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Simply transitive geodesic ball packings to $\mathbf{S}^2 \times \mathbf{R}$ space groups generated by glide reflections

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Abstract The $\mathbf{S}^2 \times \mathbf{R}$ geometry can be derived by the direct product of the spherical plane $\mathbf{S}^2$ and the real line $\mathbf{R}$. In (Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) 42:235–250, 2001), Farkas has classified and given the complete list of the space groups of $\mathbf{S}^2 \times \mathbf{R}$. The $\mathbf{S}^2 \times \mathbf{R}$ manifolds were classified by Molnár and Farkas in [2] by similarity and diffeomorphism. In Szirmai (Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) 52(2):413–430, 2011), we have studied the geodesic balls and their volumes in $\mathbf{S}^2 \times \mathbf{R}$ space; moreover, we have introduced the notion of geodesic ball packing and its density and have determined the densest geodesic ball packing for generalized Coxeter space groups of $\mathbf{S}^2 \times \mathbf{R}$. In this paper, we study the locally optimal ball packings to the $\mathbf{S}^2 \times \mathbf{R}$ space groups having Coxeter point groups, and at least one of the generators is a glide reflection. We determine the densest simply transitive geodesic ball arrangements for the above space groups; moreover, we compute their optimal densities and radii. The density of the densest packing is ≈ 0.80407553, may be surprising enough in comparison with the Euclidean result $\frac{\pi}{\sqrt{18}} \approx 0.74048$. Molnár has shown in (Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) 38(2):261–288, 1997) that the homogeneous 3-spaces have a unified interpretation in the real projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbb{R})$. In our work, we shall use this projective model of $\mathbf{S}^2 \times \mathbf{R}$ geometry.

Keywords Thurston geometries · Space groups · Geodesic ball packings

Mathematics Subject Classification (2010) 52C17 · 52C22 · 53A35 · 51M20

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1 Introduction

$\mathbf{S}^2\times\mathbf{R}$ is derived as the direct product of the spherical plane $\mathbf{S}^2$ and the real line $\mathbf{R}$. The points are described by (P, p) where $P \in \mathbf{S}^2$ and $p \in \mathbf{R}$ [2]. The isometry group $Isom(\mathbf{S}^2\times\mathbf{R})$ of $\mathbf{S}^2\times\mathbf{R}$ can be derived by the direct product of the isometry group of the sphere $Isom(\mathbf{S}^2)$ and the isometry group of the real line $Isom(\mathbf{R})$.

$$\begin{aligned}&Isom(\mathbf{S}^2) := \{A \in \mathbf{O}(3) : \mathbf{S}^2 \mapsto \mathbf{S}^2 : (P, p) \mapsto (PA, p)\} \text{ for any fixed } p.\\ &Isom(\mathbf{R}) := \{\rho : (P, p) \mapsto (P, \pm p + r)\}, \text{ for any fixed } P.\\ &\text{Here, the “}-\text{” sign provides a reflection in the point } \frac{r}{2} \in \mathbf{R},\\ &\text{and by the “+” sign, we get a translation of } \mathbf{R}.\end{aligned} \tag{1.1}$$

The structure of an isometry group $\Gamma \subset Isom(\mathbf{S}^2\times\mathbf{R})$ is the following: $\Gamma := \{(A_1 \times \rho_1), \dots (A_n \times \rho_n)\}$, where $A_i \times \rho_i := A_i \times (R_i, r_i) := (g_i, r_i)$, $(i \in \{1, 2, \dots n\}$ and $A_i \in Isom(\mathbf{S}^2)$, R_i is either the identity map $\mathbf{1_R}$ of $\mathbf{R}$ or the point reflection $\overline{\mathbf{1}}_\mathbf{R}$. $g_i := A_i \times R_i$ is called the linear part of the transformation $(A_i \times \rho_i)$ and r_i is its translation part. The multiplication formula is the following:

$$(A_1 \times R_1, r_1) \circ (A_2 \times R_2, r_2) = ((A_1A_2 \times R_1R_2, r_1R_2 + r_2). \tag{1.2}$$

Definition 1.1 A group of isometries $\Gamma \subset Isom(\mathbf{S}^2\times\mathbf{R})$ is called *space group* if the linear parts form a finite group Γ_0 called the point group of Γ; moreover, the translation parts to the identity of this point group are required to form a one-dimensional lattice L_Γ of $\mathbf{R}$.

Remark 1.2 1. It can be proved that the space group Γ has a compact fundamental domain $\mathcal{F}_\Gamma$.
2. If Γ is not assumed to have a lattice, then it may have an infinite point group Γ_0.

