0$

$$\sum_{j=1}^{n} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \| g(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \|^p_{B^s_{p,p}(\mathbb{R})} \mathrm{d}\vec{x}_j \Bigg)^{1/p}$$

can be used as an equivalent norm in $B^s_{p,p}(\mathbb{R}^n)$, see, e.g., [25, 2.5.13]. Here,

$$\mathrm{d}\vec{x}_j := \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \mathrm{d}x_\ell .$$

Under the conditions of Theorem 4 and using Theorem 2, Propositions 7-8, we derive

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n-1}} \| \partial_j \, (f \circ g)(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \|^p_{B^{s-1}_{p,p}(\mathbb{R})} \mathrm{d}\vec{x}_j \\
&\leq c \Bigg(\| (f' \circ g) \, \partial_j \, g \|^p_p \\
&\qquad + \int\limits_{\mathbb{R}^{n-1}} \| \partial_j \, (f \circ g)(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \|^p_{\mathcal{B}^{s-1}_{p,p}(\mathbb{R})} \mathrm{d}\vec{x}_j \Bigg) \\
&\leq c \Bigg(\| f' \|^p_\infty \, \| \partial_j g \|^p_p + \| f' \|^p_{\mathcal{B}^{s-1}_{p,p}} \ \big(1 + \| \partial_j g \|_\infty \big)^{\delta p} \\
&\qquad \times \int\limits_{\mathbb{R}^{n-1}} \| g(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \|^p_{B^s_{p,p}(\mathbb{R})} \, \mathrm{d}\vec{x}_j \Bigg) \\
&\leq c \, \| f' \|^p_{\mathcal{B}^{s-1}_{p,p}} \ (1 + \| \nabla g \|_\infty)^{\delta p} \ \| g \|^p_{B^s_{p,p}(\mathbb{R}^n)},
\end{aligned}$$

where we used $B^{s-1}_{p,p}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$ since $s > 1$. This completes the proof.

3.7 Proof of Theorem 5

Step 1.
Let $f \in B^{s,\ell oc}_{p,p}(\mathbb{R})$ and $g \in B^s_{p,p} \cap \dot{W}^1_\infty(\Omega)$. Now, let $\mathcal{E}g$ be an extension of g s.t. $\mathcal{E}g \in B^s_{p,p} \cap W^1_\infty(\mathbb{R}^n)$. Then, by Theorem 3, $(f - f(0)) \circ \mathcal{E}g \in B^s_{p,p}(\mathbb{R}^n)$. Obviously, $f(0)\,\mathcal{E}g \in B^s_{p,p}(\mathbb{R}^n)$. Hence, the restriction of $f \circ \mathcal{E}g$ to Ω belongs to $B^s_{p,p}(\Omega)$. This proves sufficiency.

Step 2.
Necessity. Let $x^0 \in \Omega$. Testing the operator T_f with the family of functions

$$g_a(x) = a\,(x_1 - x_1^0), \quad x \in \Omega, \quad a > 0,$$

we conclude $f \circ g_a \in B^s_{p,p}(\Omega)$ since $g_a \in B^s_{p,p}(\Omega)$. By $\mathcal{E}(f \circ g_a)$ we denote an arbitrary extension of $f \circ g_a$ and by Q a cube with side-length $\varepsilon > 0$ and center x^0 s.t. the set

$$\{x : \max_{j=1,\dots n} |x_j - x_j^0| < m\,\varepsilon\} \subset \Omega$$

for some sufficiently large integer $m > s$. Then, the characterization of $B^s_{p,p}(\mathbb{R}^n)$ by differences yields

$$\| \mathcal{E}(f \circ g_a) \|_{B^s_{p,p}(\mathbb{R}^n)} \geq c_1 \left(\int_0^{\varepsilon} [t^{-s} \sup_{|h|<t} \| \Delta_h^m \mathcal{E}(f \circ g_a) \|_{L_p(Q)}]^p \frac{\mathrm{d}t}{t} \right)^{1/p}$$

$$\geq c_2 \left(\int_0^{\varepsilon} [t^{-s} \sup_{|h|<at} \| \Delta_h^m f \|_{L_p([-a\varepsilon, a\varepsilon])}]^p \frac{\mathrm{d}t}{t} \right)^{1/p}$$

with a constant $c_2 = c_2(Q, \varepsilon, a) > 0$. But this implies $f \in B^{s,\ell oc}_{p,p}(\mathbb{R})$. The proof is complete.

4 Appendix

For the convenience of the reader, we collect here all what is needed about Besov spaces. This includes inhomogeneous, homogeneous, and modified Besov spaces [see (4)]. In case of homogeneous and modified Besov spaces, we found the existing literature not sufficient. For this reason, certain parts of this collection are with proofs.

4.1 Distributions modulo polynomials

Since the elements of the homogeneous Besov spaces are distributions modulo polynomials, we need convenient notation. For $m \in \mathbb{N}_0$, we denote by $\mathcal{P}_m(\mathbb{R}^n)$ the set of polynomials in $\mathbb{R}^n$, of degree less than m. In particular, $\mathcal{P}_0(\mathbb{R}^n)$ is reduced to $\{0\}$, and $\mathcal{P}_\infty(\mathbb{R}^n)$ the set of all polynomials on $\mathbb{R}^n$. For $m \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $\mathcal{S}_m(\mathbb{R}^n)$ the set of all $u \in \mathcal{S}(\mathbb{R}^n)$ such that $\langle f, u \rangle = 0$ for all $f \in \mathcal{P}_m(\mathbb{R}^n)$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $[f]_m$ the equivalence class of f modulo $\mathcal{P}_m(\mathbb{R}^n)$. The mapping which takes any $[f]_m$ to the restriction of f to $\mathcal{S}_m(\mathbb{R}^n)$ turns out to be a vector space isomorphism of $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m(\mathbb{R}^n)$ onto $\mathcal{S}'_m(\mathbb{R}^n)$. For this reason, $\mathcal{S}'_m(\mathbb{R}^n)$ is called the space of *distributions modulo polynomials of degree less than m*.

