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Wolff–Denjoy theorems in nonsmooth convex domains

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Abstract We give a short proof of Wolff–Denjoy theorem for (not necessarily smooth) strictly convex domains. With similar techniques we are also able to prove a Wolff–Denjoy theorem for weakly convex domains, again without any smoothness assumption on the boundary.

Keywords Wolff–Denjoy theorem · Convex domains · Holomorphic dynamics

Mathematics Subject Classification (2000) 32H50 · 37F99

1 Introduction

Studying the dynamics of a holomorphic self-map $f: \Delta \to \Delta$ of the unit disk $\Delta \subset \mathbb{C}$ one is naturally led to consider two different cases. If f has a fixed point, then Schwarz's lemma readily implies that either f is an elliptic automorphism, or the sequence $\{f^k\}$ of iterates of f converges (uniformly on compact sets) to the fixed point. The classical Wolff–Denjoy theorem [23,43] says what happens when f has no fixed points:

Theorem 1 (Wolff–Denjoy) *Let $f: \Delta \to \Delta$ be a holomorphic self-map without fixed points. Then there exists a point $\tau \in \partial\Delta$ such that the sequence $\{f^k\}$ of iterates of f converges (uniformly on compact sets) to the constant map τ.*

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Since its discovery, a lot of work has been devoted to obtain similar statements in more general situations (surveys covering different aspects of this topic are [3,25,39]). In one complex variable, there are results in multiply connected domains, multiply and infinitely connected Riemann surfaces, and even in the settings of one-parameter semigroups and of random dynamical systems (see, e.g., [10,28,37]). In several complex variables, the first Wolff–Denjoy theorems are due to Hervé [29,30]; in particular, in [30] he proved a statement identical to the one above for fixed point free self-maps of the unit ball $B^n \subset \mathbb{C}^n$. Hervé's theorem has also been generalized in various ways to open unit balls of complex Hilbert and Banach spaces (see, e.g., [21,41] and references therein).

A breakthrough occurred in 1988, when the first author (see [1]) showed how to prove a Wolff–Denjoy theorem for holomorphic self-maps of smoothly bounded strongly convex domains in $\mathbb{C}^n$. The techniques introduced there turned out to be quite effective in other contexts too (see, e.g., [5,7,12–14]); but in particular they led to Wolff–Denjoy theorems in smooth strongly pseudoconvex domains and smooth domains of finite type (see, e.g., [4,31,40]).

Two natural questions were left open by the previous results: how much does the boundary smoothness matter? And, what happens in weakly (pseudo)convex domains? As already shown by the results obtained by Hervé [29] in the bidisk, if we drop both boundary smoothness and strong convexity the situation becomes much more complicated; but most of Hervé's techniques were specific for the bidisk, and so not necessarily applicable to more general domains. On the other hand, for smooth weakly convex domains a Wolff–Denjoy theorem was obtained in [3] (but here we shall get a better result; see Corollary 3).

In 2012, Budzyńska [18] (see also [20] and [19] for infinite dimensional generalizations) proved a Wolff–Denjoy theorem for holomorphic fixed point free self-maps of a bounded strictly convex domain in $\mathbb{C}^n$, under no smoothness assumption on the boundary; but she did not deal with weakly convex domains.

In Sect. 3 of this paper (Sect. 2 is devoted to recalling a few known preliminary facts), using only tools already introduced in [1] and no additional machinery, we shall give a simpler proof of Budzyńska's result, that is, we shall prove

Theorem 2 *Let $D \subset\subset \mathbb{C}^n$ be a bounded strictly convex domain, and $f\colon D \to D$ a k_D-nonexpansive (e.g., holomorphic) self-map without fixed points. Then there exists a $x_0 \in \partial D$ such that the sequence of iterates $\{f^k\}$ converges to the constant map x_0.*

It is worth mentioning that the final proof is simpler than the proof presented in [1] for the smooth case.

In Sect. 4 we shall furthermore show how, combining our ideas with Budzyńska's new tools, one can obtain a Wolff–Denjoy theorem for weakly convex domains with no smoothness assumptions, thus addressing the second natural question mentioned above. In particular, we shall prove the following result [see Sect. 3 for the definitions of the "convex hulls" $\mathrm{ch}(E)$ and $\mathrm{Ch}(E)$ of a subset $E \subseteq \partial D$ of the boundary of a convex domain D, and Sect. 4 for the definitions of horosphere sequences and $G_z(x, 1, \mathbf{x})$]:

Theorem 3 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $f\colon D \to D$ a k_D-nonexpansive (respectively, holomorphic) self-map without fixed points. Then there exist $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{z\in D} \mathrm{ch}\big(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D\big) = \bigcap_{R>0} \mathrm{ch}\big(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D\big)$$

if f is k_D-nonexpansive, or

$$T(f) \subseteq \bigcap_{z\in D} \mathrm{Ch}\big(\overline{G_z(x,1,\mathbf{x})} \cap \partial D\big) = \bigcap_{R>0} \mathrm{Ch}\big(\overline{G_{z_0}(x,R,\mathbf{x})} \cap \partial D\big)$$

if f is holomorphic, where $T(f)$ is the union of the images of limit points of the sequence of iterates of f.

Finally, in Sect. 5 we shall specialize our results to the polydisk, and we shall see that Hervé's results imply that our statements are essentially optimal.

2 Preliminaries

In this section we shall collect a few more or less known facts on bounded convex domains in $\mathbb{C}^n$.

2.1 Euclidean geometry

Let us begin by recalling a few standard definitions and notations.

Definition 1 Given $x, y \in \mathbb{C}^n$ let

$$[x,y] = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in [0,1]\} \text{ and } (x,y) = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in (0,1)\}$$

denote the *closed*, respectively, *open*, *segment* connecting x and y. A set $D \subseteq \mathbb{C}^n$ is *convex* if $[x,y] \subseteq D$ for all $x, y \in D$; and *strictly convex* if $(x,y) \subseteq D$ for all $x, y \in \overline{D}$.

An easy but useful observation (whose elementary proof is left to the reader) is as follows:

Lemma 1 *Let $D \subset \mathbb{C}^n$ be a convex domain. Then:*

(i) $(z,w) \subset D$ for all $z \in D$ and $w \in \partial D$;
(ii) if $x, y \in \partial D$ then either $(x,y) \subset \partial D$ or $(x,y) \subset D$.

