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Capillary surfaces and floating bodies

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Abstract We investigate capillary surfaces with boundary components on a floating body. The unknowns of this problem, the free surface and the position and orientation of the floating body are determined by minimizing the corresponding energy; besides gravitation and cohesion forces, we consider also adhesion between the fluid and both the wall of the container and the floating body. Existence of a solution is shown in the class of Caccioppoli sets under suitable restrictions on the data, as well as for surfaces that are graphs of real functions.

Keywords Capillarity · Variational problem for Caccioppoli sets

Mathematics Subject Classification 76B45

1 Introduction

We consider a cylindrical container $G := \Omega \times \mathbb{R}^+$, where Ω is a bounded, simply connected domain in $\mathbb{R}^2$; G is partially filled with some liquid of density ρ. A rigid body $\mathcal{B}$ whose density ρ_0 is smaller than ρ is floating on the liquid, and it is assumed that the interface between the liquid and the air above is governed by surface tension. Besides the gravitational force,

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there is adhesion between the fluid and both the outer wall of the container and the floating body. We then look for configurations in which these forces are in equilibrium; the unknowns are the position of the floating body and the capillary surface, and they are determined by a variational problem for the energy.

A special version of this problem has been studied in [1]; as [1] is part of an investigation of free-boundary problems for the Navier–Stokes equations, we allowed only for contact angles of $\pi/2$ because in domains with edges of this type solutions to the Navier–Stokes equations are regular up to the boundary. Such a contact angle occurs only if there is no adhesion energy. In the present paper, we prove the existence of capillary surfaces under the standard assumptions, in particular with non-vanishing adhesion energy.

In Sect. 2, we study the energy functional for the case that the region occupied by the fluid is a Caccioppoli set. Let $\mathcal{B}$ denote the rigid body, and let $\mathcal{B}(c, R) := \{y = c + Rx;\ x \in \mathcal{B}\}$ where $c \in \mathbb{R}^3$ denotes a translation and $R = R(d, \alpha) \in SO(3)$ describes a rotation with respect to some axis with direction d, $\|d\| = 1$, about some angle α. The quantities c and R are restricted by requiring that the floating body is contained in G, i.e $\mathcal{B}(c, R) \subseteq G$; $\mathcal{B}(c, R)$ is assumed to be a closed set. If we denote the domain occupied by the fluid by E, we have $E \subseteq G \setminus \mathcal{B}(c, R)$ and $\mathscr{L}^3(E) = V_0$ where V_0 is the volume of the Lebesgue measurable set E. The energy functional then reads

$$\begin{aligned} \mathcal{F}(c, R; E) := \sigma \int\limits_{G\setminus\mathcal{B}(c,R)} |D\varphi_E| + \kappa \int\limits_{\partial G} \varphi_E \, \mathrm{d}\sigma + \kappa_0 \int\limits_{\partial\mathcal{B}(c,R)} \varphi_E \, \mathrm{d}\sigma \\ + \rho g \int\limits_{G\setminus\mathcal{B}(c,R)} x_3 \varphi_E \, \mathrm{d}x + \rho_0 g \int\limits_{\mathcal{B}(c,R)} x_3 \mathrm{d}x, \end{aligned} \tag{1}$$

where

$$\int\limits_{G\setminus\mathcal{B}(c,R)} |D\varphi_E| := \sup \left\{ \int\limits_{G\setminus\mathcal{B}(c,R)} \varphi_E \ \mathrm{div}\ g \ \mathrm{d}x : g \in C_c^1 \left(G \setminus \mathcal{B}(c, R);\ \mathbb{R}^3\right),\ \|g\|_{C^0} \le 1 \right\}$$

denotes the total variation of the characteristic function φ_E of E; σ is the coefficient of surface tension. The integrals $\int_{\partial G} \varphi_E \, \mathrm{d}\sigma$ and $\int_{\partial\mathcal{B}(c,R)} \varphi_E \, \mathrm{d}\sigma$ denote the area of the wetted part of the container's boundary ∂G and the boundary $\partial\mathcal{B}(c, R)$ of the floating body, respectively; the corresponding coefficients of the adhesion energy are κ and κ_0; their relative size with respect to σ determines the contact angles. The last two integrals in (1) represent the gravitational energy of the fluid and of the floating body; g is the gravitational constant. We prove that there is a minimizing configuration $(\mathcal{B}(c, R), E)$ to (1); concerning the data σ, κ and κ_0, we make no further restrictions than in the classical capillary problem without floating bodies. The functional $\mathcal{F}(c, R; E)$ is to be minimized in the class

$$\begin{aligned} \mathcal{C} = \Big\{ (c, R; E) \,:\, c \in \mathbb{R}^3,\ R \in SO(3) \text{ such that } \mathcal{B}(c, R) \subseteq G; \\ E \subset G\setminus\mathcal{B}(c, R) \ \text{ measurable with } \ \mathcal{L}^3(E) = V_0 \Big\}. \end{aligned} \tag{2}$$

In Sect. 3, we formulate the variational problem in the class of graphs. The body $\mathcal{B}$ is assumed to be strictly convex such that the non-wetted part of its boundary can be described by a graph; also the capillary surface is assumed to be a graph which is certainly appropriate in our case because of the influence of the gravitational force. Archimedes' principle characterizes the position of the floating body by an equilibrium condition. In the presence of capillary and adhesion forces, the first variation of the energy leads to an equilibrium of forces that act on $\mathcal{B}$, where the resultant of the surface forces is a vector field on the contact line that is tangential to the capillary surface. This result was first proved by McCuan [4] in the general case of parametric surfaces and deformable bodies.

The existence theorem requires more restrictions on the data compared to the variational problem in the class of Caccioppoli sets. This more special solution allows, however, to prove regularity of the minimizer as well as geometrical properties, in particular that the upper part of ∂B that is not wetted is a simply connected set. These results are contained in Sect. 4.

2 Existence of solutions: the general case

We first prove that the infimum of the energy functional $\mathcal{F}(c, R; E)$ from (1) is attained for some configuration $(c_0, R_0; E_0)$ from $\mathcal{C}$, which is defined in (2). In [1], we adopted the variational method for Caccioppoli sets to the case that a floating body is involved. If also adhesion energy is part of the functional it is well known that only the sum of surface and adhesion energy, $\sigma \int_G |D\varphi_E| + \kappa \int_{\partial G} \varphi_E \,\mathrm{d}\sigma$, is lower semi-continuous with respect to L^1-convergence; the integral over ∂G is generally not lower semi-continuous by itself. To prove semi-continuity, one uses the estimate of Emmer [2]

$$\int\limits_{\partial G} u \,\mathrm{d}\sigma \leq \sqrt{1+L^2} \int\limits_{G(\varepsilon)} |Du| + C_\varepsilon \int\limits_{G(\varepsilon)} |u|, \tag{3}$$

which holds for functions $u \in BV(G).G(\varepsilon)$ is a strip near the boundary, and L denotes the Lipschitz constant of ∂G. In our problem, we consider $G \setminus \mathcal{B}(c, R)$ instead of G; then, the boundary need not be Lipschitz continuous even if ∂G and $\partial\mathcal{B}(c, R)$ are smooth because the rigid body $\mathcal{B}$ might touch the boundary ∂G of the container. Hence, we must prove Emmer's lemma (3) for this case.