Definition 1 Let $0 \leq k < m \leq \infty$ and let E be a vector subspace of $\mathcal{S}'_m(\mathbb{R}^n)$ endowed with a quasi-norm such that the embedding $E \hookrightarrow \mathcal{S}'_m(\mathbb{R}^n)$ holds. A *realization* of E in $\mathcal{S}'_k(\mathbb{R}^n)$ is a continuous linear mapping $\sigma : E \to \mathcal{S}'_k(\mathbb{R}^n)$ such that $[\sigma(f)]_m = f$ for all $f \in E$.

In short words, a realization is a coherent way to associate to each element of E a specific representative. We need a further notion.

Definition 2 A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishes at infinity if $\lim_{\lambda \to 0} f\left(\frac{\cdot}{\lambda}\right) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$.

Here are examples of such distributions:

- functions in $L_p(\mathbb{R}^n)$, for $p < \infty$,
- derivatives of functions in $L_\infty(\mathbb{R}^n)$,
- derivatives of distributions which vanish at infinity.

This notion was first introduced in [4]. Its usefulness relies upon the following:

Lemma 2 *The only polynomial vanishing at infinity is the zero polynomial.*

Proof See [4, p. 46]. □

4.2 The Littlewood–Paley setting

Probably, the easiest way to introduce both, inhomogeneous as well as homogeneous, Besov spaces is the Fourier theoretical approach via the Littlewood–Paley decomposition. This nowadays classical approach, initiated by Peetre and Triebel, see, e.g., the monographs [22, 25], is still very popular, in particular in the "Nonlinear World," we refer to the recent monograph of Bahouri, Chemin, and Danchin [1].

We start with the cutoff function ρ introduced in the Notation. Then, we define

$$\gamma(\xi) := \rho(\xi) - \rho(2\xi), \quad \forall \xi \in \mathbb{R}^n .$$

The function γ is supported in the compact annulus $1/2 \leq |\xi| \leq 3/2$ and the following identities hold:

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

$$\rho(\xi) + \sum_{j \geq 1} \gamma(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^n .$$

For $j \in \mathbb{Z}$, we define the operator Q_j similarly to S_j, by replacing ρ by γ in the formula (3); this operator takes $\mathcal{S}'_\infty(\mathbb{R}^n)$ to the space of analytical functions of exponential type. The Littlewood–Paley decompositions of a tempered distribution are described in the following well-known statements:

Proposition 3 (i) *For every $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n)$), it holds*

$$f = \sum_{j \in \mathbb{Z}} Q_j f, \tag{35}$$

in $\mathcal{S}_\infty(\mathbb{R}^n)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n)$).

(ii) *For every $f \in \mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) and every $k \in \mathbb{Z}$, it holds*

$$f = S_k f + \sum_{j > k} Q_j f, \tag{36}$$

in $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$).

There is only a short step from the Littlewood–Paley decomposition to the definition of the Besov spaces.

Definition 3 Let $s \in \mathbb{R},\ 1 \leq p \leq \infty$ and $0 < q \leq \infty$.

(i) The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the set of $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{j \in \mathbb{Z}} (2^{sj} \, \|Q_j f\|_p)^q \Big)^{1/q} < +\infty .$$

(ii) The inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of tempered distributions f such that

$$\|f\|_{B^s_{p,q}} := \|S_0 f\|_p + \Big(\sum_{j \geq 1} (2^{sj} \|Q_j f\|_p)^q \Big)^{1/q} < +\infty .$$

$\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces for the above-defined quasi-norms, continuously embedded in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. For all $f \in \mathcal{S}'(\mathbb{R}^n)$, we define

$$\|f\|_{\dot{B}^s_{p,q}} := \|[f]_\infty\|_{\dot{B}^s_{p,q}}.$$

We will make use of the following convention: *In all the following statements, the numbers* s, p, q *will verify assumptions of* Definition 3*, unless otherwise stated.*

The dyadic decomposition, given in Proposition 3, can be replaced by continuous ones:

Proposition 4 *Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\widehat{\phi}$ *is a nonzero radial function with compact support contained in* $\mathbb{R}^n \setminus \{0\}$*. Let us set*

$$\phi_t(x) := t^{-n}\phi(x/t), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^n. \tag{37}$$

Then, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$*, it holds* $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ *if, and only if,*

$$\left(\int_0^\infty \left(\frac{1}{t^s}\,\|\phi_t * f\|_p\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} < +\infty.$$

Moreover, the above expression is an equivalent quasi-norm in $\dot{B}^s_{p,q}(\mathbb{R}^n)$*.*

Remark 2 Proposition 4 has a counterpart for inhomogeneous spaces, which we do not need in the present paper.