This suggests the following

Definition 2 Let $D \subset \mathbb{C}^n$ be a convex domain. Given $x \in \partial D$, we put

$$\mathrm{ch}(x) = \{y \in \partial D \mid [x,y] \subset \partial D\}\ ;$$

we shall say that x is a *strictly convex point* if $\mathrm{ch}(x) = \{x\}$. More generally, given $F \subseteq \partial D$ we put

$$\mathrm{ch}(F) = \bigcup_{x\in F} \mathrm{ch}(x)\ .$$

A similar construction having a more holomorphic character is the following:

Definition 3 Let $D \subset \mathbb{C}^n$ be a convex domain. A *complex supporting functional* at $x \in \partial D$ is a $\mathbb{C}$-linear map $\sigma\colon \mathbb{C}^n \to \mathbb{C}$ such that $\mathrm{Re}\,\sigma(z) < \mathrm{Re}\,\sigma(x)$ for all $z \in D$. A *complex supporting hyperplane* at $x \in \partial D$ is an affine complex hyperplane $L \subset \mathbb{C}^n$ of the form $L = x + \ker\sigma$, where σ is a complex supporting functional at x (the existence of complex supporting functionals and hyperplanes is guaranteed by the Hahn–Banach theorem). Given $x \in \partial D$, we shall denote by $\mathrm{Ch}(x)$ the intersection of $\overline{D}$ with of all complex supporting hyperplanes at x. Clearly, $\mathrm{Ch}(x)$ is a closed convex set containing x; in particular, $\mathrm{Ch}(x) \subseteq \mathrm{ch}(x)$. If $\mathrm{Ch}(x) = \{x\}$ we say that x is a *strictly* $\mathbb{C}$*-linearly convex point;* and we say that D is *strictly* $\mathbb{C}$*-linearly convex* if all points of ∂D are strictly $\mathbb{C}$-linearly convex. Finally, if $F \subset \partial D$ we set

$$\mathrm{Ch}(F) = \bigcup_{x \in F} \mathrm{Ch}(x) ;$$

clearly, $\mathrm{Ch}(F) \subseteq \mathrm{ch}(F)$.

Remark 1 If ∂D is of class C^1 then for each $x \in \partial D$ there exists a unique complex supporting hyperplane at x, and thus $\mathrm{Ch}(x)$ coincides with the intersection of this complex supporting hyperplane with ∂D. In particular, $\mathrm{Ch}(x)$ is smaller than the flat region introduced in [3, p. 277] as the intersection of ∂D with the real supporting hyperplane. But nonsmooth points can have more than one complex supporting hyperplanes; this happens for instance in the polydisk (see Sect. 5).

2.2 Intrinsic geometry

The intrinsic (complex) geometry of convex domains is conveniently described using the (intrinsic) Kobayashi distance. We refer to [3,32] and [35] for details and much more on the Kobayashi (pseudo)distance in complex manifolds; here we shall just recall what is needed for our aims. Let k_Δ denote the Poincaré distance on the unit disk $\Delta \subset \mathbb{C}$. If X is a complex manifold, the *Lempert function* $\delta_X : X \times X \to \mathbb{R}^+$ of X is

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) \mid \exists \phi : \Delta \to X \text{ holomorphic with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\}$$

for all $z, w \in X$. In general, the *Kobayashi pseudodistance* $k_X : X \times X \to \mathbb{R}^+$ of X is the largest pseudodistance on X bounded above by δ_X; when $D \subset\subset \mathbb{C}^n$ is a bounded convex domain in $\mathbb{C}^n$, Lempert [38] has proved that δ_D is an actual distance, and thus it coincides with the Kobayashi distance k_D of D.

The main property of the Kobayashi (pseudo)distance is that it is contracted by holomorphic maps: if $f : X \to Y$ is a holomorphic map, then

$$k_Y\big(f(z), f(w)\big) \leq k_X(z, w)$$

for all $z, w \in X$. In particular, biholomorphisms are isometries, and holomorphic self-maps are k_X-nonexpansive.

The Kobayashi distance of convex domains enjoys several interesting properties. For instance, it coincides with the Carathéodory distance, and it is a complete distance (see, e.g., [3] or [38]); in particular, k_D-bounded subsets of D are relatively compact in D. We shall also need the following estimates:

Lemma 2 [33,36,38] *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Then:*

(i) if $z_1, z_2, w_1, w_2 \in D$ and $s \in [0, 1]$ then

$$k_D\big(sz_1 + (1 - s)w_1, sz_2 + (1 - s)w_2\big) \leq \max\big\{k_D(z_1, z_2), k_D(w_1, w_2)\big\} ;$$

(ii) if $z, w \in D$ and $s, t \in [0, 1]$ then

$$k_D\big(sz + (1 - s)w, tz + (1 - t)w\big) \leq k_D(z, w) .$$

As a consequence we have:

Lemma 3 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, $x, y \in \partial D$, and let $\{z_\nu\}, \{w_\nu\} \subset D$ be two sequences converging to x and y, respectively. If*

$$\sup_{\nu\in\mathbb{N}} k_D(z_\nu, w_\nu) = c < +\infty$$

then $[x, y] \subset \partial D$. In particular, if x (or y) is a strictly convex point then $x = y$.

Proof By Lemma 1 we know that either $(x, y) \subset D$, or $(x, y) \subset \partial D$. Assume by contradiction that $(x, y) \subset D$. Lemma 2 yields

$$k_D\big(sz_\nu + (1-s)w_\nu, tz_\nu + (1-t)w_\nu\big) \le k_D(z_\nu, w_\nu) \le c$$

for each $\nu \in \mathbb{N}$ and for all $s, t \in (0, 1)$. Hence

$$k_D\big(sx + (1-s)y, tx + (1-t)y\big) = \lim_{\nu\to\infty} k_D\big(sz_\nu + (1-s)w_\nu, tz_\nu + (1-t)w_\nu\big) \le c$$

for all $s, t \in (0, 1)$. But this implies that (x, y) is relatively compact in D, which is impossible because $x, y \in \partial D$. □

2.3 Dynamics

In this subsection we recall a few known facts about the dynamics of holomorphic (or more generally k_D-nonexpansive) self-maps of convex domains.

When $D \subset\subset \mathbb{C}^n$ is a bounded domain, the space $\mathrm{Hol}(D, D)$ is, by Montel's theorem, relatively compact in $\mathrm{Hol}(D, \mathbb{C}^n)$. In particular, if $f \in \mathrm{Hol}(D, D)$, then every sequence $\{f^{k_j}\}$ of iterates contains a subsequence converging to a holomorphic map $h \in \mathrm{Hol}(D, \mathbb{C}^n)$. Analogously, using this time Ascoli-Arzelà theorem, if $f: D \to D$ is k_D-nonexpansive, then every sequence $\{f^{k_j}\}$ of iterates contains a subsequence converging to a continuous map $h: D \to \overline{D} \subset \mathbb{C}^n$.

Definition 4 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $f: D \to D$ a holomorphic or k_D-nonexpansive self-map. A map $h: D \to \mathbb{C}^n$ is a *limit point* of the sequence $\{f^k\}$ of iterates of f if there is a subsequence $\{f^{k_j}\}$ of iterates converging (uniformly on compact subsets) to h; we shall denote by $\Gamma(f)$ the set of all limit points of $\{f^k\}$. The *target set* $T(f)$ of f is then defined as the union of the images of limit points of the sequence of iterates:

$$T(f) = \bigcup_{h\in\Gamma(f)} h(D)\ .$$

Definition 5 A sequence $\{f_k\} \subset C(X, Y)$ of continuous maps between topological spaces is *compactly divergent* if for each pair of compact subsets $H \subseteq X$ and $K \subseteq Y$ there is $k_0 \in \mathbb{N}$ such that $f^k(H) \cap K = \emptyset$ for all $k \ge k_0$.