Definition 1.3 The $\mathbf{S}^2\times\mathbf{R}$ space groups Γ_1 and Γ_2 are geometrically equivalent, called equivariant, if there is a “similarity” transformation $\Sigma := S \times \sigma$ $(S \in Isom(\mathbf{S}^2), \sigma \in Sim(\mathbf{R}))$, such that $\Gamma_2 = \Sigma^{-1}\Gamma_1\Sigma$. Here, $\sigma(s, t) : p \to p \cdot s + t$ is a similarity of $\mathbf{R}$, that is, multiplication by $0 \neq s \in \mathbf{R}$ and then addition by $t \in \mathbf{R}$ for every $p \in \mathbf{R}$.

Remark 1.4 If Γ_1 and Γ_2 are equivariant space groups, then their factor groups Γ_1/L_{Γ_1} and Γ_2/L_{Γ_2} are also equivariant.

Thus, the structure of the space group remains invariant under a similarity in the $\mathbf{R}$-component, and the spherical part is uniquely determined up to an isometry of $\mathbf{S}^2$.

We characterize the spherical plane groups by the *Macbeath-signature* (see [3,11]).

In this paper, we deal with such a $\mathbf{S}^2\times\mathbf{R}$ *space group where the generators* $\mathbf{g}_i$, $(i = 1, 2, \dots m)$ *of its point group* Γ_0 *are reflections and at least one of the possible translation parts of the above generators is unequal to zero. These groups are called glide reflection groups.*

Remark 1.5 In [11], we have introduced the notion of *generalized Coxeter group* if the generators $\mathbf{g}_i$, $(i = 1, 2, \dots m)$ of its point group Γ_0 are reflections with translation parts $\tau_i = 0$, $(i = 1, 2, \dots m)$.

In this paper, we deal with the glide reflection space groups in $\mathbf{S}^2\times\mathbf{R}$ space which are by denotation of [1]:

1. $(+,\ 0,\ [\ ]\{(q, q)\}) \times \mathbf{1_R},\ q \geq 2,$
   $\Gamma_0 = (\mathbf{g}_1, \mathbf{g}_2 - \mathbf{g}_1^2, \mathbf{g}_2^2, (\mathbf{g}_1\mathbf{g}_2)^q)$, **2q. I. 2**: $\left(\frac{1}{2}, \frac{1}{2}\right)$; **2qe. I. 3**: $\left(0, \frac{1}{2}\right)$;

2. $(+,\ 0,\ [\ ]\ \{(2,2,q)\})\times \mathbf{1_R},\ \mathbf{q}\geq \mathbf{2}$,
$\Gamma_0=(\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3-\mathbf{g}_1^2,\mathbf{g}_2^2,\mathbf{g}_3^2,(\mathbf{g}_1\mathbf{g}_3)^2,(\mathbf{g}_2\mathbf{g}_3)^2),(\mathbf{g}_1\mathbf{g}_2)^q)$, **4q. I. 2**: $\left(0,0,\frac{1}{2}\right)$; **4q. I. 3**: $\left(\frac{1}{2},\frac{1}{2},0\right)$; **4q. I. 4**: $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$; **4qe. I. 5**: $\left(0,\frac{1}{2},0\right)$; **4qe. I. 6**: $\left(0,\frac{1}{2},\frac{1}{2}\right)$;
3. $(+,\ 0,\ [\ ]\ \{(2,3,3)\})\times \mathbf{1_R}$,
$\Gamma_0=(\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3-\mathbf{g}_1^2,\mathbf{g}_2^2,\mathbf{g}_3^2,(\mathbf{g}_1\mathbf{g}_2)^2,(\mathbf{g}_1\mathbf{g}_3)^3,(\mathbf{g}_2\mathbf{g}_3)^3)$, **11. I. 2**: $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$;
4. $(+,\ 0,\ [\ ]\ \{(2,3,4)\})\times \mathbf{1_R}$,
$\Gamma_0=(\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3-\mathbf{g}_1^2,\mathbf{g}_2^2,\mathbf{g}_3^2,(\mathbf{g}_1\mathbf{g}_2)^2,(\mathbf{g}_1\mathbf{g}_3)^3,(\mathbf{g}_2\mathbf{g}_3)^4)$, **12. I. 2**: $\left(0,\frac{1}{2},0\right)$; **12. I. 3**: $\left(\frac{1}{2},0,\frac{1}{2}\right)$; **12. I. 4**: $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$;
5. $(+,\ 0,\ [\ ]\ \{(2,3,5)\})\times \mathbf{1_R}$,
$\Gamma_0=(\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3-\mathbf{g}_1^2,\mathbf{g}_2^2,\mathbf{g}_3^2,(\mathbf{g}_1\mathbf{g}_2)^2,(\mathbf{g}_1\mathbf{g}_3)^3,(\mathbf{g}_2\mathbf{g}_3)^5)$, **13. I. 2**: $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$;