Proposition 4 has an immediate consequence, which explains the terminology used for spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$:

Corollary 1 *There is an equivalent quasi-norm* N*, in* $\dot{B}^s_{p,q}(\mathbb{R}^n)$*, which enjoys the* homogeneity property*:*

$$N\big(f(\lambda(\cdot))\big) = \lambda^{s-(n/p)}\,N(f) \quad \forall f \in \dot{B}^s_{p,q}(\mathbb{R}^n), \quad \forall \lambda > 0.$$

4.3 Properties of Besov spaces

4.3.1 Convergence of the Littlewood–Paley series

In case $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, the series expansion (35) makes sense not only in $\mathcal{S}'_\infty(\mathbb{R}^n)$ but also in $\mathcal{S}'_m(\mathbb{R}^n)$ for some minimal value of m that we introduce first. We associate to any set (s, n, p, q) of parameters a number $\nu \in \mathbb{N}_0$ defined by

$$\nu = \big([s-(n/p)]+1\big)_+ \quad \text{if} \quad s-(n/p) \notin \mathbb{N}_0 \quad \text{or} \quad q > 1;$$
$$\nu = s-(n/p) \quad \text{if} \quad s-(n/p) \in \mathbb{N}_0 \quad \text{and} \quad q \le 1.$$

The following statement explains the intrinsic meaning of ν w.r.t. the space $\dot{B}^s_{p,q}(\mathbb{R}^n)$. We refer to [10, prop. 4.6, thm. 4.2] for details.

Proposition 5 (i) *Let* $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$*. Then, the series* (35) *converges in* $\mathcal{S}'_\nu(\mathbb{R}^n)$*, and its sum in* $\mathcal{S}'_\nu(\mathbb{R}^n)$*, denoted by* $\sigma_\nu(f)$*, is the unique representative of* f *in* $\mathcal{S}'_\nu(\mathbb{R}^n)$ *whose derivatives of order* ν *vanish at infinity.*

(ii) *The mapping* σ_ν *is a realization of* $\dot{B}^s_{p,q}(\mathbb{R}^n)$ *in* $\mathcal{S}'_\nu(\mathbb{R}^n)$ *which commutes with translations, and the integer* ν *is minimal for this property: if there exists a translation commuting realization of* $\dot{B}^s_{p,q}(\mathbb{R}^n)$ *in* $\mathcal{S}'_m(\mathbb{R}^n)$*, then* $m \ge \nu$*.*

4.3.2 Various embeddings

We recall successively: (i) the embeddings between spaces with different values of s and/or p, (ii) the connection between homogeneous and inhomogeneous spaces, (iii) the behavior of Besov spaces w.r.t. differentiation.

Proposition 6 *The continuous embedding $\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s_2}_{p_2,q}(\mathbb{R}^n)$ holds for all parameters such that*

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2} \quad \textit{and} \quad p_2 \geq p_1 .$$

For a proof, we refer to Jawerth [18]. The following so-called elementary embeddings are also useful, see, e.g., [2, thm. 6.3.1] or [25, prop. 2.3.2/2].

(i) Besov spaces are monotonic with respect to the third index, i.e.,

$$\dot{B}^{s}_{p,q_0}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s}_{p,q_1}(\mathbb{R}^n), \quad B^{s}_{p,q_0}(\mathbb{R}^n) \hookrightarrow B^{s}_{p,q_1}(\mathbb{R}^n) \quad \text{if} \quad q_0 \leq q_1 . \tag{38}$$

(ii) Inhomogeneous Besov spaces are monotone with respect to the smoothness index, i.e.,

$$B^{s_0}_{p,q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p,q_1}(\mathbb{R}^n) \quad \text{if} \quad s_0 > s_1 . \tag{39}$$

To the contrary, homogeneous Besov spaces *are not* monotone with respect to the smoothness index (just as a consequence of the homogeneity property stated in Corollary 1).

(iii) Finally, we wish to mention

$$B^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{0}_{p,1}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \quad \text{if} \quad s > 0, \tag{40}$$

see [25, 2.5.7].

Proposition 7 *If $s > 0$, then it holds*

$$B^{s}_{p,q}(\mathbb{R}^n) = \Big\{ f \in L_p(\mathbb{R}^n) \ : \ [f]_\infty \in \dot{B}^{s}_{p,q}(\mathbb{R}^n) \Big\} .$$

Moreover, $\| f \|_p + \| f \|_{\dot{B}^s_{p,q}}$ is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$.

Proof *Step 1.* Let $f \in L_p(\mathbb{R}^n)$ and $[f]_\infty \in \dot{B}^{s}_{p,q}(\mathbb{R}^n)$. Since S_0 is a bounded convolution operator in L_p, it holds

$$\begin{aligned} \| S_0 f \|_p + \Big(\sum_{j \geq 1} (2^{sj} \| Q_j f \|_p)^q \Big)^{1/q} &\leq c \, \| f \|_p + \Big(\sum_{j \in \mathbb{Z}} (2^{sj} \| Q_j f \|_p)^q \Big)^{1/q} \\ &\leq c \, \| f \|_p + \| f \|_{\dot{B}^s_{p,q}} . \end{aligned}$$

Step 2. Let $f \in B^s_{p,q}(\mathbb{R}^n)$. By (40), it holds $\| f \|_p \leq c \, \| f \|_{B^s_{p,q}}$. Since $(Q_j)_{j \in \mathbb{Z}}$ is a bounded family of convolution operators in L_p, it holds

$$\Big(\sum_{j \in \mathbb{Z}} (2^{sj} \| Q_j f \|_p)^q \Big)^{1/q} \leq c \left(\sum_{j \leq 0} 2^{sjq} \right)^{1/q} \| f \|_p + \left(\sum_{j \geq 1} (2^{sj} \| Q_j f \|_p)^q \right)^{1/q} .$$

The proof is complete. ◻

Remark 3 Of course, the statement of Proposition 7 is essentially known, see [2, thm. 6.3.2]. There the identity

$$B^s_{p,q}(\mathbb{R}^n) = L_p \cap \dot{B}^s_{p,q}(\mathbb{R}^n) \quad \text{if} \quad s > 0,$$

is proved.