When D is a convex domain, the target set either is contained in D if f has a fixed point or is contained in ∂D if f has no fixed points. More precisely, we have the following statement (see [1,4,17,22,33,36]):

Theorem 4 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $f: D \to D$ a k_D-nonexpansive (e.g., holomorphic) self-map. Then the following assertions are equivalent:*

(i) f has a fixed point in D;
(ii) the sequence $\{f^k\}$ is not compactly divergent;
(iii) the sequence $\{f^k\}$ has no compactly divergent subsequences;

(iv) $\{f^k(z)\}$ *is relatively compact in* D *for all* $z \in D$*;*
(v) *there exists* $z_0 \in D$ *such that* $\{f^k(z_0)\}$ *is relatively compact in* D*;*
(vi) *there exists* $z_0 \in D$ *such that* $\{f^k(z_0)\}$ *admits a subsequence relatively compact in* D.

Remark 2 For more general taut domains (and f holomorphic) the statements (ii)–(vi) are still equivalent. For some classes of domains, these statements are equivalent to f having a periodic point (see [4,31]); however, there exist holomorphic self-maps of a taut topologically contractible smooth domain satisfying (ii)–(vi) but without fixed points (see [6]).

When the sequence of iterates of f is not compactly divergent (e.g., when f has a fixed point if D is convex), the target set of f has already been characterized [1,5,11]. In particular, using Theorem 4 and repeating word by word the proof of [3, Theorem 2.1.29] we obtain

Theorem 5 *Let* $D \subset\subset \mathbb{C}^n$ *be a bounded convex domain, and* $f: D \to D$ *a* k_D*-nonexpansive (e.g., holomorphic) self-map of* D*. Assume that* f *has a fixed point in* D*. Then* $T(f)$ *is a* k_D*-nonexpansive (respectively, holomorphic) retract of* D*. More precisely, there exists a unique* k_D*-nonexpansive (respectively, holomorphic) retraction* $\rho: D \to T(f)$ *which is a limit point of* $\{f^k\}$*, such that every limit point of* $\{f^k\}$ *is of the form* $\gamma \circ \rho$*, where* $\gamma: T(f) \to T(f)$ *is a (biholomorphic) invertible* k_D*-isometry, and* $f|_{T(f)}$ *is a (biholomorphic) invertible* k_D*-isometry.*

In this paper we instead want to describe the target set of fixed points free self-maps of bounded convex domains.

3 Strictly convex domains

Since Wolff's proof of the Wolff–Denjoy theorem [43], horospheres have been the main tool needed for the study of the dynamics of fixed point free holomorphic self-maps. Let us recall the general definitions introduced in [1,3].

Definition 6 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, $z_0 \in D$, $x \in \partial D$ and $R > 0$. The *small horosphere* $E_{z_0}(x, R)$ and the *big horosphere* $F_{z_0}(x, R)$ of *center* x, *pole* z_0 and *radius* R are defined by

$$E_{z_0}(x, R) = \left\{ z \in D \;\middle|\; \limsup_{w \to x} \left[k_D(z, w) - k_D(z_0, w) \right] < \tfrac{1}{2} \log R \right\},$$

$$F_{z_0}(x, R) = \left\{ z \in D \;\middle|\; \liminf_{w \to x} \left[k_D(z, w) - k_D(z_0, w) \right] < \tfrac{1}{2} \log R \right\}.$$

The following lemma contains some basic properties of horospheres, immediate consequence of the definition and of Lemma 2:

Lemma 4 *Let* $D \subset\subset \mathbb{C}^n$ *be a bounded domain,* $z_0 \in D$ *and* $x \in \partial D$*. Then:*

(i) $E_{z_0}(x, R) \subseteq F_{z_0}(x, R)$ *for every* $R > 0$*;*
(ii) $\overline{E_{z_0}(x, R_1)} \cap D \subseteq E_{z_0}(x, R_2)$ *and* $\overline{F_{z_0}(x, R_1)} \cap D \subseteq F_{z_0}(x, R_2)$ *for every* $0 < R_1 < R_2$*;*
(iii) $B_D\left(z_0, \frac{1}{2} \log R\right) \subseteq E_{z_0}(x, R)$ *for all* $R > 1$*, where* $B_D(z_0, r)$ *denotes the Kobayashi ball of center* z_0 *and radius* r*;*
(iv) $F_{z_0}(x, R) \cap B_D\left(z_0, -\frac{1}{2} \log R\right) = \emptyset$ *for all* $0 < R < 1$*;*
(v) $\bigcup_{R>0} E_{z_0}(x, R) = \bigcup_{R>0} F_{z_0}(x, R) = D$ *and* $\bigcap_{R>0} E_{z_0}(x, R) = \bigcap_{R>0} F_{z_0}(x, R) = \emptyset$*;*
(vi) *if moreover* D *is convex then* $E_{z_0}(x, R)$ *is convex for every* $R > 0$.

Big horospheres are not always convex, even if D is convex; an example is given by the horospheres in the polydisk (see Sect. 5). However, the first important new result of this paper is that big horospheres in convex domains are always star-shaped with respect to the center:

Lemma 5 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, $z_0 \in D$, $R > 0$ and $x \in \partial D$. Then we have $[x, z] \subset \overline{F_{z_0}(x, R)}$ for all $z \in \overline{F_{z_0}(x, R)}$. In particular, x always belongs to $\overline{F_{z_0}(x, R)}$.*

Proof Given $z \in F_{z_0}(x, R)$, choose a sequence $\{x_\nu\} \subset D$ converging to x and such that the limit of $k_D(z, x_\nu) - k_D(z_0, x_\nu)$ exists and is less than $\frac{1}{2} \log R$. Given $0 < s < 1$, let $h^s_\nu \colon D \to D$ be defined by

$$h^s_\nu(w) = sw + (1 - s)x_\nu$$

for every $w \in D$; then $h^s_\nu(x_\nu) = x_\nu$, and moreover

$$k_D\big(h^s_\nu(z_1), h^s_\nu(z_2)\big) \le k_D(z_1, z_2)$$

for every $z_1, z_2 \in D$, because h^s_ν is a holomorphic self-map of D. In particular,

$$\limsup_{\nu\to+\infty}\big[k_D\big(h^s_\nu(z), x_\nu\big) - k_D(z_0, x_\nu)\big] \le \lim_{\nu\to+\infty}\big[k_D(z, x_\nu) - k_D(z_0, x_\nu)\big] < \tfrac{1}{2}\log R\ .$$