In the analog of (3), we require $\partial\Omega$ to be of class C^2 and $\mathcal{B}(c, R)$ to have a projection $P(c, R)$ into Ω such that the curvature of $\partial P(c, R)$ is larger than the curvature of $\partial\Omega$:

$$\min_{R\in SO(3)} \min_{x'\in \partial P(c,R)} K(\partial P(c, R), x') > \max_{x'\in \partial\Omega} K(\partial\Omega, x'),$$

where $K(\gamma, x')$ denotes the curvature of a curve γ at some point $x' \in \gamma$. If then $\mathcal{B}(c, R)$ touches ∂G it does so in a single point y_0. In a neighborhood U_{ε_0} of y_0, we can describe the boundaries ∂G and $\partial\mathcal{B}(c, R)$ by the graphs of the functions

$$y_3 = \omega(y_1, y_2) \quad \text{and} \quad y_3 = \beta(y_1, y_2), \ (y_1, y_2) \in A_{\varepsilon_0} \subseteq E$$

where $(y_1, y_2) \equiv y'$ are cartesian coordinates with center $y'_0 = (0, 0)$ in the tangent plane at y_0. We have $\beta(y') > \omega(y')$ for all y' in the domain A_{ε_0} where σ and ω are defined except for $y' = (0, 0)$.

Local strips between $\partial\mathcal{B}(c, R)$ and ∂G around y_0 can be defined by

$$\mathcal{B}^*_\varepsilon(c, R) := \bigcup_{0<\delta<\varepsilon} \beta^*(\delta) \quad \text{and} \quad G^*_\varepsilon := \bigcup_{0<\delta<\varepsilon} \omega^*(\delta)$$

where

$$\beta^*(\delta) = \left\{ y = \left(y', y_3\right) : \ y_3 = \beta(y') - \delta\tau(y'), \ y' \in A_{\varepsilon_0} \right\}$$
$$\omega^*(\delta) = \left\{ y = \left(y', y_3\right) : \ y_3 = \omega(y') + \delta\tau(y'), \ y' \in A_{\varepsilon_0} \right\}$$

with

$$\tau(y') := \begin{cases} \dfrac{\beta(y') - \omega(y')}{3\,\varepsilon} & \text{if} \quad \beta(y') - \omega(y') \le 3\,\varepsilon \\ 1 & \text{if} \quad \beta(y') - \omega(y') \ge 3\,\varepsilon \end{cases}$$

These sets replace the neighborhoods $G(\varepsilon)$ of constant thickness in Emmer's proof of (3).

Lemma 2.1 *Let ∂G and $\partial \mathcal{B}(c, R)$ touch in a single point y_0 as described above. Then, there holds for $u \in BV(G \backslash \mathcal{B}(c, R))$.*

$$\int\limits_{\partial G \cap U_{\varepsilon_0}} u \, d\sigma \le \sqrt{1+L^2} \int\limits_{G_\varepsilon^*} |Du| + C_\varepsilon \int\limits_{G_\varepsilon^*} |u| \, \mathrm{d}x \tag{4}$$

$$\int\limits_{\partial \mathcal{B}(c,R) \cap U_{\varepsilon_0}} u \, d\sigma \le \sqrt{1+L^2} \int\limits_{\mathcal{B}_\varepsilon^*(c,R)} |Du| + C_\varepsilon \int\limits_{\mathcal{B}_\varepsilon^*(c,R)} |u| \, \mathrm{d}x \tag{5}$$

Here, L is an upper bound for $|D\omega|$ and $|D\beta|$ in A_{ε_0}.

Remark 2.2 **(i)** As we assume that $\partial \mathcal{B}$ and ∂G are regular surfaces, we can make L small by covering ∂G and $\partial \mathcal{B}$ by small open sets. Hence, the value of L is no extra restriction for σ and κ or κ_0 in the existence theorem.

(ii) Away from the point of contact, the original proof of Emmer can be applied because there $G \setminus \partial B(c, R)$ has a smooth boundary.

Proof To $u \in BV(G \setminus \mathcal{B}(c, R))$ we consider the trace u_δ on the surface $\omega^*(\delta)$ for some $\delta \in (0, \varepsilon)$. Then,

$$\int\limits_{\Gamma_{\varepsilon_0}} u \, \mathrm{d}\sigma \le \int\limits_{\Gamma_{\varepsilon_0}} |u - u_\delta| \, \mathrm{d}\sigma + \int\limits_{\Gamma_{\varepsilon_0}} |u_\delta| \, \mathrm{d}\sigma \tag{$*$}$$

with $\Gamma_{\varepsilon_0} = \partial G \cap U_{\varepsilon_0}$, and we estimate the difference $|u - u_\delta|$ by $|Du|$:

$$\begin{aligned} \int\limits_{\Gamma_{\varepsilon_0}} |u - u_\delta| \, \mathrm{d}\sigma &= \int\limits_{A_{\varepsilon_0}} \left| u\left(y', \omega(y')\right) - u\left(y', \omega(y') + \delta\tau(y')\right) \right| \cdot \sqrt{1 + |D\omega\,(y')|^2} \, \mathrm{d}y' \\ &\le \sqrt{1+L^2} \int\limits_{A_{\varepsilon_0}} \int\limits_{\omega(y')}^{\omega(y') + \delta\tau(y')} \left| \frac{\partial u}{\partial y_3}\,(y', t) \right| \mathrm{d}t \, \mathrm{d}y' \\ &\le \sqrt{1+L^2} \int\limits_{G_\varepsilon^*} |Du| \end{aligned}$$

$$\begin{aligned} \int\limits_{\Gamma_{\epsilon_0}} |u_\delta(y)| \, \mathrm{d}\sigma &= \int\limits_{A_{\varepsilon_0}} \left| u\left(y', \omega(y') + \delta\tau(y')\right) \right| \sqrt{1 + |D\left(\omega(y') + \delta\tau(y')\right)|^2} \, \mathrm{d}y' \\ &\quad + \int\limits_{A_{\varepsilon_0}} \left| u\left(y', \omega(y') + \delta\tau(y')\right) \right| \left[\sqrt{1 + |D\omega(y')|^2} \right. \\ &\quad \left. - \sqrt{1 + |D(\omega(y') + \delta\tau(y'))|^2} \right] \mathrm{d}y' \end{aligned}$$

Integration with respect to δ gives $\int_{G_\varepsilon^*}|u|\,\mathrm{d}y$ for the first integral and $C\int_{G_\varepsilon^*}|u|\,\mathrm{d}y$ for the second term. We now integrate (∗) with respect to $\delta \in (0,\varepsilon)$ and obtain

$$\varepsilon\int_{\Gamma_{\varepsilon_0}} u\,\mathrm{d}\sigma \;\le\; \varepsilon\sqrt{1+L^2}\int_{G_\varepsilon^*}|Du| \;+\; C\int_{G_\varepsilon^*}|u|\,\mathrm{d}x,$$

which proves (4). The case (5) can be handled in the same way. □

Theorem 2.3 *Assume that $\partial\mathcal{B}(c,R)$ and ∂G touch in at most one point, and let $\sigma,\rho,\rho_0,\kappa,\kappa_0$ be constants with $\rho>\rho_0>0$, $\sigma>0$ and $\sigma-|\kappa|\sqrt{1-L^2}<0$, $\sigma-|\kappa_0|\sqrt{1+L^2}>0$. Then, there exists an element $(c_0,R_0;E_0)\in\mathcal{C}$, such that*

$$\mathcal{F}(c_0,R_0,E_0)\;\le\;\mathcal{F}(c,R;E)\;\text{ for all }\;(c,R;E)\in\mathcal{C}. \tag{6}$$

Proof We first show that $\mathcal{F}(c,R;E)$ is bounded from below on $\mathcal{C}$. The gravitational energies are clearly positive; for the adhesion terms we have

$$\kappa_0\int_{\partial\mathcal{B}(c,R)}\varphi_E\,\mathrm{d}\sigma,\ge\;-|\kappa_0|\;|\partial\mathcal{B}|\;>\;-\infty$$

and, if $\partial\mathcal{B}(c,R)$ and ∂G touch in a point y_0,

$$\kappa\int_{\partial G\cap U_{\varepsilon_0}}\varphi_E\,\mathrm{d}\sigma\;\ge\;-|\kappa|\sqrt{1+L^2}\int_{G_\varepsilon^*}|D\varphi_E|\;-\;|\kappa|\;\;C_\varepsilon\int_{G_\varepsilon^*}\varphi_E\,\mathrm{d}x,$$

which follows from Lemma 2.1.