2 Geodesic curve and balls in $\mathbf{S}^2\times\mathbf{R}$ space

E. Molnár has shown in [4] that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbb{R})$. In our work, we shall use this projective model of $\mathbf{S}^2\times\mathbf{R}$ and the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$, $E_1^\infty(\mathbf{e}_1)$, $E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\}\subset\mathbf{V}^4$ with the unit point $E(\mathbf{e}=\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y}=c\mathbf{x}$ with $0<c\in\mathbb{R}$ (or $c\in\mathbb{R}\setminus\{0\}$) defines a point $(\mathbf{x})=(\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(\boldsymbol{e}^i)\}\subset\boldsymbol{V}_4$ describes the simplex planes, especially the plane at infinity $(\boldsymbol{e}^0)=E_1^\infty E_2^\infty E_3^\infty$, and generally, $\boldsymbol{v}=\boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u})=(\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus, $0=\mathbf{x}\boldsymbol{u}=\mathbf{y}\boldsymbol{v}$ defines the incidence of point $(\mathbf{x})=(\mathbf{y})$ and plane $(\boldsymbol{u})=(\boldsymbol{v})$, as $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ also denotes it. Thus, $\mathbf{S}^2\times\mathbf{R}$ can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

In this context, Molnár [4] has derived the well-known infinitesimal arc-length square at any point of $\mathbf{S}^2\times\mathbf{R}$ as follows:

$$(ds)^2=\frac{(dx)^2+(dy)^2+(dz)^2}{x^2+y^2+z^2}. \tag{2.1}$$

We shall apply the usual geographical coordiantes (ϕ,θ) $(-\pi<\phi\leq\pi,\ -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2})$ of the sphere with the fibre coordinate $t\in\mathbf{R}$. We describe points in the above coordinate system in our model by the following equations:

$$x^0=1,\quad x^1=e^t\cos\phi\cos\theta,\ x^2=e^t\sin\phi\cos\theta,\ x^3=e^t\sin\theta. \tag{2.2}$$

Then, we have $x=\frac{x^1}{x^0}=x^1$, $y=\frac{x^2}{x^0}=x^2$, $z=\frac{x^3}{x^0}=x^3$, that is, the usual Cartesian coordinates. We obtain by [4] that in this parametrization, the infinitesimal arc-length square at any point of $\mathbf{S}^2\times\mathbf{R}$ is the following:

$$(ds)^2=(dt)^2+(d\phi)^2\cos^2\theta+(d\theta)^2. \tag{2.3}$$

The geodesic curves of $\mathbf{S}^2\times\mathbf{R}$ are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $\gamma(t(\tau),\phi(\tau),\theta(\tau))$ in our model can be determined by the general theory of Riemann geometry (see [11]).

Then by (2.2), we get with $c = \sin v,\ \omega = \cos v$ the equation systems of a geodesic curve, visualized in Fig. 1 in our Euclidean model:

$$\begin{aligned} x(\tau) &= e^{\tau \sin v} \cos(\tau \cos v), \\ y(\tau) &= e^{\tau \sin v} \sin(\tau \cos v) \cos u, \\ z(\tau) &= e^{\tau \sin v} \sin(\tau \cos v) \sin u, \\ -\pi &< u \leq \pi,\ -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}. \end{aligned} \tag{2.4}$$

Remark 2.1 Thus, we have harmonized the scales along the fibre lines.

Definition 2.2 The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the geodesic curve from P_1 to P_2.

Definition 2.3 The geodesic sphere of radius ρ (denoted by $S_{P_1}(\rho)$) with centre at the point P_1 is defined as the set of all points P_2 in the space with the condition $d(P_1, P_2) = \rho$. Moreover, we require that the geodesic sphere is a simply connected surface without self-intersection in $\mathbf{S}^2 \times \mathbf{R}$ space.

Remark 2.4 We shall see that this last condition depends on radius ρ.

Definition 2.5 The body of the geodesic sphere of centre P_1 and of radius ρ in the $\mathbf{S}^2 \times \mathbf{R}$ space is called geodesic ball, denoted by $B_{P_1}(\rho)$, that is, $Q \in B_{P_1}(\rho)$ iff $0 \leq d(P_1, Q) \leq \rho$.

In [11], we have proved that $S(\rho)$ is a simply connected surface in $\mathbf{E}^3$ if and only if $\rho \in [0, \pi)$, because if $\rho \geq \pi$ then there is at least one $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ so that $y(\tau, v) = z(\tau, v) = 0$, that is, self-intersection would occur (see (2.4)). Thus, we obtain the following.