Furthermore we have

$$\big|k_D\big(sz + (1 - s)x, x_\nu\big) - k_D\big(h^s_\nu(z), x_\nu\big)\big| \le k_D\big(sz + (1 - s)x_\nu, sz + (1 - s)x\big) \to 0$$

as $\nu \to +\infty$. Therefore

$$\begin{aligned}
&\liminf_{w\to x}\big[k_D\big(sz + (1 - s)x, w\big) - k_D(z_0, w)\big]\\
&\quad\le \limsup_{\nu\to+\infty}\big[k_D\big(sz + (1 - s)x, x_\nu\big) - k_D(z_0, x_\nu)\big]\\
&\quad\le \limsup_{\nu\to+\infty}\big[k_D\big(h^s_\nu(z), x_\nu\big) - k_D(z_0, x_\nu)\big]\\
&\qquad+ \lim_{\nu\to+\infty}\big[k_D\big(sz + (1 - s)x, x_\nu\big) - k_D\big(h^s_\nu(z), x_\nu\big)\big]\\
&\quad< \tfrac{1}{2}\log R\ ,
\end{aligned}$$

and thus $sz + (1 - s)x \in F_{z_0}(x, R)$. Letting $s \to 1$ we get $x \in \overline{F_{z_0}(x, R)}$, and we have proved the assertion for $z \in F_{z_0}(x, R)$. If $z \in \partial F_{z_0}(x, R)$, it suffices to apply the statement to a sequence in $F_{z_0}(x, R)$ approaching z. □

One of the main points in the proof given in [1] of the Wolff–Denjoy theorem for strongly convex C^2 domains is the fact that in such domains the intersection between the closure of a big horosphere and the boundary of the domain reduces to the center of the horosphere. The following corollary will play the same rôle for not necessarily smooth convex domains:

Corollary 1 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, $z_0 \in D$, and $x \in \partial D$. Then*

$$\bigcap_{R>0} \overline{F_{z_0}(x, R)} \subseteq \mathrm{ch}(x)\ . \tag{1}$$

In particular, if x is a strictly convex point then $\bigcap_{R>0} \overline{F_{z_0}(x, R)} = \{x\}$.

Proof First of all, Lemma 5 implies that the intersection in (1) is not empty, and so, by Lemma 4, it is contained in ∂D. Take $\tilde{x} \in \bigcap_{R>0} \overline{F_{z_0}(x, R)}$ different from x. Then Lemma 5 implies that the whole segment $[x, \tilde{x}]$ is contained in the intersection, and thus in ∂D; hence $\tilde{x} \in \mathrm{ch}(x)$, and we are done. □

It is not known whether the equality in (1) always hold; in strictly convex domains and in the polydisks (see Sect. 5) it does.

Let us turn now to the study of the target set. A first step in this direction is the following:

Proposition 1 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Then:*

(i) for every connected complex manifold X and every holomorphic map $h \colon X \to \mathbb{C}^n$ such that $h(X) \subset \overline{D}$ and $h(X) \cap \partial D \neq \emptyset$ we have

$$h(X) \subseteq \bigcap_{x \in X} \mathrm{Ch}\big(h(x)\big) \subseteq \partial D\ .$$

In particular, if h is a limit point of the sequence of iterates of a holomorphic self-map of D without fixed points we have

$$h(D) \subseteq \bigcap_{z \in D} \mathrm{Ch}\big(h(z)\big)\ .$$

(ii) Let $f \colon D \to D$ be a k_D-nonexpansive self-map without fixed points, and $h \colon D \to \mathbb{C}^n$ a limit point of $\{f^k\}$. Then

$$h(D) \subseteq \bigcap_{z \in D} \mathrm{ch}\big(h(z)\big)\ .$$

Proof (i) The fact that $h(X) \subseteq \partial D$ is an immediate consequence of the maximum principle (see, e.g., [8, Lemma 2.1]).

Let now $L = h(x_0) + \ker\sigma$ be a complex supporting hyperplane at $h(x_0)$. Then $\mathrm{Re}(\sigma \circ h) \leq \mathrm{Re}\,\sigma\big(h(x_0)\big)$ on X; therefore, by the maximum principle, $\sigma \circ h \equiv \sigma\big(h(x_0)\big)$, that is, $h(X) \subset L$. Since this holds for all complex supporting hyperplanes at $h(x_0)$ the assertion follows.

(ii) Let $\{f^{k_j}\}$ be a subsequence of iterates converging to h. Since f has no fixed points, we know by Theorem 4 that $h(D) \subseteq \partial D$. Furthermore

$$k_D\big(f^{k_j}(z), f^{k_j}(w)\big) \leq k_D(z, w) < +\infty$$

for every $z, w \in D$; therefore Lemma 3 implies $[h(z), h(w)] \subset \partial D$, and the assertion follows. □

The disadvantage of these statements is that the right-hand side still depends on the given limit point of the sequence of iterates; instead we would like to determine a subset of the boundary containing the whole target set. This can be accomplished as follows:

Lemma 6 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $f \colon D \to D$ a k_D-nonexpansive (respectively, holomorphic) self-map without fixed points. Assume there exist $\emptyset \neq E \subseteq F \subset D$ such that $f^k(E) \subset F$ for all $k \in \mathbb{N}$. Then we have*

$$T(f) \subseteq \mathrm{ch}\big(\overline{F} \cap \partial D\big)$$

if f is k_D-nonexpansive, or

$$T(f) \subseteq \mathrm{Ch}\big(\overline{F} \cap \partial D\big)$$

if f is holomorphic.

Proof Let h be a limit point of the sequence of iterates of f. Since f has no fixed points, we know that $h(D) \subseteq \partial D$. Take $z_0 \in E$; by assumption, the whole orbit of z_0 is contained in F. Therefore $h(z_0) \in \overline{F} \cap \partial D$, and the assertion follows from Proposition 1. □

The Wolff lemma [1, Theorem 2.3], whose proof can easily be adapted to the case of k_D-nonexpansive maps, is exactly what we need to apply Lemma 6:

Lemma 7 *Let $D \subset\subset \mathbb{C}^n$ be a convex domain, and let $f: D \to D$ be k_D-nonexpansive and without fixed points. Then there exists $x \in \partial D$ such that for every $z_0 \in D$, $R > 0$ and $k \in \mathbb{N}$ we have*

$$f^k\big(E_{z_0}(x, R)\big) \subseteq F_{z_0}(x, R) .$$

We can now give a proof of the Wolff–Denjoy Theorem 2 for k_D-nonexpansive self-maps of strictly convex domains in the same spirit as the proof given in [1] of the Wolff–Denjoy theorem for holomorphic self-maps of C^2 strongly convex domains, without requiring the machinery introduced in [18] and [20]:

Theorem 6 *Let $D \subset\subset \mathbb{C}^n$ be a bounded strictly convex domain, and $f: D \to D$ a k_D-nonexpansive (e.g., holomorphic) self-map without fixed points. Then there exists a $x_0 \in \partial D$ such that $T(f) = \{x_0\}$, that is, the sequence of iterates $\{f^k\}$ converges to the constant map x_0.*

Proof Fix $z_0 \in D$. Lemmas 7 and 6 give $x_0 \in \partial D$ such that

$$T(f) \subseteq \bigcap_{R>0} \mathrm{ch}\big(\overline{F_{z_0}(x_0, R)} \cap \partial D\big) .$$

But D is strictly convex; therefore $\mathrm{ch}\big(\overline{F_{z_0}(x_0, R)} \cap \partial D\big) = \overline{F_{z_0}(x_0, R)} \cap \partial D$, and the assertion follows from Corollary 1. □

In [2] the first author characterized converging one-parameter semigroups of holomorphic self-maps of smooth strongly convex domains. Theorem 6 allows us to extend that characterization to not necessarily smooth strictly convex domains:

Corollary 2 *Let $D \subset\subset \mathbb{C}^n$ be a bounded strictly convex domain, and $\Phi: \mathbb{R}^+ \to \mathrm{Hol}(D, D)$ a one-parameter semigroup of holomorphic self-maps of D. Then Φ converges if and only if*

(i) either Φ has a fixed point $z_0 \in D$ and the spectral generator at z_0 of Φ has no nonzero purely imaginary eigenvalues, or
(ii) Φ has no fixed points.