For the part of the boundary that lies outside of U_{ε_0} (and that may be all of ∂G if $\partial\mathcal{B}(c,R)$ and ∂G do not touch at all), we have from Sect. 3

$$\kappa\int_{\Gamma\backslash U_{\varepsilon_0}}\varphi_E\,\mathrm{d}\sigma\ge-|\kappa|\sqrt{1+L^2}\int_{G(\varepsilon)\backslash U_{\varepsilon_0}}|D\varphi_E|-C'_\varepsilon\int_{G(\varepsilon)\backslash U_{\varepsilon_0}}\varphi_E\,\mathrm{d}x$$

with

$$G(\varepsilon)=\{x\in G:\mathrm{dist}(x,\Gamma)<\varepsilon\}.$$

The first term is majorized by $\sigma\int_{G\backslash\mathcal{B}(c,R)}|D\varphi_E|$ because of the assumption on the data, and the second one remains finite because of $\int_{G_\varepsilon^*}\varphi_E\,\mathrm{d}x\le\mathcal{L}^3(E)=V_0$.

Therefore, there exists a constant c_0 such that

$$\mathcal{F}(c,R;E)\;\ge\;c_0\;>\;-\infty\;\forall\,(c,R;E)\in\mathcal{C},$$

and a minimizing sequence $\{(c_n,R_n;E_n)\}_{n=1}^{\infty}$ from $\mathcal{C}$:

$$\lim_{n\to\infty}\mathcal{F}(c_n,R_n;E_n)\;=\;m_0\;\equiv\;\inf_{(c,R;E)\in\mathcal{C}}\mathcal{F}(c,R;E).$$

We claim that this sequence is bounded:

$$|c_n|\;+\;|R_n|+\;\|\,\varphi_{E_n}\|_{BV}\;\le\;C\qquad\forall\,n\in\mathbb{N}.$$

We may assume that

$$\mathcal{F}(c_n,R_n;E_n)\;\le\;m_0\;+\;1\quad\forall\,n\in\mathbb{N},$$

and because of $\rho g \int_G x_3 \varphi_{E_n}\, \mathrm{d}x \geq 0$, $\rho_0\, g \int_{\mathcal{B}(c_n, R_n)} x_3\, \mathrm{d}x \geq 0$ and $|\kappa_0 \int_{\partial\mathcal{B}(c_n,R_n)} \varphi_{E_n}\, \mathrm{d}\sigma| \leq c_1$ we have

$$\sigma \int_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}| + \kappa \int_{\partial G} \varphi_{E_n}\, \mathrm{d}\sigma \leq m_0 + 1 + c_1.$$

Using (4) again we get

$$\left(\sigma - |\kappa|\sqrt{1+L^2}\right) \int_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}| \leq m_0 + 1 + c_1 + C_\varepsilon \cdot V_0.$$

Because of the assumption on σ and κ and the volume constraint this gives

$$\| \varphi_{E_n} \|_{L^1(G)} + \int_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}| \leq C.$$

The area integral can be written in the form

$$\int_G \left|D\varphi_{E_n}\right| - \int_{\partial\mathcal{B}(c_n,R_n)} \varphi_{E_n}\, \mathrm{d}\sigma,$$

which gives

$$\| \varphi_{E_n} \|_{L^1(G)} + \int_G \left|D\varphi_{E_n}\right| \leq C.$$

The values of R_n belong to a compact set, hence R_n is bounded independently of $n \in \mathbb{N}$. The x_1- and the x_2 component of c_n is bounded by diam Ω, and the x_3-component is bounded because of

$$\rho_0\, g \int_{\mathcal{B}(c_n,R_n)} x_3\, \mathrm{d}x \leq m_0 + 1,$$

and, thus, we have shown that the minimizing sequence $\{(c_n, R_n; E_n)\}_{n=1}^{\infty}$ is bounded:

$$|c_n| + |R_n| + \|\varphi_{E_n}\|_{BV(G)} \leq C \quad \forall n \in \mathbb{N}.$$

This implies that there is a subsequence, which we again denote by $\{(c_n, R_n, E_n)\}_{n=1}^{\infty}$ that converges for $n \to \infty$:

$$c_n \to c_0,\ R_n \to R_0,\ \varphi_{E_n} \to \varphi_{E_0} \text{ in } L^1(G).$$

In the last steps of the proof, we show that $\mathcal{F}$ is lower semi-continuous with respect to this minimizing sequence.

The integral $\rho_0\, g \int_{\mathcal{B}(c_n,R_n)} x_3\, \mathrm{d}x$ is continuous because $\mathcal{B}(c_n, R_n)$ converges uniformly to $\mathcal{B}(c_0, R_0)$ for $n \to \infty$. The integrand $x_3\varphi_{E_n}(x)$ in the term that represents the gravitational energy of the fluid is non-negative, and therefore the integral is lower semi-continuous according to Fatou's lemma.

In order to compare the traces of φ_{E_n} and φ_{E_0} on the boundary of the rigid body, we set

$$\widetilde{E}_n := T_n\, E_n,$$

where T_n is the rigid motion that maps $\mathcal{B}(c_n, R_n)$ onto $\mathcal{B}(c_0, R_0)$; $\widetilde{E}_n$ is generally not an element of $\mathcal{C}$, but $(\varphi_{\widetilde{E}_n} - \varphi_{E_0})$ restricted to $\partial\mathcal{B}(c_0, R_0)$ can be estimated by (5). For the area energy

$$\mathscr{A}(c, R; E) := \sigma \int_{G\setminus\mathcal{B}(c,R)} |D\varphi_E| + \kappa \int_{\partial G} \varphi_E \, \mathrm{d}\sigma + \kappa_0 \int_{\partial\mathcal{B}(c,R)} \varphi_E \, \mathrm{d}\sigma$$

we get using

$$\kappa_0 \int_{\partial\mathcal{B}(c_0,R_0)} \varphi_{E_0} \, \mathrm{d}\sigma - \kappa_0 \int_{\partial\mathcal{B}(c_n,R_n)} \varphi_{E_n} \, \mathrm{d}\sigma = \kappa_0 \int_{\partial\mathcal{B}(c_0,R_0)} \varphi_{E_0} - \varphi_{\widetilde{E}_n} \, \mathrm{d}\sigma$$

and then (5) again:

$$\begin{aligned}
&\mathscr{A}(c_0, R_0; R_0) - \mathscr{A}(c_n, R_n; E_n) \\
&\le \sigma \int_{G\setminus(\mathcal{B}^*_\varepsilon(c_0,R_0)\cup G^*_\varepsilon)} |D\varphi_{E_0}| + \sigma \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} |D\varphi_{E_0}| + \sigma \int_{G^*_\varepsilon} |D\varphi_{E_0}| \\
&- \sigma \int_{G\setminus(\mathcal{B}^*_\varepsilon(c_n,R_n)\cup G^*_\varepsilon)} |D\varphi_{E_n}| - \sigma \int_{\mathcal{B}^*_\varepsilon(c_n,R_n)} \left|D\varphi_{E_n}\right| - \sigma \int_{G^*_\varepsilon} \left|D\varphi_{E_n}\right| \\
&+ |\kappa|\sqrt{1+L^2} \int_{G^*_\varepsilon} |D(\varphi_{E_0} - \varphi_{E_n})| + |\kappa|\, C_\varepsilon \int_{G^*_\varepsilon} |\varphi_{E_0} - \varphi_{E_n}| \, \mathrm{d}x \\
&+ |\kappa_0|\sqrt{1+L^2} \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} \left|D(\varphi_{E_0} - \varphi_{\widetilde{E}_n})\right| + |\kappa_0|\, C_\varepsilon \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} \left|\varphi_{E_0} - \varphi_{\widetilde{E}_n}\right| \mathrm{d}x.
\end{aligned}$$

Now we have

- **(i)** $\left(|\kappa|\sqrt{1+L^2} - \sigma\right) \int_{G^*_\varepsilon} |D\varphi_{E_n}| \le 0 \forall n \in \mathbb{N}$ because of the assumptions on the data.
- **(ii)** $|\kappa| C_\varepsilon \int_{G^*_\varepsilon} |\varphi_{E_0} - \varphi_{E_n}| \; \mathrm{d}x \to 0, n \to \infty$, for every $\varepsilon > 0$.
- **(iii)** $\int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} |\varphi_{E_0} - \varphi_{\widetilde{E}_n}| \; \mathrm{d}x \le \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} |\varphi_{E_0} - \varphi_{E_n}| \; \mathrm{d}x + \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} |\varphi_{E_n} - \varphi_{\widetilde{E}_n}| \; \mathrm{d}x$, and both integrals converge to zero for $n \to \infty$.
- **(iv)** $\int_{\mathcal{B}^*_\varepsilon} |D\varphi_{\widetilde{E}_n}| = \mathcal{H}^2(\partial\widetilde{E}_n \cap \mathcal{B}^*(c_0, R_0))$; the Hausdorff measure $\mathcal{H}^2$ is invariant under rigid motions, and therefore we have $\mathcal{H}^2(T_n(\partial E_n) \cap \mathcal{B}^*_\varepsilon(c_0, R_0)) = \mathcal{H}^2(T_n^{-1}(T_n(\partial E_n) \cap \mathcal{B}^*_\varepsilon(c_0, R_0))) = \mathcal{H}^2(\partial E_n \cap \mathcal{B}^*_\varepsilon(c_n, R_n))$ which gives $|\kappa_0|\sqrt{1+L^2} \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} |D\varphi_{\widetilde{E}_n}| - \sigma \int_{\mathcal{B}^*_\varepsilon(c_n,R_n)} |D\varphi_{E_n}| \le 0$.
- **(v)** The integrals $\sigma \int_{\mathcal{B}^*_\varepsilon(c_0,R_0)} |D\varphi_{E_0}|$ and $\sigma \int_{G^*_\varepsilon} |D\varphi_{E_0}|$ are of order $O(\varepsilon)$.
- **(vi)** $\int_{G\setminus\mathcal{B}^*_\varepsilon(c_0,R_0)} |D\varphi_{E_0}| \le \liminf_{n\to\infty} \int_{G\setminus\mathcal{B}^*_\varepsilon(c_n,R_n)} |D\varphi_{E_n}|$ has been proved in [1].

Using **(i)–(vi)** in the inequality for $\mathscr{A}(c, R; E)$, we obtain the lower semi-continuity of $\mathcal{F}$, and this concludes the proof of Theorem 2.3. □

3 Existence of solutions in the class of graphs

We now investigate capillary surfaces Σ in the presence of floating bodies under the assumption that both Σ and the upper part of $\partial\mathcal{B}(c, R)$ are graphs of real functions u and h, respectively; we use the same notation as in [1]. If we assume that $\mathcal{B} = \mathcal{B}(0, R)$ is strictly convex, there is a smallest number $\alpha = \alpha(R)$ such that

$$\partial\mathcal{B}_\alpha \equiv \partial\mathcal{B}(0,R)\cap\{x : x_3 > \alpha(R)\}$$

can be written as the graph of a real function $\overline{h} : B(\alpha) \to \mathbb{R}$, where $B(\alpha)$ is the projection of $\partial\mathcal{B}_\alpha$ onto the x'-plane. We assume that there is another number $h_0(R) > \alpha(R)$, such that

$$vol\ \ (\mathcal{B}(0,R)\cap\{x : x_3 > h_0(R)\}) = \frac{1}{3}\, vol\ (\mathcal{B})$$

and define $B(h_0) := \{x' \in B(\alpha) \colon \overline{h}(x') > h_0\},\ C(h_0) := \partial B(h_0)$.

Then, we can define the function $h\colon \Omega \to \mathbb{R}$ by

$$h(x') := \begin{cases} \overline{h}(x'), & x' \in B(h_0) \\ h_0, & x' \in \Omega \setminus \overline{B(h_0)}. \end{cases} \tag{7}$$

This function enters the variational problem as an obstacle. The surface Σ is assumed to be the graph of a real function $\overline{u} : \Omega \setminus B(h) \to \mathbb{R}$ where $B(h) \subseteq B(h_0)$ is the projection of the dry part of $\partial\mathcal{B}_\alpha$ into the x'-plane, which means that $\Gamma := \operatorname{graph} h(\gamma)$, is the contact line where $h(x') = u(x'),\ x' \in \gamma$, holds. In order to work with functions that are defined on the whole domain Ω, we set

$$u(x') := \begin{cases} \overline{u}(x'), & x' \in \Omega \setminus \overline{B(h)} \\ \overline{h}(x'), & x' \in B(h). \end{cases} \tag{8}$$

With these quantities, the energy (1) is of the form

$$\begin{aligned} \mathcal{F}(c,R;u) = \sigma \int_\Omega \sqrt{1+|Du|^2} &+ \sigma \int_{B(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\ \mathrm{d}x' - \sigma \int_{B(h_0)} \sqrt{1+|Dh|^2}\ \mathrm{d}x' \\ &+ \kappa_0 \left\{ |\partial\mathcal{B}| + \int_{\mathcal{B}(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\ \mathrm{d}x' - \int_{B(h_0)} \sqrt{1+|Dh|^2}\ \mathrm{d}x' \right\} \\ &+ \kappa \int_{\partial\Omega} u\ \mathrm{d}\sigma + \frac{\rho g}{2} \int_\Omega u^2\,\mathrm{d}x' - (\rho-\rho_0)\, g \int_{\mathcal{B}(c,R)} x_3\ \mathrm{d}x \end{aligned} \tag{9}$$

where h and h_0 depend on c and R. The first integral is defined for $u \in BV(\Omega)$, which means

$$\begin{aligned} \int_\Omega \sqrt{1+|Du|^2} :\ = \ \sup \Bigg\{ \int_\Omega g_3 \ +\ u \left(\frac{\partial g_1}{\partial x_2} + \frac{\partial g_2}{\partial x_3} \right) \mathrm{d}x' : \\ g = (g_1, g_2, g_3) \in C_c^1\left(\Omega;\mathbb{R}^3\right),\ \|\ g\ \|_{C^0(\Omega)} \le\ 1 \Bigg\}. \end{aligned}$$