Proposition 2.6 *The geodesic sphere and ball of radius ρ exists in the $\mathbf{S}^2 \times \mathbf{R}$ space if and only if $\rho \in [0, \pi]$.*

We have obtained (see [11]) the volume formula of the geodesic ball $B(\rho)$ of radius ρ by the metric tensor g_{ij} and by the Jacobian of (2.4):

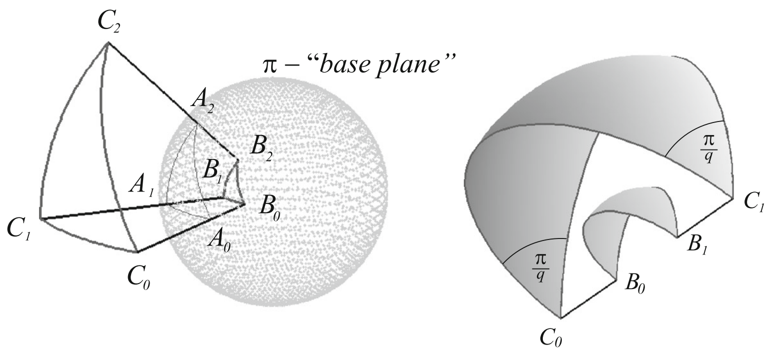

Fig. 1 Prism-like fundamental domains

Theorem 2.7

$$Vol(B(\rho)) = \int_V \frac{1}{(x^2+y^2+z^2)^{3/2}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$
$$= \int_0^{\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} |\tau \cdot \sin(\cos(v)\tau)|\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}\tau$$
$$= 2\pi \int_0^{\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\tau \cdot \sin(\cos(v)\tau)|\, \mathrm{d}v\, \mathrm{d}\tau. \tag{2.5}$$

2.1 On fundamental domains

A type of the fundamental domain of a studied space group can be combined as a fundamental domain of the corresponding spherical group with a part of a real line segment. This domain is called $\mathbf{S}^2\times\mathbf{R}$ prism (see [11]). *This notion will be important to compute the volume of the Dirichlet–Voronoi cell of a given space group because their volumes are equal and the volume of a* $\mathbf{S}^2\times\mathbf{R}$ *prism can be calculated by Theorem* 2.8.

The p-gonal faces of a prism called cover-faces, and the other faces are the side-faces. The midpoints of the side edges form a "spherical plane" denoted by Π. It can be assumed that the plane Π is the *base plane*: in our coordinate system (see (2.2)), the fibre coordinate $t=0$. From [11], we recall

Theorem 2.8 *The volume of a* $\mathbf{S}^2\times\mathbf{R}$ *trigonal prism* $\mathcal{P}_{B_0B_1B_2C_0C_1C_2}$ *and of a diagonal prism* $\mathcal{P}_{B_0B_1C_0C_1}$ *in* $\mathbf{S}^2\times\mathbf{R}$ *(see Fig. 1a, b) can be computed by the following formula:*

$$Vol(\mathcal{P}) = \mathcal{A}\cdot h \tag{2.5}$$

where $\mathcal{A}$ *is the area of the spherical triangle* $A_0A_1A_2$ *or digon* A_0A_1 *in the base plane* Π *with fibre coordinate* $t=0$*, and* $h=B_0C_0$ *is the height of the prism.*

3 Ball packings

By remark (1.2), a $\mathbf{S}^2\times\mathbf{R}$ space group Γ has a compact fundamental domain. Usually, the shape of the fundamental domain of a group of $\mathbf{S}^2$ is not determined uniquely, but the area of the domain is finite and unique by its combinatorial measure. Thus, the shape of the fundamental domain of a crystallographic group of $\mathbf{S}^2\times\mathbf{R}$ is not unique as well.

In the following, let Γ be a fixed glide reflection space group of $\mathbf{S}^2\times\mathbf{R}$. We will denote by $d(X,Y)$ the distance of two points X, Y by definition (2.2).

Definition 3.1 We say that the point set

$$\mathcal{D}(K) = \{X \in \mathbf{S}^2\times\mathbf{R} \,:\, d(K,X) \le d(K^{\mathbf{g}},X) \text{for all } \mathbf{g}\in\Gamma\}$$

is the *Dirichlet–Voronoi cell* (D-V cell) to Γ around the kernel point $K\in\mathbf{S}^2\times\mathbf{R}$.

Definition 3.2 We say that

$$\Gamma_X = \{\mathbf{g}\in\Gamma \,:\, X^{\mathbf{g}} = X\}$$

is the *stabilizer subgroup* of $X\in\mathbf{S}^2\times\mathbf{R}$ in Γ.

Definition 3.3 Assume that the stabilizer $\Gamma_K = \mathbf{I}$, that is, Γ acts simply transitively on the orbit of a point K. Then, let $\mathcal{B}_K$ denote the *greatest ball* of centre K inside the D-V cell $\mathcal{D}(K)$; moreover, let $\rho(K)$ denote the *radius* of $\mathcal{B}_K$. It is easy to see that

$$\rho(K) = \min_{\mathbf{g} \in \Gamma \setminus \mathbf{I}} \frac{1}{2} d(K, K^{\mathbf{g}}).$$

The Γ-images of $\mathcal{B}_K$ form a ball packing $\mathcal{B}_K^{\Gamma}$ with centre points $K^{\mathbf{G}}$.