Proof It follows arguing as in [2, Theorem 1.3], using Theorem 6 instead of the references to [1]. □

4 Weakly convex domains

As mentioned in the introduction, this approach works too when D is convex but not strictly convex. Let $f: D \to D$ be a k_D-nonexpansive or holomorphic self-map, without fixed points. Simply applying the same argument used to prove Theorem 6 one obtains

$$T(f) \subseteq \bigcap_{R>0} \mathrm{ch}\big(\overline{F_{z_0}(x_0, R)} \cap \partial D\big)$$

in the k_D-nonexpansive case, and

$$T(f) \subseteq \bigcap_{R>0} \mathrm{Ch}\big(\overline{F_{z_0}(x_0, R)} \cap \partial D\big) \tag{2}$$

in the holomorphic case (and it is easy to see that these intersections do not depend on $z_0 \in D$). This already can be used to strengthen the Wolff–Denjoy theorem obtained in [3, Theorem 2.4.27] for weakly convex C^2 domains. Indeed, we can prove the following:

Proposition 2 *Let $D \subset\subset \mathbb{C}^n$ be a C^2 bounded convex domain, and $x \in \partial D$. Then for every $z_0 \in D$ and $R > 0$ we have*

$$\overline{F_{z_0}(x, R)} \cap \partial D \subseteq \mathrm{Ch}(x) \ .$$

In particular, if x is a strictly $\mathbb{C}$-linearly convex point then $\overline{F_{z_0}(x, R)} \cap \partial D = \{x\}$.

Proof For every $x \in \partial D$ let $\mathbf{n}_x$ denote the unit outer normal vector to ∂D in x, and put $\sigma_x(z) = (z, \mathbf{n}_x)$, where $(\cdot\,,\cdot)$ is the canonical Hermitian product. Then σ_x is a complex supporting functional at x such that $\sigma_x(y) = \sigma_x(x)$ for some $y \in \partial D$ if and only if $y \in \mathrm{Ch}(x)$.

We can now argue as in the proof of [3, Proposition 2.4.26] replacing the P-function $\Psi \colon \partial D \times \mathbb{C}^n \to \mathbb{C}$ given by $\Psi(x, z) = \exp\big(\sigma_x(z) - \sigma_x(x)\big)$, with the P-function $\widehat{\Psi} \colon \partial D \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\widehat{\Psi}(x, z) = \frac{1}{1 - \big(\sigma_x(z) - \sigma_x(x)\big)} \ .$$

□

Corollary 3 *Let $D \subset\subset \mathbb{C}^n$ be a C^2 bounded convex domain, and $f \colon D \to D$ a holomorphic self-map without fixed points. Then there exists $x_0 \in \partial D$ such that $T(f) \subseteq \mathrm{Ch}(x_0)$. In particular, if D is strictly $\mathbb{C}$-linearly convex then the sequence of iterates $\{f^k\}$ converges to the constant map x_0.*

Proof It follows from (2), Proposition 2, and the fact that in C^2 convex domains each point in the boundary admits a unique complex supporting hyperplane. □

Remark 3 We conjecture that the final assertion of this corollary should also hold for not necessarily smooth strictly $\mathbb{C}$-linearly convex domains.

In weakly convex nonsmooth domains big horospheres might be too large, and the right-hand side of (2) might coincide with the whole boundary of the domain (see Sect. 5 for an example in the polydisk); so to get an effective statement we need to replace them with smaller sets.

Small horospheres might be too small; as shown by Frosini [26], there are holomorphic self-maps of the polydisk with no invariant small horospheres. We thus need another kind of horospheres, defined by Kapeluszny et al. [34], and studied in detail by Budzyńska [18]. To introduce them we begin with a definition:

Definition 7 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $z_0 \in D$. A sequence $\mathbf{x} = \{x_\nu\} \subset D$ converging to $x \in \partial D$ is a *horosphere sequence* at x if the limit of $k_D(z, x_\nu) - k_D(z_0, x_\nu)$ as $\nu \to +\infty$ exists for all $z \in D$.

Remark 4 It is easy to see that the notion of horosphere sequence does not depend on the point z_0.

Remark 5 In [20] it is shown that every sequence in D converging to $x \in \partial D$ contains a subsequence which is a horosphere sequence at x. In strongly convex C^3 domains all sequences converging to a boundary point are horosphere sequences (see [3, Theorem 2.6.47] and [16]); in Sect. 5 we shall give an explicit example of horosphere sequence in the polydisk.

Definition 8 Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Given $z_0 \in D$, let $\mathbf{x} = \{x_\nu\}$ be a horosphere sequence at $x \in \partial D$, and take $R > 0$. Then the *sequence horosphere* $G_{z_0}(x, R, \mathbf{x})$ is defined as

$$G_{z_0}(x, R, \mathbf{x}) = \left\{ z \in D \;\Big|\; \lim_{\nu \to +\infty} \left[k_D(z, x_\nu) - k_D(z_0, x_\nu) \right] < \tfrac{1}{2} \log R \right\} .$$

Remark 6 Actually, as mentioned in [3, p. 280], sequence horospheres are a particular instance of a general notion of horospheres valid in locally complete metric spaces; see, e.g., [24] and [9] for more details. In the latter book it is also proved that in a complete Riemannian manifold of nonpositive curvature this very general notion of horosphere coincides with the horospheres defined by taking sequences contained in a geodesic escaping to infinity. In the setting of complex geometry, horospheres defined by using (complex) geodesics are sometimes called *Busemann horospheres,* and have been used in [15,27,42].