If u is a smooth function with $u > h_0$, the first three integrals equal $\int_{\Omega\cap\{u>h\}} \sqrt{1+|Du|^2}\mathrm{d}x'$ and this is the area of the capillary surface; the quantities in brackets equal the area of the wetted part $\Sigma_{\mathcal{B}}$ of $\partial\mathcal{B}(c,R)$. We look for minimizers in the class

$$\begin{aligned} \mathcal{C}\ :=\ \Big\{(c,R;u) \in \mathbb{R}^3 \times SO(3) \times BV(\Omega) \colon \mathcal{B}(c,R)\ \subseteq\ G; \\ \int_\Omega u(x')\,\mathrm{d}x' = V_0 + vol(\mathcal{B}),\ u(x')\ \ge\ h(x')\ \text{ a.e. in }\ \Omega \Big\}. \end{aligned} \tag{10}$$

In this setup, we can calculate the first variation of $\mathcal{F}$; variations of u lead to the Euler–Lagrange equations

$$\sigma\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \rho g u + \lambda \text{ in } \Omega \setminus \overline{B(h)} \tag{11}$$

and the boundary conditions

$$\frac{Du\cdot n}{\sqrt{1+|Du|^2}} = -\frac{\kappa}{\sigma} \quad \text{on } \partial\Omega \tag{12}$$

$$\frac{1+Du\cdot Dh}{\sqrt{1+|Du|^2}\cdot\sqrt{1+|Dh|^2}} = -\frac{\kappa_0}{\sigma} \quad \text{on } \gamma. \tag{13}$$

Remark 3.1 For the proof of (13), we assume that u is smooth up to the contact line, and therefore the surface energy and the adhesion energy can be written in the form

$$I_1(u) + I_2(u) = \sigma \int\limits_{\{u>h\}} \sqrt{1+|Du|^2}\ \mathrm{d}x' + \kappa_0 \int\limits_{\{u>h\}} \sqrt{1+|Dh|^2}\ \mathrm{d}x'.$$

we consider only these two integrals because the rest of the energy terms will not be affected by the variations that determine the contact angle. For the contact line $\Gamma = \{(x', x_3) \in \partial\mathcal{B}(c,R) : x_3 = u(x')\}$, we have $u(x') = h(x')\ \ \forall x' \in \gamma$, and hence the normal to γ is

$$n(x_0') = -\frac{D(u-h)(x_0')}{\left|D(u-h)(x_0')\right|}.$$

Curves that lie in a neighborhood of γ can be described as $\gamma_\delta = \{x' \in \Omega : x' = x_0' + \delta n(x_0'),\ x_0 \in \gamma\}$. A variation $u(x') + \varepsilon\,\varphi(x')$ of u meets $\partial\mathcal{B}(c,R)$ in some curve near Γ, and it is given by $u(x') + \varepsilon\,\varphi(x') = h(x')\ \ \forall x' \in \gamma_\delta$, where $\delta = \delta(\varepsilon, x_0')$ will be determined by $u(x_0' + \delta(\varepsilon, x_0')\, n\,(x_0')) + \varepsilon\varphi(x_0' + \delta(\varepsilon, x_0')\, n(x_0')) = h(x_0' + \delta(\varepsilon, x_0')\, n(x_0'))$ which leads to

$$u(x_0') + \delta(\varepsilon, x_0')Du(x_0')\cdot n(x_0') + \varepsilon\varphi(x_0') = h(x_0') + \delta(\varepsilon, x_0')Dh(x_0')\cdot n(x_0') + \mathrm{o}(\varepsilon)$$

for $|\varepsilon|$ small; therefore, we get

$$\delta(\varepsilon, x_0') = -\frac{\varepsilon\cdot\varphi(x_0')}{D(u-h)\,(x_0')\cdot n(x_0')}.$$

The first variation of I_1 is

$$\delta I_1(u,\varphi) = \lim_{\varepsilon\to 0} \frac{\sigma}{\varepsilon}\left\{ \int\limits_{\{u+\varepsilon\varphi>h\}} \sqrt{1+|D(u+\varepsilon\varphi)|^2}\,\mathrm{d}x' - \int\limits_{\{u>h\}} \sqrt{1+|Du|^2}\,\mathrm{d}x' \right\}$$

with

$$\frac{1}{\varepsilon}\int\limits_{\{u>h\}} \sqrt{1+|D(u+\varepsilon\varphi)|^2} - \sqrt{1+|Du|^2}\ \mathrm{d}x' \longrightarrow \int\limits_{\{u>h\}} \frac{Du\cdot D\varphi}{\sqrt{1+|Du|^2}}\ \mathrm{d}x'$$

$$= -\int\limits_{\{u>h\}} \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\varphi\ \ \mathrm{d}x' + \oint\limits_{\gamma} \frac{Du\cdot n}{\sqrt{1+|Du|^2}}\,\varphi\,\mathrm{d}s$$

and

$$\frac{1}{\varepsilon}\left(\int\limits_{\{u+\varepsilon\varphi>h\}}-\int\limits_{\{u>h\}}\right)\sqrt{1+|D(u+\varepsilon\varphi)|^2}\,\mathrm{d}x'$$
$$=\frac{1}{\varepsilon}\oint\limits_{\gamma}\int\limits_{0}^{\delta(\varepsilon,x_0')}\sqrt{1+\left|D(u+\varepsilon\varphi)\,(x_0'\,+\,tn(x_0')\right|^2}\,\mathrm{d}t\,\mathrm{d}s$$
$$\longrightarrow\oint\limits_{\gamma}\sqrt{1+\left|D(u+\varepsilon\varphi)\,(x_0'\,+\,\delta(\varepsilon,x_0')\,n(x_0')\right|^2}_{|\varepsilon=0}\cdot\,\delta'(\varepsilon,x_0')_{|\varepsilon=0}\mathrm{d}s$$
$$=\oint\limits_{\gamma}\sqrt{1+|Du|^2}\cdot\left(\frac{-\varphi}{D(u-h)\cdot n}\right)\mathrm{d}s.$$

In the same way, we get

$$\delta I_2(u,\varphi)\;=\;\kappa_0\oint\limits_{\gamma}\sqrt{1+|Dh|^2}\cdot\left(\frac{-\varphi}{D(u-h)\cdot n}\right)\mathrm{d}s.$$

Finally

$$\sigma\oint\limits_{\gamma}\frac{Du\cdot n}{\sqrt{1+|Du|^2}}\cdot\varphi\,\mathrm{d}s+\sigma\oint\limits_{\gamma}\sqrt{1+|Du|^2}\cdot\left(-\frac{\varphi}{D(u-h)\cdot n}\right)\mathrm{d}s$$
$$+\kappa_0\oint\limits_{\gamma}\sqrt{1+|Dh|^2}\cdot\left(-\frac{\varphi}{D(u-h)\cdot n}\right)\mathrm{d}s=0$$

for all $\varphi\in C^0(\gamma)$ gives the equation for the contact angle:

$$\sigma\left\{\frac{Du\cdot n}{\sqrt{1+|Du|^2}}-\frac{\sqrt{1+|Du|^2}}{D(u-h)\cdot n}\right\}+\kappa_0\,\frac{-\sqrt{1+|Dh|^2}}{D(u-h)\cdot n}=0$$

which implies

$$\sigma\,\left\{-\,|Du|^2+\,Du\cdot Dh\,+\,1\,-\,|Du|^2\right\}+\kappa_0\sqrt{1+|Du|^2}\,\sqrt{1+|Dh|^2}\;=\;0$$

and thus

$$\frac{(Du,-1)}{\sqrt{1+|Du|^2}}\cdot\frac{(Dh,-1)}{\sqrt{1+|Dh|^2}}\;=\;-\frac{\kappa_0}{\sigma}$$

which means that $-\frac{\kappa_0}{\sigma}=\cos\vartheta$ gives the contact angle between the capillary surface and the floating body.