Definition 3.4 The *density* of ball packing $\mathcal{B}_K^{\Gamma}$ is

$$\delta(K) = \frac{Vol(\mathcal{B}_K)}{Vol\mathcal{D}(K)}.$$

It is clear that the orbit K^{Γ} and the ball packing $\mathcal{B}_K^{\Gamma}$ have the same symmetry group; moreover, this group contains the starting crystallographic group Γ:

$$Sym K^{\Gamma} = Sym \mathcal{B}_K^{\Gamma} \geq \Gamma.$$

Definition 3.5 We say that the orbit K^{Γ} and the ball packing $\mathcal{B}_K^{\Gamma}$ is *characteristic* if $Sym K^{\Gamma} = \Gamma$, else the orbit is not characteristic.

3.1 Simply transitive ball packings

Our problem is to find a point $K \in \mathbf{S}^2 \times \mathbf{R}$ and the orbit K^{Γ} for Γ such that $\Gamma_K = \mathbf{I}$ and the density $\delta(K)$ of the corresponding ball packing $\mathcal{B}^{\Gamma}(K)$ is maximal. In this case, the ball packing $\mathcal{B}^{\Gamma}(K)$ is said to be *optimal.*

The lattice of Γ has a free parameter $p(\Gamma)$. Then, we have to find the densest ball packing on K for fixed $p(\Gamma)$ and vary p to get the optimal ball packing.

$$\delta(\Gamma) = \max_{K,\, p(\Gamma)} (\delta(K)) \tag{3.1}$$

Let Γ be a fixed *glide reflection group*. The stabilizer of K is trivial; that is, we are looking the optimal kernel point in a 3-dimensional region, inside of a fundamental domain of Γ with free fibre parameter $p(\Gamma)$.It can be assumed by the homogeneity of $\mathbf{S}^2 \times \mathbf{R}$ that the fibre coordinate of the centre of the optimal ball is zero.

3.2 Optimal ball packing to space group **12. I. 3**

Now, we consider the following point group:

$$\begin{aligned} &(+, 0; [\ ];\ \{(2,3,4)\}) \times \mathbf{1_R}; \\ &\Gamma_0 := \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\ -\ \mathbf{g}_1^2, \mathbf{g}_2^2, \mathbf{g}_3^2, (\mathbf{g}_1\mathbf{g}_2)^2, (\mathbf{g}_1\mathbf{g}_3)^3, (\mathbf{g}_2\mathbf{g}_3)^4\}. \end{aligned} \tag{3.2}$$

This is the full isometry group of the usual cube surface, generated by the three reflections $\mathbf{g}_i$, $i = 1, 2, 3$. The possible translation parts of the generators of Γ_0 will be determined by (1.2) and by the defining relations of the point group. Finally, from the so-called Frobenius congruence relations, we obtain the four non-equivariant solutions:

$$(\tau_1, \tau_2, \tau_3) \cong (0,0,0),\ \left(0, \frac{1}{2}, 0\right),\ \left(\frac{1}{2}, 0, \frac{1}{2}\right),\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

If $(\tau_1, \tau_2, \tau_3) \cong (\frac{1}{2}, 0, \frac{1}{2})$, then we get the $\mathbf{S}^2 \times \mathbf{R}$ space group **12. I. 3**.The fundamental domain of its point group is a spherical triangle $A_0A_1A_2$ with angles $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{4}$ lying in the

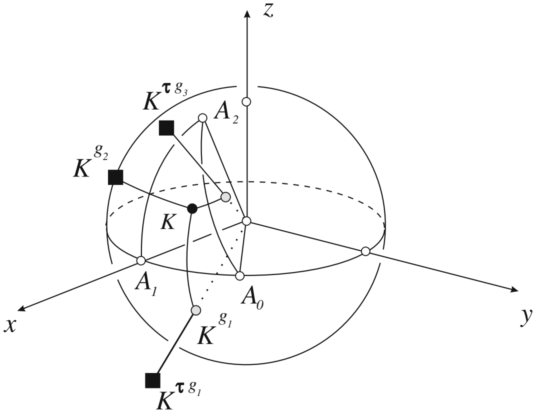

Fig. 2 Some elements of the orbit K^Γ

base plane Π (see Fig. 2). It can be assumed by the homogeneity of $\mathbf{S}^2\times\mathbf{R}$ that the fibre coordinate of the centre of the optimal ball is zero, and it is an interior point of $A_0A_1A_2$ triangle.

We shall apply the Cartesian homogeneous coordinate system introduced in Sect. 2 (see Fig. 2) and the usual geographical coordiantes (ϕ, θ), $(-\pi < \phi \le \pi,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$ of the sphere with the fibre coordinate $t \in \mathbf{R}$ (see (2.2)).