The basic properties of sequence horospheres are contained in the following:

Proposition 3 [18,20,34] *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Fix $z_0 \in D$, and let $\mathbf{x} = \{x_\nu\} \subset D$ be a horosphere sequence at $x \in \partial D$. Then:*

(i) $E_{z_0}(x, R) \subseteq G_{z_0}(x, R, \mathbf{x}) \subseteq F_{z_0}(x, R)$ *for all* $R > 0$*;*
(ii) $G_{z_0}(x, R, \mathbf{x})$ *is nonempty and convex for all* $R > 0$*;*
(iii) $\overline{G_{z_0}(x, R_1, \mathbf{x})} \cap D \subset G_{z_0}(x, R_2, \mathbf{x})$ *for all* $0 < R_1 < R_2$*;*
(iv) $B_D(z_0, \frac{1}{2} \log R) \subset G_{z_0}(x, R, \mathbf{x})$ *for all* $R > 1$*;*
(v) $B_D(z_0, -\frac{1}{2} \log R) \cap G_{z_0}(x, R, \mathbf{x}) = \emptyset$ *for all* $0 < R < 1$*;*
(vi) $\bigcup_{R>0} G_{z_0}(x, R, \mathbf{x}) = D$ *and* $\bigcap_{R>0} G_{z_0}(x, R, \mathbf{x}) = \emptyset$*.*

Remark 7 If $\mathbf{x}$ is a horosphere sequence at $x \in \partial D$ then it is not difficult to check that the family $\{G_z(x, 1, \mathbf{x})\}_{z \in D}$ and the family $\{G_{z_0}(x, R, \mathbf{x})\}_{R>0}$, with $z_0 \in D$ given, coincide.

It turns out that we can always find invariant sequence horospheres:

Lemma 8 *Let $D \subset\subset \mathbb{C}^n$ be a convex domain, and let $f : D \to D$ be k_D-nonexpansive and without fixed points. Then there exist $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that*

$$f\left(G_{z_0}(x, R, \mathbf{x})\right) \subseteq G_{z_0}(x, R, \mathbf{x})$$

for every $z_0 \in D$ and $R > 0$.

Proof Arguing as in the proof of [1, Theorem 2.3] we can find a sequence $\{f_\nu\}$ of k_D-contractions with a unique fixed point $x_\nu \in D$ such that $f_\nu \to f$ and $x_\nu \to x \in \partial D$ as $\nu \to +\infty$. Up to a subsequence, we can also assume (Remark 5) that $\mathbf{x} = \{x_\nu\}$ is a horosphere sequence at x.

Now, for every $z \in D$ we have

$$\left| k_D\big(f(z), x_\nu\big) - k_D\big(f_\nu(z), x_\nu\big) \right| \le k_D\big(f_\nu(z), f(z)\big) \to 0$$

as $\nu \to +\infty$. Therefore if $z \in G_{z_0}(x, R, \mathbf{x})$ we get

$$\begin{aligned}\lim_{\nu\to+\infty}&\big[k_D\big(f(z), x_\nu\big) - k_D(z_0, x_\nu)\big]\\ &\le \limsup_{\nu\to+\infty}\big[k_D\big(f_\nu(z), x_\nu\big) - k_D(z_0, x_\nu)\big]\\ &\quad + \limsup_{\nu\to+\infty}\big[k_D\big(f(z), x_\nu\big) - k_D\big(f_\nu(z), x_\nu\big)\big]\\ &\le \lim_{\nu\to+\infty}\big[k_D(z, x_\nu) - k_D(z_0, x_\nu)\big] < \tfrac{1}{2}\log R\end{aligned}$$

because $f_\nu(x_\nu) = x_\nu$ for all $\nu \in \mathbb{N}$, and we are done. □

Putting everything together we can at last prove the following Wolff–Denjoy theorem for (not necessarily strictly or smooth) convex domains:

Theorem 7 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $f: D \to D$ a k_D-nonexpansive (respectively, holomorphic) self-map without fixed points. Then there exist $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{z\in D} \mathrm{ch}\big(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D\big) = \bigcap_{R>0} \mathrm{ch}\big(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D\big)$$

if f is k_D-nonexpansive, or

$$T(f) \subseteq \bigcap_{z\in D} \mathrm{Ch}\big(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D\big) = \bigcap_{R>0} \mathrm{Ch}\big(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D\big)$$

if f is holomorphic.

Proof The equality of the intersections is an immediate consequence of Remark 7. Then the assertion follows from Lemmas 8 and 6. □

In the next section we shall show how this statement is essentially optimal in the polydisk; we end this section by stating a corollary valid for strictly $\mathbb{C}$-linearly convex domains:

Corollary 4 *Let $D \subset\subset \mathbb{C}^n$ be a bounded strictly $\mathbb{C}$-linearly convex domain, and $f: D \to D$ a holomorphic self-map of D without fixed points. Then there exist $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{z\in D} \overline{G_z(x, 1, \mathbf{x})} = \bigcap_{R>0} \overline{G_{z_0}(x, R, \mathbf{x})}\,.$$

Proof It follows immediately from Theorem 7 and the definition of strictly $\mathbb{C}$-linearly convex domain. □

5 The polydisk

The polydisk $\Delta^n \subset \mathbb{C}^n$ is the unit ball for the norm $\|z\| = \max\{|z_j| \mid j = 1, \ldots, n\}$, and therefore (see, e.g., [3])

$$k_{\Delta^n}(z, w) = \frac{1}{2}\log\frac{1 + \|\gamma_z(w)\|}{1 - \|\gamma_z(w)\|}$$

for every $z, w \in \Delta^n$, where

$$\gamma_z(w) = \left(\frac{w_1 - z_1}{1 - \overline{z}_1 w_1}, \dots, \frac{w_n - z_n}{1 - \overline{z}_n w_n}\right)$$

is an automorphism of the polydisk with $\gamma_z(z) = 0$.

Thanks to the homogeneity of Δ^n, we can restrict ourselves to consider only horospheres with pole z_0 at the origin, and we have (see [3, chapter 2.4.2] for detailed computations) the following description for horospheres with center $\xi \in \partial\Delta^n$ and radius $R > 0$:

$$E_O(\xi, R) = \left\{ z \in \Delta^n \;\middle|\; \max_{|\xi_j|=1} \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \right\} < R \right\}$$

and

$$F_O(\xi, R) = \left\{ z \in \Delta^n \;\middle|\; \min_{|\xi_j|=1} \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \right\} < R \right\}.$$

On the other hand, given $\xi \in \partial\Delta^n$, a not difficult computation shows that

$$\mathrm{ch}(\xi) = \bigcup_{|\xi_j|=1} \{\eta \in \partial\Delta^n \mid \eta_j = \xi_j\} \quad \text{and} \quad \mathrm{Ch}(\xi) = \bigcap_{|\xi_j|=1} \{\eta \in \partial\Delta^n \mid \eta_j = \xi_j\}\,.$$

This implies that in the polydisk big horospheres are too large to give a sensible Wolff–Denjoy theorem. Indeed we have

$$\mathrm{ch}\big(\overline{F_O(\xi, R)} \cap \partial\Delta^n\big) = \mathrm{Ch}\big(\overline{F_O(\xi, R)} \cap \partial\Delta^n\big) = \partial\Delta^n$$

as soon as ξ has at least two components of modulus 1, and

$$\begin{aligned} \mathrm{ch}\big(\overline{F_O(\xi, R)} \cap \partial\Delta^n\big) &= \mathrm{Ch}\big(\overline{F_O(\xi, R)} \cap \partial\Delta^n\big) \\ &= \partial\Delta^n \setminus \{\eta \in \partial\Delta^n \mid \eta_{j_0} \neq \xi_{j_0},\ |\eta_j| < 1 \text{ for } j \neq j_0\} \end{aligned}$$

if $|\xi_{j_0}| = 1$ and $|\xi_j| < 1$ for $j \neq j_0$.