The contact angle from (13) is restricted to some interval $[0,\vartheta_0]$ with $\vartheta_0<\pi$. This means that the tangent of graph u ranges between being parallel to the tangent of graph h (because of $u\geq h$) and pointing in the x_3-direction (because Σ is a graph). Therefore, κ_0 must be restricted, too:

$$-\,\frac{\kappa_0}{\sigma}\;\equiv\;\cos\alpha_0\;\geq\;-\,\frac{\left|Dh(R;x')\right|}{\sqrt{1+|Dh(R;x')|^2}}\quad\forall\,x'\in B(h_0),\;R\in SO(3);\qquad(14)$$

with this condition on the data, we can perform variations $T(x) = e+d \wedge x$ of the position and orientation of $\mathcal{B}(c, R)$, that is of $h(c, R; x')$, and get from [1] Theorem 3.2 the equilibrium condition

$$\oint_{\Gamma} E \cdot N \,\mathrm{d}s + pg \int_{\Sigma_{\mathcal{B}}} - E \cdot N_{\mathcal{B}}\, x_3 \,\mathrm{d}\sigma - \rho_0 g \ (e + d \wedge x_s)_3 \ vol(\mathcal{B}) = 0 \qquad (15)$$

with $E \cdot N$, the component of the variation E that is normal to the contact line Γ and tangent to graph (u), $N_{\mathcal{B}}$ the normal $\partial\mathcal{B}$ and x_s the center of gravity of $\mathcal{B}(c, R)$.

Theorem 3.2 *For $G, h = h(c, R)$, and $\sigma, \rho_0, \rho, \kappa_0, \kappa$ as above with $\sigma > |\kappa|$ the variational problem*

$$\mathcal{F}(c, R; u) \to \min \quad in\ \mathcal{C} \qquad (16)$$

has a solution $(c_0, R_0; u_0) \in \mathcal{C}$.

Proof **(i)** $\mathcal{F}(c, R; u)$ is bounded from below on $\mathcal{C}$, because $\kappa \int_{\partial G} u \,\mathrm{d}\sigma$ can be estimated by Emmer's Lemma (3) and the condition on κ. As $|\partial\mathcal{B}|$ is finite, the adhesion term $\kappa_0\{\,|\partial\mathcal{B}| - \int_{B(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\,\mathrm{d}x' + \int_{B(h_0)} \sqrt{1+|Dh|^2}\,\mathrm{d}x'\}$ can be estimated for any κ_0.

(ii) Any minimizing sequence $\{(c_n, R_n; u_n)\}_{n=1}^{\infty} \subseteq \mathcal{C}$ satisfies $\lim_{n\to\infty} \mathcal{F}(c_n, R_n; u_n) = m_0 := \inf\{\mathcal{F}(c, R; u) : (c, R; u) \in \mathcal{C}\}$. It is bounded: $|c_n| + |R_n| + ||u_n||_{BV(\Omega)} \leq C \forall n \in \mathbb{N}$. This is known for $\mathcal{E}(c, R; u)$; the adhesion energy can be treated as in **(i)**.

(iii) A sequence $\{u_n\}_{n=1}^{\infty}$ that is bounded in $BV(\Omega)$ has a subsequence which converges in $L^1(\Omega)$. Then, $\mathcal{F}$ is lower semi-continuous for a minimizing sequence $\{(c_n, R_n; u_n)\}$ with $u_n \to u_0$ in $L^1(\Omega)$, $n \to \infty$, and this proves the existence of a minimizer. □

Remark 3.3 Also in this setup, the floating body can touch the wall of the container. But as we are working with real functions u that satisfy $u \geq h_0$ on $\partial\Omega$ and a possible point of contact lies below $\{x_3 = h_0\}$ it does not affect u at all. Additional considerations in Emmer's Lemma are therefore not necessary.

4 Boundedness, regularity, and geometrical properties of the solution

We first prove that a minimizer u to the variational problem (16) is essentially bounded. In order to show regularity of the solutions, we have to make sure that u does not touch the obstacle h_0, that is, $u(x') > h_0$ a.e. in $\Omega_0 := \Omega \setminus \overline{B(h_0)}$, because then the usual variations of u lie in $\mathcal{C}$, and therefore, the standard methods from regularity theory for the mean curvature equation apply. Finally we show that the set $\{x' \in \Omega : u(x') = h(x')\}$ is simply connected.

Lemma 4.1 *Let $(c, R; u)$ be a solution to the variational problem*

$$\mathcal{F}(c, R; u) \to \min \quad in\ \mathcal{C}. \qquad (17)$$

Then, there exists a constant C, such that

$$u(x') \leq C \ a.e.\ in \quad \Omega.$$

Proof For $\kappa = \kappa_0 = 0$, the result was proved in [1], Lemma 5.1 by using the comparison function

$$v_{t,\varepsilon}(x') = \begin{cases} \min\big(u(x'), t\big) + \varepsilon & \text{for } x' \in \{u > h\} \\ u(x') & \text{for } x' \in \Omega \setminus \{u > h\}, \end{cases}$$

where ε was chosen to be

$$\varepsilon = \frac{1}{|\{u > h\}|} \int\limits_{A(t)} u - t \,\mathrm{d}x', \quad A(t) := \left\{x' \in \Omega\colon\, u(x') > t\right\}.$$

Because of this $v_{t,\varepsilon}$ satisfies the volume constraint and therefore belongs to $\mathcal{C}$. This implies

$$\mathcal{F}(c, R; u) \leq \mathcal{F}\left(c, R; v_{t,\varepsilon}\right), \tag{18}$$

in particular for large values of t, and from this inequality it was deduced that there is a $t^* > 0$ such that the measure of $A(t^*)$ vanishes, which means that u is essentially bounded.

The adhesion energy of the body $\mathcal{B}$ is

$$\kappa_0 \left\{ |\partial\mathcal{B}| + \int\limits_{B(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\,\mathrm{d}x' - \int\limits_{B(h_0)} \sqrt{1+|Dh|^2}\,\mathrm{d}x' \right\},$$

and this quantity remains unchanged if we replace u by $v_{t,\varepsilon}$. Hence, this term does not affect inequality (18).