We consider an arbitrary interior point $K(x^0, x^1, x^2, x^3)$ of spherical triangle $A_0A_1A_2$ in the above coordinate system in our model by the following equations:

$$x^0 = 1, \quad x^1 = \cos\phi\cos\theta, \quad x^2 = \sin\phi\cos\theta, \quad x^3 = \sin\theta \tag{3.3}$$

Let $\mathcal{B}_\Gamma(R)$ denote a geodesic ball packing of $\mathbf{S}^2\times\mathbf{R}$ space with balls $B(R)$ of radius R where their centres give rise to the orbit K^Γ. In the following, we consider to each ball packing the possible smallest translation part $\tau(K, R)$ (see Fig. 2) depending on Γ, K and R. A fundamental domain of Γ is its D-V cell $\mathcal{D}(K)$ around the kernel point K. It is clear that the optimal ball $\mathcal{B}_K$ has to touch some faces of its D-V cell. The volume of $D(K)$ is equal to the volume of the prism which is given by the fundamental domain of the point group Γ_0 of Γ and by the height $2\tau(R, K)$. The images of $D(K)$ by our discrete isometry group covers the $\mathbf{S}^2\times\mathbf{R}$ space without overlap. For the density of the packing, it is sufficient to relate the volume of the optimal ball to that of the solid $D(K)$ (see Definition 3.4).

It is clear that the densest ball arrangement $\mathcal{B}_\Gamma(R)$ of balls $B(R)$ has to hold the following requirements:

$$\begin{aligned}
&(a)\ d(K, K^{g_2}) = 2R = d(K, K^{\tau g_1}),\\
&(b)\ d(K, K^{g_2}) = 2R = d(K, K^{\tau g_3}),\\
&(c)\ d(K, K^{2\tau}) \ge 2R\\
&(d)\ \text{Balls of radius}\, R\, \text{with centres}\\
&K, K^{g_2}, K^{\tau g_1}, K^{\tau g_3}, K^{2\tau}\,\text{form a packing.}
\end{aligned} \tag{3.4}$$

Here, d is the distance function in the $\mathbf{S}^2\times\mathbf{R}$ space (see Definition 2.2). The equations (a) and (b) mean that the ball centres $K^{\tau g_1}$ and $K^{\tau g_3}$ lie on the equidistant geodesic surface of the points K and $K^{2\tau}$, which is a sphere in our model in this case (see [6]).

We consider two main ball arrangements:

1. We denote by $\mathcal{B}_\Gamma(R_0, K_0)$ those packing where requirements (3.4)and $d(K, K^{2\tau}) = 2R$ hold (see Fig. 3).
2. We denote by $\mathcal{B}_\Gamma(R_1, K_1)$ those packing where requirements (3.4) and $d(K^{\tau g_1}, K^{\tau g_3}) = 2R$ hold (see Fig. 4).

First, we determine the coordinates of the points K_i, $(i = 1, 2)$ (K_i is given by (3.3) with parameters ϕ and θ), the radius R of the ball, the volume of a ball $B(R)$ and the density of the packing in both main cases. We get the following solutions by systematic approximation, where the computations were carried out by *Maple V Release 10* up to 30 decimals:

$$\begin{aligned} &\phi_0 \approx 0.24389626, \quad \theta_0 \approx 0.20663860, \quad R_0 \approx 0.23860571,\\ &Vol(B(R_0) \approx 0.05668684, \quad \delta(R_0, K_0) \approx 0.45373556. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\phi_1 \approx 0.30773985, \quad \theta_1 \approx 0.17313169, \quad R_1 \approx 0.30299179,\\ &Vol(B(R_1) \approx 0.11580359, \quad \delta(R_1, K_1) \approx 0.44472930. \end{aligned} \tag{3.6}$$

We obtain by careful investigation of the density function $\delta(R, K)$ ($R \in [R_0, R_1]$) of the considered ball packing the following:

Theorem 3.6 *The ball arrangement $\mathcal{B}_\Gamma(R_0, K_0)$ (see Fig. 3) provides the densest simply transitive ball packing belonging to the* $\mathbf{S}^2 \times \mathbf{R}$ *space group* **12. I. 3**.

3.3 The densest simply transitive ball packing

We consider the following point group:

$$\begin{aligned} &(+,\ 0,\ [\ ]\ \{(q, q)\}) \times \mathbf{1_R},\ q \geq 2;\\ &\Gamma_0 = (\mathbf{g}_1, \mathbf{g}_2 - \mathbf{g}_1^2, \mathbf{g}_2^2, (\mathbf{g}_1\mathbf{g}_2)^q). \end{aligned}$$

This point group is generated by two reflections $\mathbf{g}_i$, $i = 1, 2, 3$. The possible translation parts of the generators of Γ_0 will be determined by (1.2) and by the defining relations of the point group. Finally, we obtain from the so-called Frobenius congruence relations three non-equivariant solutions:

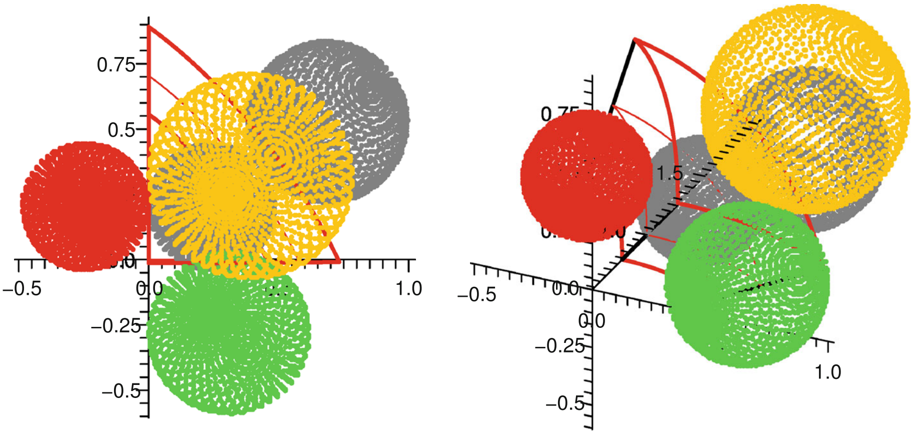

Fig. 3 The densest simply transitive ball packing $\mathcal{B}_\Gamma(R_0, K_0)$

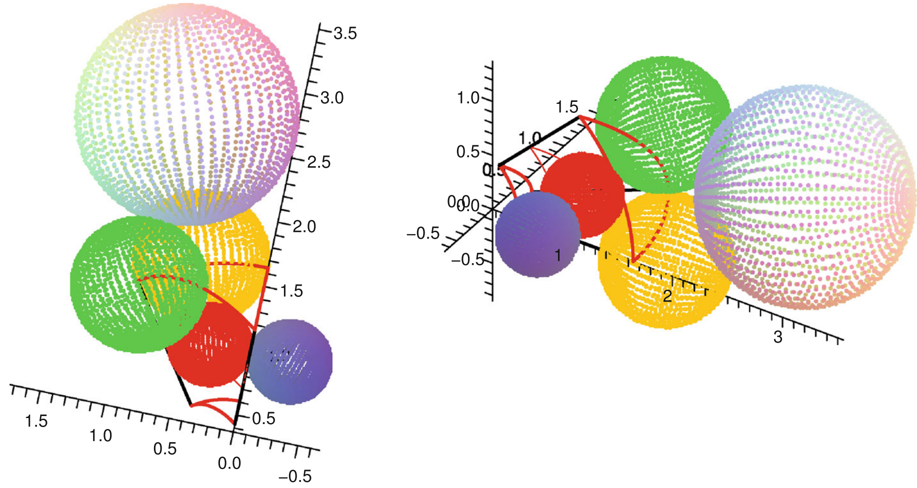

Fig. 4 The ball packing $\mathcal{B}_{\Gamma}(R_1, K_1)$

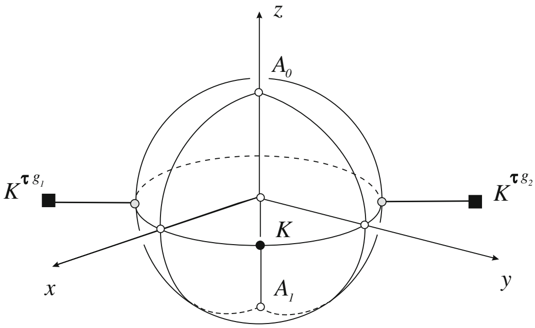

Fig. 5 Some elements of the orbit K^{Γ} by the space group $\Gamma = \mathbf{2q.I.2}$

$$(\tau_1, \tau_2) \cong (0, 0), \quad \left(0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}\right).$$

If $(\tau_1, \tau_2) \cong (\frac{1}{2}, \frac{1}{2})$, then we have obtained the $\mathbf{S}^2 \times \mathbf{R}$ space group **2q. I. 2**.

The fundamental domain of the point group of the considered space group is a spherical digon $A_0 A_1$ with angle $\frac{\pi}{q}$ in the base plane Π. Similarly to the above section, it can be assumed that the fibre coordinate of the centre of the optimal ball is zero, and it is an interior point of $A_0 A_1$ digon (see Fig. 5).

In the following, we consider ball packings belonging to $q = 2$. We use also the above introduced Cartesian homogeneous coordinate system and the usual geographical coordinates (ϕ, θ), $(-\pi < \phi \leq \pi, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$ of the sphere with the fibre coordinate $t \in \mathbf{R}$ (see (2.2)).

We consider an arbitrary interior point $K(1, x^1, x^2, x^3) = K(\phi, \theta)$ of spherical digon $A_0 A_1$ in the above coordinate system in our model (see Fig. 5).