Proposition 8 (i) *An element f of $\mathcal{S}'_\infty(\mathbb{R}^n)$ belongs to $\dot{B}^s_{p,q}(\mathbb{R}^n)$ if, and only if, its first-order derivatives $\partial_\ell f$ belong to $\dot{B}^{s-1}_{p,q}(\mathbb{R}^n)$ for $\ell = 1, \ldots, n$. Moreover, $\sum_{\ell=1}^n \|\partial_\ell f\|_{\dot{B}^{s-1}_{p,q}}$ is an equivalent quasi-norm in $\dot{B}^s_{p,q}(\mathbb{R}^n)$.*

(ii) *For all $f \in B^s_{p,q}(\mathbb{R}^n)$, its first-order derivatives $\partial_\ell f$ $(\ell = 1, \ldots, n)$ belong to $B^{s-1}_{p,q}(\mathbb{R}^n)$ and $\|\partial_\ell f\|_{B^{s-1}_{p,q}} \leq c\, \|f\|_{B^s_{p,q}}$.*

Proof The second statement is classical, see, e.g., [25, 2.3.8]. We give a sketchy proof of the first one.

Step 1. Let $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. By Bernstein inequality, see, e.g., [25, rem. 1.3.2/1], it holds

$$\|Q_j(\partial_\ell f)\|_p \leq c\, 2^j \|Q_j f\|_p$$

with a constant c independent of f and j. Hence, $\|\partial_\ell f\|_{\dot{B}^{s-1}_{p,q}} \leq c\, \|f\|_{\dot{B}^s_{p,q}}$.

Step 2. Let ρ_ℓ, $\ell = 1, \ldots, n$, be C^∞ functions on the unit sphere S^{n-1} of $\mathbb{R}^n$, such that

$$\sum_{\ell=1}^n \rho_\ell(\xi) = 1, \quad \forall \xi \in S^{n-1},$$

and such that $\xi_\ell \neq 0$ on the support of ρ_ℓ. Let us define

$$\gamma_\ell(\xi) := -\frac{\mathrm{i}}{\xi_\ell}\, \gamma(\xi) \rho_\ell\left(\frac{\xi}{|\xi|}\right).$$

Then, γ_ℓ is a C^∞ function with compact support in $\mathbb{R}^n \setminus \{0\}$, and

$$Q_j f = 2^{-j} \sum_{\ell=1}^n \gamma_\ell(2^{-j} D)(\partial_\ell f), \quad \forall f \in \mathcal{S}'_\infty(\mathbb{R}^n).$$

Hence, by a standard convolution inequality

$$\|f\|_{\dot{B}^s_{p,q}} \leq c \sum_{\ell=1}^n \|\partial_\ell f\|_{\dot{B}^{s-1}_{p,q}}$$

with c independent of f.

□

4.4 Inhomogeneous Besov spaces via differences

In this subsection, we recall the characterizations of inhomogeneous Besov spaces involving the iterated difference operators Δ^m_h. For simplicity, we introduce the following notation: if

$1 \le p, u \le \infty,\ 0 < q \le +\infty,\ s > 0,\ m \in \mathbb{N}$, and if f is any measurable function on $\mathbb{R}^n$, we put

$$M^{s,m}_{p,q}(f) := \Bigg(\int\limits_{\mathbb{R}^n} |h|^{-sq} \, \| \Delta^m_h f \|^q_p \frac{\mathrm{d}h}{|h|^n} \Bigg)^{1/q}$$

$$M^{s,m,u}_{p,q}(f) := \Bigg(\int\limits_0^\infty t^{-sq} \Bigg(\int\limits_{\mathbb{R}^n} \Big(t^{-n} \int\limits_{|h|\le t} |\Delta^m_h f(x)|^u \, \mathrm{d}h \Big)^{p/u} \mathrm{d}x \Bigg)^{q/p} \frac{\mathrm{d}t}{t} \Bigg)^{1/q}.$$

Proposition 9 *Let $s > 0$, and $m \in \mathbb{N}$ such that $s < m$.*

(i) *A tempered distribution f belongs to $B^s_{p,q}(\mathbb{R}^n)$ if, and only if, $f \in L_p(\mathbb{R}^n)$ and $M^{s,m}_{p,q}(f) < +\infty$. Moreover, the expression $\|f\|_p + M^{s,m}_{p,q}(f)$ is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$.*

(ii) *Assume further*

$$s > n\Big(\frac{1}{p} - \frac{1}{u}\Big). \tag{41}$$

Then, we can replace $M^{s,m}_{p,q}(f)$ by $M^{s,m,u}_{p,q}(f)$ in the preceding statement.

Proof The first statement is classical, see, e.g., Besov et al. [3], Nikol'skij [21], Peetre [22], and Triebel [25–27]. For the second one, we refer to Seeger [24] and Triebel [26, thm. 3.5.3, p. 194]. □

Remark 4 The above assertion remains true if, in the expressions of $M^{s,m}_{p,q}(f)$ and $M^{s,m,u}_{p,q}(f)$, one replaces integration for $h \in \mathbb{R}^n$ and $0 < t < \infty$ by integration for $|h| \le a$ and $0 < t < a$, respectively, for any fixed $a > 0$.