To control the adhesion energy of the walls of the container, we have to compare $\kappa \int\limits_{\partial\Omega} u \,\mathrm{d}\sigma$ and $\kappa \int\limits_{\partial\Omega} v_{t,\varepsilon}\,\mathrm{d}\sigma$. Then, we get

$$\begin{aligned}
\int\limits_{\partial\Omega} v_{t,\varepsilon}\,\mathrm{d}\sigma &= \int\limits_{\partial\Omega\setminus\overline{\{u>h_0\}}} v_{t,\varepsilon}\,\mathrm{d}\sigma + \int\limits_{\partial\Omega\cap\{u>h_0\}} v_{t,\varepsilon}\,\mathrm{d}\sigma \\
&= \int\limits_{\partial\Omega\setminus\overline{\{u>h_0\}}} u\,\mathrm{d}\sigma + \int\limits_{\partial\Omega\cap\{u>h_0\}} \min(u,t)+\varepsilon\,\mathrm{d}\sigma \\
&= \int\limits_{\partial\Omega\setminus\overline{\{u>h_0\}}} u\,\mathrm{d}\sigma + \int\limits_{\partial\Omega\cap\{u>h_0\}\cap A(t)} t+\varepsilon\,\mathrm{d}\sigma + \int\limits_{[\partial\Omega\cap\{u>h_0\}]\setminus A(t)} u+\varepsilon\,\mathrm{d}\sigma.
\end{aligned}$$

Recalling the choice of ε, we obtain:

$$\begin{aligned}
\int\limits_{\partial\Omega} u\,\mathrm{d}\sigma - \int\limits_{\partial\Omega} v_{t,\varepsilon}\,\mathrm{d}\sigma &= -\varepsilon\,[\,|\partial\Omega\cap\{u>h_0\}\cap A(t)| \\
&\quad + |\,[\partial\Omega\cap\{u>h_0\}]\setminus A(t)|\,] + \int\limits_{\partial\Omega\cap\{u>h_0\}\cap A(t)} u-t\,\mathrm{d}\sigma \\
&= -\frac{|\partial\Omega\cap\{u>h_0\}|}{|\{u>h\}|}\int\limits_{A(t)} u-t\,\mathrm{d}x' + \int\limits_{\partial\Omega\cap A(t)} u-t\,\mathrm{d}\sigma.
\end{aligned}$$

We now apply Emmer's lemma to the second term and get

$$\begin{aligned}
\int\limits_{\partial\Omega} u\,\mathrm{d}\sigma - \int\limits_{\partial\Omega} v_{t,\varepsilon}\,\mathrm{d}\sigma \geq &- \frac{|\partial\Omega\cap\{u>h_0\}|}{|\{u>h\}|}\int\limits_{A(t)} u-t\,\mathrm{d}x' \\
&- \sqrt{1+L^2}\int\limits_{A(t)} |Du| - C\int\limits_{A(t)} (u-t)\,\mathrm{d}x'.
\end{aligned}$$

If we use this inequality in (18) and proceed with the other terms as in the proof of Lemma 5.1 from [1], we arrive at the same inequality (33) from [1], only with a different constant. The rest of the proof follows then again from [3], pp. 211–213. □

Lemma 4.2 *Let* $(c, R; u)$ *be a solution to the variational problem*

$$\mathcal{F}(c, R; u) \to \min \qquad \textit{in } \mathcal{C}. \tag{19}$$

For the coefficients in the energy, we assume

$$|\kappa|, |\kappa_0| < \sigma \ll \rho g$$

in case κ *or* κ_0 *are negative. Then, there holds*

$$u(x') > h_0 \qquad \textit{a.e. on } \Omega_0.$$

Proof For κ and $\kappa_0 \geq 0$ the proof from [1], Lemma 5.2, applies because if the graph of u is cutoff at some height t, the parts of $\partial\Omega$ and $\partial\mathcal{B}$ that are wetted according to the function $\min(u, t)$ have smaller area than before and hence the adhesion energy decreases because of $\kappa, \kappa_0 \geq 0$. If κ or κ_0 is negative, we therefore cut off u only locally such that the adhesion energy is not affected. As this cannot be done in such a way that the area of u decreases, we must compensate a possible increase in the surface energy by an appropriate change in the gravitational energy. And this can be achieved if $\sigma/\rho g$ is sufficiently small. We now assume the proposition of the lemma not to be true, which means that there is a set $A \subseteq \Omega_0$ of positive measure such that $u(x') = h_0\ \forall\, x' \in A$. As in [1], we distinguish several cases.

(i) There is a $\delta > 0$, such that

$$u(x') > h_0 + \delta$$

for all x' of some subset of $\partial\Omega$ that has positive measure. Then, the boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{Dw}{\sqrt{1+|Dw|^2}}\right) = 0 \ \text{ in } \ \Omega_0 \\ w = h_0 \ \text{ on } \ C(h_0),\ w = \min(u, h_0 + \delta) \ \text{ on } \partial\Omega \end{cases}$$

admits a unique regular solution w_δ, if δ is small enough. This solution satisfies the strong maximum principle

$$h_0 < w_\delta(x') < h_0 + \delta \forall\, x' \in \Omega_0,$$

and therefore,

$$u_\delta := \min(u, w_\delta)$$

satisfies $u_\delta(x') > h_0$, a.e., on Ω_0. Because w_δ minimizes the area locally we have

$$\int_{\{u<w_\delta\}} \sqrt{1+|Du_\delta|^2}\,\mathrm{d}x < \int_{\{u<w_\delta\}} \sqrt{1+|Du|^2}.$$

We now change u_δ such that it includes the same volume as u and hence belongs to $\mathcal{C}$. Let s be a real number such that

$$\int_{A(s)} u(x') - s\,\mathrm{d}x' = 2 \cdot V_\delta,$$

where $V_\delta := \int_{\{u<w_\delta\}} w_\delta - u\,\mathrm{d}x'$.

Then, we choose $\varphi \in C_c^1(A(s))$ such that $\int_{A(s)} \varphi(x')\,\mathrm{d}x' = 1$. We have $u_{\delta,t}(x') \equiv u_\delta(x') - t\varphi(x') = u(x')$ for all $x' \in \partial\Omega \cup C(h)$, and the adhesion energy remains the same when we consider $u - t\varphi$ instead of u. With $t = V_\delta$ we have $\int_\Omega u_\delta - t\varphi\,\mathrm{d}x' = \int_\Omega u\,\mathrm{d}x'$, and for the area we get

$$\begin{aligned}&\int\limits_{\Omega\cap\{u>h\}} \sqrt{1 + |\,D(u - t\varphi)\,|^2}\,\mathrm{d}x' \\ &\quad = \int\limits_{\Omega\cap\{u>h\}} \sqrt{1+|Du|^2}\,\mathrm{d}x' - t \int\limits_{A(t)} \frac{Du \cdot D\varphi}{\sqrt{1+|Du|^2}}\,\mathrm{d}x' + \mathrm{O}\left(t^2\right)\end{aligned}$$

hence,

$$\int\limits_{\Omega\cap\{u>h\}} \sqrt{1+|D(u-t\varphi)|^2}\;\mathrm{d}x' \le \int\limits_{\Omega\cap\{u>h\}} \sqrt{1+|Du|^2}\;\mathrm{d}x' + V_\delta \int\limits_{A(s)} |D\varphi|.$$

The change in the gravitational energy is, up to the constant $\frac{\rho g}{2}$:

$$\begin{aligned}&\int\limits_{\Omega\cap\{u>h\}} u^2 - h^2\,\mathrm{d}x' - \int\limits_{\Omega\cap\{u>h\}} (u-t\varphi)^2 - h^2\,\mathrm{d}x' \\ &\quad = \int\limits_{A(s)} u^2 - (u-t\varphi)^2\,\mathrm{d}x' + \int\limits_{\Omega_0\cap\{u<w_\delta\}} u^2 - w_\delta^2\,\mathrm{d}x'.\end{aligned}$$

We have

$$\begin{aligned}\int\limits_{\Omega_0\cap\{u<w_\delta\}} u^2 - w_\delta^2\,\mathrm{d}x' &= \int\limits_{\Omega_0\cap\{u<w_\delta\}} (u+w_\delta)\,(u-w_\delta)\,\mathrm{d}x' \\ &\ge \min_{\Omega_0\cap\{u<w_\delta\}} (u+w_\delta) \cdot \int\limits_{\Omega_0\cap\{u<w_\delta\}} u - w_\delta\,\mathrm{d}x' \ge -\,2h_0\,V_\delta,\end{aligned}$$

and in the same way

$$\int\limits_{A(s)} u^2 - (u-t\varphi)^2\,\mathrm{d}x' \ge 2s \cdot V_\delta,$$

such that we get for the total energy

$$\mathcal{F}(R, c; u) - \mathcal{F}\left(R, c; u_{\delta,t}\right) \ge -\sigma \cdot V_\delta \cdot C + \frac{\rho g}{2}\,(2s - 2h_0)\,V_\delta > 0,$$

and this is a contradiction to u being a minimizer of $\mathcal{F}$.