Our aim is to determine the maximal radius R of the balls and the maximal density $\delta(K, R)$.

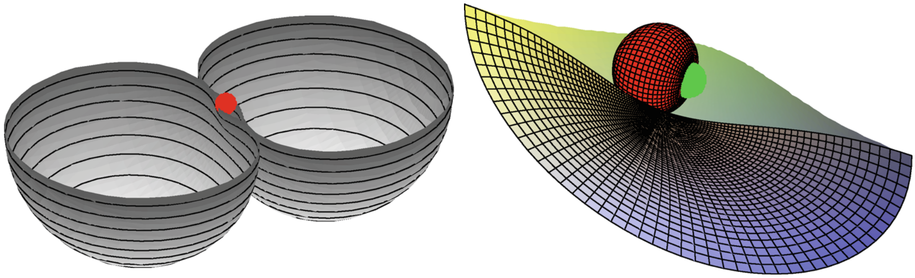

Fig. 6 The densest simply transitive geodesic ball packing

Table 1 The data of the optimal simply transitive ball packings

Space group	R	$Vol(\mathcal{B}_K(R))$	δ
2q. I. 2, $q = 2$	$\frac{\pi}{2} \approx 1.57079633$	≈ 13.74539472	≈ 0.80407553
2qe. I. 3, $q = 2$	$\frac{\pi}{2} \approx 1.57079633$	≈ 13.74539472	≈ 0.69634983
4q. I. 2, $q = 2$	≈ 0.64360446	≈ 1.08624788	≈ 0.53722971
4q. I. 3, $q = 2$	≈ 0.67517586	≈ 1.25058159	≈ 0.58958340
4q. I. 4, $q = 2$	≈ 0.95531662	≈ 3.43551438	≈ 0.74837055
4qe. I. 5, $q = 2$	≈ 0.64360446	≈ 1.08624788	≈ 0.53722971
4qe. I. 6, $q = 2$	≈ 0.67517586	≈ 1.25058159	≈ 0.58958340
11. I. 2	≈ 0.46364761	≈ 0.41154972	≈ 0.58861600
12. I. 2	≈ 0.22770028	≈ 0.04928081	≈ 0.41334779
12. I. 3	≈ 0.23860571	≈ 0.05668684	≈ 0.45373556
12. I. 4	≈ 0.31004511	≈ 0.12404486	≈ 0.53597559
13. I. 2	≈ 0.18705243	≈ 0.02735051	≈ 0.49222087

The ball arrangement $\mathcal{B}_{opt}(K, R)$ is defined by the following equations:

$$(a) d(K, K^{\tau g_1}) = 2R = d(K, K^{\tau g_2}),$$
$$(b) d(K^{\tau g_1}, K^{\tau g_2}) = 2R, \tag{3.7}$$

We can determine the coordinates of the point K, the radius R of the ball, the volume of a ball $B(R)$ and the density of this packing:

$$\phi = \frac{\pi}{4} \approx 0.78539816, \quad \theta = 0, \quad R = \frac{\pi}{2} \approx 1.57079633,$$
$$Vol(B(R) \approx 13.74539472, \quad \delta(R, K) \approx 0.80407553. \tag{3.8}$$

Similarly to the above section, we can prove the following theorem:

Theorem 3.7 *The ball arrangement $\mathcal{B}_{opt}(R, K)$ (see Fig. 6) provides the densest simply transitive ball packing belonging to the* $\mathbf{S}^2 \times \mathbf{R}$ *space group* **2q. I. 2**.

Finally, we get the next theorem (see [11]):

Theorem 3.8 *The ball arrangement $\mathcal{B}_{opt}(R, K)$ provides the densest simply transitive ball packing belonging to the generalized Coxeter and glide reflections generated* $\mathbf{S}^2 \times \mathbf{R}$ *space groups.*

By Theorems 2.7 and 2.8 and by Definitions 3.3 and 3.4, similarly to the above space groups, we have determined the data (radii, densities and volumes of optimal balls) of the optimal simply transitive ball packings to each glide reflections generated $\mathbf{S}^2\times\mathbf{R}$ space group which are summarized in Table 1.

Remark 3.9 The space groups **2q. I. 2, 2qe. I. 3, 4q. I. 2, 4q. I. 3, 4q. I. 4, 4qe. I. 5, f 4qe. I. 6** depend on parameter q; thus, their optimal ball packings depend also on q, but in the Table 1, we give only the data of the densest ball packing indicating its q parameter to each considered space group.

It is timely to arise the above question for further space groups in $\mathbf{S}^2\times\mathbf{R}$ space.

In this paper, we have mentioned only some problems in discrete geometry of $\mathbf{S}^2\times\mathbf{R}$ space, but we hope that from these it can be seen that our projective method suits to study and solve similar problems [6,10,11]. Analogous questions in other homogeneous Thurston geometries are interesting (see [5,7–9,12]).

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