4.5 Homogeneous Besov spaces via differences

Characterization of homogeneous Besov spaces by differences is given in at least three different places, we refer to Peetre [22, chap. 8, p. 160] in case $q = \infty$, Bergh and Löfström [2, thm. 6.3.1] and Triebel [25, thm. 5.2.3/2]. In the first two references, the authors identify spaces defined modulo all polynomials with spaces defined modulo polynomials of a certain degree. In the third reference only a sketch of a proof is given (which differs at least partly from our one).

Our point of departure is the following simple lemma.

Lemma 3 *Under the assumptions of* Proposition 9, *there exist constants $c_1, c_2 > 0$ such that $M^{s,m}_{p,q}(f) \le c_1 \|f\|_{\dot{B}^s_{p,q}}$ and $M^{s,m,u}_{p,q}(f) \le c_2 \|f\|_{\dot{B}^s_{p,q}}$ hold for all $f \in B^s_{p,q}(\mathbb{R}^n)$, respectively.*

Proof By Propositions 7 and 9, it holds

$$M^{s,m}_{p,q}(f) \le c \Big(\|f\|_p + \|f\|_{\dot{B}^s_{p,q}} \Big), \quad \forall f \in B^s_{p,q}(\mathbb{R}^n). \tag{42}$$

We replace now f by $f(\lambda(\cdot))$, for any $\lambda > 0$, in (42). Using Corollary 1, dividing by $\lambda^{s-(n/p)}$, and letting $\lambda \to +\infty$, we obtain the desired estimate. The same proof holds for $M^{s,m,u}_{p,q}$ under condition (41). □

Lemma 3 suggests that $M_{p,q}^{s,m}$ could be used as an equivalent quasi-norm in $\dot{B}^s_{p,q}(\mathbb{R}^n)$. But things are not that simple. Indeed, $M_{p,q}^{s,m}$ and $M_{p,q}^{s,m,u}$ are functionals with kernel $\mathcal{P}_m(\mathbb{R}^n)$. For any polynomial f of degree m, it holds $M_{p,q}^{s,m}(f) = +\infty$, while $M_{p,q}^{s,m'}(f) = \|f\|_{\dot{B}^s_{p,q}} = 0$ for all $m' > m$.

Proposition 10 *For all $s > 0$ and $m \in \mathbb{N}$ such that $s < m$, there exist constants $c_1, c_2 > 0$ depending only on n, s, p, q and m, satisfying the following:*

(i) *For all regular tempered distribution f, such that $M_{p,q}^{s,m}(f) < +\infty$, it holds $[f]_\infty \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, and*

$$\|f\|_{\dot{B}^s_{p,q}} \leq c_1\, M_{p,q}^{s,m}(f). \tag{43}$$

(ii) *Conversely, for all $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, there exists a regular tempered distribution g satisfying $[g]_\infty = f$ and*

$$M_{p,q}^{s,m}(g) \leq c_2\, \|f\|_{\dot{B}^s_{p,q}}.$$

The distribution g can be chosen so that $g^{(\alpha)}$ vanishes at infinity for all $|\alpha| = \nu$.

Under condition (41), we can replace $M_{p,q}^{s,m}$ by $M_{p,q}^{s,m,u}$ in the preceding statements.

Proof *Step 1. Proof of statement (i) in case $q \geq 1$.*

Substep 1.1. We need some auxiliary measures and functions that we first introduce. Let ϕ be a function like in Proposition 4. We can divide it by a constant, in order that

$$\int_0^\infty \widehat{\phi}(t\xi)\frac{\mathrm{d}t}{t} = 1, \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{44}$$

Let μ be the compactly supported distribution defined by

$$\langle \mu, f\rangle := \int_{S^{n-1}} \Delta_y^m f(0)\,\mathrm{d}y,\ \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Here, we integrate on the unit sphere S^{n-1} of $\mathbb{R}^n$, endowed with its canonical measure. By defining μ_t according to (37), we obtain

$$\mu_t * f = \int_{S^{n-1}} \Delta_{ty}^m f\,\mathrm{d}y,\ \forall f \in \mathcal{S}'(\mathbb{R}^n), \tag{45}$$

where the integral exists in the $*$-weak sense in $\mathcal{S}'(\mathbb{R}^n)$. The Fourier transform of μ is a radial nonzero C^∞ function on $\mathbb{R}^n$. Let us choose the function ϕ such that $\mathrm{supp}\,\widehat{\phi}$ is a compact subset of $\{\xi\ :\ \widehat{\mu}(\xi) \neq 0\}$. Then, there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi}$ has compact support included in $\mathbb{R}^n \setminus \{0\}$, and

$$\widehat{\phi}(\xi) = \widehat{\psi}(\xi)\widehat{\mu}(\xi). \tag{46}$$

Substep 1.2. Let f be a regular tempered distribution such that $M_{p,q}^{s,m}(f) < +\infty$. Using polar coordinates, we find

$$\left(\int_0^\infty \frac{1}{t^{sq}} \int_{S^{n-1}} \|\Delta_{ty}^m f\|_p^q\,\mathrm{d}y\,\frac{\mathrm{d}t}{t}\right)^{1/q} = c\, M_{p,q}^{s,m}(f), \tag{47}$$

for some $c > 0$ depending only on n and q. By assumptions $p \geq 1$ and $q \geq 1$, and by (45), it holds

$$\|\mu_t * f\|_p \leq \int_{S^{n-1}} \|\Delta_{ty}^m f\|_p \, \mathrm{d}y \leq c \left(\int_{S^{n-1}} \|\Delta_{ty}^m f\|_p^q \, \mathrm{d}y \right)^{1/q}. \tag{48}$$