(ii) There is a $\delta > 0$, such that

$$u(x') > h_0 + \delta$$

for all x' of some subset of $C(h_0)$ that has positive measure. As in [1], Lemma 5.2, we construct a function w_δ that minimizes $\int_{\Omega_0} \sqrt{1+|Dw|^2}$ among all functions in $BV(\Omega_0)$ with boundary data $w = u$ on $\partial\Omega_0$. We then correct the comparison function such that it satisfies the volume constraint; this can be done as in **(i)** above.

(iii) If $u(x') > h_0 + \delta$ holds on subsets of positive measure of $C(h_0)$ and of $\partial\Omega$, then we can argue as in **(i)**. Therefore, it remains to consider the case that $u(x') = h_0$ on all of $C(h_0)$ and $\partial\Omega$.

(iv) Now assume that $u(x') \equiv h_0$ on $\Omega_0 \cup C(h_0) \cup \partial\Omega$. Then, u cannot be minimizing because the function $u_{\alpha,t}$ has less energy, where $u_{\alpha,t}$ is constructed in the following way. We choose $\alpha \ll 1$ and set $u_\alpha := \min(u, h_0 + \alpha)$. Then, u_α is not an element of $\mathcal{C}$ because its volume exceeds that of u by $V_\alpha = \int_{\{u_\alpha > u\}} u_\alpha - u \, \mathrm{d}x'$. Hence, we correct u_α as in part **(i)** of the proof, and then, $\mathcal{F}(u_{\alpha,t}) < \mathcal{F}(u)$ follows as before.

(v) If $u = h_0$ on $C(h_0) \cup \partial\Omega$, but not identically h_0 on Ω_0, we can use the fact that u is a minimizer with respect to variations that have compact support in Ω_0. Then, the adhesion energy is not affected, and we can therefore proceed as in [1], Lemma 5.2, (iv). □

Lemma 4.3 *Let $(c, R; u)$ be a solution of*

$$\mathcal{F}(c, R; u) \to \min \textit{ in } \mathcal{C}. \tag{20}$$

(i) *If $\kappa_0 \le 0$, and if $\mathcal{B}$ is not completely wetted, that is, the set $\{x' \in B(h_0) : u(x') = h(x')\}$ has positive measure, then u is regular: $u \in C^{2+\alpha}(\Omega \cap \{u > h\}) \cap C^{1+\alpha}(\overline{\Omega \cap \{u > h\}})$. Furthermore, u meets the body $\mathcal{B}$ under a constant angle ϑ with $\cos\vartheta = -\kappa_0 / \sigma$, and the boundary γ of $\Omega \cap \{u > h\}$ is locally a $C^{1+\alpha}$-curve.*

(ii) *If $|Du(x')| < \infty$ for all $x' \in \gamma$ then the set $\Omega \cap \{u > h\}$ is connected, or equivalently the non-wetted part of $\partial\mathcal{B}$ is simply connected.*

Proof The condition on the coincidence set implies that the capillary surface meets the body $\mathcal{B}$. Then, the boundary γ is locally a $C^{1+\alpha}$-curve according to the regularity theorem of Taylor [5]. This result is proved for any κ_0 with $|\kappa_0| < \sigma$ in the context of geometric measure theory, and hence the set E that is occupied by the fluid may be any Caccioppoli set E and not necessarily the subgraph $U := \{x \in G : \ h(x') < x_3 < u(x')\}$ of some function $u \in BV(\Omega)$.

We now restrict the adhesion coefficient κ_0 in such a way that the corresponding contact angle ϑ with $\cos\vartheta = -\kappa_0 / \sigma$ can be realized in the class of graphs. If we have a priori no knowledge about the curve Γ in which $\partial\mathcal{B}$ and Σ meet, the vertex of $\partial\mathcal{B}$ might be a point of Γ and consequently the normal to $\partial\mathcal{B}$ at that element of Γ points into the x_3-direction. Then, Σ can only be a graph if $\kappa_0 \le 0$ because then ϑ ranges between 0 and $\pi/2$. If it is known that the normal vectors in points of Γ make an angle φ with the x_3-direction that is larger or equal to some $\varphi_0 > 0$ then one can allow for a contact angle between 0 and $\frac{\pi}{2} + \varphi_0$, and consequently the lemma holds also for the corresponding positive values of κ_0.

If **(ii)** were not true, there would be an open set $M \subset\subset B(h_0)$ with $u(x') > h(x') \; \forall \, x' \in M$ and $u(x') = h(x') \; \forall \, x \in \partial M$. Then, there would exist at least one point $x_0' \in M$ such that

$$h\left(x_0'\right) \le h\left(x'\right) \;\; \forall \;\; x' \in \overline{M}. \tag{$*$}$$

Clearly x_0' must lie on the boundary ∂M, because for $x_0' \in M$ there would be some point $x_1' = x_0' - \varepsilon \, Dh(x_0')$ which lies in M for ε small enough and for which $h(x_1') < h(x_0')$ holds. This follows from the fact that h is strictly concave and therefore $Dh(x_0') \neq 0$. In such a point $x_0' \in \partial M \equiv \gamma$ the tangent vector τ to the curve $\Gamma = \operatorname{graph} h(\gamma)$ satisfies $\tau_3(x_0') = 0$. Let $x' = x'(t)$, $|t| < \varepsilon$, be a representation of the curve γ with $x'(0) = x_0'$. Then, $x(t) = \left(x'(t), h\left(x'(t)\right)\right)$, $|t| < \varepsilon$, describes Γ locally around $x_0 = \left(x_0', h(x_0')\right)$. Because of $(*)$ we have in particular

$$h\left(x'(0)\right) \le h\left(x'(t)\right) \forall \;\; t \in (-\varepsilon, \varepsilon),$$

hence

$$0 = \frac{d}{\mathrm{d}t} h\left(x'(t)\right)_{|t=0} = Dh\left(x_0'\right) \cdot \frac{d}{\mathrm{d}t} x'(t)_{|t=0}$$
$$\equiv \tau_\varepsilon(x_0).$$

Now let $n = n(x_0')$ be the direction perpendicular to γ and pointing into M. Then, $h_n(x_0') := Dh(x_0') \cdot n(x_0') > 0$ and because of $u_\tau(x_0') = h_\tau(x_0') = 0$ we have

$$u_n\left(x_0'\right) = -\frac{1}{h_n(x_0')} < 0.$$

This is, however, a contradiction to $u(x') > h(x')\ \ \forall\, x' \in M$, and the lemma is proved. □

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