Let us then compute the sequence horospheres. Fix a horosphere sequence $\mathbf{x} = \{x_\nu\}$ converging to $\xi \in \partial\Delta^n$. Arguing as in [3, chapter 2.4.2], we get

$$G_O(\xi, R, \mathbf{x}) = \left\{ z \in \Delta^n \;\middle|\; \max_{|\xi_j|=1} \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \lim_{\nu \to +\infty} \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \right\} < R \right\}\,.$$

Since if $|\xi_j| = 1$ we clearly have

$$\alpha_j := \lim_{\nu \to +\infty} \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \leq 1\,,$$

we get

$$G_O(\xi, R, \mathbf{x}) = \left\{ z \in \Delta^n \;\middle|\; \max_j \left\{ \alpha_j \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |\xi_j| = 1 \right\} < R \right\}.$$

In other words, we can write $G_O(\xi, R, \mathbf{x})$ as a product

$$G_O(\xi, R, \mathbf{x}) = \prod_{j=1}^n E_j\,,$$

where, denoting by $E^\Delta(\sigma, R) \subset \Delta$ the standard horocycle of center $\sigma \in \partial\Delta$, pole the origin and radius $R > 0$, we have put

$$E_j = \begin{cases} \Delta & \text{if } |\xi_j| < 1, \\ E^\Delta(\xi_j, R/\alpha_j) & \text{if } |\xi_j| = 1. \end{cases}$$

As a consequence,

$$\mathrm{ch}\big(\overline{G_O(\xi, R, \mathbf{x})} \cap \partial\Delta^n\big) = \mathrm{Ch}\big(\overline{G_O(\xi, R, \mathbf{x})} \cap \partial\Delta^n\big) = \bigcup_{j=1}^{n} \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta} \,,$$

where

$$C_j(\xi) = \begin{cases} \{\xi_j\} & \text{if } |\xi_j| = 1, \\ \partial\Delta & \text{if } |\xi_j| < 1. \end{cases}$$

Notice that the right-hand sides do not depend either on R or on the horosphere sequence $\mathbf{x}$, but only on ξ.

So Theorem 7 in the polydisk assumes the following form:

Corollary 5 *Let $f : \Delta^n \to \Delta^n$ be a k_{Δ^n}-nonexpansive (e.g., holomorphic) self-map without fixed points. Then there exists $\xi \in \partial\Delta^n$ such that*

$$T(f) \subseteq \bigcup_{j=1}^{n} \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta} \,, \tag{3}$$

where

$$C_j(\xi) = \begin{cases} \{\xi_j\} & \textit{if } |\xi_j| = 1, \\ \partial\Delta & \textit{if } |\xi_j| < 1. \end{cases}$$

This is the best one can do, in the sense that while it might be true (for instance in the bidisk; see below) that the image of a limit point of the sequence of iterates of f is always contained in just one of the sets appearing in the right-hand side of (3), it is impossible to determine a priori in which one it is contained on the basis of the point ξ only; it is necessary to know something more about the map f. Indeed, Hervé has proved the following:

Theorem 8 [29] *Let $F = (f, g) : \Delta^2 \to \Delta^2$ be a holomorphic self-map of the bidisk, and write $f_w = f(\cdot, w)$ and $g_z = g(z, \cdot)$. Assume that F has no fixed points in Δ^2. Then one and only one of the following cases occurs:*

(i) if $g(z, w) \equiv w$ (respectively, $f(z, w) \equiv z$) then the sequence of iterates of F converges uniformly on compact sets to $h(z, w) = (\sigma, w)$, where σ is the common Wolff point of the f_w's (respectively, to $h(z, w) = (z, \tau)$, where τ is the common Wolff point of the g_z's);

(ii) if $\mathrm{Fix}(f_w) = \emptyset$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \{y(z)\} \subset \Delta$ for all $z \in \Delta$ (respectively, if $\mathrm{Fix}(f_w) = \{x(w)\}$ and $\mathrm{Fix}(g_z) = \emptyset$) then $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$, where $\sigma \in \partial\Delta$ is the common Wolff point of the f_w's (respectively, $T(f) \subseteq \overline{\Delta} \times \{\tau\}$, where τ is the common Wolff point of the g_z's);

(iii) if $\mathrm{Fix}(f_w) = \emptyset$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \emptyset$ for all $z \in \Delta$ then either $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$ or $T(f) \subseteq \overline{\Delta} \times \{\tau\}$, where $\sigma \in \partial\Delta$ is the common Wolff point of the f_w's, and $\tau \in \partial\Delta$ is the common Wolff point of the g_z;

(iv) *if* $\mathrm{Fix}(f_w) = \{x(w)\} \subset \Delta$ *for all* $w \in \Delta$ *and* $\mathrm{Fix}(g_z) = \{y(z)\} \subset \Delta$ *for all* $z \in \Delta$ *then there are* $\sigma, \tau \in \partial D$ *such that the sequence of iterates converges to the constant map* (σ, τ).

All four cases can occur: see [29].

We end this paper providing, as promised, an example of horosphere sequence. Given $\xi \in \partial\Delta^n$, put $x_\nu = (1 - 1/\nu)^{1/2}\xi$; we claim that $\mathbf{x} = \{x_\nu\}$ is a horosphere sequence. Indeed, arguing as in [3, chapter 2.4.2] we see it suffices to show that

$$\max_{|\xi_j|=1} \left\{ \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \frac{|1 - \overline{z_j} x_{\nu,j}|^2}{1 - |z_j|^2} \right\}$$

converges as $\nu \to +\infty$. But indeed with this choice of x_ν we have

$$\begin{aligned} &\max_{|\xi_j|=1} \left\{ \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \frac{|1 - \overline{z_j} x_{\nu,j}|^2}{1 - |z_j|^2} \right\} \\ &= \max_{|\xi_j|=1} \left\{ \min_h \left\{ \nu(1 - |\xi_h|^2) + |\xi_h|^2 \right\} \frac{|1 - \overline{z_j} x_{\nu,j}|^2}{1 - |z_j|^2} \right\} \\ &= \max_{|\xi_j|=1} \left\{ \frac{|1 - \overline{z_j} x_{\nu,j}|^2}{1 - |z_j|^2} \right\} \to \max_{|\xi_j|=1} \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \right\}. \end{aligned}$$

In particular, using this horosphere sequence one obtains $G_O(\xi, R, \mathbf{x}) = E_O(\xi, R)$.

Finally, it is worth mentioning that in [27] Frosini, using Hervé's Theorem 8, has established in the bidisk which points $x \in \partial\Delta^2$ admit f-invariant (sequence or Busemann) horospheres, i.e., horopsheres satisfying the statement of Lemma 8.