Combining (48) and (47), we obtain

$$\left(\int_0^\infty \left(\frac{1}{t^s} \|\psi_t * (\mu_t * f)\|_p \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \leq c \, \|\psi\|_1 \, M_{p,q}^{s,m}(f). \tag{49}$$

By the definition of ψ, there exist numbers $b > a > 0$ such that the Fourier transform of $\psi_t * (\mu_t * f)$ is supported into the annulus $at^{-1} \leq |\xi| \leq bt^{-1}$, for every $t > 0$. Then, using (49), together with a Nikol'skij representation argument, see [20, prop. 3.4], [23, prop. 2.3.2(1), p. 59], we deduce that

$$Uf := \int_0^\infty \psi_t * (\mu_t * f) \, \frac{\mathrm{d}t}{t} \tag{50}$$

exists in the $*$-weak sense in $\mathcal{S}'_\infty(\mathbb{R}^n)$, and that

$$\|Uf\|_{\dot{B}^s_{p,q}} \leq c \, M_{p,q}^{s,m}(f). \tag{51}$$

By computing the Fourier transform of Uf, and by using (46) and (44), we obtain $Uf = [f]_\infty$. By (51), we conclude that $[f]_\infty \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ with the estimate (43).

Step 2. Proof of statement (i) in case $0 < q < 1$.
We use the same ideas as in Step 1, with some technical modifications. We change first the definition of the measure μ by replacing m by $2m$. We will use the following identity for the difference operators:

$$\Delta_h^{2m} f(x) = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \Delta_h^m f(x + kh). \tag{52}$$

Let f be a regular tempered distribution such that $M_{p,q}^{s,m}(f) < +\infty$. Using (52), we obtain the following counterpart of (48):

$$\|\mu_t * f\|_p \leq c \left(\int_{S^{n-1}} \|\Delta_{ty}^m f\|_p^q \, \mathrm{d}y \right) \left(\sup_{|h| \leq t} \|\Delta_h^{2m} f\|_p^{1-q} \right).$$

By Hölder inequality with the exponents $1/q$ and $1/(1-q)$, and the identity (47), we deduce

$$\left(\int_0^\infty \left(\frac{1}{t^s}\,\|\psi_t*(\mu_t*f)\|_p\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

$$\leq c\,\|\psi\|_1 \left(\int_0^\infty \left(\frac{1}{t^{sq}}\int_{S^{n-1}} \|\Delta_{ty}^m f\|_p^q\,\mathrm{d}y\right)^q \left(\frac{1}{t^{sq}}\sup_{|h|\leq t}\|\Delta_h^{2m} f\|_p^q\right)^{1-q} \frac{\mathrm{d}t}{t}\right)^{1/q}$$

$$\leq c\left(M_{p,q}^{s,m}(f)\right)^q \left(\int_0^\infty \frac{1}{t^{sq}}\sup_{|h|\leq t}\|\Delta_h^{2m} f\|_p^q\,\frac{\mathrm{d}t}{t}\right)^{(1/q)-1}.$$

Then, we use the following estimate:

$$\left(\int_0^\infty \frac{1}{t^{sq}}\sup_{|h|\leq t}\|\Delta_h^{2m} f\|_p^q\,\frac{\mathrm{d}t}{t}\right)^{1/q} \leq c\,M_{p,q}^{s,m}(f),$$

proved by Triebel [25, proof of thm. 2.5.12/Step 3, p. 112]. We conclude that the estimate (49) still holds for $0<q<1$.

Step 3. Let f be a tempered distribution such that $[f]_\infty \in \dot{B}_{p,q}^s(\mathbb{R}^n)$.
It holds $\|Q_j f\|_p \leq 2^{-sj}\,\|f\|_{\dot{B}_{p,q}^s}$. By assumption $s>0$, by Proposition 7, and by the identity (36), it holds $f-S_0 f\in B_{p,q}^s(\mathbb{R}^n)$. By Lemma 3, we deduce

$$M_{p,q}^{s,m}(f-S_0 f)\leq c\,\|f\|_{\dot{B}_{p,q}^s}.$$

To deal with the remaining term $S_0 f$, we set $v_k := \sum_{j=-k}^{0} Q_j f$ for all $k\in\mathbb{N}_0$. Then, $v_k\in B_{p,q}^s(\mathbb{R}^n)$ for all k, and

$$\|v_k\|_{\dot{B}_{p,q}^s}\leq c\,\|f\|_{\dot{B}_{p,q}^s},\quad \forall k\geq 0.$$

Hence, by Lemma 3,

$$M_{p,q}^{s,m}(v_k)\leq c\,\|f\|_{\dot{B}_{p,q}^s},\quad \forall k\geq 0. \tag{53}$$

□

Moreover, we have the following:

Claim: *There exists a sequence $(R_k)_{k\geq 0}$ of polynomials of degree $<\nu$, such that the sequence $(v_k-R_k)_{k\geq 0}$ converges uniformly on every compact subset of $\mathbb{R}^n$.*

Proof of the Claim *Case $\nu=0$.* By Nikol'skij inequality, it holds

$$\|Q_j f\|_\infty \leq c\,2^{jn/p}\|Q_j f\|_p\leq c\,2^{j((n/p)-s)}\|f\|_{\dot{B}_{p,q}^s}$$

with c independent of f and j, see, e.g., [25, rem. 1.3.2/1]. Hence, the series $\sum_{j\leq 0} Q_j f$ converges uniformly on $\mathbb{R}^n$ in the following two cases: (i) $s<n/p$, and (ii) $s=n/p$ and $q\leq 1$.