References

1. Abate, M.: Horospheres and iterates of holomorphic maps. Math. Z. **198**, 225–238 (1988)
2. Abate, M.: Converging semigroups of holomorphic maps. Rend. Acc. Naz. Lincei **82**, 223–227 (1988)
3. Abate, M.: Iteration Theory of Holomorphic Maps on Taut Manifolds. Mediterranean Press, Cosenza (1989). http://www.dm.unipi.it/~abate/libri/libriric/libriric.html
4. Abate, M.: Iteration theory, compactly divergent sequences and commuting holomorphic maps. Ann. Scuola Norm. Sup. Pisa **18**, 167–191 (1991)
5. Abate, M.: Angular derivatives in several complex variables. In: Zaitsev, D., Zampieri, G. (eds.) Real Methods in Complex and CR Geometry, Lecturer Notes in Mathematics, vol. 1848, pp. 1–47. Springer, Berlin (2004)
6. Abate, M., Heinzner, P.: Holomorphic actions on contractible domains without fixed points. Math. Z. **211**, 547–555 (1992)
7. Abate, M., Raissy, J.: Backward iteration in strongly convex domains. Adv. Math. **228**, 2837–2854 (2011)
8. Abate, M., Vigué, J.-P.: Common fixed points in hyperbolic Riemann surfaces and convex domains. Proc. Am. Math. Soc. **112**, 503–512 (1991)
9. Balmann, W., Gromov, M., Schroeder, V.: Manifolds of Nonpositive Curvature. Birkhäuser, Basel (1985)
10. Beardon, A.F.: Repeated compositions of analytic maps. Comput. Methods Funct. Theory **1**, 235–248 (2001)
11. Bedford, E.: On the automorphism group of a Stein manifold. Math. Ann. **266**, 215–227 (1983)
12. Bracci, F.: Fixed points of commuting holomorphic mappings other than the Wolff point. Trans. Am. Math. Soc. **355**, 2569–2584 (2003)
13. Bracci, F.: Dilatation and order of contact for holomorphic self-maps of strongly convex domains. Proc. Lond. Math. Soc. **86**, 131–152 (2003)
14. Bracci, F.: A note on random holomorphic iteration in convex domains. Proc. Edinb. Math. Soc. **51**, 297–304 (2008)
15. Bracci, F., Patrizio, G.: Monge–Ampère foliations with singularities at the boundary of strongly convex domains. Math. Ann. **332**, 499–522 (2005)

16. Bracci, F., Patrizio, G., Trapani, S.: The pluricomplex Poisson kernel for strongly convex domains. Trans. Am. Math. Soc. **361**, 979–1005 (2009)
17. Budzyńska, M.: Local uniform linear convexity with respect to the Kobayashi distance. Abstr. Appl. Anal. **2003**, 367–373 (2003)
18. Budzyńska, M.: The Denjoy–Wolff theorem in $\mathbb{C}^n$. Nonlinear Anal. **75**, 22–29 (2012)
19. Budzyńska, M.: The Denjoy–Wolff theorem for condensing mappings in a bounded and strictly convex domain in a complex Banach space. Preprint (2012)
20. Budzyńska, M., Kuczumow, T., Reich, S.: Theorems of Denjoy–Wolff type. Ann. Mat. Pura Appl. (2012). doi:10.1007/s10231-011-0240-z
21. Budzyńska, M., Kuczumow, T., Słodkowski, T.: Total sets and semicontinuity of the Kobayashi distance. Nonlinear Anal. **47**, 2793–2803 (2001)
22. Całka, A.: On conditions under which isometries have bounded orbits. Colloq. Math. **48**, 219–227 (1984)
23. Denjoy, A.: Sur l'itération des fonctions analytiques. C.R. Acad. Sci. Paris **182**, 255–257 (1926)
24. Eberlein, P., O'Neill, B.: Visibility manifolds. Pac. J. Math. **46**, 45–109 (1973)
25. Elin, M., Shoikhet, D.: Linearization Models for Complex Dynamical Systems. Topics in Univalent Functions, Functional Equations and Semigroup Theory. Birkhäuser, Basel (2010)
26. Frosini, C.: Dynamics on bounded domains. In: Poggi-Corradini, P. (ed.) The p-Harmonic Equation and Recent Advances in Analysis. Contemporary Mathematics, vol. 370, pp. 99–117. American Mathematical Society, Providence, RI (2005)
27. Frosini, C.: Busemann functions and Julia–Wolff–Carathéodory theorem on polydiscs. Adv. Geom. **10**, 435–463 (2010)
28. Heins, M.H.: A theorem of Wolff–Denjoy type. In: Hersch, J., Huber, A. (eds.) Complex Analysis, pp. 81–86. Birkhäuser, Basel (1988)
29. Hervé, M.: Itération des transformations analytiques dans le bicercle-unité. Ann. Sci. Éc. Norm. Sup. **71**, 1–28 (1954)
30. Hervé, M.: Quelques propriétés des applications analytiques d'une boule à m dimensions dans elle-même. J. Math. Pures Appl. **42**, 117–147 (1963)
31. Huang, X.J.: A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings. Ann. Scuola Norm. Sup. Pisa **21**, 399–419 (1994)
32. Jarnicki, M., Pflug, P.: Invariant Distances and Metrics in Complex Analysis. Walter de Gruyter, Amsterdam (1993)
33. Kapeluszny, J., Kuczumow, T., Reich, S.: The Denjoy–Wolff theorem for condensing holomorphic mappings. J. Funct. Anal. **167**, 79–93 (1999)
34. Kapeluszny, J., Kuczumow, T., Reich, S.: The Denjoy–Wolff theorem in the open unit ball of a strictly convex Banach space. Adv. Math. **143**, 111–123 (1999)
35. Kobayashi, S.: Hyperbolic Complex Spaces. Springer, Berlin (1998)
36. Kuczumow, T., Stachura, A.: Iterates of holomorphic and k_D-nonexpansive mappings in convex domains in $\mathbb{C}^n$. Adv. Math. **81**, 90–98 (1990)
37. Lárusson, F.: A Wolff–Denjoy theorem for infinitely connected Riemann surfaces. Proc. Am. Math. Soc. **124**, 2745–2750 (1996)
38. Lempert, L.: La métrique de Kobayashi et la réprésentation des domaines sur la boule. Bull. Soc. Math. Fr. **109**, 427–474 (1981)
39. Reich, S., Shoikhet, D.: Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Saces. Imperial College Press, London (2005)
40. Ren, F., Zhang, W.: Dynamics on weakly pseudoconvex domains. Chin. Ann. Math. Ser. B **16**, 467–476 (1995)
41. Stachura, A.: Iterates of holomorphic self-maps of the unit ball in Hilbert space. Proc. Am. Math. Soc. **93**, 88–90 (1985)
42. Trapani, S.: Dual maps and Kobayashi distance of bounded convex domains in $\mathbb{C}^n$. In: Bokan, N., Djorić, M., Fomenko, A.T., Rakić, Z., Wess, J. (eds.) Contemporary Geometry and Related Topics, pp. 389–406. World Scientific, River Edge, NJ (2004)
43. Wolff, J.: Sur une généralisation d'un théorème de Schwarz. C.R. Acad. Sci. Paris **182**, 918–920 (1926)