Case $\nu > 0$. If we set

$$r_j(x) := \sum_{|\alpha|<\nu} (Q_j f)^{(\alpha)}(0)\frac{x^\alpha}{\alpha!},$$

then the series $\sum_{j\le 0}(Q_j f - r_j)$ converges uniformly on every compact subset of $\mathbb{R}^n$, see [10, prop. 4.8, rem. 4.9]. Then, we define $R_k := \sum_{j=-k}^{0} r_j$.
Let v be the limit of the sequence $(v_k - R_k)_{k\ge 0}$. Since $m > s \ge \nu$, $\Delta_h^m(v_k) = \Delta_h^m(v_k - R_k)$ converges pointwise to $\Delta_h^m v$ for all h. Now, applying twice the Fatou lemma in estimate (53), we obtain

$$M_{p,q}^{s,m}(v) \le c\,\|f\|_{\dot{B}^s_{p,q}}.$$

By setting $g := v + (f - S_0 f)$, we obtain a function such that

$$[g]_\infty = [f]_\infty \quad \text{and} \quad M_{p,q}^{s,m}(g) \le c\,\|f\|_{\dot{B}^s_{p,q}}.$$

By definition of g, it holds $[g]_\nu = \sigma_\nu([f]_\infty)$. By Proposition 5, it follows that $g^{(\alpha)}$ vanishes at infinity for all $|\alpha| = \nu$.
Step 4. Now, we justify the replacement of $M_{p,q}^{s,m}$ by $M_{p,q}^{s,m,u}$ under assumption (41).
Substep 4.1. Let f be a regular tempered distribution such that $M_{p,q}^{s,m,u}(f) < +\infty$. Noticing that $u \mapsto M_{p,q}^{s,m,u}(f)$ is an increasing function on $[1, +\infty[$, we may consider only the case $u = 1$. We modify the notation of Step 1, by setting

$$\langle \mu, f\rangle := \int_{|h|\le 1} \Delta_h^m f(0)\,\mathrm{d}h,\ \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Here, we integrate on the unit ball of $\mathbb{R}^n$. We define also μ_t according to (37). It holds

$$\mu_t * f = \int_{|h|\le 1} \Delta_{th}^m f\,\mathrm{d}h,\ \forall f \in \mathcal{S}'(\mathbb{R}^n).$$

Then, the estimate (49) holds, with $M_{p,q}^{s,m}$ replaced by $M_{p,q}^{s,m,1}$ in the right-hand side. The remaining of Step 1 is unchanged, with $q \in]0, +\infty]$ instead of $q \ge 1$.
Substep 4.2. Clearly, $M_{p,q}^{s,m}$ can be replaced by $M_{p,q}^{s,m,u}$ in all Step 3.

□

4.6 The modified Besov space $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$

Some of the notions and results of this subsection are taken from the paper of Moussai [20, sect. 3]. In his paper, Moussai deals only with $q \ge 1$, but his results extend without difficulty to any $q > 0$.

4.6.1 Definition and main properties

The modified Besov space has been defined in (4). Its main properties are the following.

Proposition 11 *For $s > 0$, $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach algebra for the pointwise product. Moreover, it holds*

$$\|fg\|_{\mathcal{B}^s_{p,q}} \leq c\Big(\|f\|_\infty \|g\|_{\mathcal{B}^s_{p,q}} + \|g\|_\infty \|f\|_{\mathcal{B}^s_{p,q}}\Big) \tag{54}$$

for all f, g in $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$.

Proof See Moussai [20, thm. 3.26]. The precise estimate (54) occurs in the proof given in [20]. □

Proposition 12 *For $s > 1 + (n/p)$, it holds*

$$\mathcal{B}^s_{p,q}(\mathbb{R}^n) = \{f \in L_\infty(\mathbb{R}^n) : \partial_\ell f \in \mathcal{B}^{s-1}_{p,q}(\mathbb{R}^n), \ell = 1, \dots, n\}$$

and $\|f\|_\infty + \sum_{\ell=1}^n \|\partial_\ell f\|_{\mathcal{B}^{s-1}_{p,q}}$ is an equivalent quasi-norm in $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$.

Proof See [20, prop. 3.21]. □

We will also use the following embeddings, where we limit ourselves to the case $n = 1$.

(i) For $s > 1/p$, it holds

$$B^s_{p,q}(\mathbb{R}) \hookrightarrow \mathcal{B}^s_{p,q}(\mathbb{R}). \tag{55}$$

(ii) Let $1 \leq p < v \leq \infty$. Then, it holds

$$\mathcal{B}^s_{p,q}(\mathbb{R}) \hookrightarrow \mathcal{B}^{s-\frac{1}{p}+\frac{1}{v}}_{v,q}(\mathbb{R}). \tag{56}$$

Since $\mathcal{B}^s_{\infty,q}(\mathbb{R}) = B^s_{\infty,q}(\mathbb{R})$ for all $s > 0$, it follows that

$$\mathcal{B}^s_{p,q}(\mathbb{R}) \hookrightarrow B^{s-\frac{1}{p}}_{\infty,q}(\mathbb{R}) \quad \text{for} \quad s > 1/p. \tag{57}$$

(iii) Let $s > 1,\ 1 \leq p < \infty,\ 0 < q \leq \infty$. Let v be a real number s.t.

$$\max\left(1 + \frac{1}{p} - s, 0\right) < \frac{1}{v} < \frac{1}{sp}. \tag{58}$$

Then, there exists a constant c s.t.

$$\|f'\|_v \leq c\,\|f\|_{\mathcal{B}^s_{p,q}} \tag{59}$$

holds for all $f \in \mathcal{B}^s_{p,q}(\mathbb{R})$.

Property (ii) is a direct consequence of the Sobolev embedding for homogeneous Besov spaces, see Proposition 6. The proof of (iii) is a bit more complicated, we refer to [20, prop. 3.23].

4.6.2 Characterization by differences

Proposition 13 *Let $s > 0$ and $m \in \mathbb{N}$ be such that $s < m$. Then, a regular tempered distribution f belongs to $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ if and only if*

$$\|f\|_\infty + M^{s,m}_{p,q}(f) < +\infty. \tag{60}$$

Moreover, the above expression is an equivalent quasi-norm on $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$. The same result holds with $M^{s,m}_{p,q}$ replaced by $M^{s,m,u}_{p,q}$ under condition (41).

Proof Under condition (60), the first assertion of Proposition 10 yields

$$\|f\|_\infty + \|f\|_{\dot{B}^s_{p,q}} \le c\big(\|f\|_\infty + M^{s,m}_{p,q}(f)\big).$$

Assume conversely that $f \in L_\infty$ and $[f]_\infty \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. By Proposition 10, there exists a regular distribution g satisfying the following conditions:

- $f - g$ is a polynomial,
- $M^{s,m}_{p,q}(g) \le c\,\|f\|_{\dot{B}^s_{p,q}}$,
- $g^{(\alpha)}$ vanishes at infinity for all $|\alpha| = \nu$.

Let α a multi-index such that $|\alpha| = m$. By assumptions $f \in L_\infty(\mathbb{R}^n)$ and $\nu \le m$, we deduce that $(f - g)^{(\alpha)}$ vanishes at infinity. By Lemma 2, it follows that $f - g \in \mathcal{P}_m(\mathbb{R}^n)$. Hence, $M^{s,m}_{p,q}(g) = M^{s,m}_{p,q}(f)$. This ends up the proof. □

Proposition 14 *Let $s > 1$ and $m \in \mathbb{N}$ be such that $s - 1 < m$. It holds*

$$M^{s-1,m}_{p,q}(\partial_\ell f) \le c\,\|f\|_{\dot{B}^s_{p,q}}$$

for all $f \in \mathcal{B}^s_{p,q}(\mathbb{R}^n)$ and all $\ell = 1, \ldots, n$. The same result holds with $M^{s-1,m}_{p,q}$ replaced by $M^{s-1,m,u}_{p,q}$ under condition $s - 1 > n\left(p^{-1} - u^{-1}\right)$.

Proof The statement is a consequence of Propositions 8 and 10. The argument is similar to that used in the preceding proof. □

4.7 Besov spaces on bounded domains

The most convenient way to introduce Besov spaces on domains is to consider them as quotient spaces, see [25–27]. We use the classical notion of restriction of a distribution on $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

Definition 4 Let $1 \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

(i) Then, we define $B^s_{p,q}(\Omega)$ to be the collection of all the restrictions to Ω of elements of $B^s_{p,q}(\mathbb{R}^n)$. For $f \in B^s_{p,q}(\Omega)$, we define $\|\, f\,\|_{B^s_{p,q}(\Omega)}$ as the infimum of $\|\, g\,\|_{B^s_{p,q}(\mathbb{R}^n)}$, for all g s.t. f is the restriction of g to Ω.

(ii) By $B^s_{p,q} \cap \dot{W}^1_\infty(\Omega)$, we mean the collection of all Lipschitz continuous functions f on Ω which are restrictions to Ω of elements of $B^s_{p,q} \cap \dot{W}^1_\infty(\mathbb{R}^n)$.

The advantage of such a definition is obvious: several facts immediately carry over from the spaces defined on $\mathbb{R}^n$ to the spaces defined on Ω. The disadvantage is also clear. We do not have intrinsic characterizations. One of the assertions which carry over is the following, see [25, thm. 2.3.8].

Lemma 4 *Let Ω be an open set in $\mathbb{R}^n$. Let $1 \le p \le \infty$ and $0 < m < s$ for some integer m. Then, $f \in L_p(\Omega)$ belongs to $B^s_{p,q}(\Omega)$ if, and only if, $D^\alpha f \in B^{s-m}_{p,q}(\Omega)$ for all α, $|\alpha| = m$.*

Under certain restrictions on the quality of the domain Ω intrinsic characterizations are known, we refer to Dispa [17] and Triebel [27, thm. 1.118].

Proposition 15 *Let* $1 \le p \le \infty$ *and let* s *be a positive real number, but not an integer. Let* m *be an integer s.t.* $0 < s - m < 1$. *Furthermore, let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$. *A real-valued function* f *belongs to* $B^s_{p,p}(\Omega)$ *if, and only if,* $f \in W^m_p(\Omega)$ *and*

$$\| f \|_{W^m_p(\Omega)} + \sum_{|\alpha|=m} \left(\int_\Omega \int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{(s-m)p+n}} \, dx \, dy \right)^{1/p} < +\infty . \tag{61}$$

Moreover, the above expression generates an equivalent norm on $B^s_{p,p}(\Omega)$.

Proof In the above-mentioned references, one only can find the case $m = 0$. However, by means of Lemma 4, one can extend this to all natural numbers m, $m < s$. □

Remark 5 The spaces with the norm defined by the expression in (61) are usually called Slobodeckij spaces.

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